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Class

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According to SLO Based
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MATHEMATICS

According to NEW NATIONAL CURRICULUM
for Session 2025-26

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UNIQUE NOTES, MATHEMATICS 9th

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Chairman's Message

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Our vision is clear and resolute: to inspire excellence, foster innovation, and cultivate a generation of well-rounded individuals who will not only achieve personal success but also contribute meaningfully to the nation's progress.

Together, we work toward the ideal of "Better Education, Better Pakistan" — a mission that defines us, drives us, and keeps us striving for greater heights.

Abdul Munim Khurram
Chairman UGI





A Word of Thanks

It is a moment of immense pride and satisfaction for me to declare that this book is not merely a helping guide but a testament to the **Unique Group of Institutions' (UGI)** unwavering commitment to empowering students to perform at their very best. This edition stands as a reflection of the tireless efforts and unparalleled dedication of our esteemed teachers, whose professional expertise, subject mastery, and student-centered approach have truly brought this work to life.

The exceptional diversity demonstrated by our educators in creating this edition reflects both their individual passion and professional commitment to their respective disciplines. Every effort has been made to ensure that the content is **accessible, comprehensive, and aligned with the desired learning objectives**, keeping in view the individual learning needs of students. Equal emphasis has been placed on formative and summative evaluations, ensuring that assessment and learning outcomes are both measurable and achievable.

This book has been meticulously designed with a focus on **clarity, brevity, and precision**, while its carefully curated content serves as a true "knowledge treasure" for students. Undoubtedly, **Unique Notes** are a proven key to success, empowering students to excel academically and achieve their dreams. I deeply appreciate the **untiring efforts** and profound commitment of our distinguished teachers who have worked diligently to make this edition possible. Their contributions reflect the very essence of **excellence, innovation, and purpose-driven education** that UGI stands for.

Prof. Anjani AS Khan
Rector UGI



Foreword

Unique Notes have been meticulously prepared in alignment with the recommendations of the **National Education Policy** and modern pedagogical requirements, ensuring that they meet the highest standards of contemporary education. The contents included within this material are carefully designed to foster not only academic excellence but also the **mental, personal, and intellectual** development of students. Each section has been crafted to spark critical thinking, enhance problem-solving skills, and cultivate a deep understanding of the subjects.

We firmly believe that achieving the dream of **educational advancement** demands continuous effort, innovation, and an unwavering commitment to excellence. With this realization, we are indefatigably striving to create resources that empower learners and inspire educators. In this pursuit, we warmly invite **valuable opinions and constructive suggestions** from scholars, esteemed educators, students, parents, and all stakeholders involved. Your feedback is a cornerstone of our progress and will enable us to continuously improve and evolve our mission to deliver the finest educational resources.

Together, let us join our hands and unite our efforts to envision and create a stronger, smarter, and more enlightened future for our students and our nation—a future that stands as a beacon of progress, knowledge, excellence, and inspiration for generations to come.

Prof. Muhammad Abdullah
Additional Director UGI



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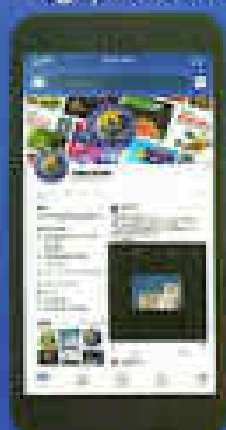


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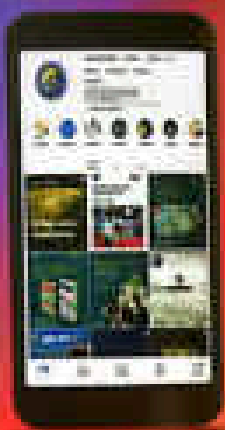


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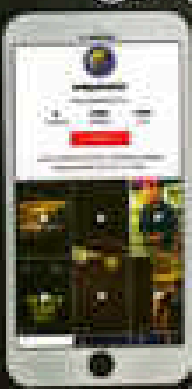


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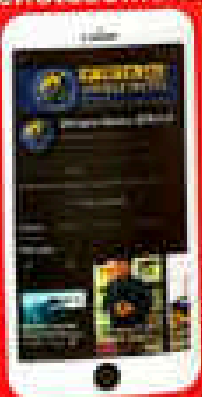
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UNIT 01

Real Numbers

Introduction to Real Numbers

Earlier mathematicians particularly Richard Dedekind have precisely defined the concept of real numbers which include both rational numbers such as $\frac{2}{3}$ and irrational numbers such as $\sqrt{2}$. The real numbers are used to solve real life problems such as finding velocity, speed or distance, the profit or loss of a business, the difference in stock market etc.

Q	Q'
Set of Rational Numbers	Set of Irrational Numbers

Key fact

Set of rational numbers (Q) and set of irrational numbers (Q') are disjoint i.e., $Q \cap Q' = \emptyset$. But $R = Q \cup Q'$, then Q and Q' are called exhaustive sets.



Properties of real numbers

(i) Closure property

N.C.T. (1)

If $a, b \in R$ then $a + b \in R$

e.g. If $4, 6 \in R$ then $4 + 6 = 10 \in R$

N.C.T. (2)

If $a, b \in R$ then $a \times b \in R$

e.g. If $4, 6 \in R$ then $4 \times 6 = 24 \in R$

(ii) Commutative property

N.C.T. (1)

If $a, b \in R$ then $a + b = b + a$

e.g. If $4, 7 \in R$ then $4 + 7 = 7 + 4$

$$11 = 11$$

N.C.T. (2)

If $a, b \in R$ then $a \times b = b \times a$

e.g. If $4, 7 \in R$ then $4 \times 7 = 7 \times 4$

$$28 = 28$$

(iii) Associative property

N.C.T. (1)

If $a, b, c \in R$ then $a + (b + c) = (a + b) + c$

e.g. If $4, 6, 8 \in R$ then $4 + (6 + 8) = (4 + 6) + 8$

$$4 + 14 = 10 + 8$$

$$18 = 18$$

N.C.T. (2)

If $a, b, c \in R$ then $a \times (b \times c) = (a \times b) \times c$

e.g. If $4, 6, 8 \in R$ then $4 \times (6 \times 8) = (4 \times 6) \times 8$

$$4 \times 48 = 24 \times 8$$

$$192 = 192$$

Note

Subtraction and division do not possess commutative and associative properties.

(iv) Identity

N.C.T. (1)

If $a \in R$ then there exist a number $0 \in R$

such that $a + 0 = a$ or $0 + a = a$

e.g. $0, 6 \in R$ then $0 + 6 = 6 + 0$

$$6 = 6$$



0 is additive identity of $\mathbb{R}$

THEOREM 1.2

If $a \in \mathbb{R}$ then there exist a number $1 \in \mathbb{R}$ such that $a \cdot 1 = a$ or $1 \cdot a = a$

e.g. If $1 \in \mathbb{R}$ then $1 \cdot 6 = 6 \cdot 1 = 6$

$$6 \cdot 6 = 36$$

1 is multiplicative identity of $\mathbb{R}$

(v) Inverse property

THEOREM 1.3

If $a \in \mathbb{R}$ then there exist a number $-a \in \mathbb{R}$ such that $a + (-a) = 0$ or $-a + a = 0$

e.g. If $6 \in \mathbb{R}$ then $6 + (-6) = -6 + 6 = 0$

THEOREM 1.4

If $a \in \mathbb{R}$, $a \neq 0$ then there exist a number

$$\frac{1}{a} \in \mathbb{R} \text{ such that } a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

$$\text{e.g. If } 6 \in \mathbb{R} \text{ then } 6 \cdot \frac{1}{6} = \frac{1}{6} \cdot 6 = 1$$

multiplicative inverse of 0 does not exist

(vi) Distributive property of (+)

THEOREM 1.5

If $a, b, c \in \mathbb{R}$ then $a(b+c) = ab+ac$

e.g. If $4, 6, 8 \in \mathbb{R}$ then $4(6+8) = 4 \cdot 6 + 4 \cdot 8$

$$4(14) = 56 + 32$$

$$56 = 56$$

THEOREM 1.6

If $a, b, c \in \mathbb{R}$ then $a(b-c) = ab-ac$

e.g. If $4, 6, 8 \in \mathbb{R}$ then $4(6-2) = 4 \cdot 6 - 4 \cdot 2$

$$4(4) = 16 - 8$$

$$8 = 8$$

Examples 1

If $a = 5$, $b = 6$ and $c = 9$, then verify that

$$a(b+c) = ab+ac$$

Answer

Solution:

$$\text{L.H.S.} = a(b+c) = 5(6+9) = 5(15) = 75$$

$$\text{R.H.S.} = ab+ac = (5 \cdot 6) + (5 \cdot 9) = 30 + 45 = 75$$

$$\text{Thus L.H.S.} = \text{R.H.S.} \therefore a(b+c) = ab+ac$$

Examples 2

Use the distributive property to simplify

$$32\left(\frac{1}{8} + \frac{1}{4}\right)$$

Answer

Solution

$$\text{Distributive property is } a(b+c) = ab+ac$$

$$32\left(\frac{1}{8} + \frac{1}{4}\right) = 32 \times \frac{1}{8} + 32 \times \frac{1}{4}$$

$$= 4 + 8 = 12$$

Properties of Equality and Inequality of Real numbers

There are three symbols which can be used to show the possible relations between any two real numbers 'a' and 'b'. These are '<', '>' and '=', where '=' is equality symbol (and '<' and '>' are inequality symbols).

Following table shows the use of these symbols.

Read	Write
a is less than b	$a < b$
a is equal to b	$a = b$
a is greater than b	$a > b$

If a and b real numbers, then only one of the following statement is true:

$$(i) a < b \quad (ii) a = b \quad (iii) a > b$$

This property is known as Trichotomy Property.

A mathematical statement with the equality sign is called an 'equality'. A mathematical statement in which we do not use the symbol of equality but other symbols like '<' or '>' at both, is called an 'inequality'.

Properties of equality of real numbers

Set	Properties
i	Reflexive $a = a, \quad \forall a \in \mathbb{R}$
ii	Symmetric If $a = b$ then $b = a, \quad \forall a, b \in \mathbb{R}$
iii	Transitive If $a = b$ and $b = c$ then $a = c$ $\forall a, b, c \in \mathbb{R}$
iv	Additive If $a = b$ then $a + c = b + c$ $\forall a, b, c \in \mathbb{R}$
v	Multiplicative $\forall a, b, c \in \mathbb{R}, a \neq 0, b \neq 0$



	If $a = b$ then $ac = bc$
vi	Cancellation w.r.t addition If $a + c = b + c$ then $a = b$
vii	Cancellation w.r.t multiplication If $ac = bc$ then $a = b$ where $c \neq 0$

Properties of inequality of real numbers:

Set	Properties
i	Trichotomy $\forall a, b \in \mathbb{R}$ either $a > b$ or $a = b$ or $a < b$
ii	Transitive $\forall a, b, c \in \mathbb{R}$ (i) If $a > b$ and $b > c$ then $a > c$ (ii) If $a < b$ and $b < c$ then $a < c$
iii	Additive $\forall a, b, c \in \mathbb{R}$ (i) If $a > b$ then $a + c > b + c$ (ii) If $a < b$ then $a + c < b + c$
iv	Multiplicative $\forall a, b, c \in \mathbb{R}, a > 0, b > 0$ (i) If $a > b$ then $ac > bc$ (ii) If $a < b$ then $ac < bc$
v	Multiplicative $\forall a, b, c \in \mathbb{R}, a < 0, b < 0$ (i) If $a > b$ then $ac < bc$ (ii) If $a < b$ then $ac > bc$ * The inequality sign is reversed if both sides of an inequality are multiplied by a negative number.
vi	Cancellation w.r.t addition (i) If $a + c > b + c$ then $a > b$ (ii) If $a + c < b + c$ then $a < b$
vii	Cancellation w.r.t multiplication $\forall a, b, c \in \mathbb{R}, a > 0, b > 0$ (i) If $ac > bc$ then $a > b$ (ii) If $ac < bc$ then $a < b$
viii	$\forall a, b, c \in \mathbb{R}, a < 0, b < 0$ (i) If $ac < bc$ then $a > b$ (ii) If $ac > bc$ then $a < b$
ix	Multiplicative inverse $\forall a, b \in \mathbb{R}$ where $a \neq 0, b \neq 0$ (i) If $a > b$ then $\frac{1}{a} < \frac{1}{b}$ (When a & b both have same sign)

$$(ii) \text{ If } a > b \text{ then } \frac{1}{a} < \frac{1}{b}$$

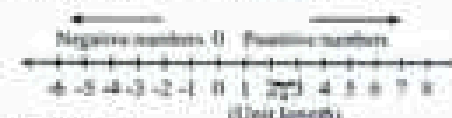
(When a & b both have different sign)

Note: Taking the reciprocal of both sides of an inequality reverse the direction of inequality if values have the same sign and maintain the direction if they have opposite signs.

Number Line

We use a number line to visualize real numbers and their relation to each other. To construct a number line, we choose a point corresponding

to the number '0'. Points at equally spaced intervals are then associated with the integers. The positive integers are to the right of '0' and negative integers are to the left of 0. All other real numbers are associated with a point which is called **coordinate of that point**. The point associated with zero is called the **origin**. Usually few integers are shown in a number line to indicate the unit length of the line, that is, the distance between any two consecutive integers. A number line is shown in figure below.



Examples 2

Show the following numbers on a number line:

(a) 4, (b) $\frac{7}{8}$, (c) $-\frac{1}{5}$

(d) $\sqrt{2}$, (e) $-\pi$

Solution:



(a) A is representing 4 on number line.

(b) $\frac{7}{8} = 0.875$ is between 0 and 1, which is a terminating decimal. Point B is the figure is representing $\frac{7}{8}$ on the number



- line.
- (ci) $-\frac{1}{3} = -0.333...$ or $-\frac{1}{3}$ is between 0 and -1, which is a repeating decimal. Point C in the figure is representing $-\frac{1}{3}$ on the number line.
- (di) Since $\sqrt{2} = 1.414213...$ is between 1 and 2. Point D in the figure is the location of $\sqrt{2}$.
- (ei) Since $-x = -1.14159...$ is between -3 and -4. Point E in the figure is the location of $-x$.

Example 4

Represent the following sets of real numbers on a number line.

- (a) $x \leq 2$ (b) $-6 < x < 4$
(c) $x \geq -4$ (d) $-3 \leq x < 1$

Solution: (a) $x \leq 2$ means
The inequality $x \leq 2$ specifies all real numbers less than or equal to 2.



A filled circle indicates that 2 is included in the set.

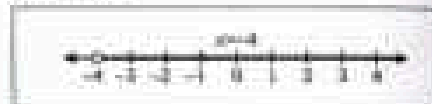
Solution: (b) $-6 < x < 4$ means
The inequality $-6 < x < 4$ specifies all

real numbers between -6 and 4, as shown on the number line.



We use hollow circle to indicate that both -6 and 4 are not included in the set.

Solution: (c) $x \geq -4$ means
Here, $x \geq -4$ specifies all real numbers greater than -4.



We use hollow circle to indicate that -4 is not included in the set.

Solution: (d) $-2 \leq x < 1$ means
 $-2 \leq x < 1$ is the set of all real numbers between -2 and including -2 but excluding 1.



A filled circle indicates that -2 is included but hollow circle shows that 1 is excluded.

Exercise 1.1

Q.1: Represent each number on the number line.

- (i) $\frac{3}{4}$ means (ii) $-\frac{1}{3}$ means
(iii) $4\frac{1}{2}$ means (iv) $-\sqrt{2}$ means
(v) $\sqrt{6}$ means (vi) $-4\frac{1}{2}$ means
(vii) $\frac{1}{3}$ means (viii) $-\frac{7}{8}$ means

Solution

(i) $\frac{3}{4}$

$\frac{3}{4} = 0.75$ is between 0 and 1, which is a terminating decimal. Point P in the figure is representing $\frac{3}{4}$ on the number line.



- (ii) $-\frac{1}{3}$
 $-\frac{1}{3} = -0.333...$ is between 0 and -1. Point P in the figure is representing $-\frac{1}{3}$ on the number line.



- (iii) $4\frac{1}{2}$
 $4\frac{1}{2} = 4.5$ is between 4 and 5. Point P in the figure is representing $4\frac{1}{2}$ on the number line.



- (iv) $-\sqrt{2}$
Since $-\sqrt{2} = -1.414213...$ is between -2 and -1. Point P in the figure is the location of $-\sqrt{2}$ on the number line.



- (v) $\sqrt{6}$
Since $\sqrt{6} = 2.449...$ is between 2 and 3. Point P in the figure is the location of $\sqrt{6}$ on the number line.



- (vi) $-4\frac{1}{2}$
 $-4\frac{1}{2} = -4.5$ is between -4 and -5. Point P in the figure is representing $-4\frac{1}{2}$ on the number line.



- (vii) $\frac{1}{3}$
 $\frac{1}{3} = 0.333...$ is between 0 and 1. Point P in the figure is representing $\frac{1}{3}$ on the number line.



- (viii) $-\frac{7}{8}$
 $-\frac{7}{8} = -0.875$ is between 0 and -1, which is a terminating decimal. Point P in the figure is representing $-\frac{7}{8}$ on the number line.



Q.2: Identify the property that justifies.

Solution:
1. $1 \times (y - 2) = y - 2$ means
Multiplicative identity



ii	$(8 \cdot 2) \cdot 3 = 1$ Multiplicative inverse	correct
iii	$(x + 2) + y = y + (x + 2)$ Commutative property w.r.t addition	correct
iv	$-(3x) + (3x) = 0$ Additive inverse	correct
v	$(x + 3) - 1 = x + (3 - 1)$ Associative property with respect to addition	correct
vi	$-3(x + y) = -3x - 3y$ Distributive property of multiplication over subtraction	correct

Q.2: Represent the following on the number line.

Solution:

- (i) $x = 0$ correct

- (ii) $-\frac{3}{2} < x < \frac{3}{2}$ correct

- (iii) $x \leq -8$ correct

- (iv) $x > 0$ correct

- (v) $x < -3$ correct

- (vi) $-4 < x \leq 8$ correct


Q.3: Identify the properties of equality and inequality of real numbers that justifies the statement.

Solution:

Ask	Questions and answers	
i.	$9x = 9x$	correct
Ans.	Reflexive property	
ii.	If $x + 2 = y$ and $y = 2x - 3$, then $x + 2 = 2x - 3$	correct
Ans.	Transitive property of equality	
iii.	If $2x + 3 = x$, then $x = 2x + 3$	correct
Ans.	Symmetric property	
iv.	If $3 < 4$, then $-3 > -4$	correct
Ans.	Multiplicative property of inequality	
v.	If $2p + 2 = q$ and $p = 50$, then $2x + 2 = 90$	correct
Ans.	Transitive property of equality	
vi.	If $x + 4 > y + 4$, then $x > y$	correct
Ans.	Subtraction property w.r.t addition	
vii.	If $2 < 3$ and $3 < 4$ then $2 < 4$	correct
Ans.	Transitive property of inequality	
viii.	If $-10 < -16$, then $9 > 3$	correct
Ans.	Cancellation property w.r.t multiplication	

Radical and Radicand

A square root of a positive number 'a' is another number 'm' whose square is 'a'. Any positive number has two square roots, which are additive inverses of each other.

For example, 4 is a square root of 16, because $(4)^2 = 16$ and -4 is also a square root of 16, because $(-4)^2 = 16$. Therefore, the two square roots of 16 are -4 and 4 , which are additive inverses of each other.

Principal Square Root

Positive square roots are called 'principal square roots'. e.g. $\sqrt{25} = 5$, $\sqrt{81} = 9$ etc. In expressions like $\sqrt{25}$ entire $\sqrt{25}$ is called a square radical or radical. The symbol $\sqrt{\quad}$ is called a radical sign and the number '25' under radical sign is called the radicand.



Definition of nth Root

For any real numbers 'a' and 'b' and any positive integer $n > 1$, if $a^n = b$, then $a = b^{\frac{1}{n}}$, where 'n' is the index of the radical.

For example, $9^2 = 81$, then $9 = 81^{\frac{1}{2}}$

Here 2 is the index of radical and 9 is the radicand.

Radical Form	Index of the Radical	Radicand
$\sqrt[3]{18}$	3	18
$\sqrt{\frac{22}{7}}$	2	$\frac{22}{7}$
$\sqrt[3]{-4xyz^2}$	3	$-4xyz^2$

Key-Point

- i. $\sqrt{a} = \sqrt[2]{a}$ i.e. $\sqrt{\quad}$ are equivalent.
 ii. There is no real number that is a square root of a negative number.
 e.g. $\sqrt{-16} = 4$, since $(+4)^2 = +16$. In Mathematics 'imaginary numbers' are defined to handle the square root of negative numbers.

Properties of radicals

For any $n \neq 1$ and for all real numbers 'a' and 'b' for which operations are defined.

- (a) Product rule for radicals: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
 (b) Quotient rule for radicals: $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
 (c) $\sqrt[n]{a^n} = a$ (d) $\sqrt[n]{a^n} = \sqrt[n]{a^n}$ (e) $\sqrt[n]{a^n} = (\sqrt[n]{a})^n$

Example 5

Use the product rule for radicals to simplify the following.

- (i) $\sqrt{2x} \cdot \sqrt{3x}$ (ii) $\sqrt{\frac{1}{x}} \cdot \sqrt{\frac{2}{x}}$ (iii) $\sqrt{5} \cdot \sqrt{5}$

Solution:

- (i) $\sqrt{2x} \cdot \sqrt{3x} = \sqrt{6x^2}$
 (ii) $\sqrt{\frac{1}{x}} \cdot \sqrt{\frac{2}{x}} = \sqrt{\frac{2}{x^2}}$
 (iii) $\sqrt{5} \cdot \sqrt{5} = 5$

- (iii) The product rule for radicals does not apply to $\sqrt{5} \cdot \sqrt{2}$, because the radicands are not same.

Simplifying the Index

If the index of the radical and the exponent of the radicand have a common factor, the expression can be written with a smaller index. We will explain it with the help of an example.

Consider $\sqrt[12]{9}$, here the index of the radical is 12 and exponent of radicand is 6. Also 6 is the common factor of 12 and 6, so we can write the expression with a smaller index as follows.

$$\sqrt[12]{9} = 9^{\frac{1}{12}} = 9^{\frac{1}{2 \cdot 6}} = \sqrt[2]{9} = 3$$

Example 7

Write the given expression with a smaller index. Assume that variable 'x' represents positive numbers $\sqrt[12]{x^6}$.

$$\text{Solution: } \sqrt[12]{x^6} = x^{\frac{6}{12}} = x^{\frac{1}{2}} = \sqrt{x}$$

Laws of Exponents / Radicals

We use exponent to indicate the repeated multiplication of the same factors. The exponent indicates that how many times a factor, called the base, occurs in the multiplication form.

e.g.

$$3 \cdot 3 \cdot 3 = 3^3$$

Exponent

Base

The expression 3^x is an exponential form read as 'three to the fourth power' where $3 \cdot 3 \cdot 3 \cdot 3$ is the factorial form.

Note: The words 'square' and 'cube' are sometimes used for exponents '2' and '3' respectively, rather than 'to the second power' and 'to the third power'.

Rational Exponents

When we perform operations with exponents, we have to define a zero exponent and a negative exponent. This may lead us to define a rational exponent.



Definition of $b^{\frac{1}{n}}$

If b is a real number and n is a positive integer, then $b^{\frac{1}{n}} = \sqrt[n]{b}$ e.g. $8^{\frac{1}{3}} = \sqrt[3]{8}$.

Definition of $b^{\frac{m}{n}}$

If m and n are positive integers with no common factor except 1 and $n \neq 0$, then

$$b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{b}\right)^m \text{ for all real numbers } b$$

$$\text{e.g. } 36^{\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^3.$$

The procedure for evaluating $b^{\frac{m}{n}}$ can be summarized as follows:

- Determine the n th root of b .
- Raise the result to the m power.

Examples 8

In the following table:

- Perform the operation to column A and compare the result to the value of radical in column B.

Column-A Exponential form	Column-B Radical form
$8^{\frac{1}{3}}$	$\sqrt[3]{8}$
$64^{\frac{1}{4}}$	$\sqrt[4]{64}$
$81^{\frac{1}{4}}$	$\sqrt[4]{81}$
$125^{\frac{1}{3}}$	$\sqrt[3]{125}$

- What do you observe about the denominator of the exponent and the index of the radical?

Solution

$$(i) 8^{\frac{1}{3}} = 2 \text{ and } \sqrt[3]{8} = 2$$

$$(ii) 64^{\frac{1}{4}} = (2^3)^{\frac{1}{4}} = 4 \text{ and } \sqrt[4]{64} = 4$$

$$(iii) 81^{\frac{1}{4}} = (3^2)^{\frac{1}{4}} = 3 \text{ and } \sqrt[4]{81} = 3$$

$$(iv) 125^{\frac{1}{3}} = 5 \text{ and } \sqrt[3]{125} = 5$$

- In each part, the denominator of the exponent is the same as the index of the

radical.

Examples 9: Simplify

$$(a) 8^{\frac{1}{3}} \quad (b) 36^{\frac{1}{2}} \text{ answer}$$

Solution

$$(a) 8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^1 = 2 \times 2 = 4$$

$$(b) 36^{\frac{1}{2}} = (6^2)^{\frac{1}{2}} = 6^{2 \times \frac{1}{2}} = 6^1 = 6$$

Negative Rational Exponent

For integral exponents, we define:

$$a^{-n} = \frac{1}{a^n} \text{ provided } a \neq 0 \text{ e.g. } 8^{-1} = \frac{1}{8}.$$

We can extend his definition to negative exponents.

If m and n are any two integers such that one of them is negative and they have no common factor other than 1 and $n \neq 0$, then

$$b^{\frac{m}{n}} = \frac{1}{b^{\frac{-m}{n}}} \text{ for all } b \neq 0, \text{ for which } b^{\frac{-m}{n}} \text{ is}$$

defined.

Examples 10: Simplify

$$(a) 16^{-\frac{3}{2}} \quad (b) \left(\frac{16}{25}\right)^{-\frac{3}{2}} \text{ answer}$$

Solution

$$(a) 16^{-\frac{3}{2}} = \frac{1}{(16)^{\frac{3}{2}}} = \frac{1}{(2^4)^{\frac{3}{2}}} = \frac{1}{2^6} = \frac{1}{64} \text{ answer}$$

$$(b) \left(\frac{16}{25}\right)^{-\frac{3}{2}} = \left(\frac{24}{16}\right)^{\frac{3}{2}} = \frac{\sqrt[3]{24}}{\sqrt[3]{16}} = \frac{\sqrt[3]{24}}{\sqrt[3]{2^4}} = \frac{2}{4} \text{ answer}$$

Examples 11

Write exponential expressions as an equivalent radical expression.

$$(a) (-7)^{\frac{3}{4}} \quad (b) (2)^{\frac{1}{3}}$$

Solution

$$(a) (-7)^{\frac{3}{4}} = (-7)^{3 \times \frac{1}{4}} = (-7)^{\frac{3}{4}} = \sqrt[4]{-7^3} \text{ answer}$$

$$(b) (2)^{\frac{1}{3}} = (2^1)^{\frac{1}{3}} = (2^{\frac{1}{3}}) = \sqrt[3]{2} \text{ answer}$$

Properties of exponents

	Name	Properties
i	Product rule	$a^m \times a^n = a^{m+n}$
ii	Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
iii	Power to power rule	$(a^m)^n = a^{m \times n}$
iv	Power of product rule	$(a \times b)^m = a^m \times b^m$
v	Power of quotient rule	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
vi	Negative exponent law	$a^{-m} = \frac{1}{a^m}$
vii	Negative power of a quotient rule	$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$
viii	Zero exponent law	$a^0 = 1$

Examples 12

Use the properties of exponents to evaluate each of the following.

$$(a) (2^3)^{\frac{1}{2}} \text{ answer} \quad (b) 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} \text{ answer}$$

Solution

$$(a) (2^3)^{\frac{1}{2}} = 2^{3 \times \frac{1}{2}} = 2^{\frac{3}{2}} = 2 \times 2 = 4$$

$$(b) 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = (2 \times 2)^{\frac{1}{2}} = (4)^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2$$

Examples 13

Use properties of exponents to simplify each of the following. Assume that all variables represent positive numbers. (Write all results with positive exponents).

Exercise 12

Q.1: By using the property of product and quotient rule for radicals, write each expression as a single radical and simplify.

Solution

$$(i) \sqrt[3]{8} \times \sqrt[3]{5} = \sqrt[3]{8 \times 5} = \sqrt[3]{40} = 16^{\frac{1}{3}} = 2 \text{ answer}$$

$$(ii) \sqrt[3]{4} \times \sqrt[3]{8} = \sqrt[3]{4 \times 8} = \sqrt[3]{32} = 2 \text{ answer}$$

$$(iii) \sqrt[3]{8} \times \sqrt[3]{2} = \sqrt[3]{8 \times 2} = \sqrt[3]{16} = 2 \text{ answer}$$

$$(a) a^{\frac{1}{2}}(a^{\frac{1}{2}} - a^{\frac{1}{2}}) \text{ answer} \quad (b) \left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right)^{\frac{1}{2}} \text{ answer}$$

Solution

$$(a) a^{\frac{1}{2}}(a^{\frac{1}{2}} - a^{\frac{1}{2}})$$

$$= a^{\frac{1}{2}} \times a^{\frac{1}{2}} - a^{\frac{1}{2}} \times a^{\frac{1}{2}} \text{ (distributive law)}$$

$$= a^{\frac{1}{2} + \frac{1}{2}} - a^{\frac{1}{2} + \frac{1}{2}} \text{ (} a^m \times a^n = a^{m+n} \text{)}$$

$$= a^1 - a^1$$

$$= a - a$$

$$= 0$$

$$= 0$$

$$= 0$$

$$= \frac{a^{\frac{1}{2}} \times a^{\frac{1}{2}} - 1}{a^{\frac{1}{2}}} \text{ (L.C.M.)}$$

$$= \frac{a^{\frac{1}{2} + \frac{1}{2}} - 1}{a^{\frac{1}{2}}} = \frac{a^1 - 1}{a^{\frac{1}{2}}} = \frac{a - 1}{a^{\frac{1}{2}}} = a^{\frac{1}{2}} \times a^{-\frac{1}{2}}$$

$$(b) \left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right)^{\frac{1}{2}} = \frac{(a^{\frac{1}{2}})^{\frac{1}{2}}}{(b^{\frac{1}{2}})^{\frac{1}{2}}} = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}$$



$$= \sqrt[4]{(2 \times 1000)}$$

$$= \sqrt[4]{21000}$$

$$(v) \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \sqrt[3]{\frac{x}{x}} = \sqrt[3]{x^{1-1}} = \sqrt[3]{x^0} = \sqrt[3]{1} = 1$$

$$(vi) \frac{\sqrt[3]{1000}}{\sqrt[3]{25}} = \sqrt[3]{\frac{1 \times 1000}{25}} = \sqrt[3]{\frac{1000}{25}} = \sqrt[3]{40} = 10$$

$$(vii) \frac{\sqrt[3]{1000}}{\sqrt[3]{25}} = \sqrt[3]{\frac{1 \times 1000}{25}} = \sqrt[3]{\frac{1000}{25}} = \sqrt[3]{40} = 10$$

$$(viii) \sqrt[3]{10} \cdot \sqrt[3]{7} = \sqrt[3]{10 \times 7} = \sqrt[3]{70} = (10)^{\frac{1}{3}} \cdot (7)^{\frac{1}{3}} = (10)^{\frac{1}{3}} \cdot (7)^{\frac{1}{3}} = (10 \times 7)^{\frac{1}{3}} = (70)^{\frac{1}{3}} = \sqrt[3]{70}$$

Q.2. Write each exponential expression as an equivalent radical expression and simplify if possible.

Solution

$$(i) (216)^{\frac{1}{3}} = \sqrt[3]{216} = \sqrt[3]{(2 \times 108)} = \sqrt[3]{2 \times (2 \times 27)} = \sqrt[3]{2 \times 2 \times 27} = \sqrt[3]{2^2 \times 3^3} = 2$$

$$(ii) (27)^{\frac{1}{3}} = \sqrt[3]{27} = 3$$

$$(iii) \left(\frac{1}{32}\right)^{\frac{1}{5}} = \sqrt[5]{\frac{1}{32}} = \sqrt[5]{\frac{1}{2^5}} = \frac{1}{2}$$

$$(iv) (216)^{-\frac{1}{3}} = \frac{1}{(216)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{216}} = \frac{1}{2}$$

$$= \frac{1}{\sqrt[3]{216}} = \frac{1}{2}$$

$$(v) (1000)^{\frac{1}{3}} = \sqrt[3]{1000} = \sqrt[3]{2^3 \times 5^3} = \sqrt[3]{2^3 \times 5^3} = 10$$

$$(vi) \left(\frac{1}{32}\right)^{\frac{1}{5}} = \left(\frac{1}{2^5}\right)^{\frac{1}{5}} = (2^{-5})^{\frac{1}{5}} = 2^{-1} = \frac{1}{2}$$

Q.3. Write each radical expression as an equivalent exponential expression and simplify if possible.

Solution

$$(i) (\sqrt[3]{5})^2 = (5^{\frac{1}{3}})^2 = 5^{\frac{2}{3}}$$

$$(ii) (\sqrt[3]{10})^2 = (10^{\frac{1}{3}})^2 = 10^{\frac{2}{3}} = 10^{\frac{2}{3}} = 10^{\frac{2}{3}}$$

$$(iii) -(\sqrt[3]{6})^2 = -(6^{\frac{1}{3}})^2 = -(6^{\frac{2}{3}}) = -(6^{\frac{2}{3}}) = -6^{\frac{2}{3}}$$

$$(iv) (\sqrt[3]{6})^2 = (6^{\frac{1}{3}})^2 = 6^{\frac{2}{3}} = 6^{\frac{2}{3}} = 6^{\frac{2}{3}}$$

$$(v) -(\sqrt[3]{5})^2 = -(5^{\frac{1}{3}})^2 = -5^{\frac{2}{3}}$$

$$(vi) -(\sqrt[3]{10})^2$$

$$= -(10^{\frac{1}{3}})^2 = -(10^{\frac{2}{3}})$$

$$= -(10^{\frac{2}{3}}) = -(10^{\frac{2}{3}}) = -10^{\frac{2}{3}}$$

Q.4. Use the properties of exponents to simplify each of the following. Assume that all variables represent positive numbers. (Write all results with positive exponents.)

Solution

$$(i) \frac{16^{\frac{1}{2}} \cdot 16^{\frac{1}{2}}}{16^{\frac{1}{2}}} = 16^{\frac{1}{2} + \frac{1}{2} - \frac{1}{2}} = 16^{\frac{1}{2}} = 4$$

Solution

$$\frac{16^{\frac{1}{2}} \cdot 16^{\frac{1}{2}}}{16^{\frac{1}{2}}} = \frac{16^{\frac{1}{2} + \frac{1}{2}}}{16^{\frac{1}{2}}} = \frac{16^1}{16^{\frac{1}{2}}} = 16^{\frac{1}{2}} = 4$$

$$= \frac{16^{\frac{1}{2} + \frac{1}{2}}}{16^{\frac{1}{2}}} = \frac{16^1}{16^{\frac{1}{2}}} = 16^{\frac{1}{2}} = 4$$

$$= 16^{\frac{1}{2} + \frac{1}{2} - \frac{1}{2}} = 16^{\frac{1}{2}} = 4$$

$$= 16^{\frac{1}{2}} = 4$$

$$= 16^{\frac{1}{2}} = 4$$

$$= (2^4)^{\frac{1}{2}} = 2^2 = 4$$

$$= 2^2 = 4$$

$$= 2^2 = 4$$

$$(ii) 7^{\frac{1}{2}} (7^{\frac{1}{2}} - 7^{\frac{1}{2}}) = 7^{\frac{1}{2}} \cdot 0 = 0$$

Solution

$$7^{\frac{1}{2}} (7^{\frac{1}{2}} - 7^{\frac{1}{2}}) = 7^{\frac{1}{2}} \cdot 0 = 0$$

$$= 7^{\frac{1}{2}} \cdot 0 = 0$$

$$= 7^{\frac{1}{2}} \cdot 0 = 0$$

$$= 7^{\frac{1}{2}} \cdot 0 = 0$$

$$= 7^{\frac{1}{2}} \cdot 0 = 0$$

$$= 7^{\frac{1}{2}} \cdot 0 = 0$$

$$= 7^{\frac{1}{2}} \cdot 0 = 0$$

$$(iii) \frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2} + \frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^1}{2^{\frac{1}{2}}} = 2^{\frac{1}{2}} = \sqrt{2}$$

Solution

$$\frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2} + \frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^1}{2^{\frac{1}{2}}} = 2^{\frac{1}{2}} = \sqrt{2}$$

$$= \frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2} + \frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^1}{2^{\frac{1}{2}}} = 2^{\frac{1}{2}} = \sqrt{2}$$

$$= 2^{\frac{1}{2}} = \sqrt{2}$$



$$(iv) \frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2} + \frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^1}{2^{\frac{1}{2}}} = 2^{\frac{1}{2}} = \sqrt{2}$$

Solution

$$\frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2} + \frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^1}{2^{\frac{1}{2}}} = 2^{\frac{1}{2}} = \sqrt{2}$$

$$(v) \left(\frac{36^{\frac{1}{2}} \cdot 4^{\frac{1}{2}}}{8^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

Solution

$$\left(\frac{36^{\frac{1}{2}} \cdot 4^{\frac{1}{2}}}{8^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}}\right)^{\frac{1}{2}} = \left(\frac{6 \cdot 2}{2 \cdot 3}\right)^{\frac{1}{2}} = \left(\frac{6}{3}\right)^{\frac{1}{2}} = 2^{\frac{1}{2}} = \sqrt{2}$$

$$= \frac{36^{\frac{1}{2}} \cdot 4^{\frac{1}{2}}}{8^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}} = \frac{(6^2)^{\frac{1}{2}} \cdot (2^2)^{\frac{1}{2}}}{(2^3)^{\frac{1}{2}} \cdot (3^3)^{\frac{1}{2}}} = \frac{6 \cdot 2}{2 \cdot 3} = \frac{6}{3} = 2$$

$$= \frac{(6^2)^{\frac{1}{2}} \cdot (4^2)^{\frac{1}{2}}}{(8^2)^{\frac{1}{2}} \cdot (27^2)^{\frac{1}{2}}} = \frac{6 \cdot 2}{2 \cdot 3} = \frac{6}{3} = 2$$

$$= \frac{(6^2)^{\frac{1}{2}} \cdot (4^2)^{\frac{1}{2}}}{(8^2)^{\frac{1}{2}} \cdot (27^2)^{\frac{1}{2}}} = \frac{6 \cdot 2}{2 \cdot 3} = \frac{6}{3} = 2$$

$$(vi) \left(\frac{2187 \cdot 4 \cdot 2^{\frac{1}{2}}}{8^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}}\right)^{\frac{1}{2}} = \left(\frac{3^7 \cdot 2^2 \cdot 2^{\frac{1}{2}}}{2^{\frac{1}{2}} \cdot 3^3}\right)^{\frac{1}{2}} = \left(\frac{3^4 \cdot 2^{\frac{3}{2}}}{2^{\frac{1}{2}} \cdot 3^3}\right)^{\frac{1}{2}} = \left(\frac{3 \cdot 2}{1}\right)^{\frac{1}{2}} = \sqrt{6}$$

$$= \left(\frac{2187 \cdot 4 \cdot 2^{\frac{1}{2}}}{8^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}}\right)^{\frac{1}{2}} = \left(\frac{3^7 \cdot 2^2 \cdot 2^{\frac{1}{2}}}{2^{\frac{1}{2}} \cdot 3^3}\right)^{\frac{1}{2}} = \left(\frac{3^4 \cdot 2^{\frac{3}{2}}}{2^{\frac{1}{2}} \cdot 3^3}\right)^{\frac{1}{2}} = \left(\frac{3 \cdot 2}{1}\right)^{\frac{1}{2}} = \sqrt{6}$$

$$\left(\frac{2187 \cdot 4 \cdot 2^{\frac{1}{2}}}{8^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}}\right)^{\frac{1}{2}} = \left(\frac{3^7 \cdot 2^2 \cdot 2^{\frac{1}{2}}}{2^{\frac{1}{2}} \cdot 3^3}\right)^{\frac{1}{2}} = \left(\frac{3^4 \cdot 2^{\frac{3}{2}}}{2^{\frac{1}{2}} \cdot 3^3}\right)^{\frac{1}{2}} = \left(\frac{3 \cdot 2}{1}\right)^{\frac{1}{2}} = \sqrt{6}$$



$$\begin{aligned}
 &= \left(\frac{a^m \cdot b^{n-1}}{a^{m-1} \cdot b^n} \right) \\
 &= \left(\frac{a^m \cdot b^{n-1}}{a^{m-1} \cdot b^n} \right) \\
 &= \frac{a^m \cdot b^{n-1}}{a^{m-1} \cdot b^n} \\
 &= \frac{a^m}{a^{m-1}} \cdot \frac{b^{n-1}}{b^n} \\
 &= a^{m-(m-1)} \cdot b^{(n-1)-n} \\
 &= a^1 \cdot b^{-1} \\
 &= a \cdot \frac{1}{b} \\
 &= \frac{a}{b}
 \end{aligned}$$

Solution

$$\begin{aligned}
 &\frac{a^m}{a^m} = \frac{a^m}{a^m} = \frac{a^m}{a^m} \\
 &= \frac{a^m}{a^m} = \frac{a^m}{a^m} = \frac{a^m}{a^m} \\
 &= \frac{a^m}{a^m} = \frac{a^m}{a^m} = \frac{a^m}{a^m}
 \end{aligned}$$

$$= \left(a^m \cdot b^{n-1} \cdot c^{p-1} \right)$$

$$= \left(a^m \cdot b^{n-1} \cdot c^{p-1} \right)$$

$$= \left(1 \times 1 \times 1 \right) \quad (\text{zero exponent law})$$

$$= (1)$$

$$= 1$$

Q.5: Use suitable laws of exponents to show that

$$\left(\frac{a}{b} \right)^m = \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m$$

Solution

$$\text{L.H.S.} = \left(\frac{a}{b} \right)^m = \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m \cdot \left(\frac{a}{b} \right)^m$$

$$= \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right) \cdot \left(a^m \cdot b^{n-1} \right)$$

$$= a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1}$$

$$= a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1} \cdot b^{n-1}$$

$$\text{L.H.S.} = \text{R.H.S.}$$

3	2187
3	729
3	243
3	81
3	27
3	9
3	3
3	1

Answer

Application of Real Numbers in daily life

Example 14: A cooking oil company produces four types of oils in packing of 1 litre, 5 litre & 10 litre. The inventory is shown in the table.

Answer

Name packing size	1 litre	5 litre	10 litre
Cooking oil-I	5000	2500	1000
Cooking oil-II	5000	2500	1000
Cooking oil-III	5000	2500	1000
Cooking oil-IV	5000	2500	1000

After removal of 40% of an item, it is to be replenished. The daily removal of 1 litre cooking oil-II packing is 20% and for 10 litre cooking oil-IV packing, daily removal is 2%. Find

(a) Number of daily removed 1 litre cooking oil-II packing.

(b) After how many day, 1 litre cooking oil-II are to be replenished?

(c) Number of daily removed 10 litre cooking oil-IV packing.

(d) After how many day, 10 litre cooking oil-IV are to be replenished?

Solution

Total 1 litre cooking oil-II packing in inventory = 5000

(a) Number of daily removed 1 litre cooking oil-II packing = 20% of 5000

$$= \frac{20}{100} \times 5000 = 20 \times 50 = 1000$$

(b) After 40% removal, replenished is to be made.

$$\text{Rem } 40\% \text{ of } 5000 = 2000$$

After two days the replenishment is due.

(c) Number of daily removed 10 litre cooking oil-IV packing = 2% of 100

$$= \frac{2}{100} \times 1000 = 20 \text{ litres}$$



(d) Rem. 40% of 1000 = 400 litre packs removed per day are 30.
Therefore

Number of days = $400 \div 30 = 13$
After 13 days the replenishment is due.

Exercise 1.5

Q.1: On his last bank statement, Qasim had a balance of Rs. 1,75,000 in his checking account. He wrote one cheque for Rs. 45,790 and another for Rs. 112,921. What is his current balance?

Answer

Solution

Balance amount = Rs. 1,75,000

Amount of one cheque = Rs. 45,790

Amount of other cheque = Rs. 112,921

Total amount of cheques

$$= \text{Rs. } 45,790 + 112,921$$

$$= \text{Rs. } 158,711$$

Current balance

$$= \text{Rs. } 1,75,000 - \text{Rs. } 158,711$$

$$= \text{Rs. } 16,289$$

Q.2: Last week Waqif drove 283.4 km on 16.2 litres of petrol. He says that he averaged about 1.75 km/litre. Is his answer reasonable? Explain.

Answer

Solution

Total distance = 283.4 km

Petrol consumed = 16.2 litres

$$\text{Average of fuel} = \frac{283.4}{16.2}$$

$$= 17.49 \text{ km/litre}$$

Hence answer of average of 1.75 km/litre is not reasonable.

Q.3: Salma bought 3.2 yard of fabric for a total price of Rs. 139.2. How much did the fabric cost per yard?

Answer

Solution

Total length of cloth = 3.2 yard

Total price = Rs. 139.2

Cost of 3.2 yard of cloth = Rs. 139.2

$$\text{Cost of 1 yard of cloth} = \text{Rs. } \frac{139.2}{3.2}$$

$$= \text{Rs. } 43.2$$

Q.4: Momina walks 3.5 km/h. She took a 12 h walk. How far did she walk?

Answer

Solution

Speed of Momina = 3.5 km/h

Time duration = 12h

Distance covered = ?

$$\boxed{\text{Distance} = \text{Speed} \times \text{time}} \quad \left(\because S = \frac{D}{T} \right)$$

$$\text{Distance} = 3.5 \times 12$$

$$\text{Distance} = 42 \text{ km}$$

Q.5: The hiking club went on a 7 day trip. Each day they hiked between 5.5 and 7.5 miles. It is reasonable to assume that climbing the days the club hiked.

Answer

a. Less than 35 miles

b. Between 35 and 55 miles

c. Exactly 35 miles

d. More than 55 miles

Solution

1 day hiking = between 5.5 and 7.5 miles

7 days hiking = between 7×5.5 and 7×7.5 miles

7 days hiking = between 38.5 and 52.5 miles.

Answer: Option (b) is correct

Q.6: For a class party the students council purchased 42 balloons at Rs. 1.85 each. What is the total amount the student council paid for the balloons?

Answer

Solution

No. of balloons purchased = 42

Price of each balloon = Rs. 1.85

Total price of balloons = Rs. 1.85×42

$$= \text{Rs. } 77.7$$

Q.7: A group of friends made 4-yard long rectangular banner. They paid Rs. 3.75 per yard for the fabric and Rs. 9 for the firm to go around the banner, 16-yard perimeter. What was the width of the banner?

Answer

Solution

Length of banner = L = 4 yard

Perimeter of banner = P = 16 yard

Width of banner = W = ?



Perimeter of rectangular banner is given as:

$$2(L + W) = P$$

$$2(4 + W) = 10$$

$$4 + W = \frac{10}{2}$$

$$4 + W = 5$$

$$W = 5 - 4$$

$$W = 1$$

Width of banner = 1 yard

Q.8: A shoe factory has an asset for Rs.

2,000,000 of which $\frac{3}{5}$ is the capital and rest

is the debt. Find the amount of capital and debt.

(Asset = capital + debt).

Solution

Asset of factory = Rs. 2,000,000

Capital = $\frac{3}{5}$ of asset

$$= \frac{3}{5} \times \text{Rs. } 2,000,000$$

$$= 3 \times \text{Rs. } 400,000$$

$$= \text{Rs. } 1,200,000$$

Debt = Asset - Capital

$$= \text{Rs. } (2,000,000 - 1,200,000)$$

$$= \text{Rs. } 800,000$$

Q.9: World lowest temperature in past 100 years was recorded to be -89.2°C at Vostok, Antarctica on July 21, 1983. Convert this temperature into Fahrenheit and Kelvin scales.

$$(F = \frac{9}{5} C + 32, \quad K = C + 273)$$

Solution

Temperature in $^\circ\text{C}$ = -89.2°C

In Fahrenheit

$$F = \frac{9}{5} C + 32$$

$$F = \frac{9}{5} (-89.2) + 32$$

$$F = -160.56 + 32$$

$$F = -128.56^\circ\text{F}$$

is Kelvin

$$K = ^\circ\text{C} + 273$$

$$K = -89.2 + 273$$

$$K = 183.8\text{ K}$$

Q.10: A company was nationalised by the government and for low quality production. If the company has 3 share holders, Farah, Maryam and Tehreem investing in the ratio of 1 : 2 : 3 and the amount of penalty is Rs. 456,868.97. Find the amount of penalty paid by each of 3 share holders.

Solution

Amount of penalty = Rs. 456,868.97

Ratio of share = 1 : 2 : 3

Sum of ratio = $1 + 2 + 3 = 6$

Farah's penalty amount

$$= \frac{1}{6} \times \text{Rs. } 456,868.97$$

$$= \text{Rs. } 76,144.83$$

Maryam's penalty amount

$$= \frac{2}{6} \times \text{Rs. } 456,868.97$$

$$= \text{Rs. } 152,289.66$$

Tehreem's penalty amount

$$= \frac{3}{6} \times \text{Rs. } 456,868.97$$

$$= \text{Rs. } 228,434.48$$

Multiple Choice Questions

Q.1: Encircle the correct option in the following:

1. $a(b + c + d)$ equals

(a) $a(b + c + d)$ (b) $a + b + c + d$

(c) $ab + ac + ad$ (d) $ab + bc + cd$

2. $x^2 \cdot x^3 = x^5$

(a) x^6

(b) x^4

(c) x^7

(d) $\frac{x^5}{x^2}$



3. $\sqrt{ab}$ is equal to:

(a) $\sqrt{ab}$

(b) $\sqrt{ab}$

(c) $\sqrt{ab}$

(d) $\sqrt{ab}$

4. Which number is self-multiplicative inverse?

(a) 3

(b) -3

(c) -1

(d) 0

5. If $x > 0$, then $\sqrt{x}$ is:

(a) real

(b) integer

(c) irrational

(d) rational

6. If $a + b = a$, what is value of b?

(a) 1

(b) -1

(c) a

(d) 0

7. If $ab = 1$, what is value of b?

(a) 1

(b) $\frac{1}{b}$

(c) $\frac{1}{a}$

(d) -1

8. According to reflexive property:

$$x^2 = 17 \Rightarrow 17 = x^2$$

$$(a) x^2 = 17$$

$$(b) x = 17$$

$$(c) x^2 = 17$$

$$(d) 17 = x^2$$

9. If $a \cdot b = a$, what is value of b?

$$(a) \frac{1}{a}$$

$$(b) 1$$

$$(c) a$$

$$(d) -1$$

10. If $a \cdot b = 1$, what is b called?

(a) multiplicative inverse of a

(b) additive identity

(c) multiplicative identity

(d) self-multiplicative inverse

11. Commutative property does not hold with respect to:

(a) addition

(b) multiplication

(c) subtraction

(d) both a and b

Additional MCQs

12. Which of the following sets have the closure property w.r.t. multiplication?

(a) $\{-1, -2\}$

(b) $\{0, -1\}$

(c) $\{0, 1\}$

(d) $\{1, \sqrt{2}\}$

13. $2(3+4) = 2 \times 3 + 2 \times 4$, here the property used is:

(a) commutative

(b) associative

(c) distributive

(d) closure

14. Which of the following represent the number greater than -3 but less than 6?

(a) $\{x : -3 < x < 6\}$

(b) $\{x : -3 \leq x \leq 6\}$

(c) $\{x : -2 < x < 6\}$

(d) $\{x : -2 \leq x \leq 6\}$

15. Name the property of real numbers used in

$$\left(\frac{1}{2}\right) \cdot 1 = 1$$

(a) Additive identity

(b) Additive inverse

(c) Multiplicative identity

(d) Multiplicative inverse

16. Which of the following is an integer but not a whole number?

(a) 0

(b) 3

(c) 2

(d) -6

17. Which of the following is an integer but not a natural number?

(a) 0

(b) 1

(c) 2

(d) 3

18. If $x < 0$ and $x < y$, then:

(a) $x < y$

(b) $x > y$

(c) $x < y$

(d) $x < y$

19. If $a, b \in \mathbb{R}$ then only one of $a < b$ or $a > b$ or $a = b$ holds is called... property.

(a) Trichotomy

(b) Transitive

(c) Additive

(d) Multiplicative

20. Non-terminating, non-recurring decimal represents... number.

(a) A natural

(b) A rational

(c) A irrational

(d) A prime



21. The union of the set of rational numbers and irrational numbers is known as set of _____ number. answer
 (a) Rational (b) Irrational
 (c) Real (d) Whole

22. For each prime number P , $\sqrt{P}$ is an _____ number. answer
 (a) Irrational (b) Rational
 (c) Real (d) Whole

23. Square roots of all positive non-square integers are: answer

- (a) Irrational (b) Rational
 (c) Real (d) Whole
 24. π is an _____ number. answer
 (a) Irrational (b) Rational
 (c) Real (d) None

25. $\forall$ subsets M of $a < b$ and $b < c$ then $a < c$ is _____ property. answer
 (a) Transitive (b) Trichotomy
 (c) Additive (d) Multiplicative

26. Name the property of real numbers used if $x = y$ then $y = x$. answer
 (a) Trichotomy (b) Transitive
 (c) Reflexive (d) Symmetric

27. Name the property of real numbers used in $a - (-a) = 0$. answer
 (a) Additive inverse
 (b) Multiplicative inverse
 (c) Additive identity
 (d) Multiplicative identity

28. $\sqrt{2}, \sqrt{3}$ is a _____ number. answer
 (a) Rational (b) Irrational
 (c) decimal (d) fractional

29. $\sqrt[3]{16} = \dots$ answer
 (a) $\sqrt[3]{2}$ (b) $-\sqrt[3]{2}$
 (c) $-\sqrt[3]{8}$ (d) $\sqrt[3]{8}$

30. $-8^{\frac{1}{2}} = \dots$ answer
 (a) $(-8)^{\frac{1}{2}}$ (b) $-\sqrt{8}$
 (c) $\sqrt{-8}$ (d) $\sqrt{-3}$

31. $\sqrt[3]{x^2} \cdot \sqrt[3]{x^2} = \dots$ answer
 (a) $\sqrt[3]{x^2}$ (b) $\sqrt[3]{x^4}$
 (c) x^2 (d) $\sqrt[3]{2x}$

32. $(25x^4)^{\frac{1}{2}} = \dots$ answer
 (a) $\frac{\sqrt{25}}{5}$ (b) $\frac{\sqrt{5}}{5}$
 (c) $\frac{\sqrt{5}}{5}$ (d) $\frac{\sqrt{5}}{5}$

33. Write $\sqrt[3]{x^3}$ in exponential form. answer
 (a) $x^{\frac{1}{3}}$ (b) $x^{\frac{1}{2}}$
 (c) $x^{\frac{1}{3}}$ (d) $x^{\frac{1}{2}}$

34. Write $4^{\frac{1}{2}}$ with radical sign. answer
 (a) $\sqrt[3]{4}$ (b) $\sqrt[3]{4}$
 (c) $\sqrt[3]{4}$ (d) $\sqrt[3]{4}$

35. In $\sqrt[3]{25}$ the radicand is: answer
 (a) 3 (b) $\frac{2}{3}$

- (c) 33 (d) $(33)^{\frac{1}{3}}$

36. $\left(\frac{25}{16}\right)^{\frac{1}{2}} = \dots$ answer
 (a) $\frac{5}{4}$ (b) $\frac{4}{5}$
 (c) $\frac{5}{4}$ (d) $\frac{4}{5}$

37. The additive inverse of $\sqrt{3}$ is: answer
 (a) $-\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$
 (c) $\sqrt{-3}$ (d) 3

38. If $n = 3$ and $16 = 2^n + 4^{n-2}$, then $2n - 2 = \dots$ answer
 (a) -4 (b) -2
 (c) 0 (d) 8



39. The simplified form of $(x^{\frac{1}{2}})^2 + x^{\frac{1}{2}} \cdot 4$ is: answer

- (a) $\sqrt{x}$ (b) x^2
 (c) $\frac{1}{x}$ (d) $\frac{1}{x}$

40. $\left(\frac{8}{x}\right)^{\frac{1}{2}} + \left(\frac{x}{8}\right)^{\frac{1}{2}} = \dots$ answer
 (a) $\left(\frac{8}{x}\right)^{\frac{1}{2}}$ (b) $\left(\frac{8}{x}\right)^{\frac{1}{2}}$
 (c) $\left(\frac{x}{8}\right)^{\frac{1}{2}}$ (d) $\left(\frac{x}{8}\right)^{\frac{1}{2}}$

Answer Key

1	b	2	d	3	d	4	c	5	a
6	d	7	c	8	c	9	b	10	a
11	c	12	c	13	c	14	c	15	d
16	d	17	a	18	b	19	a	20	c
21	c	22	a	23	a	24	a	25	a
26	d	27	a	28	a	29	d	30	b
31	c	32	a	33	c	34	a	35	c
36	b	37	a	38	a	39	d	40	a

Q.2: Represent each number on the number line.

- (i) $-5\frac{1}{3}$ answer (ii) $\frac{17}{3}$ answer
 (iii) $-2 < x < 4$ answer (iv) $x \geq 6$ answer

Solution
 (i) $-5\frac{1}{3}$ answer
 $-5\frac{1}{3} = -5.2$ is between -5 and -6 , which is a terminating decimal. Point P is the figure is representing $-5\frac{1}{3}$ on the number line.



- (ii) $\frac{17}{3} = 5\frac{2}{3}$ answer
 $\frac{17}{3} = 5\frac{2}{3}$ is between 5 and 6. Point P in the figure is representing $\frac{17}{3} = 5\frac{2}{3}$ on the number line.



Q.3: Write each exponential expression as an equivalent radical expression and simplify if possible. answer

- Solution
 (i) $(-2)^4 = \sqrt[4]{(-2)^4} = \sqrt{16}$
 (ii) $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$
 (iii) $(\sqrt{16})^2 = (\sqrt{2^4})^2 = (4)^2 = (2^2)^2 = 2^4 = 16$
 (iv) $(\sqrt{-8})^3 = (\sqrt[3]{(-2)^3})^3 = (-2)^3 = -8$
 (v) $(x^{-2})^{\frac{1}{2}} = \sqrt{x^{-2}} = \frac{1}{\sqrt{x^2}}$



$$= x^{-2} \cdot x^2 = x^{-2+2} = x^0 = \frac{1}{x^0}$$

Q.4: Use the properties of exponents to simplify each of the following.

(i) $\frac{(-2)^7 \cdot (-2)^{-5} \cdot (-2)}{(-2)^{-1}}$ answer

Solution

$$\frac{(-2)^7 \cdot (-2)^{-5} \cdot (-2)}{(-2)^{-1}} = \frac{(-2)^{7-5+1}}{(-2)^{-1}}$$

$$= \frac{(-2)^3}{(-2)^{-1}}$$

$$= (-2)^{3+1}$$

$$= (-2)^4$$

$$= 16$$

(ii) $\frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}}}{2^{\frac{1}{4}}} \div \frac{3 \cdot 3^{\frac{1}{2}}}{3^{\frac{1}{3}}}$ answer

Solution

$$\frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}}}{2^{\frac{1}{4}}} \div \frac{3 \cdot 3^{\frac{1}{2}}}{3^{\frac{1}{3}}}$$

Additional Questions

Q.5: Use the properties of exponents to simplify each of the following.

(i) $\frac{(ab)^{\frac{1}{2}}}{\left(\frac{1}{ab}\right)^{\frac{1}{3}}}$ answer

Solution

$$\frac{(ab)^{\frac{1}{2}}}{\left(\frac{1}{ab}\right)^{\frac{1}{3}}} = (ab)^{\frac{1}{2}} \div \left(\frac{1}{ab}\right)^{\frac{1}{3}}$$

$$= 2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}$$

$$= 2^{\frac{3}{2}+\frac{1}{2}} = 2^{\frac{4}{2}}$$

$$= 2^2 \times 2^{\frac{2-1}{2}}$$

$$= 2^2 \times 2^{\frac{1}{2}}$$

$$= 2^2 \times \sqrt{2}$$

$$= 2\sqrt{2}$$

Q.6: Determine whether each statement is true or false. If false, give an example of a number that shows the statement is true.

- Every rational number is an integer.
- Every real number is an irrational.
- Every irrational number is a real number.
- Every integer number is rational number.
- Every real number is either rational number or an irrational number.

Answers

(a)	F	(b)	F	(c)	T	(d)	T	(e)	T
-----	---	-----	---	-----	---	-----	---	-----	---

$$= (ab)^{\frac{1}{2}} \div \frac{(ab)^{\frac{1}{3}}}{1} = (ab)^{\frac{1}{2}-\frac{1}{3}}$$

$$= (ab)^{\frac{3-2}{6}} = a^{\frac{1}{6}} \cdot b^{\frac{1}{6}}$$

(ii) $\frac{(3xy)^{\frac{1}{2}} \cdot \sqrt{x^2 y^2}}{(2x^2 y^2)^{\frac{1}{3}}}$ answer



Solution

$$\frac{(3xy)^{\frac{1}{2}} \cdot \sqrt{x^2 y^2}}{(2x^2 y^2)^{\frac{1}{3}}}$$

$$= \frac{3^{\frac{1}{2}} x^{\frac{1}{2}} y^{\frac{1}{2}} \cdot (x^2 y^2)^{\frac{1}{2}}}{2^{\frac{1}{3}} x^{\frac{2}{3}} y^{\frac{2}{3}}}$$

$$= \frac{3^{\frac{1}{2}} x^{\frac{1}{2}} \cdot (x^2 y^2)^{\frac{1}{2}}}{4x^{\frac{2}{3}} y^{\frac{2}{3}}}$$

$$= \frac{3^{\frac{1}{2}} x^{\frac{1}{2}} \cdot x^1 y^1}{4x^{\frac{2}{3}} y^{\frac{2}{3}}}$$

$$= \frac{3^{\frac{1}{2}} x^{\frac{3}{2}} y^1}{4x^{\frac{2}{3}} y^{\frac{2}{3}}}$$

$$= \frac{3^{\frac{1}{2}} x^{\frac{3}{2}-\frac{2}{3}} y^{1-\frac{2}{3}}}{4}$$

$$= \frac{3^{\frac{1}{2}} x^{\frac{9-4}{6}} y^{\frac{3-2}{3}}}{4}$$

$$= \frac{3^{\frac{1}{2}} x^{\frac{5}{6}} y^{\frac{1}{3}}}{4}$$

(iii) $\frac{2^{m+1} 3^{m+1} 5^{m+1} 6^m}{6^m 10^{m+1} 15^m}$ answer

Solution

$$\frac{2^{m+1} 3^{m+1} 5^{m+1} 6^m}{6^m 10^{m+1} 15^m}$$

$$= \frac{2^{m+1} 3^{m+1} 5^{m+1} (2 \times 3)^m}{(2 \times 3)^m (2 \times 5)^{m+1} (3 \times 5)^m}$$

$$= \frac{2^{m+1} 3^{m+1} 5^{m+1} 2^m 3^m}{2^m \times 3^m \times 2^{m+1} \times 5^{m+1} \times 3^m \times 5^m}$$

$$= \frac{2^{m+1+1} 3^{m+1+1} 5^{m+1}}{2^{m+m+1} 3^{m+m+1} 5^{m+m+1}}$$

$$= \frac{2^{m+2} 3^{m+2} 5^{m+1}}{2^{2m+1} 3^{2m+1} 5^{2m+1}}$$

$$= \frac{2^1 3^1 5^1}{2^{2m+1-1} 3^{2m+1-1} 5^{2m+1-1}}$$

$$= \frac{2^1 3^1 5^1}{2^{2m} 3^{2m} 5^{2m}}$$

$$= \frac{2^1 3^1 5^1}{2^{2m} 3^{2m} 5^{2m}}$$

$$= \frac{1}{2 \times 3} \quad (m=0)$$

$$= \frac{1}{6}$$

(iv) $\left(\frac{x}{x^2}\right)^{-3} \div \left(\frac{x}{x^2}\right)^{-4} \div \left(\frac{x}{x^2}\right)^{-5}$ answer

Solution

$$\left(\frac{x}{x^2}\right)^{-3} \div \left(\frac{x}{x^2}\right)^{-4} \div \left(\frac{x}{x^2}\right)^{-5}$$

$$= (x^{1-2})^{-3} \div (x^{1-2})^{-4} \div (x^{1-2})^{-5}$$

$$= x^{-3} \div x^{-4} \div x^{-5}$$

$$= x^{-3+4+5}$$

$$= x^6$$

$$= 1$$

Q.6: Prove that $\left[\frac{a^2 + 6a^2 + 2^2}{2^2 \times (120)^2}\right]^{\frac{1}{2}} = 2$ answer

Solution

$$\text{L.H.S} = \left[\frac{a^2 + 6a^2 + 2^2}{2^2 \times (120)^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{(2^2)^2 + (2^2)^2 + 2^2}{(2^2)^2 \times (2^2)^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2^4 + 2^4 + 2^2}{2^4 \times 2^4}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2^4 + 2^4 + 2^2}{2^8}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2^4}{2^8}\right]^{\frac{1}{2}}$$



$$= (2^{1+2})^3$$

$$= (2^3)^3$$

$$= 2^{9^3}$$

$$L.H.S = R.H.S$$

Hence proved.

Q.7: Use laws of exponents to simplify.

$$\frac{(81)^{\frac{1}{2}} \cdot 3^4 + (3)^{4+2} \cdot (243)}{(9^3)(3^2)}$$

Solution:

$$\frac{(81)^{\frac{1}{2}} \cdot 3^4 + (3)^{4+2} \cdot (243)}{(9^3)(3^2)}$$

$$= \frac{(3^4)^{\frac{1}{2}} \cdot 3^4 + (3)^{6+2} \cdot (3^5)}{(3^2)^3(3^2)}$$

$$= \frac{(3^2) \cdot 3^4 + (3)^{12+5}}{(3^6)(3^2)}$$

$$= \frac{3^{2+4} + 3^{17}}{3^{6+2}}$$

$$= \frac{3^{6+17}}{3^{8+2}}$$

$$= 3^{6+17-8-2}$$

$$= 3^{13}$$

$$= 3^3 + 3^3$$

$$= 9 + 9$$

$$= 18$$

Q.8: Use laws of exponents to simplify.

$$(i) \left(\frac{32x^{-4}y^{-4}z}{625x^2y^2z^{-4}} \right)^{\frac{1}{5}}$$

Solution:

3	625
5	125
5	25
5	5
5	1

32	625
2	125
2	25
2	5
2	1

3	32
2	16
2	8
2	4
2	2
2	1

$$\left(\frac{32x^{-4}y^{-4}z}{625x^2y^2z^{-4}} \right)^{\frac{1}{5}}$$

$$= \left(\frac{2^5x^{-4}y^{-4}z^1}{5^4x^2y^2z^{-4}} \right)^{\frac{1}{5}}$$

$$= \left(\frac{2^5x^{-4}y^{-4}z^1}{5^4x^2y^2z^{-4}} \right)^{\frac{1}{5}}$$

$$= \left(\frac{2^{\frac{5}{5}}x^{\frac{-4}{5}}y^{\frac{-4}{5}}z^{\frac{1}{5}}}{5^{\frac{4}{5}}x^{\frac{2}{5}}y^{\frac{2}{5}}z^{\frac{-4}{5}}} \right)^{\frac{1}{5}}$$

$$= \frac{2^1x^{-\frac{4}{5}}y^{-\frac{4}{5}}z^{\frac{1}{5}}}{5^{\frac{4}{5}}x^{\frac{2}{5}}y^{\frac{2}{5}}z^{\frac{-4}{5}}}$$

$$= \frac{4x^2}{5^{\frac{4}{5}}x^{\frac{2}{5}}y^{\frac{2}{5}}z^{\frac{-4}{5}}} \quad \left(1 \div \frac{8}{5} = 1 \div \frac{2}{5} = 1 + \frac{3}{5} = \frac{8}{5} \right)$$

$$= \frac{4x^2}{5x^{\frac{2}{5}}y^{\frac{2}{5}}z^{\frac{-4}{5}}}$$

$$(ii) \frac{(243)^{-\frac{1}{3}}(32)^{-\frac{1}{5}}}{\sqrt{(196)^{-1}}}$$

Solution:

$$\frac{(243)^{-\frac{1}{3}}(32)^{-\frac{1}{5}}}{\sqrt{(196)^{-1}}}$$

$$= \frac{(243)^{-\frac{1}{3}}(32)^{-\frac{1}{5}}}{\sqrt{(196)^{-1}}}$$

$$= \frac{(3^5)^{-\frac{1}{3}}(2^5)^{-\frac{1}{5}}}{(3^2)^{-\frac{1}{2}}(2^2)^{-\frac{1}{2}}}$$

$$= \frac{3^{-\frac{5}{3}} \cdot 2^{-1}}{3^{-\frac{1}{2}} \cdot 2^{-1}}$$

$$= \frac{3^{-\frac{5}{3} + \frac{1}{2}} \cdot 2^{-1 + 1}}{3^{-\frac{1}{2} + \frac{1}{2}} \cdot 2^{-\frac{1}{2} + \frac{1}{2}}}$$



$$= \frac{2 \times 2^{2+1}}{2^{\frac{1}{2}}}$$

$$= \frac{2 \times 2^3}{2^{\frac{1}{2}}} \quad \left(2 \div \frac{10}{3} = 3 \div \frac{1}{3} = 3 + \frac{1}{3} \right)$$

$$= \frac{2 \times 8}{2^{\frac{1}{2}}}$$

$$= \frac{16}{2^{\frac{1}{2}}}$$

$$(iii) \frac{2^{\frac{1}{2}} \times (27)^{\frac{1}{3}} \times (60)^{\frac{1}{2}}}{(180)^{\frac{1}{2}} \times (4)^{\frac{1}{2}} \times (9)^{\frac{1}{2}}}$$

Solution:

$$\frac{2^{\frac{1}{2}} \times (27)^{\frac{1}{3}} \times (60)^{\frac{1}{2}}}{(180)^{\frac{1}{2}} \times (4)^{\frac{1}{2}} \times (9)^{\frac{1}{2}}}$$

$$= \frac{2^{\frac{1}{2}} \times (3^3)^{\frac{1}{3}} \times (2^2 \times 3 \times 5)^{\frac{1}{2}}}{(2^2 \times 3^2 \times 5)^{\frac{1}{2}} \times (2^2)^{\frac{1}{2}} \times (3^2)^{\frac{1}{2}}}$$

$$= \frac{2^{\frac{1}{2}} \times 3^1 \times 2^1 \times 3^{\frac{1}{2}} \times 5^{\frac{1}{2}}}{2^1 \times 3^1 \times 5^{\frac{1}{2}} \times 2^1 \times 3^1}$$

$$= \frac{2^{\frac{1}{2}+1+1} \times 3^{1+\frac{1}{2}+1} \times 5^{\frac{1}{2}}}{2^{1+1+1} \times 3^{1+1+1} \times 5^{\frac{1}{2}}}$$

$$= \frac{2^{\frac{3}{2}+2} \times 3^{\frac{3}{2}+2} \times 5^{\frac{1}{2}}}{2^{3+2} \times 3^{3+2} \times 5^{\frac{1}{2}}}$$

$$= \frac{2^{\frac{3}{2}+2-3-2} \times 3^{\frac{3}{2}+2-3-2} \times 5^{\frac{1}{2}-\frac{1}{2}}}{2^{\frac{3}{2}+2-3-2} \times 3^{\frac{3}{2}+2-3-2} \times 5^{\frac{1}{2}-\frac{1}{2}}}$$

$$= 1$$

$$(iv) \sqrt{\frac{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}}}{(.04)^{\frac{1}{2}}}}$$

$$= \sqrt{\frac{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}}}{(.04)^{\frac{1}{2}}}}$$

Solution:

$$\sqrt{\frac{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}}}{(.04)^{\frac{1}{2}}}}$$

$$= \sqrt{\frac{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}} \times (.04)^{\frac{1}{2}}}{(.04)^{\frac{1}{2}}}}$$

$$= \sqrt{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}} \times \left(\frac{.04}{100} \right)^{\frac{1}{2}}}$$

$$= \sqrt{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}} \times \left(\frac{2^2}{10^2} \right)^{\frac{1}{2}}}$$

$$= \sqrt{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}} \times \left(\frac{2}{10} \right)^{\frac{1}{2}}}$$

$$= \sqrt{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}} \times \left(\frac{1}{5} \right)^{\frac{1}{2}}}$$

$$= \sqrt{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}} \times \left(\frac{1}{5} \right)^{\frac{1}{2}}}$$

$$= \sqrt{(216)^{\frac{1}{3}} \times (25)^{\frac{1}{2}} \times \left(\frac{1}{5} \right)^{\frac{1}{2}}}$$

$$(v) \sqrt{\frac{2^2}{3^2}} \times \sqrt{\frac{2^2}{3^2}} \times \sqrt{\frac{2^2}{3^2}}$$

$$\sqrt{\frac{2^2}{3^2}} \times \sqrt{\frac{2^2}{3^2}} \times \sqrt{\frac{2^2}{3^2}}$$

$$= \sqrt{\frac{2^2}{3^2} \times \frac{2^2}{3^2} \times \frac{2^2}{3^2}}$$

$$= \sqrt{(2^2)^3 \times (3^2)^3}$$

$$= \sqrt{(2^6 \times 3^6)}$$

$$= (2^3 \times 3^3)^{\frac{1}{2}}$$



$$\begin{aligned} \left(\frac{x^2}{y^3}\right)^4 &= \frac{x^{2 \times 4}}{y^{3 \times 4}} = \frac{x^8}{y^{12}} \\ \frac{x^{2 \times 4}}{y^{3 \times 4}} &= \frac{x^8}{y^{12}} \\ \frac{x^8}{y^{12}} &= \frac{x^8}{y^{12}} \end{aligned}$$

Since bases are equal so exponents are also equal.

$$2x - y = 2 \quad \text{--- (i)}$$

$$x + y - 9 = 4$$

$$x + y = 4 + 9$$

$$x + y = 13 \quad \text{--- (ii)}$$

Adding eq. (i) and (ii) we get

$$2x - y + x + y = 2 + 13$$

$$3x = 15$$

$$x = \frac{15}{3} = \boxed{x = 5}$$

Put $x = 5$ in eq. (i)

$$5 + y = 13$$

$$y = 13 - 5$$

$$\boxed{y = 8}$$

Q.15: Find the value of x

$$(i) 2^{2x-1} = 128$$

EXERCISE

Solution

$$2^{2x-1} = 128$$

$$2^{2x-1} = 2^7$$

Since bases are equal so exponents are also equal.

$$\Rightarrow 2x - 1 = 7$$

$$\Rightarrow 2x = 7 + 1$$

$$\Rightarrow 2x = 8$$

$$\Rightarrow x = \frac{8}{2} = \boxed{4}$$

$$(ii) 5^{x-2} = \frac{1}{4}$$

Solution

$$5^{x-2} = \frac{1}{4}$$

$$\left(2^{\frac{1}{2}}\right)^{x-2} = \frac{1}{2}$$

$$2^{\frac{x-2}{2}} = 2^{-1}$$

As bases are equal so exponents are also equal.

$$\Rightarrow -\frac{x-2}{2} = -1$$

$$\Rightarrow -x + 2 = -2 + 2$$

$$\Rightarrow -x = -4$$

$$\Rightarrow x = 4 \Rightarrow \boxed{x = \frac{14}{5}}$$

Q.16: Solve the following expression for y

$$\frac{16^{3x-1}}{6^{2x}} = \frac{6^{2x-1}}{216^{x-1}}$$

EXERCISE

Solution

$$\frac{16^{3x-1}}{6^{2x}} = \frac{6^{2x-1}}{216^{x-1}}$$

$$\left(\frac{2^4}{3}\right)^{3x-1} = \frac{6^{2x-1}}{(2 \cdot 3^3)^{x-1}}$$

$$\frac{2^{4(3x-1)}}{3^{3x-1}} = \frac{2^{2x-1} \cdot 3^{2x-1}}{2^{x-1} \cdot 3^{3(x-1)}}$$

$$\frac{2^{12x-4}}{3^{3x-1}} = \frac{2^{2x-1} \cdot 3^{2x-1}}{2^{x-1} \cdot 3^{3x-3}}$$

$$2^{12x-4} = 2^{2x-1} \cdot 3^{2x-1} \cdot 2^{1-3x}$$

$$2^{12x-4} = 2^{2x-1} \cdot 3^{2x-1} \cdot 2^{1-3x}$$

As bases are equal so exponents are also equal.

$$\Rightarrow x - 10 = -x - 19$$

$$\Rightarrow x + x = -19 + 10$$

$$\Rightarrow 2x = -9$$

$$\Rightarrow x = \frac{-9}{2}$$



UNIT 02

Logarithms

Scientific notation

A number 'x' is said to be in scientific notation if it is written as $x = d \times 10^n$ where $1 \leq d < 10$ and $n \in \mathbb{Z}$.

For example: 3.3×10^5 and 3.4×10^{-2} are written in scientific notation.

Writing the number in scientific notation:

i	Identify the place of the decimal point. It is at the extreme right of the number if it is not mentioned.
ii	Identify the reference position. It is always a position after the 1 st non-zero digit on the left side of the number.
iii	Move the decimal point to the reference position and count the number of digits moved. This is the absolute value of n .
iv	If the decimal point is moved to the left, the sign of n is positive.
v	If the decimal point is moved to the right, the sign of n is negative.

Example 1: Convert the following into scientific notation

Solution

(a) One light year: 9,460,800,000,000 miles

$$9,460,800,000,000 = 9.46 \times 10^{12}$$

- The decimal point is at extreme right.
- The reference position is after 9.
- The decimal point is moved 12 digits to the left so power of 10 is +12.

(b) Mass of smallest insect = 0.00000492 g

$$0.00000492 = 4.92 \times 10^{-6} \text{ g}$$

- The decimal point is at extreme right.
- The reference position is after 4.

- The decimal point is moved 6 digits to the right, so power of 10 is +6.

Note:

If the number is greater than 10 then the exponent of 10 is positive.

If the number is $1 \leq \text{Number} < 10$ then exponent of 10 is zero.

If the number is less than 1 then exponent of 10 is negative.

Standard Notation

The number already written in scientific notation, can be converted to standard notation by the multiplication of its two factors.

Example 2: Convert the following into standard notation

(a) Density of hydrogen = $8.99 \times 10^{-5} \text{ g/cm}^3$

$$\begin{aligned} 8.99 \times 10^{-5} &= 0.0000899 \times 10^{-5} \\ &= 0.0000899 \times 10^{-5} \\ &= 0.0000899 \times 10^{-5} \\ &= 0.0000899 \times 10^{-5} \\ &= 0.0000899 \times 10^{-5} \end{aligned}$$

Note: Exponent is negative so the number is smaller than 1.

(b) Number of air sacs in lungs = 3.5×10^7

$$\begin{aligned} 3.5 \times 10^7 &= 3,500,000,000 \times 10^7 \\ &= 3,500,000,000 \times 10^7 \\ &= 3,500,000,000 \times 10^7 \\ &= 3,500,000,000 \times 10^7 \end{aligned}$$



$$= 350000000 \times 1 \quad (10^0 = 1)$$

$$= 350000000$$

Note: Exponent is positive as the number is greater than 10.

Example 3: The closest star to the Earth (other than Sun) is Alpha Centauri, 4.24 light years from Earth. How many kilometers from Earth is Alpha Centauri? If one light year = 9460820 million km. Write the answer in scientific notation.

Solution:
One light year = $9,460,820 \times 10^6$ km
Distance between Earth and Alpha Centauri
= 4.24 light years
= $4.24 \times 9,460,820 \times 10^6$ km
= $41,133,962 \times 10^6$ km
= $4.1133962 \times 10^7 \times 10^6$ km
= 4.1133962×10^{13} km

Example 4: The speed of light is approximately 3×10^8 km/s and distance

between earth and sun is approximately 1.5×10^8 km. If the sun is suddenly to burn out, how long would it take for earthlings to know about it? Write the answer in standard notation.

Solution:
Formula for finding time, if the speed and distance are given, is
Time = Distance/Speed
Here, speed of light = 3×10^8 km/s
Distance between Earth and Sun = 1.5×10^8 km.

$$\text{Time} = \frac{1.5 \times 10^8 \text{ km}}{3 \times 10^8 \text{ km/s}}$$

$$\text{Time} = 0.5 \times 10^0 \text{ sec}$$

$$\text{Time} = 0.5 \times 10^1 \text{ sec}$$

$$\text{Time} = 0.5 \times 1000 \text{ sec}$$

$$\text{Time} = 500 \text{ sec}$$

Or
Time = 8 min 20 sec

$$\begin{array}{|l} 400 + 20 \text{ sec} \\ 420 \\ \hline 80 \end{array} \text{ min} = 20 \text{ min}$$

Exercise 2-1

Q.1: Write the following in scientific notation.

- Solution**
- i. 0.00053407 answer
In Scientific Notation
 $0.00053407 = 5.3407 \times 10^{-4}$
Move decimal point 4 places to the right
- ii. 33490000 answer
In Scientific Notation
 $33490000 = 3.349 \times 10^7$
Move decimal point 7 places to the left
- iii. 0.000000000012 answer
In Scientific Notation
 $0.000000000012 = 1.2 \times 10^{-11}$
Move decimal point 11 places to the right
- iv. 2.5326 answer
In Scientific Notation
 $2.5326 = 2.5326 \times 10^0$
No need to move decimal point

Q.2: Write the following in the standard notation.

- (i) 5.067×10^4 answer
Solution
 5.067×10^4
= $0.00000007 \times 10^7 \times 10^7$
= $0.00000007 \times 10^{14}$
= 0.00000007×10^8
= 0.00000007×10^8
= 0.00000007
= 0.00000007
- (ii) 5.64×10^2 answer
Solution
 $5.64 \times 10^2 = 5.64 \times 10^2$
(iii) 6.53×10^{-4} answer
Solution
 6.53×10^{-4}
= $0.00000653 \times 10^5 \times 10^{-9}$
= $0.00000653 \times 10^{-4}$
= $0.00000653 \times 10^{-4}$



= 0.00000653×1
= 0.00000653
(iv) 3.1415×10^2 answer
Solution
 3.1415×10^2
= $3141500000.0 \times 10^{-6} \times 10^2$
= $3141500000.0 \times 10^{-4}$
= 3141500000×10^2
= 3141500000×1
= 3141500000
Q.3: Simplify the following by converting into the form indicated.

Solution
(i) $865.71 \times 10^{-4} \times 2.54 \times 10^2$ answer
Solution

$$865.71 \times 10^{-4} \times 2.54 \times 10^2$$

$$= 8.6571 \times 10^2 \times 10^{-4} \times 2.54 \times 10^2$$

$$= 8.6571 \times 2.54 \times 10^{-2}$$

$$= 14.313234 \times 10^2$$

$$= 1.4313234 \times 10^2 \times 10^2$$

$$= 1.4313234 \times 10^4$$

(ii) $\frac{0.023 \times 10^4}{10^{-4}}$ Standard notation answer
Solution

$$\frac{0.023 \times 10^4}{10^{-4}}$$

$$= \frac{23}{1000} \times 10^{-4} = \frac{23}{10^3} \times 10^4 = 23 \times 10^{4-3}$$

$$= 23 \times 10^1 = 23 \times 100000 = 2300000$$

(iii) $\frac{2.549 \times 5067 \times 10^{-2}}{10^2}$ Scientific notation answer
Solution

$$\frac{2.549 \times 5067 \times 10^{-2}}{10^2}$$

$$= 2.549 \times 5.067 \times 10^2 \times 10^{-2} \times 10^{-2}$$

$$= 12.913783 \times 10^{2-2-2}$$

$$= 1.2913783 \times 10^{-2}$$

(iv) 0.0000000×10^8 Standard notation answer
Solution

0.0000000×10^8
= $9999999 \times 10^{-7} \times 10^8$
= $9999999 \times 10^{-7+8}$
= 9999999×10^1
= 9999999×1 ($\because 10^1 = 1$)
= 9999999
Q.4: If it takes 5 seconds to write 'Kalma Pak' once, how many hours will it take to write 'Kalma Pak' one million times. Convert hours into days and write the answer in standard form. Round off the answer, discarding the decimal part.
Solution
One million = $1,000,000 = 10^6$
Time taken to write one time = 5 sec.
Time taken to write one million = 5×10^6 sec.
= $5 \times 1,000,000$ sec.
= $5,000,000$ sec.
 $\because 1 \text{ hr} = 3600 \text{ s}$

$$\text{Time in hours} = \frac{5,000,000}{3600} = 1389 \text{ hours (app.)}$$

$$\text{Time in days} = \frac{1389}{24} = 58 \text{ days (app.)}$$

Q.5: The distance between Earth and Sun is 9.3225600×10^7 miles. If speed of light is approximately 1.86×10^8 miles per second, how long does it take for light to reach the Earth. Convert the answer in minutes writing in standard form.
Solution

Distance of Earth from Sun = 9.32256×10^7 miles
Speed of light = $c = 1.86 \times 10^8$ miles/s
Time = $t = ?$
We know that
 $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$
Time = $\frac{\text{Distance}}{\text{Speed}}$



Putting the values

$$t = \frac{9.33234 \times 10^7}{1.88 \times 10^8}$$

$$t = 5.0(\times 10^{-1})$$

$$t = 5.0(\times 10^2 \text{ sec.})$$

$$t = 5.0(\times 100 \text{ sec.})$$

$$t = 500 \text{ sec.}$$

$$(\times 60 \text{ sec.} = 1 \text{ min.})$$

$$t = \frac{500}{60} = 8.33 \text{ min.}$$

Definition of Logarithm

If $b^y = x$ where $x, y, b \in \mathbb{R}; b > 0, x > 0$ and $b \neq 1$, then y is the logarithm of x with base b , written as $y = \log_b x$ or $b^y = x$.

While evaluating logarithms, remember that a logarithm is an exponent, e.g. if $\log_9 81 = 2$, then 2 is the logarithm of 81 with base 9, since 9 raised to power 2 gives 81.

Example 5: Convert the following exponential equations to logarithmic equations and the logarithmic equations to exponential equations.

Solution:

(a) $2^7 = 128$,

Here base = 2, exponent = 7 and $x = 128$.



(b) $7^{-3} = \frac{1}{343} \Rightarrow \log_{7^{\frac{1}{343}}} = -3$

(c) $125^{\frac{1}{3}} = 5 \Rightarrow \log_{125^{\frac{1}{3}}} 5 = \frac{1}{3}$



(d) $\log_{\frac{1}{64}} \frac{1}{4} = -5 \Rightarrow 2^{-5} = \frac{1}{64}$

(e) $\log_{\frac{1}{3}} \frac{1}{4} = -\frac{1}{4} \Rightarrow (34)^{\frac{1}{3}} = \frac{1}{3}$

Key Fact

• $\log_b x$ is defined only for positive x .

• $\log_b 1 = 0 \quad \because b^0 = 1$

• $\log_b b = 1 \quad \because b^1 = b$

• $\log_b a^x = x \cdot \log_b a = x'$

• $\log_b a = \log_a a \Rightarrow a = a$

Example 6: Check whether these logs are defined or not?

(a) $\log_2 2 = y \Rightarrow 2^y = 2$

None of the exponents of 2 can give answer 2, so $\log_2 2$ is undefined.

(b) $\log_2 (-1) = y \Rightarrow 2^y = -1$

None of the exponents of 2 can give answer -1 , so $\log_2 (-1)$ is undefined.

Note: log of negative number is not defined.

(c) $\log_2 0 = y \Rightarrow 2^y = 0$

None of the exponent of 2, can give answer 0, so $\log_2 0$ is not defined.

Note: log of zero with any base is not defined.

(d) Is $\log_2 (4 - 2x) = y$ true or not for

$x = 0, 1, 2$.

$\log_2 (4 - 2x) = y \Rightarrow 2^y = 4 - 2x$

If $x = 0$, then $2^y = 4$ is true for $y = 2$.

If $x = 1$, then $2^y = 2$ is true for $y = 1$.

If $x = 2$, then $2^y = 0$ is not true for any value of y .

Example 7: Find the value of unknowns by converting logarithmic form to exponential form.

(a) $\log_a x = 4$

Solution

$\log_a x = 4$

$\Rightarrow a^4 = x$

$\Rightarrow x = a^4$

$\Rightarrow x = 2 \times 2 \times 2 \times 2$

$\Rightarrow x = 16$

(b) $\log_{16} x = \frac{-4}{3}$

Solution

$\log_{16} x = \frac{-4}{3}$

$\Rightarrow (64)^{\frac{1}{3}} = x$

$\Rightarrow (64)^{\frac{1}{3}} = x$

$\Rightarrow (64)^{\frac{1}{3}} = x$

$\Rightarrow (64)^{\frac{1}{3}} = x$

$\Rightarrow (64)^{\frac{1}{3}} = x$

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$\Rightarrow (64)^{\frac{1}{3}} = x$

$\Rightarrow (64)^{\frac{1}{3}} = x$

$\Rightarrow (64)^{\frac{1}{3}} = x$



$\Rightarrow x = (4^{\frac{1}{3}})^{-3}$

$\Rightarrow x = 4^{-1} = \frac{1}{4} = \frac{1}{2^2} = \frac{1}{2^2}$

(c) $\log_{\frac{1}{128}} = -7$

Solution

$\Rightarrow b^{-7} = \frac{1}{128}$

$b^{-7} = \frac{1}{128}$

$b^{-7} = \frac{1}{128}$

$b^{-7} = \frac{1}{128}$

$b^{-7} = \frac{1}{128}$

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$b^{-7} = \frac{1}{128}$

$b^{-7} = \frac{1}{128}$

$b^{-7} = \frac{1}{128}$

$\Rightarrow 25^y = 1$

$\Rightarrow (3^y)^2 = 1$

$\Rightarrow 3^y = 1$

$\Rightarrow 3^y = 1$

As bases are same so exponents must be same, i.e. $3y = 1$

$\Rightarrow y = \frac{1}{3}$

Example 8: Find y if $\log_9 (y^2 + 1) = \log_9 28$

Solution

$\log_9 (y^2 + 1) = \log_9 28$

As bases of log are same so numbers must be same.

$\Rightarrow y^2 + 1 = 28$

$\Rightarrow y^2 = 28 - 1$

$\Rightarrow y^2 = 27$

Taking cube root of both sides

$\sqrt[3]{y^2} = \sqrt[3]{27}$

$y = 3$

$y = 3$

$y = 3$

$y = 3$

$y = 3$

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Exercise 2.2

Q.1: Check whether $\log_2 (7 - x)$ is defined for:

(i) $x = 0$ (ii) $x = 1$ (iii) $x = 6$ (iv) $x \geq 7$

Solution

(i) For $x = 0$

If $x = 0$ then $\log_2 (7 - 0) = \log_2 7$

None of the exponent of 0 can give 7 so $\log_2 (7 - x)$ is undefined for $x = 0$.

(ii) For $x = 1$

If $x = 1$ then $\log_2 (7 - 1) = \log_2 6$

None of the exponent of 1 can give 6 so $\log_2 (7 - x)$ is undefined for $x = 1$.

(iii) For $x = 6$

If $x = 6$ then $\log_2 (7 - 6) = \log_2 1$

Log of unity to any valid base is 0, i.e. 0 exponent of 0 can give 1 so $\log_2 (7 - x)$ is defined for $x = 6$.

(iv) For $x \geq 7$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

$\log_2 (7 - x)$

(a) If $x = 7$ then $\log_2 (7 - 7) = \log_2 0$

None of the exponent of 7 can give 0 so $\log_2 (7 - x)$ is undefined for $x = 7$.

(b) If $x > 7$ then $(7 - x) < 0$. None of the exponent of a positive number can give negative value so $\log_2 (7 - x)$ is undefined for $x > 7$.

Ultimately we can conclude that $\log_2 (7 - x)$ is undefined for $x \geq 7$.

Q.2: Convert the form of following question i.e. from exponential to logarithmic form and vice versa.

Solution: (a)

(i) $7^4 = 2401$ $\log_7 2401 = 4$

(ii) $5^{-\frac{1}{4}} = \frac{1}{25}$ $\log_5 \frac{1}{25} = -\frac{1}{4}$

(iii) $(125)^{\frac{1}{3}} = 5$ $\log_{125} 5 = \frac{1}{3}$

(iv) $2^x = 16$ $\log_2 16 = x$

(v) $3^y = 27$ $\log_3 27 = y$

(vi) $4^z = 64$ $\log_4 64 = z$

(vii) $9^w = 81$ $\log_9 81 = w$

(viii) $16^v = 256$ $\log_{16} 256 = v$

(ix) $25^u = 625$ $\log_{25} 625 = u$

(x) $81^t = 6561$ $\log_{81} 6561 = t$



(viii)	$(256)^{\frac{1}{2}} = \frac{1}{64}$	$\log_{256} \frac{1}{64} = \frac{x}{4}$
(ix)	$2x+1 = 2^x$	$\log_2(2x+1) = x$

Solution: (ii)

No.	Logarithmic to	Exponential
(i)	$\log_2 216 = 3$	$2^3 = 216$
(iii)	$\log_2 x = 3$	$2^3 = x$
(vii)	$\log_{10} 10^x = y$	$10^y = 10^x$
(viii)	$\log_2(x^2+1) = 2$	$2^2 = (x^2+1)$
(ix)	$\log_2(2x-3) = 1$	$2^1 = (2x-3)$

Q.3: Find the value of x from the following questions.

(i) $\log_2 3 = 1$ answer

Solution

$\log_2 3 = 1$

In exponential form

$2^1 = 3$

$x = 3$

(ii) $\log_{10} 9 = 2$ answer

Solution

$\log_{10} 9 = 2$

In exponential form

$(x+1)^2 = 9$

$\sqrt{x+1} = \pm 3$

i.e.

$x+1 = \pm 3$

$x+1 = 3$ or $x+1 = -3$

$x = 2$ or $x = -4$

$x = 2$ or $x = -4$

Since $\log$ is not defined for negative base so required value of x is 2 i.e. $x = 2$

(iii) $\log_2 81 = x$ answer

Solution

$\log_2 81 = x$

In exponential form

$2^x = 81$

$2^x = 2^x$

As bases are same so exponents must be same.

i.e.

$x = 4$

(iv) $\log_2 64 = x+1$ answer

Solution

$\log_2 64 = x+1$

In exponential form

$2^{x+1} = 64$

$2^{x+1} = 2^6$

As bases are same so exponents must be same.

i.e.

$x+1 = 6$

$x = 5$

(v) $\log_2 x = 4$ answer

Solution

$\log_2 x = 4$

In exponential form

$\log_2 x = 4$

$2^4 = x$

$16 = x$

$x = 16$

(vi) $\log_2(x^2-1) = 3$ answer

Solution

$\log_2(x^2-1) = 3$

In exponential form

$2^3 = x^2-1$

$8 = x^2-1$

$8+1 = x^2$

$9 = x^2$

$\sqrt{9} = \pm 3$

$x = \pm 3$

Q.4: Find the unknown appeared in the question 1.

(i) $\log_2 x = 3$ answer

Solution

$\log_2 x = 3$

In exponential form

$2^3 = x$

$8 = x$



$3x, 5x, 7x, 9x, 2 = x$

$3125 = x$

$x = 3125$

(ii) $h^{-\frac{1}{2}} = \frac{1}{25}$ answer

Solution

$h^{-\frac{1}{2}} = \frac{1}{25}$

$h^{-\frac{1}{2}} = \frac{1}{25}$

$h^{-\frac{1}{2}} = \frac{1}{25}$

$h^{-\frac{1}{2}} = 2^{-4}$

$h^{-\frac{1}{2}} = 2^{-4}$

$h^{-\frac{1}{2}} = 2^{-4}$

$h^{-\frac{1}{2}} = 2^{-4}$

$h^{-\frac{1}{2}} = 2^{-4}$

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$h^{-\frac{1}{2}} = 2^{-4}$

(vii) $(256)^{\frac{1}{2}} = \frac{1}{64}$ answer

Solution

$(256)^{\frac{1}{2}} = \frac{1}{64}$

$(2^8)^{\frac{1}{2}} = \frac{1}{2^6}$

$2^4 = 2^{-6}$

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(ii) $2x+1=2^7$

Solution

$$2x+1=2^7$$

$$2x+1=8$$

$$2x=8-1$$

$$2x=7$$

$$x = \frac{7}{2} = 3.5$$

Common Logarithm

There are two most commonly used bases for logarithm i.e. '10' and 'e = 2.71828' (an irrational number). Base 10 was used by Henry Briggs.

If $10^y = x$, for $x > 0$, then y is called common log of x i.e. $10^y = x \Rightarrow \log_{10} x = y$

These logarithms are also called Briggs's logarithms, denoted by $\log_{10} x$ or simply $\log x$. If none of the base is mentioned with $\log$ then it is obviously a common logarithm e.g. $\log_{10}$ it can be simply written as $\log 10$.

Logarithm of a number

= characteristic + Mantissa.

Mantissa

Decimal part or the fractional part of logarithm is called mantissa.

Mantissa is always a non-negative number less than 1, i.e. it can be either zero or positive but never negative.

Characteristic

Integral part of the logarithm is called characteristic. Characteristic is an integer. It is either the integral power of 10. When the

number is written in scientific notation e.g. characteristic of $\log 3.3 \times 10^2$ is 2.

Example 9

Find (a) $\log 156.3$ (b) $\log 0.0123$ (answer)

Solution

(a) $\log 156.3$

$$= \log(1.563 \times 10^2)$$

$$\text{Characteristic} = 2$$

$$\text{Mantissa} = .1939 \text{ or } = 0.1939$$

$$\log 156.3 = [\text{Characteristic} + \text{Mantissa}]$$

$$\log 156.3 = 2 + .1939$$

$$\log 156.3 = 2.1939$$

(b) $\log 0.0123$ (answer)

$$\log 0.0123 = \log(1.23 \times 10^{-2})$$

$$\text{Characteristic} = -2 \text{ or } \bar{2}$$

$$\text{Mantissa} = .0899$$

$$\log 0.0123 = [\text{Characteristic} + \text{Mantissa}]$$

$$\log 0.0123 = \bar{2} + 0.0899$$

$$\log 0.0123 = \bar{2}.0899$$

$$\log 156.3 = \bar{2}.0899$$

$$\log 156.3 = \bar{2}.0899$$

$$\text{Note: } \bar{2}.0899 \neq -2.0899.$$

They are different numbers.

Example 10

Find $\log 1000$ (answer)

Solution

$$\log 1000 = \log(1.000 \times 10^3)$$

$$\text{characteristic} = 3$$

$$\text{mantissa} = 0.0039 (= .39)$$

$$\therefore \log 1000 = 3.0039$$

Exercise 2.1

Find logarithm of following numbers if possible.

1. 5013 (answer)

$$\log 5.013 = \log(5.013 \times 10^0)$$

$$\text{Characteristic} = 0$$

$$\text{Mantissa} = .7213$$

$$\log 5.013 = 0.7213$$

2. 4888 (answer)

$$\log 4.888 = \log(4.888 \times 10^0)$$

$$\text{Characteristic} = 0$$

$$\text{Mantissa} = .6889$$

$$\log 4793 = 3.6809$$

3. 9.613 (answer)

$$\log 9.613 = \log(9.613 \times 10^0)$$

$$\text{Characteristic} = 0$$

$$\text{Mantissa} = .9828$$

$$\log 9.613 = 0.9828$$

4. 110.9 (answer)

$$\log 110.9 = \log(1.109 \times 10^2)$$

$$\text{Characteristic} = 2$$

$$\text{Mantissa} = .0449 \quad [0414 + 33 = 0449]$$

$$\log 110.9 = 2.0449$$



5. 52.39 (answer)

$$\log 52.39 = \log(5.239 \times 10^1)$$

$$\text{Characteristic} = 1$$

$$\text{Mantissa} = .7182$$

$$\log 52.39 = 1.7182$$

6. 0.01207 (answer)

$$\log 0.01207 = \log(1.207 \times 10^{-2})$$

$$\text{Characteristic} = -2 \text{ or } \bar{2}$$

$$\text{Mantissa} = .0816$$

$$\log 0.01207 = \bar{2}.0816$$

7. 0.0893 (answer)

$$\log 0.0893 = \log(8.93 \times 10^{-2})$$

$$\text{Characteristic} = -2 \text{ or } \bar{2}$$

$$\text{Mantissa} = .9503$$

$$\log 0.0893 = \bar{2}.9503$$

8. 1 Trillion (answer)

$$1 \text{ Trillion} = 10^{12}$$

$$\log(1 \text{ Trillion}) = \log 10^{12}$$

$$= \log(1 \times 10^{12})$$

$$\text{Characteristic} = 12$$

$$\text{Mantissa} = 0.0000$$

$$\log 10^{12} = 12 + 0.0000 = 12.0000$$

9. 0.00004 (answer)

$$\log 0.00004 = \log(4.0 \times 10^{-5})$$

$$\text{Characteristic} = -5 \text{ or } \bar{5}$$

$$\text{Mantissa} = .6021$$

$$\log 0.00004 = \bar{5}.6021$$

10. 4 (answer)

$$\log 4 = \log(4 \times 10^0)$$

$$\text{Characteristic} = 0$$

$$\text{Mantissa} = .6021$$

$$\log 4 = 0.6021$$

11. 4000 (answer)

$$\log 4000 = \log(4 \times 10^3)$$

$$\text{Characteristic} = 3$$

$$\text{Mantissa} = .6021$$

$$\log 4000 = 3.6021$$

12. 54 (answer)

$$\log 54 = \log(5.4 \times 10^1)$$

$$\text{Characteristic} = 1$$

$$\text{Mantissa} = .7324$$

$$\log 54 = 1.7324$$

13. 1.009 (answer)

$$\log 1.009 = \log(1.009 \times 10^0)$$

$$\text{Characteristic} = 0$$

$$\text{Mantissa} = .0038$$

$$\log 1.009 = 0.0038$$

14. 0.1009 (answer)

$$\log 0.1009 = \log(1.009 \times 10^{-1})$$

$$\text{Characteristic} = -1 \text{ or } \bar{1}$$

$$\text{Mantissa} = .0038$$

$$\log 0.1009 = \bar{1}.0038$$

15. -4 (answer)

$$\log(-4) \text{ is not possible}$$

log is undefined for negative numbers.

Antilogarithm

The inverse operation of taking log is called antilog. An antilog is used to cancel the effect of log.

If $\log x = y$ then x is called the antilog of y or $x = \text{antilog } y$.

Example 11: (a) Find antilog 2.4541 (answer)

Solution: 2.4541 is obviously log of some number suppose that number be x (answer)

$$\log x = 2.4541$$

$$\text{Characteristic} = 2 \quad (\text{Integral part})$$

$$\text{mantissa} = .4541 \quad (\text{Fractional part})$$

$$\text{antilog}(\log x) = \text{antilog}(2.4541)$$

$$\therefore x = 284.5$$

(b) If $\log x = \bar{2}.000$, then find x . (answer)

Solution

$$\log x = \bar{2}.0000$$

$$\text{characteristic} = \bar{2}$$

$$\text{mantissa} = .0000$$

$$(\text{taking antilog on both sides})$$

$$\text{antilog}(\log x) = \text{antilog}(\bar{2}.0000)$$

$$x = \text{antilog}(\bar{2}.0000)$$

$$\therefore x = 0.01000 = 0.0000$$

Example 12: Find antilog of $\bar{1}.4900$ (answer)

Solution

$$\text{antilog } \bar{1}.4900$$

$$\text{characteristic} = \bar{1}$$

$$\text{mantissa} = .4900$$

$$\text{antilog}(\bar{1}.4900) = (10^{-1} \times 30) = 0.03000$$



Exercise 2:

Find the antilog of following numbers.

1. 2.4324 answer

Solution
Let $\log x = 2.4324$
Characteristic = 2
Mantissa = .4324
 $x = \text{anti-log}(2.4324)$
 $x = 270.7$

2. 1.5890 answer

Solution
Let $\log x = 1.5890$
Characteristic = 1
Mantissa = .5890
 $x = \text{anti-log}(1.5890)$
 $x = 38.82$ (3882)

3. 0.2425 answer

Solution
Let $\log x = 0.2425$
Characteristic = 0
Mantissa = .2425
 $x = \text{anti-log}(0.2425)$
 $x = 1.744$ (1746+2+1744)

4. 3.5636 answer

Solution
Let $\log x = 3.5636$
Characteristic = 3
Mantissa = .5636
 $x = \text{anti-log}(3.5636)$
 $x = 3661.0$ (3658+5+3661)
 $x = 3661$

5. 4 answer

Solution
Let $\log x = 4.0000$
Characteristic = 4
Mantissa = .0000
 $x = \text{anti-log}(4.0000)$
 $x = 1.0000$

Put decimal point after 4 digits.

 $x = 10000$

6. 0.0009 answer

Solution
Let $\log x = 0.0009$
Characteristic = 0

Mantissa = .0009
 $x = \text{anti-log}(0.0009)$
 $x = 1.009$ (1007+2+1009)
 $x = 1.009$

7. 1.2429 answer

Solution
Let $\log x = 1.2429$
Characteristic = 1
Mantissa = .2429
 $x = \text{anti-log}(1.2429)$
 $x = 0.1.750$ (1746+4+1750)

8. 1.9281 answer

Solution
Let $\log x = 1.9281$
Characteristic = 1
Mantissa = .9281
 $x = \text{anti-log}(1.9281)$
 $x = 0.08.474$ (8472+2+8474)
 $x = 0.08474$

9. 1.5219 answer

Solution
Let $\log x = 1.5219$
Characteristic = 1
Mantissa = .5219
 $x = \text{anti-log}(1.5219)$
 $x = 0.03.324$ (3219+7+3224)

10. 0.0000 answer

Solution:
Let $\log x = 0.0000$
Characteristic = 0
Mantissa = .0000
 $x = \text{anti-log}(0.0000)$
 $x = 1.000 = 1.000 = 1$

11. -3 answer

Solution
-1.0000 or 1.0000
Let $\log x = 1.0000$



Characteristic = 1
Mantissa = .0000
 $x = \text{anti-log}(1.0000)$
 $x = 0.001.000 = 0.0010$

12. 5.9990 answer

Solution
Let $\log x = 5.9990$
Characteristic = 5
Mantissa = .9990
 $x = \text{anti-log}(5.9990)$
 $x = 0.01100.0$ (9977)

We have to put decimal point after 3 digits from reference position as we write two extra zeros.

13. 1.4900 answer

Solution
Let $\log x = 1.4900$
Characteristic = 1
Mantissa = .4900

$x = \text{anti-log}(2.4900)$
 $x = 3.09.0 = 309$ (3090)

14. 0.49000 answer

Solution
 $x = 0.49000$
Let $\log x = 0.49000$
Characteristic = 0
Mantissa = .4900
 $x = \text{anti-log}(0.4900)$
 $x = 3.090 = 3.090$

15. 2.34 answer

Solution
Let $\log x = 2.34$
Characteristic = 2
Mantissa = .34
 $x = \text{anti-log}(2.34)$
 $x = 2.18.8 = 218.8$

Common and Natural Logarithms

John Napier started developing log tables with base e , so the logarithms with base e are called natural logarithms or Napierian logarithms, represented by ' $\ln$ '.

If $a^x = x$, for positive values of x then x is called natural log of x i.e. $a^x = x$ so $x = \ln a$.

The difference between common and natural logarithms is depicted below.

Log Type	Representation	Base	Nature of base	Properties
Common (Briggs)	$\log x$	10	Rational	$\log 1 = 0$ $\log 10 = 1$ $\log 10^x = x$
Natural (Napierian)	$\log_e x$	$e = 2.71828$	Irrational	$\log_e 1 = 0$ $\ln e = 1$ $\log_e e^x = x$

Laws of logarithm:

(1) Product Law of logarithm:

For any positive real numbers m , a , b where $b \neq 1$, $\log_b m = \log_b m + \log_b a$

Proof

Let $\log_b m = x$ and $\log_b a = y$

In exponential form,

$b^x = m$ and $b^y = a$

Multiplying eq.(iii) and (iv)

$b^x \cdot b^y = m \cdot a$

$b^{x+y} = m \cdot a$ (product law of exponents)

In logarithmic form,

$\log_b m \cdot a = x + y$



Putting the value of x and y from (i) and (ii)

$$\log_e m = \log_e n + \log_e a$$

The logarithm of a product of two (or more) numbers is equal to the sum of their logarithms, provided that all the logarithms are defined.

Example 13

(a) Use the product rule to expand $\log(x^2 y^3 z)$. *****

Solution

$$\log_e(x^2 y^3 z) = \log_e x^2 + \log_e y^3 + \log_e z$$

(b) Use product rule to combine

$\log a + \log \sqrt{b} + \log c$ in a single logarithmic term.

$$\text{Solution: } \log a + \log \sqrt{b} + \log c = \log(a \sqrt{b} c^2)$$

(2) Quotient law of logarithm

For any positive real numbers m , n and b

$$\text{where } b \neq 1, \log_b \frac{m}{n} = \log_b m - \log_b n$$

Proof

$$\text{Let } \log_b m = x \text{ (i) and } \log_b n = y \text{ (ii)}$$

In exponential form,

$$b^x = m \text{ (iii) and } b^y = n \text{ (iv)}$$

Dividing eq.(iii) by (iv)

$$\frac{b^x}{b^y} = \frac{m}{n}$$

$$b^{x-y} = \frac{m}{n} \quad (\text{Quotient law of exponents})$$

In logarithmic form,

$$\log_b \frac{m}{n} = x - y$$

Putting the value of x and y from (i) and (ii)

$$\log_b \frac{m}{n} = \log_b m - \log_b n$$

The logarithm of a quotient of two numbers is equal to the difference of their logarithms, provided that all the logarithms are defined.

Example 14: (a) Use quotient law of logarithm to expand $\log \frac{13}{76}$. *****

Solution: (a)

$$\log \frac{13}{76} = \log 13 - \log 76 = \log 13 - (\log 7 + \log 10)$$

$$= \log 13 - \log 7 - \log 10$$

(b) Use quotient law of logarithm to combine $\log 39 - \log 1 - \log 6$ into single logarithmic term.

Solution: (b)

$$\log 39 - \log 1 - \log 6$$

$$= \log 39 - (\log 1 + \log 6)$$

$$= \log 39 - \log 6$$

$$= \log \frac{39}{6}$$

(3) Power law of logarithm

For any real numbers m , a , b where $b \neq 1$,

$$\log_b m^a = a \log_b m$$

Proof

$$\text{Let } \log_b m = x \text{ (i)}$$

In exponential form,

$$b^x = m \text{ (ii)}$$

Taking a th power of both sides

$$(b^x)^a = (m)^a$$

$$b^{ax} = m^a$$

In logarithmic form,

$$\log_b m^a = ax$$

Putting the value of x from (i)

$$\log_b m^a = a \log_b m$$

The logarithm of power of a number is equal to the power times the logarithm of the number, provided that all the logarithms are defined.

Example 15

(a) Use power law of logarithm to expand $\log_3 100^3$. *****

Solution: (a)

$$\log_3 100^3 = 3 \log_3 100$$

(b) Use power law of logarithm to combine

$$-\frac{4}{3} \log_3 7$$

Solution: (b)

Using power law of log

$$-\frac{4}{3} \log_3 7 = \log_3 (7)^{-\frac{4}{3}}$$

(4) Change of base law of logarithm

If a, b and m are positive real numbers and

$a \neq 1, b \neq 1$ then $\log_a m = \log_b m \times \log_a b$

$$\text{or } \log_a m = \frac{\log_b m}{\log_b a}$$

Proof:

$$\text{Let } \log_a m = x \text{ (i)}$$

In exponential form,

$$a^x = m \text{ (ii)}$$

Taking log of both sides with base a

$$\log_a a^x = \log_a m$$

By power law of log,

$$x \cdot \log_a a = \log_a m$$

Putting the value of x from (i)

$$\log_a m = \log_a b \times \log_a a$$

$$\text{or } \log_a m = \frac{\log_b m}{\log_b a}$$

The change of base law is taking the log of the argument number divided by the log of the base number provided that all the logarithms are defined.

Example 16

Convert the base of $\log_5 526$ into 10. *****

Solution:

Using change of base law of log

$$\log_5 526 = \frac{\log_{10} 526}{\log_{10} 5} \quad \text{or} \quad \frac{\log 526}{\log 5}$$

Example 17

(a) Use laws of logarithms to expand $\log$

$$\frac{3x^2 y^3}{4\sqrt{x}}$$

Solution

$$\log \frac{3x^2 y^3}{4\sqrt{x}}$$

Using quotient law of log

$$= \log(3x^2 y^3) - \log(4\sqrt{x})$$

Using product law of log

$$= \log 3 + \log x^2 + \log y^3 - (\log 4 + \log \sqrt{x} + \log 1)$$

Using power law of log

$$= \log 3 + 2 \log x + \frac{1}{2} \log y + \log 4 - \frac{1}{2} \log x - \log 1$$

(b) Use laws of logarithms to evaluate

$$(i) \log_2 5\sqrt{3}$$

Solution

$$\log_2 5\sqrt{3} = \frac{\log_{10} 5\sqrt{3}}{\log_{10} 2} \quad \leftarrow \text{change of base law}$$

$$= \left(\log_{10} 5 + \log_{10} \sqrt{3} \right) \div (\log 2)$$

$$= \left(\log_{10} 5 + \frac{1}{2} \log_{10} 3 \right) \div (\log 2)$$

$$= \left(0.6990 + \frac{1}{2}(0.4771) \right) \div (0.3010)$$

$$= (0.6990 + 0.23855) \div (0.3010)$$

$$= (0.93755) \div (0.3010)$$

$$= 3.1148$$

(ii) Evaluate *****

$$\log_2 (\log_2 8^3) - \log_{10} 27 + \log_{10} 10$$

Solution:

$$\log_2 (\log_2 8^3) - \log_{10} 27 + \log_{10} 10$$

$$= \log_2 \left(\log_2 (2^3)^3 \right) - \log_{10} \left((3^3)^3 \right) + \log_{10} (\sqrt{10})^2$$

$$= \log_2 \left(\log_2 2^9 - \log_{10} (\sqrt{3})^9 + \log_{10} (\sqrt{10})^2 \right)$$

$$= \log_2 (6 \log_2 2 - 6 \log_{10} \sqrt{3} + 2 \log_{10} \sqrt{10})$$

$$= \log_2 [6(1) - 6(1) + 2(1)] \quad (\because \log_a b = 1)$$

$$= \log_2 (6 - 6 + 2)$$

$$= \log_2 2 \quad (\because \log_a b = 1)$$

$$= 1$$

(c) If $\log_2 2 = 0.3010$, $\log_2 3 = 0.4771$ and

$\log_2 5 = 0.6990$, then evaluate $\log_2 0.0036$.

applying laws of logarithms. *****

Solution

$$\log_2 0.0036$$

$$= \log_2 \left(\frac{36}{10000} \right)$$



$$= \log_2 \left(\frac{2^4 \times 2^1}{2^2 \times 2^2} \right) \quad (\text{By Factorizing 16 \& 10000})$$

$$= \log_2 \left(\frac{2^4}{2^2 \times 2^2} \right) \quad (\text{By simplifying})$$

$$= \log_2 2^2 - \log_2 (2^2 \times 2^2)$$

$$= \log_2 2^2 - (\log_2 2^2 + \log_2 2^2)$$

$$= \log_2 2^2 - \log_2 2^2 - \log_2 2^2$$

$$= 2 \log_2 2 - 2 \log_2 2 - 4 \log_2 2$$

$$= 2(0.4771) - 2(0.3010) - 4(0.6990)$$

$$= 0.9542 - 0.6020 - 2.796$$

$$= -2.4438$$

$$= -3 + 3 - 2.4438 \quad (\text{making mantissa positive})$$

$$= \overline{7}.1162$$

Important results deduced from laws of Logarithm

$$\text{i) } \log_a a = \log_a b = 1$$

$$\text{ii) } \log_a a = \frac{1}{\log_a a}$$

$$\text{iii) } \log_a a \cdot \log_a b = \log_a a$$

$$\text{iv) } \log_a a' = \log_a a'' = \log_a a' = \log_a a''$$

$$\text{v) } \log_a a = \log_a a = \frac{1}{\log_a a} = 1$$

$$\text{vi) } \log_a a \cdot \log_a b = \log_a a = 1$$

Exercise 2.1

Q.1 Use laws of logarithms to expand followings.

(i) $\log 94$ answer

Solution

$$\log 94 = \log 9 + \log 4$$

$$= \log 2^2 + \log 4$$

$$= 2 \log 2 + \log 4$$

(ii) $\log \frac{59}{5}$ answer

Solution

$$\log \frac{59}{5} = \log 59 - \log 5$$

(iii) $\log \left(\frac{5pq^2}{r^3} \right)$ answer

Solution

$$\log \left(\frac{5pq^2}{r^3} \right)$$

$$= \log(5pq^2) - \log(r^3)$$

$$= \log 5 + \log p + \log q^2 - (\log r + \log r^2)$$

$$= \log 5 + \log p + 2 \log q - \log r - 2 \log r$$

(iv) $\log \sqrt{\frac{53.3}{46.4}}$ answer

Solution

$$\log \sqrt{\frac{53.3}{46.4}}$$

$$= \log \left(\frac{53.3}{46.4} \right)^{\frac{1}{2}}$$

$$= \frac{1}{2} \log \left(\frac{53.3}{46.4} \right)$$

$$= \frac{1}{2} (\log 53.3 - \log 46.4)$$

$$= \frac{1}{2} \log 53.3 - \frac{1}{2} \log 46.4$$

(v) $\log \left(\frac{q^2 r^2 a^3}{(4.4a^2 b^2)} \right)$ answer

Solution

$$\log \left(\frac{q^2 r^2 a^3}{(4.4a^2 b^2)} \right) = \log(q^2 r^2 a^3) - \log(4.4a^2 b^2)$$



$$= \log q^2 + \log r^2 + \log a^3 - \left[\log(4.4a^2) + \log b^2 \right]$$

$$= \log q^2 + \log r^2 + \log a^3 - \left[\frac{1}{2} \log(4.4a^2) + \log b^2 \right]$$

$$= 2 \log q + 2 \log r + \frac{1}{2} \log a - \left[\frac{1}{2} (\log 4.4 + \log a^2) + \log b^2 \right]$$

$$= 2 \log q + 2 \log r + \frac{1}{2} \log a - \frac{1}{2} \log 4.4 - \frac{1}{2} \log a - 2 \log b$$

(vi) $\log \left(\frac{7^2 r^2 p}{d^2 b} \right)$ answer

Solution

$$\log \left(\frac{7^2 r^2 p}{d^2 b} \right) = \log \left(\frac{7^2 r^2 p}{d^2 b} \right)$$

$$= \frac{1}{2} \log \left(\frac{7^2 r^2 p}{d^2 b} \right)$$

$$= \frac{1}{2} [\log(7^2 r^2 p) - \log(d^2 b)]$$

$$= \frac{1}{2} [\log 7^2 + \log r^2 + \log p - (\log d^2 + \log b^2)]$$

$$= \frac{1}{2} [2 \log 7 + 2 \log r + \log p - 2 \log d - 2 \log b]$$

$$= \frac{2}{2} \log 7 + \frac{2}{2} \log r + \frac{1}{2} \log p - \frac{2}{2} \log d - \frac{2}{2} \log b$$

(vii) $\log \left(\frac{\sqrt{5.312 \text{ pm}^2}}{95.9 \text{ a}^2 \text{ b}} \right)$ answer

Solution

$$\log \left(\frac{\sqrt{5.312 \text{ pm}^2}}{95.9 \text{ a}^2 \text{ b}} \right)$$

$$= \left[\log \frac{\sqrt{5.312 \text{ pm}^2}}{95.9 \text{ a}^2 \text{ b}} \right]$$

$$= \left[\log \sqrt{5.312 \text{ pm}^2} - \log 95.9 \text{ a}^2 \text{ b} \right]$$

$$= \left[\log(5.312)^{\frac{1}{2}} + \log p + \log m^2 \right]$$

$$= \left[\log(5.312)^{\frac{1}{2}} + \log p + \log b \right]$$

$$= \left[\left(\frac{1}{2} \log 5.312 \right) + \log p + \frac{1}{2} \log m - \frac{1}{2} \log(9.91) - 2 \log a - \log b \right]$$

$$= \frac{1}{2} \log 5.312 + \log p + \frac{1}{2} \log m - \frac{1}{2} \log 9.91 - 2 \log a - \log b$$

Q.2: Use laws of logarithms to combine the followings into single logarithmic terms.

(i) $3 \log x - 5 \log y$ answer

Solution

$$3 \log x - 5 \log y$$

$$= \log x^3 - \log y^5$$

$$= \log \frac{x^3}{y^5}$$

(ii) $\frac{1}{2} \log x + \frac{1}{3} \log r - \frac{1}{5} \log s$ answer

Solution

$$\frac{1}{2} \log x + \frac{1}{3} \log r - \frac{1}{5} \log s$$

$$= \log x^{\frac{1}{2}} + \log r^{\frac{1}{3}} - \log s^{\frac{1}{5}}$$

$$= \log \sqrt{x} + \log \sqrt[3]{r} - \log \sqrt[5]{s}$$

$$= \log \left(\sqrt{x} \cdot \sqrt[3]{r} \right) - \log \sqrt[5]{s}$$

$$= \log \left(\frac{\sqrt{x} \cdot \sqrt[3]{r}}{\sqrt[5]{s}} \right)$$

(iii) $\frac{1}{7} \log 57.7 - 3 \log 9.24 + 4 \log 36.6 - 2 \log 23.3$ answer

Solution

$$= \frac{1}{7} \log 57.7 - 3 \log 9.24 + 4 \log 36.6 - 2 \log 23.3$$

$$= \frac{1}{7} \log 57.7 - 4 \log 9.24 - 2 \log 23.3 + 4 \log 36.6$$

$$= \frac{1}{7} \left[\log 57.7 + \log(36.6^4) - \log(23.3^2) - \log(9.24^4) \right]$$



$$= \frac{1}{3} \left[\log 57.7 + (\log 36.6)^2 - \log (23.3)^2 + (\log 9.24)^2 \right]$$

$$= \frac{1}{3} \log \left(\frac{57.7 \times (36.6)^2}{(23.3)^2 \times (9.24)^2} \right)$$

$$= \log \left(\frac{57.7 \times (36.6)^2}{(23.3)^2 \times (9.24)^2} \right)^{\frac{1}{3}}$$

$$= \log \left(\frac{57.7 \times (36.6)^2}{(23.3)^2 \times (9.24)^2} \right)^{\frac{1}{3}}$$

(iv)

$$5 \log 6 - 7 \log 9.42 + \frac{1}{3} \log 4 - \frac{1}{2} \log 32.2 + \frac{2}{3} \log 8$$

answer

Solution

$$5 \log 6 - 7 \log 9.42 + \frac{1}{3} \log 4 - \frac{1}{2} \log 32.2 + \frac{2}{3} \log 8$$

$$= 5 \log 6 - \frac{1}{3} \log 4 + \frac{2}{3} \log 8 - \frac{1}{2} \log 32.2 - 7 \log 9.42$$

$$= \log (6)^5 + \log (4)^{\frac{2}{3}} + \log (8)^{\frac{2}{3}} - \log (32.2)^{\frac{1}{2}} - \log (9.42)^7$$

$$= \log (6)^5 + \log (4)^{\frac{2}{3}} + \log (8)^{\frac{2}{3}} - \left(\log \sqrt{32.2} + \log (9.42)^7 \right)$$

$$= \log \left(6^5 \times \sqrt[3]{4} \times \sqrt[3]{8} \right) - \log \left(\sqrt{32.2} \times (9.42)^7 \right)$$

$$= \log \left(\frac{6^5 \times \sqrt[3]{4} \times \sqrt[3]{8}}{\sqrt{32.2} \times (9.42)^7} \right)$$

$$(v) \frac{5}{4} \log 37.74 - \frac{1}{4} \log 53.71 + \frac{1}{4} \log 28.83$$

answer

Solution

$$\frac{5}{4} \log 37.74 - \frac{1}{4} \log 53.71 + \frac{1}{4} \log 28.83$$

$$= \frac{5}{4} \log 37.74 + \frac{1}{4} \log 28.83 - \frac{1}{4} \log 53.71$$

$$= \log (37.74)^{\frac{5}{4}} + \log (28.83)^{\frac{1}{4}} - \log (53.71)^{\frac{1}{4}}$$

$$= \log \left[\frac{(37.74)^5 \times (28.83)}{(53.71)} \right]^{\frac{1}{4}}$$

$$= \log \left(\frac{(37.74)^5 \times (28.83)}{(53.71)} \right)^{\frac{1}{4}}$$

$$= \log \left(\frac{(37.74)^5 \times (28.83)}{53.71} \right)^{\frac{1}{4}}$$

$$= \log \left(\frac{(37.74)^5 \times (28.83)}{53.71} \right)^{\frac{1}{4}}$$

Q.3: Use laws of logarithm to evaluate the following

(i) $\log_2 15$

answer

Solution

$$\log_2 15 = \frac{\log 15}{\log 2} = \frac{1.1761}{0.3010} = 3.9077$$

(ii) $\log_5 \sqrt[3]{9}$

answer

Solution

$$\text{Let } \log_5 \sqrt[3]{9} = x$$

In exponential form

$$9^x = \sqrt[3]{9}$$

$$9^x = 9^{\frac{1}{3}}$$

$$\Rightarrow x = \frac{1}{3}$$

$$\text{Thus } \log_5 \sqrt[3]{9} = \frac{1}{3}$$

(iii) $\log_3 48$

answer

Solution

$$\log_3 48 = \frac{\log 48}{\log 3} = \frac{1.6829}{0.4771} = 3.7998$$

(iv) $\log_4 72.34$

answer

Solution

$$\log_4 72.34 = \frac{\log 72.34}{\log 4} = \frac{\log 72.34}{\log (2^2)}$$

$$= \frac{\log 72.34}{\frac{1}{2} \log 2} = \frac{1.8593}{\frac{1}{2} (0.4771)}$$

$$= \frac{1.8593}{0.23855} = 7.7941$$

(v) $\log_{27} 343$

answer

Solution

$$\text{Let } \log_{27} 343 = x$$

In exponential form

$$(\sqrt[3]{27})^x = 343$$

$$\left(\frac{1}{3} \right)^x = 3^7$$

$$3^{-x} = 3^7$$

$$\Rightarrow \frac{1}{x} = 7$$

$$x = 3 = 2$$

$$x = 6$$

Q.4: If $\log_2 2 = 0.3010$, $\log_2 3 = 0.4771$, $\log_2 5 = 0.6990$ then evaluate the following with laws of logarithm.

(i) $\log_2 \frac{6}{5}$

answer

Solution

$$\log_2 \frac{6}{5} = 0.3010 + 0.4771 - 0.6990$$

$$= \log_2 6 - \log_2 5 \quad \text{Using quotient law}$$

$$= \log_2 (2 \times 3) - \log_2 5 \quad \text{Factorizing 6}$$

$$= \log_2 2 + \log_2 3 - \log_2 5 \quad \text{Using product law}$$

$$= 0.7781 - 0.6990$$

$$= 0.0791$$

(ii) $\log_9 \frac{100}{9}$

answer

Solution

$$\log_9 \frac{100}{9}$$

$$= \log_9 100 - \log_9 9 \quad \text{(Using quotient law)}$$

$$= \log_9 (2^2 \times 5^2) - \log_9 3^2 \quad \text{(Factorizing 100 & 9)}$$

$$= \log_9 2^2 + \log_9 5^2 - \log_9 3^2 \quad \text{(Using product law)}$$

$$= 2 \log_9 2 + 2 \log_9 5 - 2 \log_9 3 \quad \text{(Using power law)}$$

$$= 2(\log_9 2 + \log_9 5 - \log_9 3) \quad \text{(Taking 2 as common)}$$

$$= \log_9 2^2 + \log_9 5^2 - \log_9 3^2 \quad \text{(Using product law)}$$

$$= 2 \log_9 2 + 2 \log_9 5 - 2 \log_9 3 \quad \text{(Using power law)}$$



$$= 2(\log_9 2 + \log_9 5 - \log_9 3)$$

(Taking 2 as common)

$$= 2(0.3010 + 0.6990 - 0.4771)$$

$$= 2(1 - 0.4771)$$

$$= 2(0.5229)$$

$$= 1.0458$$

$$(iii) \log_8 \frac{\sqrt[3]{450}}{\sqrt[3]{27}}$$

answer

Solution

$$\log_8 \frac{\sqrt[3]{450}}{\sqrt[3]{27}}$$

$$= \log_8 \sqrt[3]{450} - \log_8 \sqrt[3]{27} \quad \text{(Using quotient law)}$$

$$= \log_8 (450)^{\frac{1}{3}} - \log_8 (27)^{\frac{1}{3}}$$

$$= \frac{1}{3} \log_8 450 - \frac{1}{3} \log_8 27 \quad \text{(Using power law)}$$

$$= \frac{1}{3} \log_8 (2 \times 3^2 \times 5^2) - \frac{1}{3} \log_8 3^3 \quad \text{(Factorizing 450 & 27)}$$

$$= \frac{1}{3} (\log_8 2 + \log_8 3^2 + \log_8 5^2) - \frac{1}{3} \log_8 3^3$$

$$= \frac{1}{3} (\log_8 2 + 2 \log_8 3 + 2 \log_8 5) - \frac{1}{3} (3 \log_8 3)$$

$$= \frac{1}{3} (\log_8 2 + 2 \log_8 3 + 2 \log_8 5) - \log_8 3$$

By putting values we get

$$= \frac{1}{3} [0.3010 + 2(0.4771) + 2(0.6990)] - \log_8 3$$

$$= \frac{1}{3} [0.3010 + 0.9542 + 1.3980] - \frac{1}{3} (1.4313)$$

$$= \frac{1}{3} [2.6532] - \frac{1}{3} (1.4313)$$

$$= 0.8844 - 0.4771$$

$$= 0.4073$$

$$= 0.4073$$

(iv) $\log_8 0.024$

answer

Solution

$$\log_8 0.024$$

$$\log_8 0.024$$

$$= \log_8 \frac{24}{1000}$$

$$= \log_8 \left(\frac{2^3 \times 3}{2^3 \times 5^3} \right) \quad \text{(Factorizing 2400 & 24)}$$



$$= \log_2 \frac{3}{2}$$

$$= \log_2 3 - \log_2 2 \quad (\text{Using quotient law})$$

$$= \log_2 3 - 1 \log_2 2 \quad (\text{Using power law})$$

By putting the values we get,

$$= 0.4771 - 1(0.6990)$$

$$= 0.4771 - 0.6990$$

$$= -1.0199$$

$$= -2 + 2 - 1.0199 \quad (\text{making numbers positive})$$

$$= 2 - 0.0199$$

$$= 1.9801$$

$$\left(\log_2 \sqrt{\frac{3}{2}} \right) \quad \text{answer}$$

Solution

$$\log_2 \sqrt{\frac{3}{2}}$$

$$= \log_2 \left(\frac{27}{8} \right)^{\frac{1}{2}} = \frac{1}{2} \log_2 \left(\frac{27}{8} \right) \quad (\text{Using power law})$$

$$= \frac{1}{2} [\log_2 27 - \log_2 8] \quad (\text{Using quotient law})$$

$$= \frac{1}{2} [\log_2 3^3 - \log_2 2^3] \quad (\text{Factorizing 27})$$

$$= \frac{1}{2} [3 \log_2 3 - \log_2 8] \quad (\text{Using power law})$$

$$= \frac{1}{2} [3(0.4771) - (0.6990)] \quad (\text{Putting the values})$$

$$= \frac{1}{2} [1.4313 - 0.6990]$$

$$= \frac{1}{2} [0.7323]$$

$$= 0.3661$$

Applications of logarithm

Example 18: An earthquake that occurred in Pakistan in 2005, measured 7.6 on the Richter scale. In 1972, an earthquake in China measured 8.2 on the Richter scale. How many times, the China's earthquake was stronger than the Pakistan's earthquake.

Solution: Let I_1 be the intensity of Pakistan's

earthquake and I_2 be the intensity of China's earthquake.

$$A_1, M = \log \left(\frac{I_1}{I_0} \right)$$

magnitude of Pakistan's earthquake

$$\text{Then, } 7.6 = \log \left(\frac{I_1}{I_0} \right) = \log I_1 - \log I_0 \dots \dots \dots (i)$$

magnitude of China's earthquake

$$8.2 = \log \left(\frac{I_2}{I_0} \right) = \log I_2 - \log I_0 \dots \dots \dots (ii)$$

Subtraction (i) from (ii)

$$8.2 - 7.6 = (\log I_2 - \log I_0) - (\log I_1 - \log I_0)$$

$$0.6 = \log I_2 - \log I_0 - \log I_1 + \log I_0$$

$$\Rightarrow \log I_2 - \log I_1 = 0.6$$

$$\Rightarrow \log \left(\frac{I_2}{I_1} \right) = 0.6$$

(taking antilog on both sides)

$$\text{antilog } \log \left(\frac{I_2}{I_1} \right) = \text{antilog } 0.6$$

$$\frac{I_2}{I_1} = 3.98 \approx 4$$

$$I_2 = 4I_1$$

$\therefore$ China's earthquake is nearly 4 times stronger than Pakistan's earthquake.

Example 19

Use laws of logarithm to evaluate.

$$(a) \frac{\sqrt{5.39} \times (55.6)^7}{2.51 \times \sqrt{2.12}} \quad \text{answer}$$

$$\text{Solution: (a) let } x = \frac{\sqrt{5.39} \times (55.6)^7}{2.51 \times \sqrt{2.12}}$$

Then, by taking common log on both sides

$$\log x = \log \frac{\sqrt{5.39} \times (55.6)^7}{2.51 \times \sqrt{2.12}}$$

$$= \log(5.39)^{\frac{1}{2}} + \log(55.6)^7 - \log(2.51) - \log(2.12)^{\frac{1}{2}}$$



$$= \frac{1}{2} \log 5.39 + 7 \log 55.6 - \log 2.51 - \frac{1}{2} \log(2.12)$$

$$= \frac{1}{2} (0.0340) + 7(1.7451) - (0.3997) - \frac{1}{2} (0.3261)$$

$$= 0.3170 + 12.2157 - 0.3997 - 0.1630$$

$$\log x = 12.2386$$

Taking antilog on both sides

$$\text{antilog } \log x = \text{antilog } 12.2386$$

$$x = 1732.21$$

$$= \frac{\sqrt{5.39} \times (55.6)^7}{2.51 \times \sqrt{2.12}} = 1732.21$$

$$(b) \text{ Solution: } \sqrt{\frac{5}{3}} \times \sqrt{\frac{22}{7}} \quad \text{answer}$$

$$\text{Let } x = \sqrt{\frac{5}{3}} \times \sqrt{\frac{22}{7}}$$

Taking common log on both sides

$$\log x = \log \left(\sqrt{\frac{5}{3}} \times \sqrt{\frac{22}{7}} \right)$$

$$\log x = \log \left(\frac{10}{3} \right)^{\frac{1}{2}} + \log \left(\frac{22}{7} \right)^{\frac{1}{2}}$$

$$\log x = \frac{1}{2} \log \left(\frac{10}{3} \right) + \frac{1}{2} \log \left(\frac{22}{7} \right)$$

$$\log x = \frac{1}{2} (\log 10 - \log 3) + \frac{1}{2} (\log 22 - \log 7)$$

$$\log x = \frac{1}{2} \log 10 - \frac{1}{2} \log 3 - \frac{1}{2} \log 7 + \frac{1}{2} \log 22$$

$$\log x = \frac{1}{2} (1.0000) - \frac{1}{2} \log(0.4771) - \frac{1}{2} \log(1.3010) + \frac{1}{2} (0.3424)$$

$$\log x = 0.4014 - 0.1385 - 0.0712 + 0.1712$$

$$\log x = 0.0929$$

$$\log x = (-1 + 1 + 0.0929) \text{ making numbers positive}$$

$$\log x = \bar{1}.0929$$

Taking antilog on both sides

$$\text{antilog } \log x = \text{antilog } (\bar{1}.0929)$$

$$x = 0.9858$$

Example 20: Find the number of digits in 9^m . answer

Solution:

The number of digits in a whole number is 1 more than the characteristic of the log of that number.

$$\text{Now } \log 9^m = m \log 9 \quad (\text{Using power law})$$

$$= 10 \times (0.9542)$$

$$= 9.542$$

$$\text{Characteristic} = 9$$

$$\text{Number of digits in } 9^m = \text{Characteristic} + 1 \\ = 9 + 1 = 10$$

Exercise 2.6

Q.1: Find the number of digits in:

(i) 3^{20} answer

Solution

$$\log 3^{20} = 20 \log 3 = 20(0.4771) = 9.542$$

$$\text{Characteristic of } \log 3^{20} = 9$$

$$\text{Number of digits in } 3^{20} = \text{Characteristic} + 1 \\ = 9 + 1 = 10$$

Hence number of digits in 3^{20} are 10.

(ii) 100^{100} answer

Solution

$$\log 100^{100} = 100 \log 100 = 100(2.0000) = 200.0$$

$$\text{Characteristic of } \log 100^{100} = 200$$

$$\text{Number of digits in } 100^{100} = \text{Characteristic} + 1 \\ = 200 + 1 = 201$$

Hence number of digits in 100^{100} are 201.

(iii) 2^{10} answer

Solution

$$\log 2^{10} = 10 \log 2 = 10(0.3010) = 3.010$$

$$\text{Characteristic of } \log 2^{10} = 3$$

$$\text{Number of digits in } 2^{10} = \text{Characteristic} + 1 \\ = 3 + 1 = 4$$

Hence number of digits in 2^{10} are 4.



(iii) 5^{17}

Answer

Solution

$$\log 5^{17} = 17 \log 5 = 17 (0.6989) = 11.8813$$

$$\text{Number of digits in } 5^{17} = \text{Characteristics} + 1 \\ = 11 + 1 = 12$$

Hence number of digits in 5^{17} are 12.

(iv) 529^{10}

Answer

Solution

$$\log 529^{10} = 10 \log 529 = 10 (2.7234) = 27.234$$

$$\text{Characteristics of } \log 529^{10} = 27$$

$$\text{Number of digits in } 529^{10} = \text{Characteristics} + 1 \\ = 27 + 1 = 28$$

Hence number of digits in 529^{10} are 28.

(v) 23^{17}

Answer

Solution

$$\log 23^{17} = 17 \log 23 = 17 (1.3617) = 23.1489$$

$$\text{Characteristics of } \log 23^{17} = 23$$

$$\text{Number of digits in } 23^{17} = \text{Characteristics} + 1 \\ = 23 + 1 = 24$$

Hence number of digits in 23^{17} are 24.

Q.3 Evaluate applying laws of logarithms.

(i) 23.57×5.967

Answer

Solution

$$\text{Let } x = 23.57 \times 5.967$$

Taking log of both sides

$$\log x = \log(23.57 \times 5.967)$$

$$\log x = \log 23.57 + \log 5.967$$

$$\log x = 1.3724 + 0.7757$$

$$\log x = 2.1481$$

$$\log x = 2.1481$$

Taking antilog we get

$$\text{Ch.} = 2$$

$$\text{Mantissa} = 0.1481$$

$$x = \text{antilog } 2.1481$$

$$x = 140.6 \times 140.6$$

$$\text{Thus } 23.57 \times 5.967 = 140.6$$

$$\text{Thus } 23.57 \times 5.967 = 140.6$$

$$\text{(ii) } \frac{65.89}{7.392}$$

Answer

Solution

$$\text{Let } x = \frac{65.89}{7.392}$$

$$\text{Let } x = \frac{65.89}{7.392}$$

Taking log of both sides

$$\log x = \log \frac{65.89}{7.392}$$

$$\log x = \log 65.89 - \log 7.392$$

$$\log x = 1.8188 - 0.8688$$

$$\log x = 0.9500$$

$$\log x = 0.9500$$

Taking antilog we get

$$\text{Ch.} = 0$$

$$\text{Mantissa} = 0.9500$$

$$x = \text{antilog } 0.9500$$

$$x = 8.9125 \times 8.9125$$

$$\text{Thus } \frac{65.89}{7.392} = 8.9125$$

$$\text{(iii) } \frac{47.27 \times 5.321}{9.712 \times 4.171}$$

Answer

$$\text{Let } x = \frac{47.27 \times 5.321}{9.712 \times 4.171}$$

$$\text{Let } x = \frac{47.27 \times 5.321}{9.712 \times 4.171}$$

$$\text{Let } x = \frac{47.27 \times 5.321}{9.712 \times 4.171}$$

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$$\text{Let } x = \frac{47.27 \times 5.321}{9.712 \times 4.171}$$



Taking log of both sides

$$\log x = \log \frac{\sqrt[3]{27.98}}{\sqrt[3]{28.73}}$$

$$\log x = \log \frac{\sqrt[3]{27.98}}{\sqrt[3]{28.73}}$$

$$\log x = \log \sqrt[3]{27.98} - \log \sqrt[3]{28.73}$$

$$\log x = \log(27.98)^{\frac{1}{3}} - \log(28.73)^{\frac{1}{3}}$$

$$\log x = \frac{1}{3} \log(27.98) - \frac{1}{3} \log(28.73)$$

$$\log x = \frac{1}{3} (1.4469) - \frac{1}{3} (1.4584)$$

$$\log x = 0.4823 - 0.4861$$

$$\log x = -0.0038$$

$$\log x = -1 + 1 - 0.0038 \leftarrow (\text{Making mantissa} + \text{Ve})$$

$$\log x = \bar{1}.9961$$

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$$\log x = \bar{1}.9961$$

$$\log x = \bar{1}.9961$$

$$\log x = \bar{1}.9961$$

Taking antilog we get

$$\text{Ch.} = \bar{1}$$

$$\text{Mantissa} = .9961$$

$$x = \text{antilog } (\bar{1}.9961)$$

$$x = 0.0546 \times 0.0546$$

$$\text{Thus } \frac{\sqrt[3]{27.98}}{\sqrt[3]{28.73}} = 0.0546$$

$$\text{(vi) } \frac{\sqrt{38.24} \times \sqrt{1.931}}{964.4 \times \sqrt{23.91}}$$

$$\text{Answer}$$

$$\text{Let } x = \frac{\sqrt{38.24} \times \sqrt{1.931}}{964.4 \times \sqrt{23.91}}$$

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$$\text{Let } x = \frac{\sqrt{38.24} \times \sqrt{1.931}}{964.4 \times \sqrt{23.91}}$$

$$\text{Let } x = \frac{\sqrt{38.24} \times \sqrt{1.931}}{964.4 \times \sqrt{23.91}}$$



$$(14) \sqrt{\frac{16 \frac{3}{4}}{0.23}}$$

Solution:

$$\text{Let } x = \sqrt{\frac{16 \frac{3}{4}}{0.23}}$$

Taking log of both side:

$$\log x = \log \sqrt{\frac{16 \frac{3}{4}}{0.23}}$$

$$\log x = \log \sqrt{16 \frac{3}{4}} - \log \sqrt{0.23}$$

$$\log x = \log \left(\frac{67}{4} \right) - \log(0.23)$$

$$\log x = \frac{1}{2} \log \frac{67}{4} - \frac{1}{2} \log 0.23$$

$$\log x = \frac{1}{2} (\log 67 - \log 4) - \frac{1}{2} \log 0.23$$

$$\log x = \frac{1}{2} (1.8261 - 0.6021) - \frac{1}{2} (1.7241)$$

$$\log x = \frac{1}{2} (1.2240) - \frac{1}{2} (1.7241)$$

$$\log x = 0.6120 - 0.8620$$

$$\log x = -0.2500$$

Taking antilog we get

$$x = 0.562$$

$$\text{Magnitude} = 0.562$$

$$x = \text{antilog}(-0.2500)$$

$$x = 0.562$$

$$\text{Then } \sqrt{\frac{16 \frac{3}{4}}{0.23}} = 0.562$$

$$(15) \frac{(27.98)^2}{(28.73)^2}$$

Solution:

$$\text{Let } x = \frac{(27.98)^2}{(28.73)^2}$$

Taking log of both sides

$$\log x = \log \frac{(27.98)^2}{(28.73)^2}$$

$$\log x = \log(27.98)^2 - \log(28.73)^2$$

$$\log x = 2 \log(27.98) - 2 \log(28.73)$$

$$\log x = 2(1.4469) - 2(1.4584)$$

$$\log x = 2.8938 - 2.9168$$

$$\log x = -0.0230$$

$$\log x = -2 + 2 - 0.0230 = - (\text{Making mantissa } +ve)$$

$$\log x = \bar{2}.9769$$

Taking antilog we get:

$$x = 0.937$$

$$\text{Magnitude} = 0.937$$

$$x = \text{antilog}(\bar{2}.9769)$$

$$x = 0.02301 = 0.02301$$

$$\text{Thus } \frac{(27.98)^2}{(28.73)^2} = 0.02301$$

Q.2: The Kauei, China earthquake of 1920 was measured 8.5 on Richter Scale and the Tokyo, Japan earthquake of 1923 was measured 7.8 on that scale, how many times stronger was the 1920 earthquake than 1923 earthquake?

Solution:

If I is intensity of shock waves and I_0 is a constant then its magnitude is given as

$$M = \log \left(\frac{I}{I_0} \right)$$

Let I_1 be the intensity of China's earthquake of 1920 and I_2 be the intensity of Japan's earthquake of 1923.

Magnitude of China's earthquake is 8.5 so

$$8.5 = \log \left(\frac{I_1}{I_0} \right)$$

$$8.5 = \log I_1 - \log I_0 \quad \text{.....(i)}$$



Magnitude of Japan's earthquake is 7.8 so

$$7.8 = \log \left(\frac{I_2}{I_0} \right)$$

$$7.8 = \log I_2 - \log I_0 \quad \text{.....(ii)}$$

Subtracting eq. (i) from eq. (ii)

$$8.5 - 7.8 = (\log I_1 - \log I_0) - (\log I_2 - \log I_0)$$

$$0.7 = \log I_1 - \log I_2 - \log I_0 + \log I_0$$

$$0.7 = \log I_1 - \log I_2$$

$$0.7 = \log \left(\frac{I_1}{I_2} \right)$$

$$\log x = \log \frac{a}{b} = \log \left(\frac{a}{b} \right)$$

$$\log \left(\frac{I_1}{I_2} \right) = 0.7$$

taking antilog of both sides

$$\frac{I_1}{I_2} = \text{antilog}(0.7)$$

$$\frac{I_1}{I_2} = 5.011 \approx 5$$

$$I_1 = 5 I_2$$

Thus the China's earthquake of 1920 was nearly 5 times stronger than the Japan's earthquake of 1923.

Miscellaneous Exercise 2

1. Encircle the correct option in the following:

1. If $a = b = 10^x$ is written in scientific notation then: correct

(a) $0 < x < 10$ (b) $0 < x < 10$

(c) $1 < x < 10$ (d) $1 < x < 10$

2. In 0.337, reference position is: correct

(a) after 0 (b) after 7

(c) after 3 (d) before 7

3. $\log(100)$ is: correct

(a) 2 (b) -2

(c) 0 (d) impossible

4. If $\log(x-3) = \log(13x-4)$ then x is: correct

(a) 6.5 (b) 7

(c) 17 (d) 2

5. $\log(7^2) = \log_7 4^2$ is: correct

(a) 3 (b) -3

(c) 9 (d) -9

6. For the log 0.00327, characteristic is: correct

(a) -2 (b) -3

(c) 3 (d) 0

7. $\log_5(M \times N)$ is: correct

(a) $\log_5 M \times \log_5 N$ (b) $\log_5 M + \log_5 N$

(c) both a and b (d) none of these

8. $\log_2 8$ is: correct

(a) $\log_2 8$ (b) $\log_2(8)$

(c) $(\log_2 8) \times 8$ (d) $8 \log_2 8$

9. $\log_5 M - \log_5 N$ is: correct

$$(a) \frac{\log_5 M}{\log_5 N}$$

$$(b) \log_5 \frac{M}{N}$$

$$(c) \log_5 M$$

$$(d) \frac{\log_5 N}{\log_5 M}$$

10. $\log_{10} 100^2$ is: correct

$$(a) 2$$

$$(b) 1$$

$$(c) 4$$

$$(d) 8$$

11. $\log 18$ is: correct

$$(a) 3 \log 2 + \log 3$$

$$(b) \log 2 + 2 \log 3$$

$$(c) 2 \log 3 + 2 \log 2$$

$$(d) 2 \log 3 + 3 \log 2$$

12. $\log 5 - \log 8 + \log 2 - \log 2$ is: correct

$$(a) \log \frac{5 \times 2}{8 \times 2}$$

$$(b) \log \frac{15}{16}$$

$$(c) \log \frac{10}{8}$$

$$(d) \log -2$$

13. $\log_{10} 100^2$ is: correct

$$(a) 2$$

$$(b) 0$$

$$(c) 1$$

$$(d) impossible$$

14. Scientific notation of 6.25 is: correct

$$(a) 6.25 \times 10^2$$

$$(b) 6.25 \times 10^0$$

$$(c) 6.25 \times 10$$

$$(d) 0.625 \times 10^2$$

15. Base of natural logarithm is: correct

$$(a) \text{rational number}$$

$$(b) \text{integer}$$

$$(c) \text{irrational number}$$

$$(d) 10$$



16. If $\log_{10} 25 = x$ then x is: answer
 (a) $+1$ (b) -3
 (c) 25 (d) impossible
17. $\log_{10} 10^5 = \log_{10} 10$ is: answer
 (a) $\log_{10} \frac{10^5}{10}$
 (b) $\log_{10} 10^5 - \log_{10} 10$

- (c) 4 (d) $\log_{10} (10^5 - 10)$
 18. $3 \log 2 - 2 \log 5$ is: answer
 (a) $\frac{(\log 2)^3}{(\log 5)^2}$ (b) $\frac{\log 2^3}{\log 5^2}$
 (c) $\log \frac{2^3}{5^2}$ (d) $\frac{3 \log 2}{2 \log 5}$

Additional MCQs

19. If $a^x = a$, then: answer
 (a) $x = \log_a a$ (b) $x = \log_a a$
 (c) $x = \log_a a$ (d) $x = \log_a a$
20. The relation of $y = \log_a x$ implies: answer
 (a) $x^y = x$ (b) $x^y = x$
 (c) $x^y = y$ (d) $y^x = x$
21. The logarithm of unity in any base is: answer
 (a) 1 (b) 10
 (c) e (d) 0
22. The logarithm of any number to itself as base is: answer
 (a) 1 (b) 0
 (c) -1 (d) 10
23. $\log e = ______$ where $e = 2.718$: answer
 (a) 0 (b) 0.4343
 (c) ∞ (d) 1
24. $\log_{10} 10 = ______$ where $e = 2.718$: answer
 (a) 2.3026 (b) 0.4343
 (c) e^{18} (d) 10
25. $\log_a a = \log_a b$ can be written as: answer
 (a) $\log_a a$ (b) $\log_a a$
 (c) $\log_a b$ (d) $\log_a c$
26. Base of common log is: answer
 (a) 10 (b) e
 (c) π (d) 3
27. Base of natural logarithm is: answer
 (a) 10 (b) e
 (c) π (d) 1
28. Characteristic of $\log 4.9 \times 10^{-5}$ is:

- (a) -5 (b) 10
 (c) 4 (d) 9
29. $\log_a x$ will be equal to: answer
 (a) $\frac{\log_a x}{\log_a x}$ (b) $\frac{\log_a x}{\log_a x}$
 (c) $\frac{\log_a x}{\log_a y}$ (d) $\frac{\log_a x}{\log_a x}$
30. $\log \left(\frac{1}{a} \right) = ______$: answer
 (a) $\log 1$ (b) $\log a$
 (c) $\log (1 - a)$ (d) $-\log a$
31. $\log_a \frac{1}{a^2} = ______$: answer
 (a) -1 (b) -2
 (c) 2 (d) does not exist
32. If $\log_2^2 = x$, then $x = ______$: answer
 (a) 64 (b) 2^2
 (c) 3 (d) 2^6
33. If $\log_2^2 = 5$, then $x = ______$: answer
 (a) 25 (b) 32
 (c) 10 (d) 2^{25}
34. If $\log 27 = 1.431$, then value of $\log 9$ is: answer
 (a) 0.934 (b) 0.943
 (c) 0.934 (d) 0.938
35. If $\log \frac{a}{b} + \log \frac{b}{a} = \log(a + b)$, then: answer
 (a) $a + b = 1$ (b) $a - b = 1$
 (c) $a = b$ (d) $a^2 - b^2 = 1$



36. If $\log_{10} 70 = x$, then $\log_{10} \left(\frac{1}{70} \right) = ______$: answer
 (a) $-(1+x)$ (b) $(1+x)^{-1}$
 (c) $\frac{x}{10}$ (d) $-x$
37. The value of $\log 4 + \log 25$ is: answer
 (a) 2 (b) 3
 (c) 4 (d) 5
38. $\log_e 7$ is: answer
 (a) an integer
 (b) a rational number
 (c) an irrational number
 (d) a prime number
39. If $9 + e^{2x+1} = 10$ then the value of x is: answer
 (a) 2 (b) 3
 (c) 4 (d) 5
40. $\log \sqrt{10} = ______$: answer
 (a) -1 (b) $-\frac{1}{2}$
 (c) $\frac{1}{2}$ (d) 2
41. For any non-zero value of x , $x^0 = ______$: answer
 (a) 2 (b) 1
 (c) 0 (d) 10
42. Evaluate $\log_e \frac{1}{\sqrt{e}}$: answer
 (a) -1 (b) $-\frac{1}{2}$

- (c) $\frac{1}{2}$ (d) 2
 43. Evaluate $\log_e \sqrt[3]{81} = ______$: answer
 (a) $\frac{2}{3}$ (b) $\frac{1}{3}$
 (c) $\frac{2}{9}$ (d) $\frac{4}{9}$
44. Simplifying $10^2 \times 10^3 = 10^5$ we get: answer
 (a) 10^{20} (b) 10^{25}
 (c) 10^6 (d) 10^7
45. If $\log_a x = 4$ and $\log_a y = 2$ then value of $\log_a (x^2 y^3)$ is: answer
 (a) 3 (b) 7
 (c) 9 (d) 10
46. $\log x = 2$ $\log y = ______$: answer
 (a) $\log \frac{x}{y}$ (b) $\log xy$
 (c) $\log \frac{x^2}{y}$ (d) $\log \frac{x}{y^2}$
47. Number of digits in 10^5 is: answer
 (a) 32 (b) 34
 (c) 35 (d) 33
48. Which of the following is equivalent to "y is natural log of x"? answer
 (a) $e^x = x$ (b) $e^x = y$
 (c) $y^x = x$ (d) $x^y = y$

Answer Key

1	d	3	c	5	d	7	a	9	a
6	b	7	d	8	c	9	b	10	d
11	b	12	b	13	b	14	b	15	c
16	a	17	c	18	c	19	c	20	b
21	d	22	a	23	b	24	a	25	a
26	a	27	b	28	a	29	c	30	d
31	b	32	c	33	b	34	c	35	a
36	d	37	a	38	c	39	a	40	c
41	b	42	b	43	d	44	c	45	d
46	d	47	c	48	a				



Q.7: Simplify with the help of laws of

$$\text{logarithm: } \left(\frac{0.5327 \times \sqrt{42.97}}{0.0039} \right)^2 = \dots$$

Solution

$$\text{Let } x = \left(\frac{0.5327 \times \sqrt{42.97}}{0.0039} \right)^2$$

$$\text{Or } x = \frac{(0.5327)^2 \times (\sqrt{42.97})^2}{(0.0039)^2}$$

Taking log both sides

$$\log x = \log \frac{(0.5327)^2 \times 42.97}{(0.0039)^2}$$

$$\log x = \log(0.5327)^2 + \log 42.97 - \log(0.0039)^2$$

$$\log x = 2 \log(0.5327) + \log 42.97 - 2 \log(0.0039)$$

$$\log x = 2(-1.7263) + 1.6332 - 2(-3.7709)$$

$$\log x = 2(-1 + .7265) + 1.6332 - 2(-3 + .7709)$$

$$\log x = 2(-0.2735) + 1.6332 - 2(-2.2291)$$

$$\log x = -0.5470 + 1.6332 + 4.4582$$

$$\log x = 5.5444$$

Taking antilog we get

$$\text{Ch.} = 7$$

$$\text{Mantissa} = .5444$$

$$\text{antilog}(\log x) = \text{antilog}(5.5444)$$

$$x = 3,542,2776.6$$

$$\text{Thus } \left(\frac{0.5327 \times \sqrt{42.97}}{0.0039} \right)^2 = 3,542,2776.6$$

Q.8: Prove that $\log_e U = \log_e V + \log_e W = 1$

Solution

$$\text{L.H.S} = \log_e U = \log_e V + \log_e W$$

By using change of Base law

$$\text{L.H.S} = \frac{\log U}{\log V} + \frac{\log V}{\log W} + \frac{\log W}{\log U}$$

$$\text{L.H.S} = \frac{\log U}{\log U} + \frac{\log V}{\log V} + \frac{\log W}{\log W}$$

$$\text{L.H.S} = 1 + 1 + 1$$

$$\text{L.H.S} = 3$$

$$\text{L.H.S} = \text{R.H.S}$$

Hence prove

UNIT 03

Sets and Relations

Set

A set is a well-defined collection of distinct objects. The term well-defined, means that the objects follow a given discipline with which presence or absence of some object in the set is checked.

Example 1

- A set of Prime numbers which are also even i.e. {2}
- A set of Pakistani currency i.e. {5, 10, 20, 50, 100, 500, 1000, 5000}
- A set of flowers in my garden i.e. {Pansy, Lily, Daisy, Jasmine, Tulip, Rose, Hibiscus}
- A set of Natural numbers less than 1 i.e. { }
- A set of all letters present in the word 'set' i.e. {s, e, t}
- A set of Natural numbers among 1, -0.2, 3, 4, 2 i.e. {2, 3, 4}

Note

The repeating elements are taken once only, since repetition is not allowed in a set.

Set Builder Form (Rule Method)

In set builder form, all the elements of a set are not listed, however we write the set by its defining rule. While writing a set in this method, some variable say x is chosen which represents all the elements of that set according to the defining rule.

e.g. $A = \text{Set of all integers}$, can be written in set builder form as $A = \{x | x \in \mathbb{Z}\}$ or read as " A is the set of all elements x such that x belongs to $\mathbb{Z}$ "

Example 2

Write the following sets in the set builder form.

- $B = \text{Set of Prime numbers less than 17}$
 $B = \{x | x \in P \wedge x < 17\}$
- $C = \text{Set of multiples of 4 greater than or equal to 40}$
 $C = \{4x | x \in \mathbb{N} \wedge x \geq 10\}$
 $\text{or } C = \{12, 16, 20, \dots\}$
- $D = \{x | x \text{ is a factor of 6}\}$

Example 3

- The set $G = \{15x | x \in \mathbb{Z} \wedge x \geq 1\}$, in descriptive form is written as
 $G = \text{Set of integral multiples of 15, greater than or equal to 15}$
- The set $H = \{x | x \in \mathbb{W} \wedge x^2 - 1 = 7\}$, in tabular form is $H = \{2\}$
- The set $I = \{x | x \in \mathbb{W} \wedge -3 > x > -5\}$, in tabular form is $I = \{-4\}$
- The set $J = \left\{ \frac{1}{x} | x \in \mathbb{R} \wedge x > 0 \right\}$, in tabular form is $J = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$
- The set $K = \{y | y \in \mathbb{Z}^+ \wedge y^2 = 9\}$, in descriptive method is
 $K = \text{Set of positive integers containing '3' only}$

Most Commonly used Sets of Numbers

- Set of Natural numbers
 $\mathbb{N} = \{1, 2, 3, 4, \dots\}$
- Set of Whole numbers
 $\mathbb{W} = \{0, 1, 2, 3, 4, \dots\}$



iii. Set of Integers

$$Z = \{x | x \in \{1, 2, 3, \dots\}\}$$

iv. Set of Even numbers

$$E = \{x | x \in \{2, 4, 6, \dots\}\}$$

v. Set of Odd numbers

$$O = \{x | x \in \{1, 3, 5, \dots\}\}$$

vi. Set of Prime numbers

$$P = \{2, 3, 5, 7, 11, 13, \dots\}$$

vii. Set of Rational numbers

$$Q = \left\{ \frac{p}{q} \mid p, q \in Z, q \neq 0 \right\}$$

viii. Set of Irrational numbers

$$Q' = \left\{ x \mid x \neq \frac{p}{q}, p, q \in Z, q \neq 0 \right\}$$

ix. Set of Real numbers

$$R = \{x \mid x \in Q \vee x \in Q'\}$$

Note

The above mentioned sets from (i)-(vi) are first written in descriptive method and then in tabular form (Roster method).

However, the sets from (vii)-(ix) are first written in descriptive method then in set builder form.

Set Operations

Union of Sets

Let A and B be two given sets. Then union of A and B is the set of all those elements which are taken either from A or from B or from both.

The union of A and B is denoted by ' $A \cup B$ ' and

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in \text{both}\}$$

e.g. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6\}$, then:

$$A \cup B = \{1, 2, 3, 4, 5, 6\}$$

Intersection of Sets

Let A and B be given sets then the intersection of A and B is the set of elements which belong to both A and B.

The intersection of A and B is denoted by ' $A \cap B$ ' and

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

e.g. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 7, 8\}$, then: $A \cap B = \{3, 4, 5\}$

Difference of Sets

Difference of two sets A and B, denoted by $A - B$ is a collection of those elements of A which are not present in B. For the two sets A and B:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 7, 8\}$, then: $A - B = \{1, 2, 3, 4, 5\} - \{3, 4, 5, 7, 8\} = \{1, 2, 3\}$

In general, $A - B \neq B - A$

Complement of a Set

Let $A \subset U$ (i.e. A is proper subset of the universal set). Then the set of all elements of U, which are not in A, is called complement of A. Complement of A is denoted by A' or A^c and is defined as:

$$A' = \{x \mid x \in U \text{ and } x \notin A\}$$

e.g. If $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6\}$, then: $A' = U - A = \{1, 3, 5, 7\}$

Disjoint Sets

Two sets A and B are said to be disjoint if they have no common elements, i.e. $A \cap B = \emptyset$

For example,

$A = \{1, 2, 3\}$ and $B = \{4, 5\}$ are disjoint sets as both sets have no common element, i.e.

$$A \cap B = \{1, 2, 3\} \cap \{4, 5\} = \emptyset$$

Overlapping Sets

Two sets A and B are called overlapping if:

(i) There is at least one element common to both the sets.

(ii) Neither of the sets is a subset of other set.

For example, the sets

$A = \{2, 3, 5, 7, 9\}$ and $B = \{3, 5, 7, 8\}$ are overlapping as $A \cap B = \{3, 5, 7\} \neq \emptyset$ and A and B are not subsets of each other.



Solved

Set A is called subset of a set B if every element of set A is also an element of B.

For Example, the Set $A = \{a, b, c\}$ is a subset of $B = \{a, b, c, d, \dots\}$ as all elements of set A are also elements of Set B.

Example 1

Represent sets $A = \{0, 1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{6, 7, 8, 9\}$ through Venn diagram.

Solution:

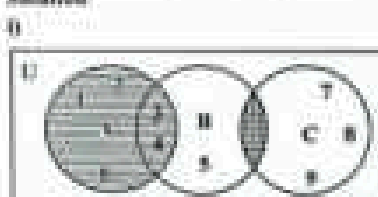


Example 2

If $A = \{0, 1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{6, 7, 8, 9\}$, find $A \cup (B \cap C)$ and $A \cap (B \cup C)$ through Venn diagram.

Solution:

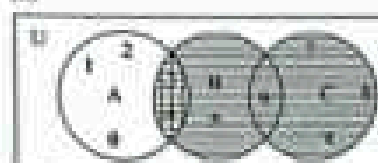
Solution:



$B \cap C = \{6\}$ = Region of

$A \cup (B \cap C) = \{0, 1, 2, 3, 4, 6\}$ = Region of

(ii)



$B \cup C = \{3, 4, 5, 6, 7, 8, 9\}$ = Region of

$A \cap (B \cup C) = \{3, 4\}$ = Region of

Example 3

If $P = \{a, b, c, d, e\}$, $Q = \{c, d, e, f\}$, $R = \{d, e, f, g, h\}$ find

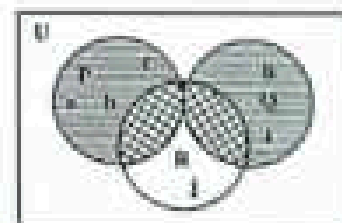
(i) $P \cup Q \cap R$

(ii) $(P \cap Q) \cap R$

Through Venn diagram.

Solution:

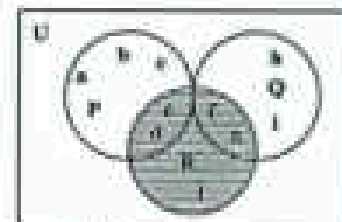
(i)



$P \cup Q = \{a, b, c, d, e, f\}$ = Region of

$(P \cup Q) \cap R = \{d, e\}$ = Region of

(ii)



$P \cap Q = \{c, d, e\}$ (No Region)

$(P \cap Q) \cap R = \{d, e\}$ = Region of

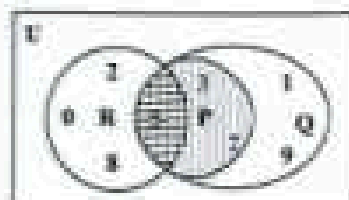
Example 7

If $P = \{3, 5, 7\}$, $Q = \{1, 3, 5, 7, 9\}$, $R = \{0, 2, 4, 6\}$, then find $P \cup (Q \cap R)$ using

Venn diagram.

Solution:

$P = \{3, 5, 7\}$, $Q = \{1, 3, 5, 7, 9\}$, $R = \{0, 2, 4, 6\}$



$Q-R = (3)$ = Regions of

$P \cup (Q-R)$ = Regions of and

Verification of Associative Laws Using Venn Diagram

Associative Property of Union

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof

Let $x \in (A \cup B) \cup C$

$\Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C$

$\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C)$

$\Rightarrow x \in A \cup (B \cup C)$

$\Rightarrow (A \cup B) \cup C \subseteq A \cup (B \cup C) \dots (i)$

Similarly, we can prove that

$$(A \cup B) \cup C \supseteq A \cup (B \cup C) \dots (ii)$$

From (i) and (ii), we have:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Example 8

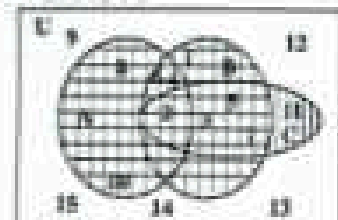
If $U = \{1, 2, 3, 4, 5, \dots, 15\}$, $A = \{2, 4, 6, 8, 10\}$,

$B = \{1, 2, 3, 4, 5, 6, 7\}$ and $C = \{2, 3, 5, 7, 11\}$,

then verify the associative property of union using Venn diagram.

Solution

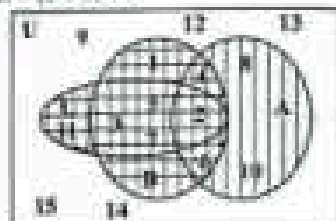
$$L.H.S = (A \cup B) \cup C$$



$(A \cup B) \cup C$ = Regions of

$(A \cup B) \cup C$ = Regions of

$$R.H.S = (B \cup C) \cup A$$



$(B \cup C) \cup A$ = Regions of

$A \cup (B \cup C)$ = Regions of

From both the figures, it is observed that same regions are shaded.

∴ $(A \cup B) \cup C = A \cup (B \cup C)$

Associative Property of Intersection

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Proof

Let $x \in (A \cap B) \cap C$

$\Rightarrow x \in A \cap B$ and $x \in C$

$\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C$

$\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \in C)$

$\Rightarrow x \in A \text{ and } x \in (B \cap C)$

$\Rightarrow x \in A \cap (B \cap C)$

$\Rightarrow (A \cap B) \cap C \subseteq A \cap (B \cap C) \dots (i)$

Similarly, we can prove that:

$$(A \cap B) \cap C \supseteq A \cap (B \cap C) \dots (ii)$$

From (i) and (ii), we have:

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Example 9

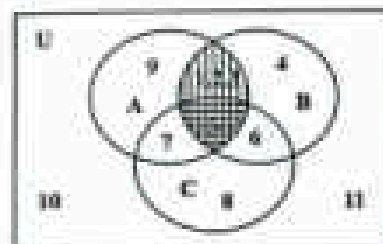
If $U = \{1, 2, 3, \dots, 11\}$, $A = \{1, 2, 3, 5, 7, 9\}$,

$B = \{1, 2, 3, 4, 5, 8\}$ and $C = \{3, 6, 7, 8\}$ then

verify the associative property of intersection.

Solution

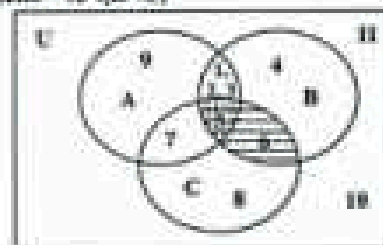
$$L.H.S = (A \cap B) \cap C$$



$A \cap B$ = Regions of

$(A \cap B) \cap C$ = Region of

$$R.H.S = A \cap (B \cap C)$$



$B \cap C$ = Regions of

$A \cap (B \cap C)$ = Region of

From both the figures it is observed that same regions are shaded.

∴ $(A \cap B) \cap C = A \cap (B \cap C)$

∴ Associative property of intersection is verified.

Verification of Distributive Laws Using Venn Diagram

(a) Distributive Property of Union over Intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or $(x \in B \text{ and } x \in C)$

$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$

$\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \dots (i)$

Similarly, we can prove that:

$$A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C) \dots (ii)$$

Again, let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow (y \in A \text{ or } y \in B) \text{ and } (y \in A \text{ or } y \in C)$

$\Rightarrow y \in A \text{ or } (y \in B \text{ and } y \in C)$

$\Rightarrow y \in A \text{ or } y \in (B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$

$\Rightarrow (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \dots (iii)$

From (i) and (iii), we have:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(b) Distributive Property of Intersection over Union

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof

Let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B \text{ or } x \in C)$

$\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \dots (i)$

Again, let $y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow y \in (A \cap B) \text{ or } y \in (A \cap C)$

$\Rightarrow (y \in A \text{ and } y \in B) \text{ or } (y \in A \text{ and } y \in C)$

$\Rightarrow y \in A \text{ and } (y \in B \text{ or } y \in C)$

$\Rightarrow y \in A \text{ and } y \in (B \cup C)$

$\Rightarrow y \in A \cap (B \cup C)$

$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \dots (ii)$

From (i) and (ii), we have:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example 10

Verify through Venn diagram

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Where $U = \{x : x \in W \text{ and } x \in Z\}$,

$A = \{x : x \in W \text{ and } x \in Z\}$,

$B = \{x : x \in W \text{ and } x \in Z\}$ and

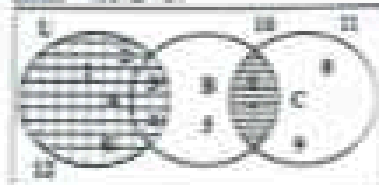
$C = \{x : x \in W \text{ and } x \in Z\}$

Solution

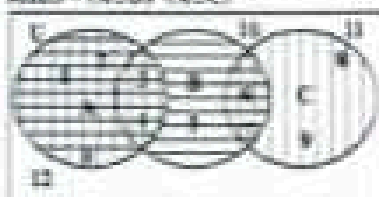
(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



$$B \cap C = A \cup (B \cap C)$$

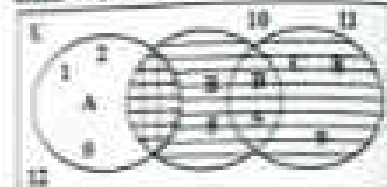


$B \cap C$ = Region of B and C
 $A \cup (B \cap C)$ = Region of A and $\text{B} \cap \text{C}$
 $B \cap C = A \cup (B \cap C)$

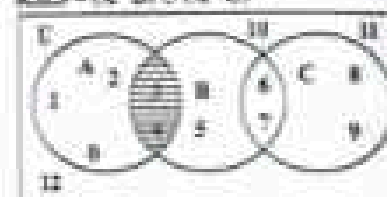


$A \cup B$ = Region of A and B
 $A \cup C$ = Region of A and C
 $(A \cup B) \cap (A \cup C)$ = Region of A
 From both the figures it is observed that same regions are shaded.
 So, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$\text{Let } A \cap (B \cap C) = A \cap (B \cap C) \cap A^c$$



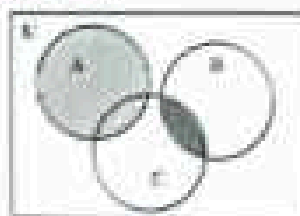
$B \cap C$ = Region of B and C
 $A \cap (B \cap C)$ = Region of A
 $B \cap C = A \cap (B \cap C) \cup (A^c \cap (B \cap C))$



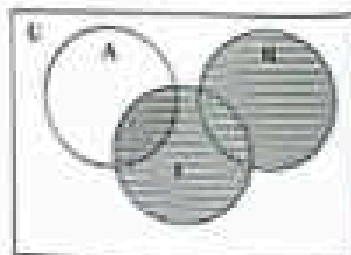
$A \cap B$ = Region of A and B
 $A \cap C$ = (No region)
 $(A \cap B) \cup (A \cap C)$ = Region of A
 From both the figures it is observed that same regions are shaded. So,
 $A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$

Exercise 1.1

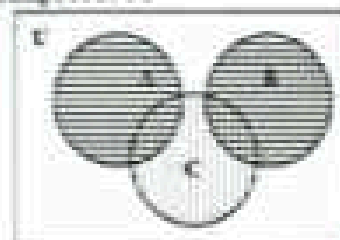
- Q.1 Shade $A \cup (B \cap C)$, $A \cap (B \cup C)$, $(A \cup B) \cap C$ and $A \cap (B \cap C)$ using following Venn diagrams.
- (i) Solution (i)
 Shading $A \cup (B \cap C)$



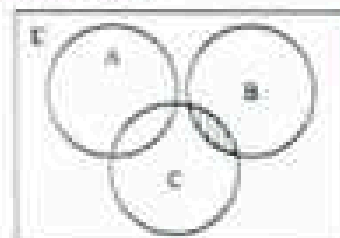
- (i) The region of B shows $B \cap C$.
 (ii) The regions of B and C show $A \cup (B \cap C)$.
 Shading $A \cap (B \cup C)$



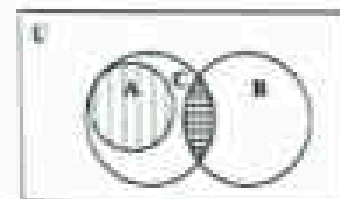
$B \cup C$ = Region of B and C
 $A \cap (B \cup C)$ = Region of A
 Shading $A \cup (B \cap C)$



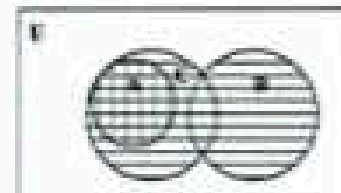
$A \cup B$ = Region of A and B
 $(A \cup B) \cap C$ = Region of C and $\text{A} \cup \text{B}$
 Shading $A \cap (B \cap C)$



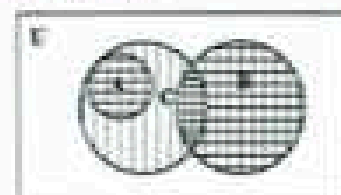
$B \cap C$ = Region of B and C
 $A \cap (B \cap C)$ = (No region)
 (ii) Solution
 Shading $A \cup (B \cap C)$



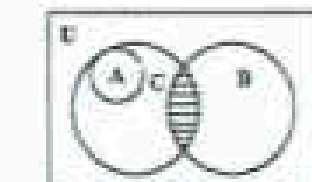
$B \cap C$ = Region of B and C
 $A \cup (B \cap C)$ = Region of A and $\text{B} \cap \text{C}$
 Shading $A \cap (B \cup C)$



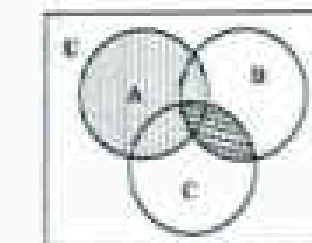
$B \cup C$ = Region of B and C
 $A \cap (B \cup C)$ = Region of A
 Shading $A \cup (B \cap C)$



$A \cup B$ = Region of A and B
 $(A \cup B) \cap C$ = Region of C and $\text{A} \cup \text{B}$
 Shading $A \cap (B \cap C)$



$B \cap C$ = Region of B and C
 $A \cap (B \cap C)$ = (No region)
 (iii) Solution
 Shading $A \cup (B \cap C)$

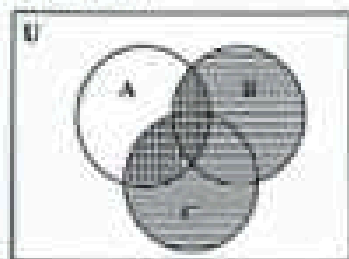




$(B \cap C)$ = Region of

$A \cup (B \cap C)$ = Region of and

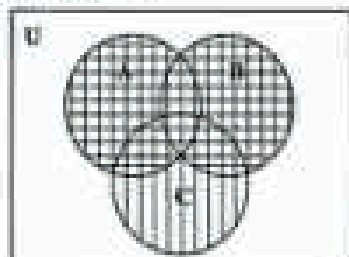
Shading $A \cap (B \cup C)$



$(B \cup C)$ = Region of and

$A \cap (B \cup C)$ = Region of and

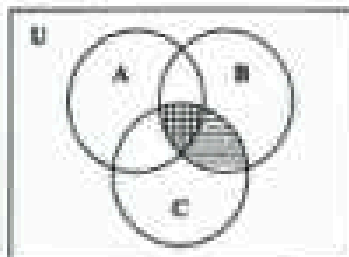
Shading $(A \cup B) \cup C$



$(A \cup B)$ = Region of and

$(A \cup B) \cup C$ = Region of and

Shading $A \cap (B \cap C)$



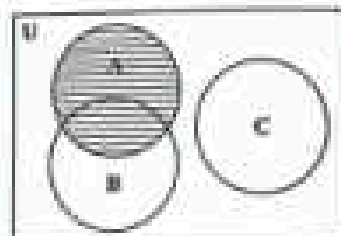
$(B \cap C)$ = Region of and

$A \cap (B \cap C)$ = Region of and

(iv) Solution

Shading $A \cup (B \cap C)$

INTERSECTION

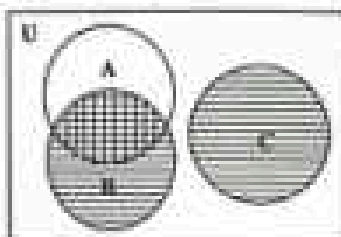


$(B \cap C) = \emptyset$ (No region)

$A \cup (B \cap C)$ = Region of and

Hint: $A \cap \emptyset = A$

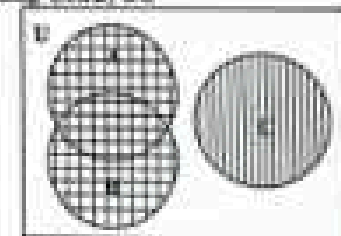
Shading $A \cap (B \cup C)$



$(B \cup C)$ = Region of and

$A \cap (B \cup C)$ = Region of and

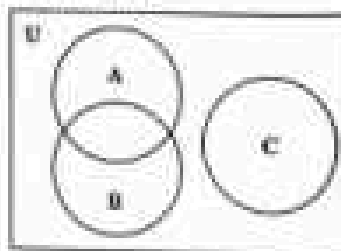
Shading $(A \cup B) \cup C$



$(A \cup B)$ = Region of and

$(A \cup B) \cup C$ = Region of and

Shading $A \cap (B \cap C)$



$(B \cap C) = \emptyset$ (No region)

$A \cap (B \cap C) = \emptyset$ (No region)

Q.2 If $X = \{a, b, c, d, e\}$, $Y = \{a, c, e\}$, $Z = \{g, h, i, j\}$ then, find the following using venn diagram.

i) $(X \cup Y) \cup Z$

INTERSECTION

Solution

$X = \{a, b, c, d, e\}$

$Y = \{a, c, e\}$

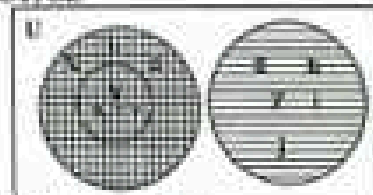
$Z = \{g, h, i, j\}$

We observe that $Y \subset X$ (Y is subset of X)

$X \cap Z = \emptyset$ (X and Z are disjoint sets)

$Y \cap Z = \emptyset$ (Y and Z are disjoint sets)

$(X \cup Y) \cup Z$



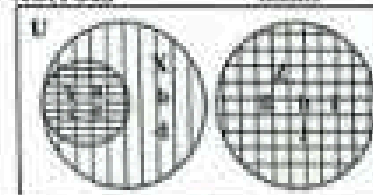
$(X \cup Y)$ = Region of and

$(X \cup Y) \cup Z$ = Region of and

$(X \cup Y) \cup Z = \{a, b, c, d, e, g, h, i, j\}$

ii) $X \cap (Y \cup Z)$

INTERSECTION



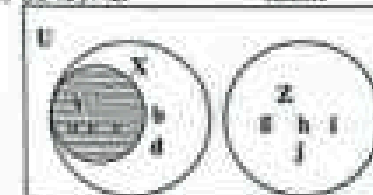
$(Y \cup Z)$ = Region of and

$X \cap (Y \cup Z)$ = Region of and

$X \cap (Y \cup Z) = \{a, b, c, d, e, g, h, i, j\}$

iii) $(X \cap Y) \cup Z$

INTERSECTION

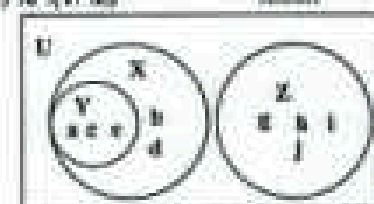


$(X \cap Y)$ = Region of = $Y = \{a, c, e\}$

$(X \cap Y) \cup Z = \emptyset$ (No region)

iv) $X \cap (Y \cap Z)$

INTERSECTION

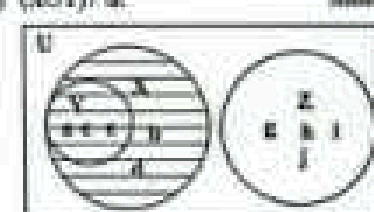


$(Y \cap Z) = \emptyset$ (No Region)

$X \cap (Y \cap Z) = \emptyset$ (No Region)

v) $(X \cup Y) \cap Z$

INTERSECTION

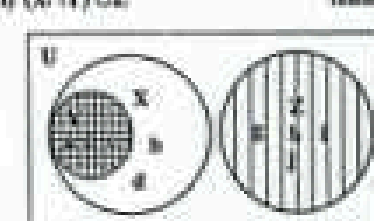


$(X \cup Y)$ = Region of and

$(X \cup Y) \cap Z = \emptyset$ (No Region)

vi) $(X \cap Y) \cup Z$

INTERSECTION



$(X \cap Y)$ = Region of = $Y = \{a, c, e\}$

$(X \cap Y) \cup Z$ = Region of and = $\{a, c, e, g, h, i, j\}$

Q.3 Verify associative law of union and intersection by using diagrams of question 1.

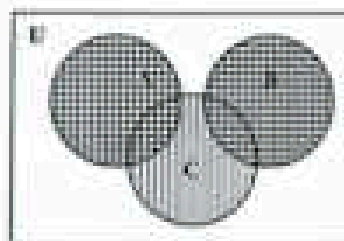
i) Solution Figure 1

(ii) Associative law of union

INTERSECTION

$(A \cup B) \cup C = A \cup (B \cup C)$

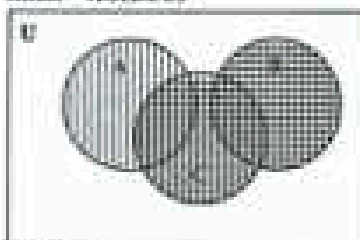
L.H.S = $(A \cup B) \cup C$



$$A \cup B = \text{Region of } \text{shaded area}$$

$$(A \cap B) \cup C = \text{shaded area and } \text{shaded area}$$

$$R.H.S = A \cup (B \cap C)$$



$$(B \cap C) = \text{shaded area}$$

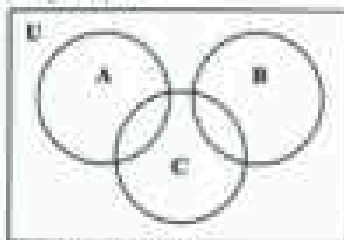
$$A \cup (B \cap C) = \text{shaded area and } \text{shaded area}$$

From both the figures, it is observed that the regions showing $(A \cap B) \cup C$ and $A \cup (B \cap C)$ are same i.e. $(A \cap B) \cup C = A \cup (B \cap C)$. Hence, associative law is verified.

(b) Associative law of intersection

$$(A \cap B) \cap C = A \cap (B \cap C)$$

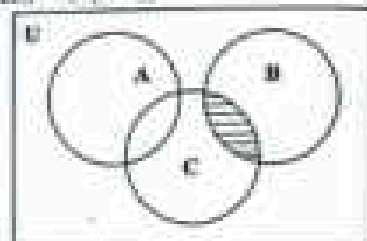
$$L.H.S = (A \cap B) \cap C$$



$$(A \cap B) = \text{No Region}$$

$$(A \cap B) \cap C = \text{No Region}$$

$$L.H.S = A \cap (B \cap C)$$



$$(B \cap C) = \text{Region of } \text{shaded area}$$

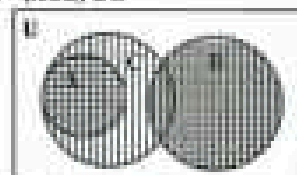
$$A \cap (B \cap C) = \text{shaded area}$$

From both the figures, we observe that $(A \cap B) \cap C = A \cap (B \cap C)$. Hence, associative law of intersection is verified.

(c) Associative law of union

$$(A \cup B) \cup C = A \cup (B \cup C)$$

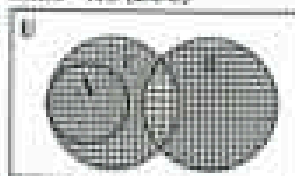
$$L.H.S = (A \cup B) \cup C$$



$$(A \cup B) = \text{Region of } \text{shaded area}$$

$$(A \cup B) \cup C = \text{Region of } \text{shaded area and } \text{shaded area}$$

$$R.H.S = A \cup (B \cup C)$$



$$(B \cup C) = \text{Region of } \text{shaded area}$$

$$A \cup (B \cup C) = \text{Region of } \text{shaded area and } \text{shaded area}$$

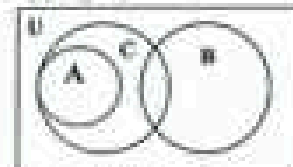
From both the figures, regions showing $(A \cup B) \cup C$ and $A \cup (B \cup C)$ are same i.e. $(A \cup B) \cup C = A \cup (B \cup C)$. Hence, associative law of union is verified.



(iv) Associative law of intersection

$$(A \cap B) \cap C = A \cap (B \cap C)$$

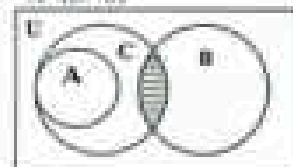
$$L.H.S = (A \cap B) \cap C$$



$$(A \cap B) = \text{No Region}$$

$$(A \cap B) \cap C = \text{No Region}$$

$$R.H.S = A \cap (B \cap C)$$



$$(B \cap C) = \text{Region of } \text{shaded area}$$

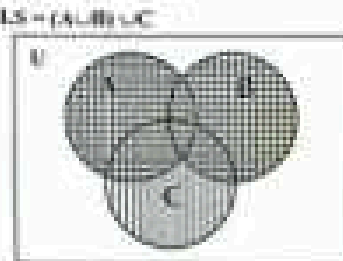
$$A \cap (B \cap C) = \text{No Region}$$

From both the figures, it is observed that $(A \cap B) \cap C = A \cap (B \cap C)$ are same. Hence, Associative law of intersection is verified.

(v) Associative law of union

$$(A \cup B) \cup C = A \cup (B \cup C)$$

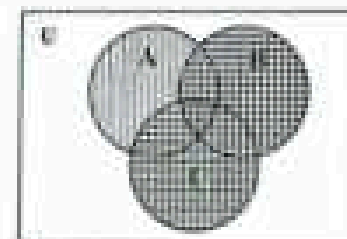
$$L.H.S = (A \cup B) \cup C$$



$$(B \cup C) = \text{Region of } \text{shaded area}$$

$$A \cup (B \cup C) = \text{Region of } \text{shaded area and } \text{shaded area}$$

$$R.H.S = A \cup (B \cup C)$$



$$(B \cup C) = \text{Region of } \text{shaded area}$$

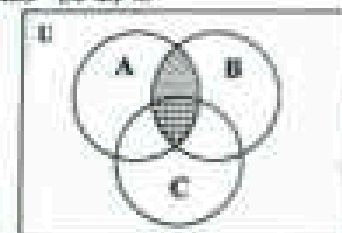
$$(A \cup B) \cup C = \text{Region of } \text{shaded area and } \text{shaded area}$$

From both the figures, it is observed that regions showing $(A \cup B) \cup C$ and $A \cup (B \cup C)$ are same i.e. $(A \cup B) \cup C = A \cup (B \cup C)$. Hence, associative law of union is verified.

(vi) Associative law of intersection

$$(A \cap B) \cap C = A \cap (B \cap C)$$

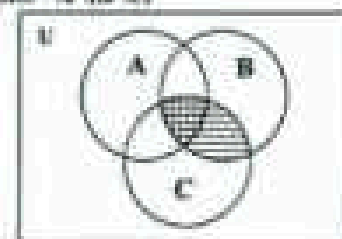
$$L.H.S = (A \cap B) \cap C$$



$$(A \cap B) = \text{Region of } \text{shaded area and } \text{shaded area}$$

$$(A \cap B) \cap C = \text{Region of } \text{shaded area}$$

$$R.H.S = A \cap (B \cap C)$$



$$(B \cap C) = \text{Region of } \text{shaded area and } \text{shaded area}$$

$$A \cap (B \cap C) = \text{Region of } \text{shaded area}$$



From both the figures, regions showing $(A \cap B) \cap C$ and $A \cap (B \cap C)$ are same i.e.

$(A \cap B) \cap C = A \cap (B \cap C)$. Hence associative law of intersection is verified.

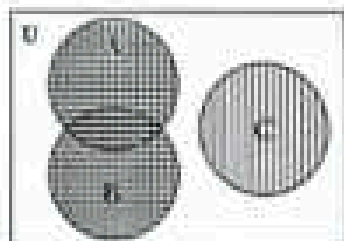
iv) **Solution Figure IV**

(a) Associative law of union

same

$$(A \cup B) \cup C = A \cup (B \cup C)$$

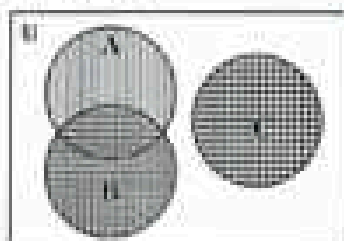
$$L.H.S = (A \cup B) \cup C$$



$(A \cup B) =$ Region of A and B

$(A \cup B) \cup C =$ Region of A , B and C

$$R.H.S = A \cup (B \cup C)$$



$(B \cup C) =$ Region of B and C

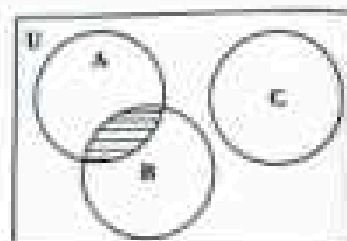
$A \cup (B \cup C) =$ region of A , B and C

From the figures the regions showing $(A \cup B) \cup C$ and $A \cup (B \cup C)$ are same i.e. $(A \cup B) \cup C = A \cup (B \cup C)$. Hence associative law of union is verified.

(b) Associative law of intersection **same**

$$(A \cap B) \cap C = A \cap (B \cap C)$$

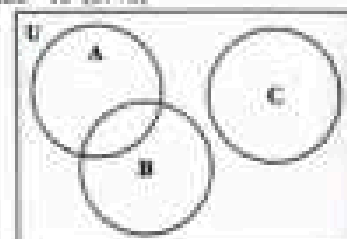
$$L.H.S = (A \cap B) \cap C$$



$(A \cap B) =$ Region of A and B

$(A \cap B) \cap C =$ (No Region)

$$R.H.S = A \cap (B \cap C)$$



$(B \cap C) =$ (No Region)

$A \cap (B \cap C) =$ (No Region)

From both the figures the Regions $(A \cap B) \cap C$ and $A \cap (B \cap C)$ are same. Hence associative law of intersection is verified.

$$(A \cap B) \cap C = A \cap (B \cap C)$$

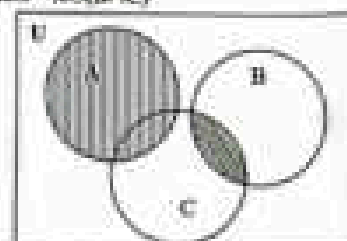
Q.4 Verify

Solution Figure - I

i) Distributive property of union over intersection **same**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

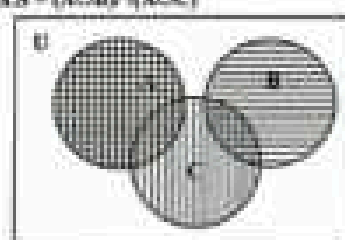
$$L.H.S = A \cup (B \cap C)$$



$(B \cap C) =$ Region of B and C

$A \cup (B \cap C) =$ Region of A and $(B \cap C)$

$$R.H.S = (A \cup B) \cap (A \cup C)$$



$(A \cup B) =$ Region of A and B

$(A \cup C) =$ Region of A and C

$(A \cup B) \cap (A \cup C) =$ Region of A

From both figures we observe that regions showing $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are same i.e.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Hence distributive law of union over intersection is verified.

ii) Distributive property of intersection over union **by using diagrams of position I.**

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

same

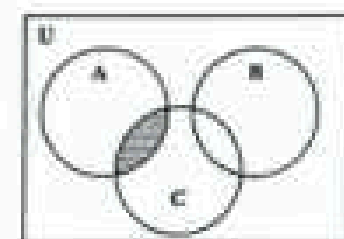
$$L.H.S = A \cap (B \cup C)$$



$(B \cup C) =$ Region of B and C

$A \cap (B \cup C) =$ Region of A

$$R.H.S = (A \cap B) \cup (A \cap C)$$



$(A \cap B) =$ (No Region)

$(A \cap C) =$ Region of A and C

$(A \cap B) \cup (A \cap C) =$ Region of A

From both figures we observe that regions showing $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are same i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Hence distributive law of intersection over union is verified.

Solution Figure - II

i) Distributive property of union over intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

same

$$L.H.S = A \cup (B \cap C)$$



$(B \cap C) =$ Region of B and C

$A \cup (B \cap C) =$ Region of A and $(B \cap C)$

$$R.H.S = (A \cup B) \cap (A \cup C)$$



$(A \cup B) =$ Region of A and B

$(A \cup C) =$ Region of A and C

$(A \cup B) \cap (A \cup C) =$ Region of A



From both the figures, regions showing $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are same i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Hence proved.

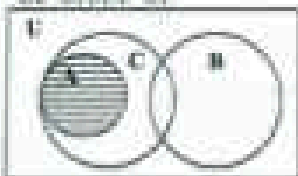
ii) Distributive property of intersection over union.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



$(B \cup C) =$ Regions of ii and iii
 $A \cap (B \cup C) =$ Region of ii

$$\text{R.H.S} = (A \cap B) \cup (A \cap C)$$



$$A \cap B = \phi$$

$A \cap C =$ Region of iii

$(A \cap B) \cup (A \cap C) =$ Region of iii

From both the figures, the regions showing $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are same i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

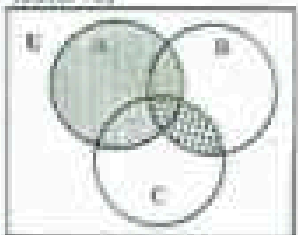
Hence proved.

Solution Figure - iii

ii) Distributive property of union over intersection.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

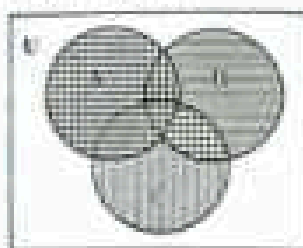
$$\text{L.H.S} = A \cup (B \cap C)$$



$(B \cap C) =$ Region of ii

$A \cup (B \cap C) =$ Region of ii and iii

$$\text{R.H.S} = (A \cup B) \cap (A \cup C)$$



$(A \cup B) =$ Regions of ii and iii

$(A \cup C) =$ Regions of ii and iii

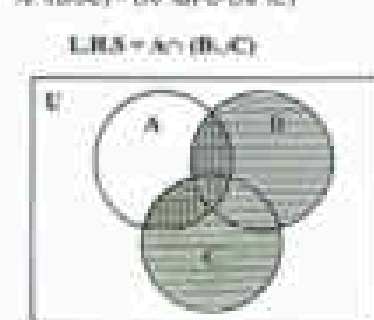
$(A \cup B) \cap (A \cup C) =$ Region of ii

From both figures, it is observed that regions showing $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are same i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Hence proved.

iii) Distributive property of intersection over union.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$\text{L.H.S} = A \cap (B \cup C)$$

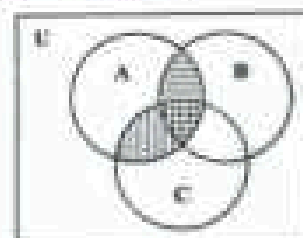


$(B \cup C) =$ Regions of ii and iii

$A \cap (B \cup C) =$ Region of ii

Now,

$$\text{R.H.S} = (A \cap B) \cup (A \cap C)$$



$(A \cap B) =$ Region of ii and iii

$(A \cap C) =$ Region of ii and iii

$(A \cap B) \cup (A \cap C) =$ Region of ii , iii and iv

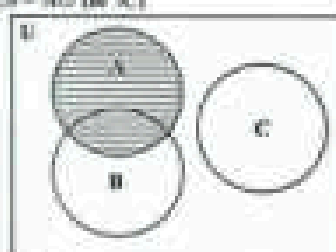
From both figures, we observe that regions showing $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are same i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ Hence proved.

Solution Figure IV

ii) Distributive property of union over intersection.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\text{L.H.S} = A \cup (B \cap C)$$



$(B \cap C) = \phi$

$A \cup (B \cap C) =$ Region of ii

$$\text{R.H.S} = (A \cup B) \cap (A \cup C)$$



$(A \cup B) =$ Regions of ii and iii

$(A \cup C) =$ Regions of ii and iii

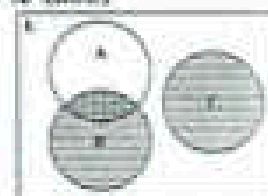
$(A \cup B) \cap (A \cup C) =$ Region of ii

From both the figures, regions showing $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are same i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Hence proved.

iii) Distributive property of intersection over union.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

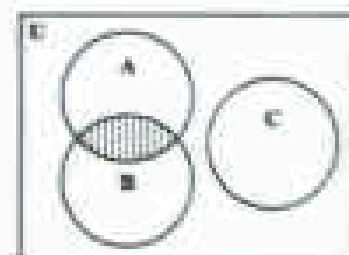
$$\text{L.H.S} = A \cap (B \cup C)$$



$(B \cup C) =$ Regions of ii and iii

$A \cap (B \cup C) =$ Region of ii

$$\text{R.H.S} = (A \cap B) \cup (A \cap C)$$



$(A \cap B) =$ Region of ii

$(A \cap C) = \phi$

$(A \cap B) \cup (A \cap C) =$ Region of ii

From both the figures, the regions showing $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are same i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ Hence proved.



Q.5 Prove by using Venn diagram.

$$(a) (P \cup Q) \cup R = P \cup (Q \cup R)$$

Solution:

$$P = \{0, 1, 2, 3\}, Q = \{2, 3, 4, 5, 6\}$$

$$R = \{3, 5, 7, 8, 9\}$$

P and Q are overlapping sets

Q and R are overlapping sets

$$(a) (P \cup Q) \cup R = P \cup (Q \cup R)$$

$$(b) (P \cap Q) \cap R = P \cap (Q \cap R)$$

When (i) $P = \{0, 1, 2, 3\}$,

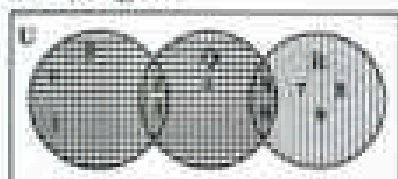
$$Q = \{2, 3, 4, 5, 6\}, R = \{5, 6, 7, 8, 9\}$$

$$(ii) P = \{0, 1, 2, 3, 4, 5\}, Q = \{3, 4, 5, 6\},$$

$$R = \{5, 6, 7, 8\}$$

$$(a) (P \cup Q) \cup R = P \cup (Q \cup R)$$

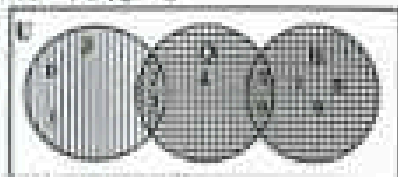
$$L.H.S = (P \cup Q) \cup R$$



$$P \cup Q = \text{Regions of } \text{[shaded area]}$$

$$(P \cup Q) \cup R = \text{Regions of } \text{[shaded area]} \text{ and } \text{[shaded area]} \text{ --- (i)}$$

$$R.H.S = P \cup (Q \cup R)$$



$$(Q \cup R) = \text{Regions of } \text{[shaded area]}$$

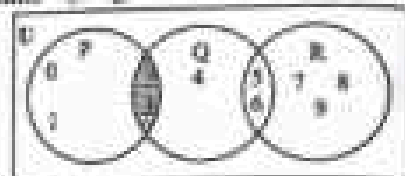
$$P \cup (Q \cup R) = \text{Regions of } \text{[shaded area]} \text{ and } \text{[shaded area]} \text{ --- (ii)}$$

From both the figures we observe that the regions showing $(P \cup Q) \cup R$ and $P \cup (Q \cup R)$ are same. i.e. $(P \cup Q) \cup R = P \cup (Q \cup R)$

Hence proved.

$$(b) (P \cap Q) \cap R = P \cap (Q \cap R)$$

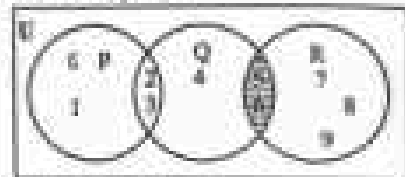
$$L.H.S = (P \cap Q) \cap R$$



$$(P \cap Q) = \text{Regions of } \text{[shaded area]}$$

$$(P \cap Q) \cap R = \text{No Region} = \emptyset \text{ --- (i)}$$

$$R.H.S = P \cap (Q \cap R)$$



$$(Q \cap R) = \text{Region of } \text{[shaded area]}$$

$$P \cap (Q \cap R) = \emptyset \text{ (No region) --- (ii)}$$

From both the figures, we observe that $L.H.S = R.H.S$.

$$(P \cap Q) \cap R = P \cap (Q \cap R)$$

Hence proved.

$$(i) P = \{0, 1, 2, 3, 4, 5\}$$

$$Q = \{3, 4, 5, 6\}$$

$$R = \{5, 6, 7, 8\}$$

We observe that

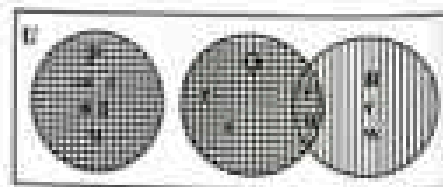
P and Q are disjoint sets

Q and R are overlapping sets.

P and R are disjoint sets.

$$(a) (P \cap Q) \cap R = P \cap (Q \cap R)$$

$$L.H.S = (P \cap Q) \cap R$$

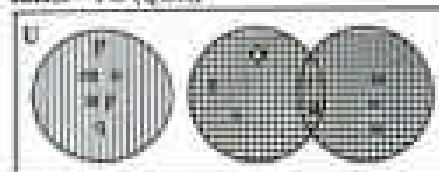


$$(P \cap Q) = \text{Regions of } \text{[shaded area]}$$

$$(P \cap Q) \cap R = \text{Regions of } \text{[shaded area]} \text{ and } \text{[shaded area]}$$



$$R.H.S = P \cup (Q \cap R)$$

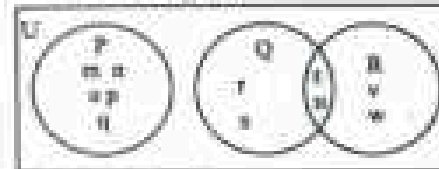


$$(Q \cap R) = \text{Regions of } \text{[shaded area]}$$

$$P \cup (Q \cap R) = \text{Regions of } \text{[shaded area]} \text{ and } \text{[shaded area]} \text{ --- (i)}$$

$$(b) (P \cap Q) \cap R = P \cap (Q \cap R)$$

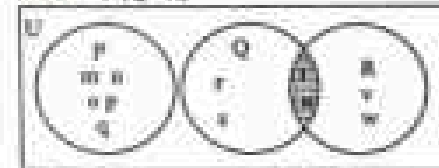
$$L.H.S = (P \cap Q) \cap R$$



$$(P \cap Q) = \text{Region} = \emptyset$$

$$(P \cap Q) \cap R = \emptyset \text{ --- (ii)}$$

$$R.H.S = P \cap (Q \cap R)$$



$$(Q \cap R) = \text{Region of } \text{[shaded area]}$$

$$P \cap (Q \cap R) = \emptyset \text{ --- (ii)}$$

From both the figures we observe that

$$(P \cap Q) \cap R = P \cap (Q \cap R)$$

Hence proved.

Q.6 Verify $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ using Venn diagrams for the following sets.

$$X = \{-1, -2, -3\}, Y = \{0, 1, 2, 3\},$$

$$Z = \{0, \pm 1, \pm 2, \pm 3\}$$

Solution

$$X = \{-1, -2, -3\}$$

$$Y = \{0, 1, 2, 3\}$$

$$Z = \{0, \pm 1, \pm 2, \pm 3\}$$

We observe that

X and Y are disjoint sets.

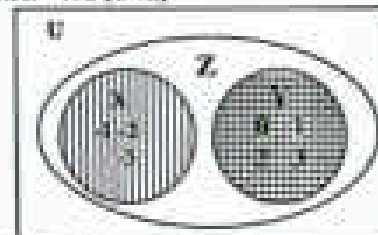
X is proper subset of Z

Y is proper subset of Z.

(i) Distributive property of union over intersection.

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

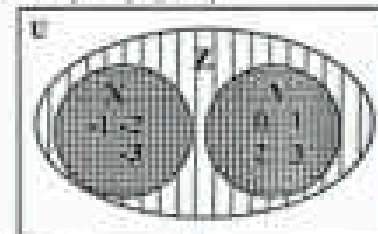
$$L.H.S = X \cup (Y \cap Z)$$



$$(Y \cap Z) = \text{Region of } \text{[shaded area]}$$

$$X \cup (Y \cap Z) = \text{Regions of } \text{[shaded area]} \text{ and } \text{[shaded area]} \text{ --- (i)}$$

$$R.H.S = (X \cup Y) \cap (X \cup Z)$$



$$(X \cup Y) = \text{Regions of } \text{[shaded area]} \text{ and } \text{[shaded area]}$$

$$(X \cup Z) = \text{Regions of } \text{[shaded area]} \text{ and } \text{[shaded area]}$$

$$(X \cup Y) \cap (X \cup Z) = \text{Regions of } \text{[shaded area]} \text{ --- (ii)}$$

From both the figures showing $X \cup (Y \cap Z)$ and $(X \cup Y) \cap (X \cup Z)$ are same i.e.

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

$$Q.7 \text{ Verify } X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$

X = Set of first three vowels.

Y = Set of letters of the word "energy"

Z = Set of letters of the word "alpha"

Solution

$$X = \{a, e, i\}$$

$$Y = \{e, n, r, g, y\}$$



$Z = \{c, d, e, f, g, h, i\}$

We observe that

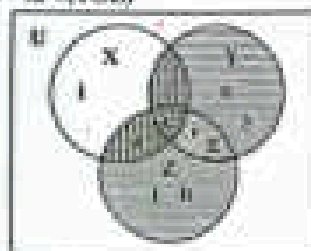
X and Y are overlapping sets

Y and Z are overlapping sets

Z and X are overlapping sets

$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

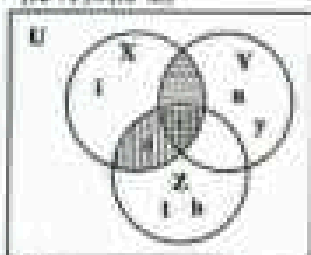
$L.R.S = X \cap (Y \cup Z)$



$(Y \cup Z) =$ Regions of --- and ---

$X \cap (Y \cup Z) =$ Regions of --- and --- (i)

$L.R.S = (X \cap Y) \cup (X \cap Z)$



$(X \cap Y) =$ Regions of --- and ---

$(X \cap Z) =$ Regions of --- and ---

$(X \cap Y) \cup (X \cap Z) =$ Regions of --- , --- , --- , and ---

From both the figures we observe that regions showing $X \cap (Y \cup Z)$ and $(X \cap Y) \cup (X \cap Z)$ are same i.e.

$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

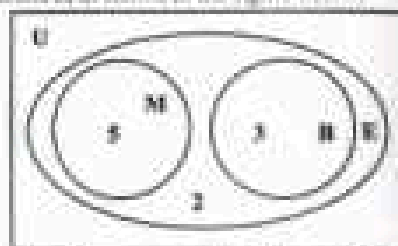
Hence proved.

Application of Venn Diagram

Example 11: Among the ten teachers of a secondary school, five teach Mathematics and three teach Biology. However, all these teachers also teach English. Show the data by

Venn diagram. Also find how many teachers teach only English.

Solution: If E represents set of English Teachers, M represents set of Mathematics Teacher, B represents the set of Biology Teachers and then both M and B are its subsets E , as shown in the figure below.

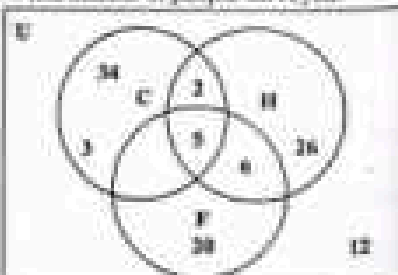


From the Venn diagram it is observed that two of the teachers only teach English.

Example 12: In a survey, people were asked whether they like cricket, hockey or football. Using the Venn diagram find the number of people playing:

- Only cricket;
- Hockey and football;
- All three games;
- Either of the three games;
- Neither of the three games.

Also find number of people surveyed.



Solution:

From the Venn diagram, it is clear that:

- Number of people who play only cricket = 34

Which shows $n(C - C \cap H \cap F)$

- Number of people who play both hockey and football = $6 + 5 = 11$

Which shows $n(H \cap F)$

- Number of people who play all three games = 4

Which shows $n(C \cap H \cap F)$

- Number of people who play either of the games = $34 + 3 + 6 + 20 + 20 = 91$

Which shows $n(C \cup H \cup F)$

- Number of people who do not play any game = 12

Which shows $n(U - C \cup H \cup F)$

Total number of people =

$$n(U) = 34 + 3 + 6 + 20 + 20 + 12 = 103$$

Application of set theory

Applications of set theory are most frequently used in science and mathematics fields like biology, chemistry and physics as well as in computer and electrical engineering. These applications range from forming logical foundations for all branches of mathematics. Therefore, understanding formulae are helpful in the set theory.

- For any two overlapping sets A and B :

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A \cup B) - n(B)$
- $n(A - B) = n(A) - n(A \cap B)$

Exercise 3.4

Q.1 Let A and B be two finite sets such that $n(A) = 24$, $n(B) = 18$ and $n(A \cap B) = 31$.

Find $n(A \cap B)$.

Solution:

Given: $n(A) = 24$

$n(B) = 18$

$n(A \cup B) = 31$

To find: $n(A \cap B) = ?$

We know that for overlapping sets

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\bullet \quad n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$$

or For any two sets A and B that are disjoint

- $n(A \cup B) = n(A) + n(B)$
- $n(A - B) = n(A)$

or For any three sets A , B and C :

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) \\ &\quad - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) \end{aligned}$$

Example 13:

In a class of 80 students, 40 like English, 34 like Mathematics and 9 like both. How many students like either of both subjects and how many like neither?

Solution:

Total number of students = $n(T) = 80$

Number of students that like English = $n(E) = 40$

Number of students that like Mathematics =

$$n(M) = 34$$

Therefore, total number of students that like both subjects is:

$$\begin{aligned} n(E \cup M) &= n(E) + n(M) - n(E \cap M) \\ &= 40 + 34 - 9 = 65 \end{aligned}$$

Number of students that do not like both subjects is:

$$n(T) - n(E \cup M) = 80 - 65 = 15$$



$$n(A \cup B) = 44$$

$$n(A \cap B) = 2 \quad (\text{overlapping sets})$$

$$\text{To find: } n(B - A) = ?$$

$$n(B) = ?$$

(i) We know that

$$n(A \cup B) = n(A - B) + n(A \cap B)$$

$$\Rightarrow 23 = 44 - n(B)$$

$$\Rightarrow n(B) = 44 - 23$$

$$n(B) = 21$$

(ii) We also know that

$$n(B - A) = n(B) - n(A \cap B)$$

$$n(B - A) = 21 - 2$$

$$n(B - A) = 19$$

Q.3 In a group of 30 Mathematics students, 20 like Algebra and 15 like both Geometry and Algebra. Show the data by Venn diagram. Also find how many students like Geometry. (3 MARKS)

Solution

Group of 30 Mathematics students

$$\Rightarrow n(A \cup G) = 30$$

$$20 \text{ like Algebra} \Rightarrow n(A) = 20$$

$$15 \text{ like both Geometry and Algebra} \Rightarrow n(A \cap G) = 15$$

$$n(A \cap G) = 15$$

$$\text{No. of students who like geometry} \Rightarrow n(G) = ?$$

We know that for overlapping sets

$$n(A \cup G) = n(A) + n(G) - n(A \cap G)$$

$$30 = 20 + n(G) - 15$$

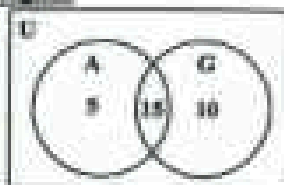
$$30 = 5 + n(G)$$

$$30 - 5 = n(G)$$

$$25 = n(G)$$

$$\Rightarrow n(G) = 25$$

Venn Diagram



$$n(G) = 15 + 10 = 25$$

25 students like geometry.

Q.4 In a street with 50 houses, 25 houses have lawns, 32 houses have car porch and 15 houses have both lawn and car porch. Show the data by Venn diagram. Also find how many houses have neither lawn nor porch. (3 MARKS)

Solution

$$\text{Total houses} = n(U) = 50$$

$$25 \text{ houses have lawn} \Rightarrow n(L) = 25$$

$$32 \text{ houses have car porch} \Rightarrow n(P) = 32$$

$$15 \text{ houses have both} \Rightarrow n(L \cap P) = 15$$

Houses having neither lawn nor porch =

$$n(L \cup P)$$

Houses having at least one

$$n(L \cup P) = n(L) + n(P) - n(L \cap P)$$

$$n(L \cup P) = 25 + 32 - 15$$

$$n(L \cup P) = 42 - 15$$

$$n(L \cup P) = 42$$

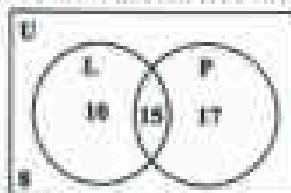
Houses having neither lawn nor porch

$$n(L \cup P)^c = n(U) - n(L \cup P)$$

$$n(L \cup P)^c = 50 - 42 = 8$$

Venn Diagram

$$n(L \cap P) = 15 \text{ shows that sets are overlapping}$$



Verification

$$n(L) = 10 + 15 = 25$$

$$n(P) = 15 + 17 = 32$$

$$n(L \cap P) = 15$$

$$n(U) = 8 + 10 + 15 + 17 = 50$$

Q.5 In a survey of 940 children, 400 students were found studying at primary level, 240 students at elementary and 175 at secondary level. Create a Venn diagram to



illustrate this information. How many children were found out of school? (3 MARKS)

Solution

$$\text{Total children surveyed} = 940$$

$$\text{Students at primary level} = n(P) = 400$$

$$\text{Students at elementary level} = n(E) = 240$$

$$\text{Students at secondary level} = n(S) = 175$$

As no student can study at more than one level so sets are disjoint.

Total students studying in school

$$n(P \cup E \cup S) = n(P) + n(E) + n(S)$$

$$n(P \cup E \cup S) = 400 + 240 + 175$$

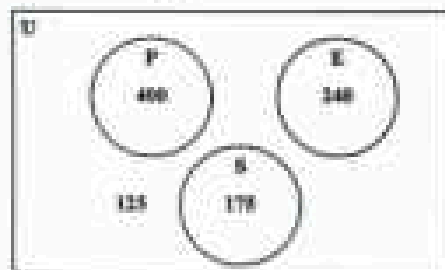
$$= 815$$

Children out of school

$$n(P \cup E \cup S)^c = n(U) - n(P \cup E \cup S)$$

$$= 940 - 815$$

$$= 125$$



Q.6 ABC Dairy polls its customers on their favorite flavor: chocolate, vanilla or mango? 100 customers said they like mango flavor, 90 customers said they like vanilla, 40 polled for chocolate, 20 customers liked both mango and vanilla while 14 liked both chocolate and vanilla. How many customers said they like:

i) only mango? (1 MARK)

ii) only vanilla (1 MARK)

iii) only chocolate (1 MARK)

Solution

$$100 \text{ like mango flavor} = n(M) = 100$$

$$90 \text{ like vanilla flavor} = n(V) = 90$$

$$40 \text{ like chocolate flavor} = n(C) = 40$$

$$20 \text{ like both mango and vanilla} = n(M \cap V) = 20$$

$$14 \text{ like both chocolate and vanilla}$$

$$= n(C \cap V) = 14$$

i) Customers liking only mango flavor

$$= n(M) - n(M \cap V)$$

$$= 100 - 20$$

$$= 80$$

ii) Customers liking only Vanilla flavor

$$= n(V) - n(M \cap V) - n(C \cap V)$$

$$= 90 - 20 - 14$$

$$= 90 - 34$$

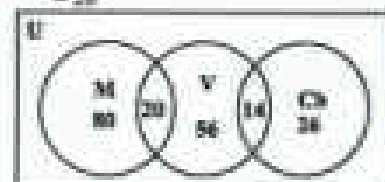
$$= 56$$

iii) Customers liking only chocolate

$$= n(C) - n(C \cap V)$$

$$= 40 - 14$$

$$= 26$$



Q.7 In a survey of university 200 students were interviewed. It was found that:

42 students have laptops, 80 students have cell phones, 100 students have iPads, 23 students have both a laptop and a cell phone, 10 students have both a laptop and iPad, 14 students have both cell phone and iPad and 8 students have all three items.

a) How many students have only cell phone? (1 MARK)

b) How many students have none of the three items? (1 MARK)

c) How many students have both iPad and laptop but not cell phone? (1 MARK)



Solution

$$\text{Total students} = n(U) = 280$$

$$\text{Students having laptops} = n(L) = 42$$

$$\text{Students having cell phone} = n(C) = 40$$

$$\text{Students having iPods} = n(I) = 100$$

$$\begin{aligned} \text{Students having both laptops and cell phone} \\ = n(L \cap C) = 23 \end{aligned}$$

$$\begin{aligned} \text{Students having laptops and iPod} \\ = n(L \cap I) = 10 \end{aligned}$$

$$\begin{aligned} \text{Students having cell phone and iPod} \\ = n(C \cap I) = 14 \end{aligned}$$

$$\begin{aligned} \text{Students having all three items: laptops,} \\ \text{cell phone and iPod} = n(L \cap C \cap I) = 8 \end{aligned}$$

$$\begin{aligned} \text{(a) Students having only cell phones} \\ = n(C) - n(L \cap C) - n(C \cap I) + n(L \cap C \cap I) \\ = 40 - 23 - 14 + 8 \\ = 11 \end{aligned}$$

$$\begin{aligned} \text{(b) Students having none of three items} \\ \text{First we find students having at least one item.} \end{aligned}$$

$$\begin{aligned} n(L \cup C \cup I) &= n(L) + n(C) + n(I) - n(L \cap C) \\ &\quad - n(L \cap I) - n(C \cap I) + n(L \cap C \cap I) \\ &= 42 + 40 + 100 - 23 - 10 - 14 + 8 \\ &= 243 \end{aligned}$$

$$\begin{aligned} \text{No, we find students having none of three items.} \\ n(L \cup C \cup I)' = n(U) - n(L \cup C \cup I) \\ = 280 - 243 \\ = 37 \end{aligned}$$

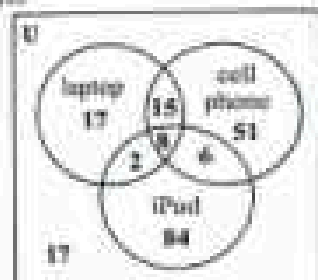
$$\begin{aligned} \text{(c) To find students who have both an iPod and} \\ \text{laptop but not cell phone, we calculate:} \\ 84 + 17 + 2 = 103 \end{aligned}$$

Therefore,

$$\text{(i) Students who have only a cell phone} = 11$$

$$\text{(ii) Students who have none of the three items} = 37$$

$$\begin{aligned} \text{(ii) Students who have both an iPod and a} \\ \text{laptop but not a cell phone} = n(L \cap I \text{ only}) \\ = 103 \end{aligned}$$



Q.8 In a girls college, every student plays either badminton or table tennis or both. If 350 students play badminton, 280 play table tennis and 150 play both. Find how many students are there in the college? *answer*

Solution

$$\text{Students playing badminton} = n(B) = 350$$

$$\text{Students playing table tennis} = n(T) = 280$$

$$\text{Students playing both} = n(B \cap T) = 150$$

$$\text{Total students in college} = n(B \cup T)$$

We know that

$$n(B \cup T) = n(B) + n(T) - n(B \cap T)$$

$$n(B \cup T) = 350 + 280 - 150$$

$$= 480 - 150$$

$$= 330$$

There are 480 students in the college.

Q.9 Among 50 students, 8 are learning both English and Chinese languages. A total of 24 students are learning English if every student is learning at least one language. how many students are learning Chinese? *answer*

Solution

$$\text{Total students} = n(B \cup C) = 50$$

$$\text{Students learning English} = n(E) = 26$$

$$\text{Students learning both} = n(E \cap C) = 8$$

$$\text{Students learning Chinese} = n(C) = ?$$

We know that

$$n(B \cup C) = n(E) + n(C) - n(E \cap C)$$

$$50 = 26 + n(C) - 8$$

$$50 = 18 + n(C)$$

$$50 - 18 = n(C)$$

$$32 = n(C)$$

$$\Rightarrow n(C) = 32$$

Thus 32 students are learning Chinese language.

Q.10 Out of 70 people, 48 like tea and 40 like coffee and each person likes at least one of the two drinks. How many like both tea and coffee? *answer*

Solution

$$\text{Total people} = 70$$

$$48 \text{ like tea} = n(T) = 48$$

$$40 \text{ like coffee} = n(C) = 40$$

$$\text{People liking both tea and coffee} = n(T \cap C)$$

We know that

$$n(T \cup C) = n(T) + n(C) - n(T \cap C)$$

$$70 = 48 + 40 - n(T \cap C)$$

$$30 = 88 - n(T \cap C)$$

$$\Rightarrow n(T \cap C) = 88 - 30$$

$$n(T \cap C) = 18$$

Thus 18 people like both tea and coffee.

Q.11 There are 46 students in science group and 58 students in arts group. Finding the number of students who are either in science or arts group. *answer*

Solution

$$\text{Students in science group} = n(S) = 46$$

$$\text{Students in Arts group} = n(A) = 58$$

$$\text{Total number of students} = n(S \cup A) = ?$$

For disjoint sets,

$$n(S \cup A) = n(S) + n(A)$$

$$= 46 + 58 = 104$$

Thus there are 104 students who are either in science or arts group.

Q.12 In a group of people, 52 people can speak Arabic and 112 can speak French. How many can speak Arabic only? How

many can speak French only if 12 of them can speak both languages? How many people were in the group? *answer*

Solution

$$\text{Arabic speaking} = n(A) = 52$$

$$\text{French speaking} = n(F) = 112$$

$$\text{Both language speaking} = n(A \cap F) = 12$$

$$\begin{aligned} \text{(i) Arabic speaking only} &= n(A) - n(A \cap F) \\ &= 52 - 12 \\ &= 40 \end{aligned}$$

$$\begin{aligned} \text{(ii) French speaking only} &= n(F) - n(A \cap F) \\ &= 112 - 12 \\ &= 100 \end{aligned}$$

$$\begin{aligned} \text{(iii) Total people in group} \\ n(A \cup F) &= n(A) + n(F) - n(A \cap F) \\ &= 52 + 112 - 12 \\ &= 152 + 100 \\ &= 252 \end{aligned}$$

Thus there were 252 people in the group.

Q.13 In a high school, 260 students like reading story books, 170 like practical activities and 150 like both. Find

- The number of students who like reading story books only.
- The number of students who like only practical activities.
- The total number of students in the school. *answer*

Solution

$$\text{Students like reading story books} = n(R) = 260$$

$$\text{Students like practical activity} = n(P) = 170$$

$$\text{Students like both} = n(R \cap P) = 150$$

$$\begin{aligned} \text{(i) Sets are overlapping} \\ \text{Students like reading only} \\ &= n(R) - n(R \cap P) \\ &= 260 - 150 \\ &= 110 \end{aligned}$$

$$\begin{aligned} \text{(ii) Students like practical only} \\ &= n(P) - n(R \cap P) \\ &= 170 - 150 \\ &= 20 \end{aligned}$$



iii) Total number of students

$$\begin{aligned} &\Rightarrow n(B \cup P) = n(B) + n(P) - n(B \cap P) \\ &= 160 + 170 - 150 \\ &= 330 - 150 \\ &= 180 \end{aligned}$$

Q.18 In a survey of 60 people, it was found that 25 people watch channel A, 16 watch channel B, 13 watch channel C, 4 watch both A and B, 7 watch both B and C, 8 watch both A and C, 3 watch all three channels. Find the number of people who watch at least one of the channels? Also find number of people who do not watch these channels.

Solution

$$\text{Total people} = n(U) = 60$$

$$\text{People like to watch}$$

$$\text{Channel A} = n(A) = 25$$

$$\text{Channel B} = n(B) = 16$$

$$\text{Channel C} = n(C) = 13$$

$$\text{Both channels A and B} = n(A \cap B) = 4$$

$$\text{Both channels B and C} = n(B \cap C) = 7$$

$$\text{Both channels A and C} = n(A \cap C) = 8$$

$$\text{All three channels} = n(A \cap B \cap C) = 3$$

i) People who watch atleast one channel.

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) \\ &= 25 + 16 + 13 - 4 - 7 - 8 + 3 \\ &= 37 - 19 \\ &= 18 \end{aligned}$$

ii) People who do not watch any channel.

$$\begin{aligned} n(A \cup B \cup C)^c &= n(U) - n(A \cup B \cup C) \\ &= 60 - 18 \\ &= 22 \end{aligned}$$

Thus, 22 people do not watch any channel.

Ordered Pair

Pair of two numbers in which order of numbers is not invertible, is called an ordered pair. The numbers in an ordered pair are written within small brackets (parenthesis) and are separated by comma.

For example, (a, b) is an ordered pair in which a is called first element and b is called second element.

Equality of Two Ordered Pairs

Two ordered pairs (a, b) and (c, d) are equal if

$$a = c \text{ and } b = d$$

Example 15

Find the values of x and y when

$$(x - 3y, 5x + 1) = (4, 6)$$

Solution: By the equality of ordered pairs, we have:

$$x - 3y = 4 \quad \text{..... (i)}$$

$$5x + 1 = 6 \quad \text{..... (ii)}$$

From equation (ii), $5x = 6 - 1$

$$5x = 5 \Rightarrow \boxed{x = 1}$$

Substituting $x = 1$ in equation (i), we get:

$$\begin{aligned} 1 - 3y + 4 &= -3y + 5 \\ \Rightarrow y &= -1 \end{aligned}$$

Cartesian Product of Sets

If A and B are two non-empty sets, then Cartesian product $A \times B$ is the set of ordered pairs (x, y) such that $x \in A$ and $y \in B$. Mathematically:

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}$$

Note

$$\bullet n(A \times B) = n(A) \times n(B)$$

$$\bullet A \times B = \phi, \text{ if either}$$

$$A = \phi \text{ or } B = \phi$$

Example 16

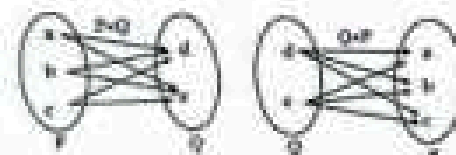
Exhibit $P \times Q$ and $Q \times P$ by arrow diagram when $P = \{a, b, c\}$ and $Q = \{d, e\}$.



Solution

$$P \times Q = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$$

$$Q \times P = \{(d, a), (d, b), (d, c), (e, a), (e, b), (e, c)\}$$



Exercise 3.3

Q.1 Find the values of unknown when:

$$(a, -b) = (7, 1) \quad \text{answer}$$

Solution

$$(a, -b) = (7, 1)$$

By the equality of ordered pairs, we have

$$\boxed{a = 7} \text{ and } -b = 1$$

$$\Rightarrow \boxed{b = -1}$$

$$\text{ii) } (2a, 2b+3) = (-10, -6) \quad \text{answer}$$

Solution

$$(2a, 2b+3) = (-10, -6)$$

By the equality of ordered pairs, we have

$$2a = -10 \quad \text{and} \quad 2b + 3 = -6$$

$$a = \frac{-10}{2} \quad \text{and} \quad 2b + 3 = -6$$

$$\boxed{a = -5} \quad \text{and} \quad 2b = -9$$

$$b = \frac{-9}{2}$$

$$\Rightarrow \boxed{b = -\frac{9}{2}}$$

$$\text{iii) } (2a-4, 6) = (8, -b+1) \quad \text{answer}$$

Solution

$$(2a-4, 6) = (8, -b+1)$$

By the equality of ordered pairs, we get

$$2a - 4 = 8 \quad \text{and} \quad 6 = -b + 1$$

$$2a = 8 + 4 \quad \text{and} \quad b = 1 - 6$$

$$2a = 12 \quad \text{and} \quad \boxed{b = -5}$$

$$a = \frac{12}{2}$$

$$\boxed{a = 6}$$

$$\text{iv) } (x+2y, y-3) = (2, 5) \quad \text{answer}$$

Solution

$$(x+2y, y-3) = (2, 5)$$

By the equality of ordered pairs, we have

$$x + 2y = 2 \quad \text{..... (i)}$$

$$y - 3 = 5 \quad \text{..... (ii)}$$

From equation (ii)

$$\Rightarrow y = 5 + 3$$

$$\boxed{y = 8}$$

Put $y = 8$ in eq. (i)

$$x + 2(8) = 2$$

$$x + 16 = 2$$

$$x = 2 - 16$$

$$\boxed{x = -14}$$

$$\text{v) } (2x-y, y-3x) = (4, 2) \quad \text{answer}$$

Solution

$$(2x-y, y-3x) = (4, 2)$$

By the equality of ordered pairs, we have

$$2x - y = 4 \quad \text{..... (i)}$$

$$y - 3x = 2 \quad \text{..... (ii)}$$

Adding eq. (i) and (ii), we get

$$2x - y + y - 3x = 4 + 2$$

$$-x = 6$$

$$\Rightarrow \boxed{x = -6}$$

Put $x = -6$ in eq. (ii)

$$y - 3(-6) = 2$$

$$y + 18 = 2$$

$$y = 2 - 18$$

$$\boxed{y = -16}$$

$$\text{vi) } (4x+6y, x-12y) = (8, -3) \quad \text{answer}$$



Solution

$$(4x + 6y, x - 12y) = (8, -3)$$

By the equality of ordered pairs, we have

$$4x + 6y = 8 \quad \text{..... (i)}$$

$$x - 12y = -3 \quad \text{..... (ii)}$$

Multiplying eq. (ii) by 4, we get

$$4(x - 12y) = 4(-3)$$

$$4x - 48y = -12 \quad \text{..... (iii)}$$

Subtracting (iii) from (i), we get

$$4x + 6y = 8$$

$$-4x + 48y = 12$$

$$54y = 20$$

$$y = \frac{10}{27}$$

$$\Rightarrow \boxed{y = \frac{10}{27}}$$

Putting $y = \frac{10}{27}$ in eq. (ii) we get

$$x - 12\left(\frac{10}{27}\right) = -3$$

$$x - 4 = -3$$

$$x = -3 + 4$$

$$\boxed{x = 1}$$

$$\text{vii) } (5x + y, -x + y) = (8, 1)$$

Solution

$$(5x + y, -x + y) = (8, 1)$$

By the equality of ordered pairs, we get

$$5x + y = 8 \quad \text{..... (i)}$$

$$-x + y = 1 \quad \text{..... (ii)}$$

Subtracting eq. (ii) from (i)

$$5x + y = 8$$

$$-x + y = 1$$

$$6x = 7$$

$$\boxed{x = \frac{7}{6}}$$

Put it the eq. (i),

$$\frac{-5}{6} + y = 1$$

$$y = 1 + \frac{5}{6}$$

$$y = \frac{11}{6}$$

$$\boxed{y = \frac{11}{6}}$$

Q.2 Let $A = \{1, 4, 8\}$ and $B = \{1, 0\}$. Find.

i) $A \times B$

Solution

$$A \times B = \{1, 4, 8\} \times \{1, 0\}$$

$$A \times B = \{(1, 1), (1, 0), (4, 1), (4, 0), (8, 1), (8, 0)\}$$

$$\text{Number of elements in } A \times B = n(A \times B)$$

$$= 3 \times 2 = 6$$

ii) $B \times A$

Solution

$$B \times A = \{1, 0\} \times \{1, 4, 8\}$$

$$B \times A = \{(1, 1), (1, 4), (1, 8), (0, 1), (0, 4), (0, 8)\}$$

$$\text{Number of elements in } B \times A = n(B \times A)$$

$$= 2 \times 3 = 6$$

iii) $A \times A$

Solution

$$A \times A = \{1, 4, 8\} \times \{1, 4, 8\}$$

$$A \times A = \{(1, 1), (1, 4), (1, 8), (4, 1), (4, 4), (4, 8),$$

$$(8, 1), (8, 4), (8, 8)\}$$

$$\text{Number of elements in } A \times A = n(A \times A) = 3 \times 3$$

$$= 9$$

iv) $B \times B$

Solution

$$B \times B = \{1, 0\} \times \{1, 0\}$$

$$B \times B = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$$

$$\text{Number of elements in } B \times B = n(B \times B) = 2 \times 2 = 4$$

Q.3 Let $E = \{1, 2\}$ and $F = \{4, 6, 8\}$. Express

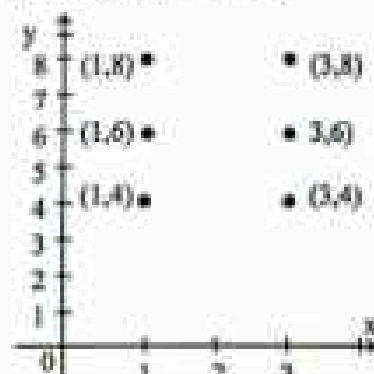
$E \times F$, $F \times E$, $E \times E = F \times F$ graphically.

Solution

$$\text{i) } E \times F = \{1, 2\} \times \{4, 6, 8\}$$

$$E \times F = \{(1, 4), (1, 6), (1, 8), (2, 4), (2, 6), (2, 8)\}$$

Graphical Representation of $E \times F$



$$\text{ii) } F \times E = \{4, 6, 8\} \times \{1, 2\}$$

$$F \times E = \{(4, 1), (4, 2), (6, 1), (6, 2), (8, 1), (8, 2)\}$$

Graphical representation of $F \times E$



$$\text{iii) } E \times E = \{1, 2\} \times \{1, 2\}$$

$$E \times E = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

Graphical representation of $E \times E$

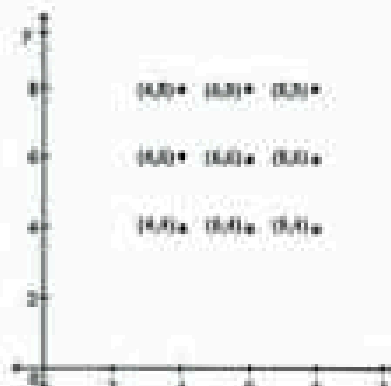


$$\text{iv) } F \times F = \{4, 6, 8\} \times \{4, 6, 8\}$$

$$F \times F = \{(4, 4), (4, 6), (4, 8), (6, 4),$$

$$(6, 6), (6, 8), (8, 4), (8, 6), (8, 8)\}$$

Graphical Representation of $F \times F$



Q.4 If $L = M = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$, then find sets L , M and $M \times L$.

Solution

$$L = M = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$$

$$L = M = \{(0, 1) + (2, 3, 4)\}$$

By comparing sets we get

$$\Rightarrow L = \{0, 1\}$$

$$M = \{2, 3, 4\}$$

Find $M \times L$:

$$M \times L = \{(2, 3, 4) \times \{0, 1\}\}$$

$$M \times L = \{(2, 0), (2, 1), (3, 0), (3, 1), (4, 0), (4, 1)\}$$

Q.5 Given that $A = \{1, 3, 5\}$, $B = \{2, 4\}$, $C = \{6, 7\}$.

i) Find $A \times (B \times C)$

Solution

$$A = \{1, 3, 5\}, B = \{2, 4\}, C = \{6, 7\}$$

ii) Finding $A \times (B \times C)$

$$(B \times C) = \{(2, 4) \times \{6, 7\}\}$$

$$(B \times C) = \{(2, 6), (2, 7),$$

Now,

$$A \times (B \times C) = \{1, 3, 5\} \times \{(2, 6), (2, 7),$$

$$(3, 6), (3, 7), (5, 6), (5, 7), (5, 8), (5, 9)\}$$

$$(3, 6), (3, 7), (5, 6), (5, 7), (5, 8), (5, 9)\}$$

iii) Find $(A \times B) \cup (A \times C)$

Solution

$$(A \times B) \cup (A \times C)$$

$$= \{1, 3, 5\} \times \{2, 4\} \cup \{1, 3, 5\} \times \{6, 7\}$$

$$= \{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4)\} \cup$$

$$\{(1, 6), (1, 7), (3, 6), (3, 7), (5, 6), (5, 7)\}$$



$$= \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4), (1,6), (1,7), (3,6), (3,7), (5,6), (5,7)\}$$

iii) Verify $A \times (B \cap C) = (A \times B) \cap (A \times C)$ answer

Solution

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$\text{L.H.S} = A \times (\{2,4\} \cap \{6,7\})$$

$$= \{(1,2), (2,2), (3,2), (5,2)\}$$

$$= \{(1,2), (1,4), (1,6), (1,7), (2,2), (2,4), (3,6), (3,7), (5,6), (5,7)\} \quad \text{--- (i)}$$

$$\text{R.H.S} = (A \times B) \cap (A \times C)$$

$$= (\{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4), (1,6), (1,7), (3,6), (3,7), (5,6), (5,7)\} \cap \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4), (1,6), (1,7), (3,6), (3,7), (5,6), (5,7)\})$$

$$= \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4), (1,6), (1,7), (3,6), (3,7), (5,6), (5,7)\} \quad \text{--- (ii)}$$

From (i) and (ii) L.H.S = R.H.S

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Q.6 Given that $D = \{a, e, i\}$, $E = \{a, e\}$, $F = \{e, i\}$.

i) Find $D \times (E \cap F)$ answer

Solution

$$D \times (E \cap F)$$

$$D \times \{a, e, i\} \cap \{a, e\} \cap \{e, i\}$$

$$D \times (E \cap F) = D \times (\{a, e\} \cap \{e, i\})$$

$$= \{a, e, i\} \times \{e\}$$

$$= \{(a, e), (e, e), (i, e)\}$$

ii) Find $(D \times E) \cap (D \times F)$ answer

Solution

$$(D \times E) \cap (D \times F)$$

$$= (\{a, e, i\} \times \{a, e\}) \cap (\{a, e, i\} \times \{e, i\})$$

$$= \{(a, a), (a, e), (e, a), (e, e), (i, a), (i, e), (i, i)\} \cap \{(a, e), (a, i), (e, e), (e, i), (i, e), (i, i)\}$$

$$= \{(a, e), (e, e), (i, i)\}$$

iii) Verify $D \times (E \cap F) = (D \times E) \cap (D \times F)$ answer

Solution

$$D \times (E \cap F) = (D \times E) \cap (D \times F)$$

$$\text{L.H.S} = D \times (E \cap F)$$

$$= D \times (\{a, e\} \cap \{e, i\})$$

$$= \{a, e, i\} \times \{e\}$$

$$= \{(a, e), (e, e), (i, e)\} \quad \text{--- (i)}$$

$$\text{R.H.S} = (D \times E) \cap (D \times F)$$

$$= (\{a, e, i\} \times \{a, e\}) \cap (\{a, e, i\} \times \{e, i\})$$

$$= \{(a, a), (a, e), (e, a), (e, e), (i, a), (i, e), (i, i)\} \cap \{(a, e), (a, i), (e, e), (e, i), (i, e), (i, i)\}$$

$$= \{(a, e), (e, e), (i, i)\} \quad \text{--- (ii)}$$

From (i) and (ii) L.H.S = R.H.S

$$D \times (E \cap F) = (D \times E) \cap (D \times F)$$

Hence verified.

Q.7 Given that $A = \{x/x \in N, x < 3\}$,

$$B = \{y/y \in W, y < 2\}, C = \{0, 2, 4\}$$

i) Verify $A \times (B \cap C) = (A \times B) \cap (A \times C)$ answer

Solution

$$A = \{x/x \in N, x < 3\}$$

$$B = \{y/y \in W, y < 2\}$$

$$C = \{0, 2, 4\}$$

Tabular form of sets:

$$A = \{1, 2\}$$

$$B = \{0, 1\}$$

$$C = \{0, 2, 4\}$$

ii) Verification of $A \times (B \cap C) = (A \times B) \cap (A \times C)$ answer

$$\text{L.H.S} = A \times (B \cap C)$$

$$= A \times (\{0, 1\} \cap \{0, 2, 4\})$$

$$= \{1, 2\} \times \{0\}$$

$$= \{(1, 0), (2, 0)\} \quad \text{--- (i)}$$

$$\text{R.H.S} = (A \times B) \cap (A \times C)$$

$$= (\{1, 2\} \times \{0, 1\}) \cap (\{1, 2\} \times \{0, 2, 4\})$$

$$= \{(1, 0), (1, 1), (2, 0), (2, 1)\} \cap \{(1, 0), (1, 2), (1, 4), (2, 0), (2, 2), (2, 4)\}$$

$$= \{(1, 0), (2, 0)\} \quad \text{--- (ii)}$$

From (i) and (ii) L.H.S = R.H.S

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Hence verified.

ii) Verify $(A \times B) \cap C = (A \times C) \cap (B \times C)$ answer

Solution

$$\text{L.H.S} = (A \times B) \cap C$$

$$= (\{1, 2\} \times \{0, 1\}) \cap C$$

$$= \{(2, 0), (2, 1), (2, 4)\} \quad \text{--- (i)}$$

$$\text{R.H.S} = (A \times C) \cap (B \times C)$$

$$= (\{1, 2\} \times \{0, 2, 4\}) \cap (\{0, 1\} \times \{0, 2, 4\})$$

$$= \{(1, 0), (1, 2), (1, 4), (2, 0), (2, 2), (2, 4)\} \cap \{(0, 0), (0, 2), (0, 4), (1, 0), (1, 2), (1, 4)\}$$

$$= \{(2, 0), (2, 2), (2, 4)\} \quad \text{--- (ii)}$$

From (i) and (ii) L.H.S = R.H.S

It is verified that $(A \times B) \cap C = (A \times C) \cap (B \times C)$.

Q.8 Let $x = \{x/x \in W, x \leq 2\}$ and $Y = \{-1, -2, -3\}$. Exhibit $X \times Y$ and $Y \times X$ by arrow diagram. answer

Solution

$$X = \{x/x \in W, x \leq 2\}$$

$$Y = \{-1, -2, -3\}$$

In tabular form:

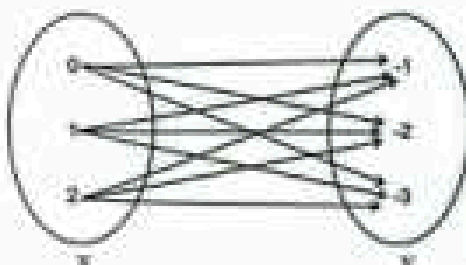
$$X = \{0, 1, 2\}$$

$$Y = \{-1, -2, -3\}$$

$$(i) X \times Y = \{0, 1, 2\} \times \{-1, -2, -3\}$$

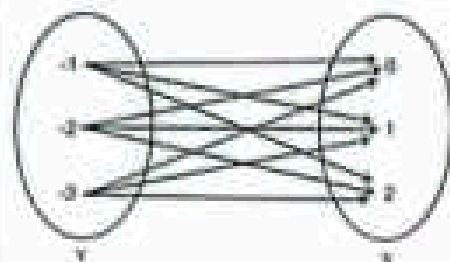
$$X \times Y = \{(0, -1), (0, -2), (0, -3), (1, -1), (1, -2), (1, -3), (2, -1), (2, -2), (2, -3)\}$$

Exhibition of X, Y by arrow diagram.



$$(ii) Y \times X = \{-1, -2, -3\} \times \{0, 1, 2\}$$

$$Y \times X = \{(-1, 0), (-1, 1), (-1, 2), (-2, 0), (-2, 1), (-2, 2), (-3, 0), (-3, 1), (-3, 2)\}$$



Binary Relation

A binary relation R in the set $A \times B$ is a subset of the Cartesian product $A \times B$. Symbolically R is the relation in a set $A \times B$ if and only if $R \subseteq A \times B$.

If R is relation from A to B , then

$$R = \{(a, b)/a \in A, b \in B\}$$

A binary relation can also be taken from only one set after taking Cartesian product of the set with itself e.g. from $A \times A$.

If R_1 is relation from A to A , then

$$R_1 = \{(a, b)/a \in A, b \in A\}$$

Example 17

If $A = \{5, 10, 15, 20, 25\}$ then find the number of binary relations in A . answer

Solution: Number of elements in $A = n(A) = 5$

Number of elements in

$$A \times A = n(A \times A) = 5 \times 5 = 25$$

Number of binary relations in A (or $A \times A$) = 2^{25}

Example 18

If $P = \{2, 3\}$, $Q = \left\{\frac{1}{2}, \frac{1}{3}\right\}$, then find all possible

binary relations in $P \times Q$. answer

Solution:

Number of elements in $P = n(P) = 2$

Number of elements in $Q = n(Q) = 2$

Number of elements in

$$P \times Q = n(P \times Q) = n(P) \times n(Q) = 2 \times 2 = 4$$

Number of binary relations in $P \times Q = 2^{n(P \times Q)}$



$$\text{Now } P \times Q = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right), \left(2, \frac{1}{4} \right), \left(2, \frac{1}{5} \right) \right\}$$

here,

$$R_1 = P \times R_2 = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_2 = \left\{ \left(2, \frac{1}{3} \right) \right\}$$

$$R_3 = \left\{ \left(2, \frac{1}{2} \right) \right\}, R_4 = \left\{ \left(2, \frac{1}{3} \right) \right\}$$

$$R_5 = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_6 = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}$$

$$R_7 = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_8 = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}$$

$$R_9 = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_{10} = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}$$

$$R_{11} = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_{12} = \left\{ \left(2, \frac{1}{2} \right) \right\}$$

$$R_{13} = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_{14} = \left\{ \left(2, \frac{1}{2} \right) \right\}$$

$$R_{15} = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_{16} = \left\{ \left(2, \frac{1}{2} \right) \right\}$$

$$R_{17} = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_{18} = \left\{ \left(2, \frac{1}{2} \right) \right\}$$

$$R_{19} = \left\{ \left(2, \frac{1}{2} \right), \left(2, \frac{1}{3} \right) \right\}, R_{20} = \left\{ \left(2, \frac{1}{2} \right) \right\}$$

Example 19

Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$ and $R = \{(a, b) : a \in A, b \in B \text{ and } a > b\}$, then find R and show it by arrow diagram.

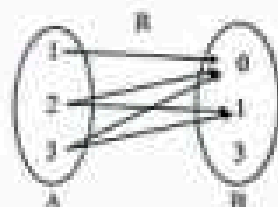
Solution

$$A = \{1, 2, 3\}, B = \{0, 1, 2\}$$

$$A \times B = \{(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$$

$$\text{Now } R = \{(a, b) : a \in A, b \in B \text{ and } a > b\}$$

$$R = \{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1)\}$$



Domain of Binary Relation

Set of all first elements of ordered pairs in a binary relation is called domain of that binary relation.

Range of Binary Relation

Set of all the second elements of ordered pairs in a binary relation is called range of that binary relation.

Example 20

(a) If $A =$ Set of Natural numbers and $R = \{(x, y) : x \in A, y \in A\}$ i.e. $R \subseteq A \times A$. Then find the domain and range of R .

(b) If $T = \{0, \pm 1, \pm 2\}$ and

$R_1 = \{(x, y) : x \in T, y \in T, x + y = 0\}$, then find the Dom R_1 and Range R_1 .

(c) If $E = \{2, 4, 6\}$, $F = \{0, 1, 2\}$ and

$R_2 = \{(x, y) : x \in E, y \in F, x + y = 6\}$, then

(i) Write $E \times F$ (ii) Write R_2 in tabular form

(iii) Find Dom R_2 and Range R_2 .

Solution

(a) $R = \{(1, 1), (1, 2), (1, 3), \dots, (2, 1), (2, 2), (2, 3), \dots, (3, 2), (3, 3), \dots\}$

So, Domain of $R = \{1, 2, 3, \dots\} =$ Set of Natural numbers

Range of $R = \{1, 2, 3, \dots\} =$ Set of Natural numbers

$\therefore$ Dom $R =$ Range $R = A$

(b) $R_1 = \{(0, 0), (-1, 1), (1, -1), (-2, 2), (2, -2)\}$

Dom $R_1 = \{0, \pm 1, \pm 2\}$, Range $R_1 = \{0, \pm 1, \pm 2\}$

(c) $E \times F = \{(2, 0), (2, 1), (2, 2), (4, 0), (4, 1), (4, 2), (6, 0), (6, 1), (6, 2)\}$



$$R_2 = \{(4, 2), (6, 0)\}$$

$$\text{Dom } R_2 = \{4, 6\}, \text{ Range } R_2 = \{0, 2\}$$

Inverse Relation

Let $R = \{(a, b) : a \in A, b \in B\}$ be a relation from A to B then the inverse of R is defined by:

$$R^{-1} = \{(b, a) : (a, b) \in R, a \in A\}$$

Example 21

Given that $A = \{0, 2, 3\}$, $B = \{0, 2, 4, 6, 8, 16\}$

and $R = \{(x, y) : x \in A, y \in B, x^2 = y\}$, answer

Verify: Dom $R^{-1} =$ Range R and

Range $R^{-1} =$ Dom R .

Solution

Given $A = \{0, 2, 3\}$, $B = \{0, 2, 4, 6, 8, 16\}$ and

$$R = \{(x, y) : x \in A, y \in B, x^2 = y\}$$

R in tabular form is:

$$R = \{(0, 0), (2, 4), (3, 9)\}$$

Dom $R = \{0, 2, 3\}$ and Range $R = \{0, 4, 9\}$

Inverse of R is:

$$R^{-1} = \{(0, 0), (4, 2), (9, 3)\}$$

Dom $R^{-1} = \{0, 4, 9\}$ and Range $R^{-1} = \{0, 2, 3\}$

Which shows that:

Dom $R^{-1} = \{0, 4, 9\} =$ Range R and

Range $R^{-1} = \{0, 2, 3\} =$ Dom R .

Exercise 3.4

Q.1 Find the number of binary relations in the following cases.

(i) $A = \{1, 3\}$, $B = \{0, 2, 4\}$

Solution

$$A = \{1, 3\}, B = \{0, 2, 4\}$$

$$\text{Number of elements in } A = n(A) = 2$$

$$\text{Number of elements in } B = n(B) = 3$$

$$\text{Number of elements in } A \times B = n(A \times B)$$

$$= n(A) \times n(B)$$

$$= 2 \times 3 = 6$$

$$\text{Number of binary relations in } A \times B = 2^{n(A \times B)}$$

$$= 2^6 = 64$$

(ii) $n(C) = 7$

Solution

$$n(C) = 7$$

$$\text{Number of elements in } C \text{ or } (C \times C) = 7 \times 7 = 49$$

$$\text{Number of binary relations in } C = 2^{49}$$

(iii) $D = \{1, 3, 5\}$

Solution

$$D = \{1, 3, 5\}$$

$$\text{Number of elements in } D = n(D) = 3$$

$$\text{Number of elements in } D \text{ or } (D \times D) = 3 \times 3 = 9$$

$$\text{Number of binary relations in } D \text{ or } (D \times D) = 2^9$$

Q.2 Find all possible binary relations in the following cases mentioning the number of binary relations in each case.

(i) $A = \{\sqrt{2}, \sqrt{3}, \sqrt{5}\}$, $B = \{\sqrt{5}\}$

Solution

$$A = \{\sqrt{2}, \sqrt{3}, \sqrt{5}\}, B = \{\sqrt{5}\}$$

$$\text{Number of elements in } A = n(A) = 3$$

$$\text{Number of elements in } B = n(B) = 1$$

$$\text{Number of elements in } A \times B = n(A \times B)$$

$$= 3 \times 1 = 3$$

$$\text{Number of binary relations in } A \times B = 2^3 = 8$$

$$A \times B = \{(\sqrt{2}, \sqrt{5}), (\sqrt{3}, \sqrt{5}), (\sqrt{5}, \sqrt{5})\}$$

Binary Relations

$$R_1 = \emptyset$$

$$R_2 = \{(\sqrt{2}, \sqrt{5})\}$$

$$R_3 = \{(\sqrt{3}, \sqrt{5})\}$$

$$R_4 = \{(\sqrt{5}, \sqrt{5})\}$$

$$R_5 = \{(\sqrt{2}, \sqrt{5}), (\sqrt{3}, \sqrt{5})\}$$

$$R_6 = \{(\sqrt{2}, \sqrt{5}), (\sqrt{5}, \sqrt{5})\}$$

$$R_7 = \{(\sqrt{3}, \sqrt{5}), (\sqrt{5}, \sqrt{5})\}$$

$$R_8 = \{(\sqrt{2}, \sqrt{5}), (\sqrt{3}, \sqrt{5}), (\sqrt{5}, \sqrt{5})\}$$



ii) $C = \{x, y\}$

Solution

$C = \{x, y\}$

Number of elements in $C = n(C) = 2$

Number of elements in $C \times C = n(C \times C) = 2 \times 2 = 4$

Number of binary relations in C (or $C \times C$) = $2^4 = 16$

$C \times C = \{x, x\}, \{x, y\}, \{y, x\}, \{y, y\}$

Binary Relations

$R_1 = \emptyset$

$R_2 = \{(x, x)\}$

$R_3 = \{(x, y)\}$

$R_4 = \{(y, x)\}$

$R_5 = \{(x, y)\}$

$R_6 = \{(x, x), (x, y)\}$

$R_7 = \{(x, x), (y, y)\}$

$R_8 = \{(x, y), (x, x)\}$

$R_9 = \{(x, y), (y, x)\}$

$R_{10} = \{(x, x), (y, y)\}$

$R_{11} = \{(x, x), (x, y), (y, y)\}$

$R_{12} = \{(x, y), (x, x), (y, y)\}$

$R_{13} = \{(x, y), (x, x), (y, x)\}$

$R_{14} = \{(x, y), (x, x), (x, y)\}$

$R_{15} = \{(x, y), (y, x), (y, y)\}$

$R_{16} = \{(x, x), (x, y), (y, x), (y, y)\}$

iii) $D = \{5\}$, $E = \{1, 10\}$

Number of elements in $D = n(D) = 1$

Number of elements in $E = n(E) = 2$

Number of elements in $D \times E = n(D \times E) = 1 \times 2 = 2$

Number of binary relations in $D \times E = 2^2 = 4$

$D \times E = \{5\} \times \{1, 10\}$

$D \times E = \{(5, 1), (5, 10)\}$

Binary Relation

$R_1 = \emptyset$

$R_2 = \{(5, 1)\}$

$R_3 = \{(5, 10)\}$

$R_4 = \{(5, 1), (5, 10)\}$

Q.3 If $P = \{7, 8, 9\}$ then find 2 binary relations from P to P . Also find domain and range of each relation.

Solution

$P = \{7, 8, 9\}$

$P \times P = \{7, 8, 9\} \times \{7, 8, 9\}$

$P \times P = \{(7, 7), (7, 8), (7, 9), (8, 7), (8, 8), (8, 9), (9, 7), (9, 8), (9, 9)\}$

Two Binary Relations

$R_1 = \{(7, 7), (7, 8), (7, 9), (8, 7)\}$

Dom(R_1) = $\{7, 8\}$

Range(R_1) = $\{7, 8, 9\}$

$R_2 = \{(8, 8), (8, 9), (9, 7), (9, 8)\}$

Dom(R_2) = $\{8, 9\}$

Range(R_2) = $\{8, 9, 7\}$

Q.4 Let $H = \{5, 6, 7, 8, 9\}$ and $G = \{5, 7, 9, 11\}$. Write the following relations from H to G in tabular form.

Solution

$H \times G = \{5, 6, 7, 8, 9\} \times \{5, 7, 9, 11\}$
 $= \{(5, 5), (5, 7), (5, 9), (5, 11), (6, 5), (6, 7), (6, 9), (6, 11), (7, 5), (7, 7), (7, 9), (7, 11), (8, 5), (8, 7), (8, 9), (8, 11), (9, 5), (9, 7), (9, 9), (9, 11)\}$

Binary Relations

i) 'is equal to'

$R_1 = \{(5, 5), (7, 7), (9, 9)\}$

ii) 'is less than'

$R_2 = \{(5, 7), (5, 9), (5, 11), (6, 7), (6, 9), (6, 11), (7, 9), (7, 11), (8, 9), (8, 11), (9, 11)\}$

iii) 'is greater than'

$R_3 = \{(6, 5), (7, 5), (8, 5), (8, 7), (9, 5), (9, 7)\}$

iv) 'is one less than'

$R_4 = \{(5, 7), (8, 9)\}$

v) 'is one greater than'

$R_5 = \{(6, 5), (8, 7)\}$

vi) 'is two less than'

$R_6 = \{(5, 7), (7, 9), (9, 11)\}$

Q.5 Let $C = \{2, 4, 6\}$, $D = \{4, 6, 8, 9, 12\}$ and $R = \{(x, y) | x \in C, y \in D \wedge x \text{ is factor of } y\}$.

i) Write R in tabular form.

Solution

$C = \{2, 4, 6\}$, $D = \{4, 6, 8, 9, 12\}$

$R = \{(x, y) | x \in C, y \in D \wedge x \text{ is factor of } y\}$

$C \times D = \{2, 4, 6\} \times \{4, 6, 8, 9, 12\}$
 $= \{(2, 4), (2, 6), (2, 8), (2, 9), (2, 12), (4, 4), (4, 6), (4, 8), (4, 9), (4, 12), (6, 4), (6, 6), (6, 8), (6, 9), (6, 12)\}$

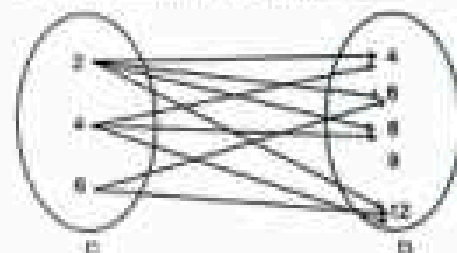
i) $R = \{(2, 4), (2, 6), (2, 8), (2, 12), (4, 4), (4, 8), (4, 12), (6, 6), (6, 12)\}$

ii) Dom(R) = $\{2, 4, 6\}$

Range(R) = $\{4, 6, 8, 12\}$

iii) $R^{-1} = \{(4, 2), (6, 2), (8, 2), (12, 2), (4, 4), (8, 4), (12, 4), (6, 6), (12, 6)\}$

iv) Representation of R by arrow diagram



Q.6 Let $R = \{(2, 0), (4, 2), (6, 4), (8, 6), (10, 8)\}$

Solution

i) Write R in set builder form.

Set builder form

$R = \{(x, y) | x = y + 2\}$

ii) Find domain and range of R .

Dom(R) = $\{2, 4, 6, 8, 10\}$

Range(R) = $\{0, 2, 4, 6, 8\}$

iii) Write R^{-1} in tabular and set builder form.

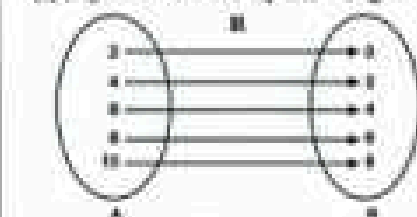
$R^{-1} = \{(0, 2), (2, 4), (4, 6), (6, 8), (8, 10)\}$

$R^{-1} = \{(x, y) | y = x + 2\}$

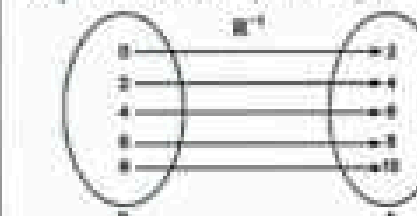


iv) Represent R and R^{-1} by arrow diagram.

(a) Representation of R by arrow diagram



Representation of R^{-1} by arrow diagram



Q.7 Let $A = \{0, 1, 3\}$ and $B = \{1, 2, 3, 5, 7\}$. Write $R = \{(x, y) | x \in A, y \in B \wedge y = 2x + 1\}$ in tabular form. Also find R^{-1} .

Solution

$A = \{0, 1, 3\}$

$B = \{1, 2, 3, 5, 7\}$

$R = \{(x, y) | x \in A, y \in B \wedge y = 2x + 1\}$

$A \times B = \{0, 1, 3\} \times \{1, 2, 3, 5, 7\}$

Tabular form

$R = \{(0, 1), (1, 3), (3, 7)\}$

Now $R^{-1} = \{(1, 0), (3, 1), (7, 3)\}$

Q.8 If $S = \{1, 2, 4, 8\}$, $T = \{3^0, 3^1, 3^2\}$, then write the following binary relations in tabular form.

$S \times T = \{1, 2, 4, 8\} \times \{3^0, 3^1, 3^2\}$

Tabular form of given relations

i) $R_1 = \{(x, y) | x \in S, y \in T \wedge x = y\}$

$R_1 = \{(1, 3^0)\}$, Dom $R_1 = \{1\}$, Range $R_1 = \{3^0\}$

ii) $R_2 = \{(x, y) | x \in S, y \in T \wedge y = x\}$

$R_2 = \{(2, 3^1), (4, 3^2), (8, 3^3), (3, 3^0)\}$

Dom $R_2 = \{2, 4, 8\}$

Range $R_2 = \{3^0, 3^1\}$



8) $R_3 = \{(x, y) | x \in S, y \in T \wedge x + y \in E\}$

answer

$R_3 = \{(1, 2^2), (1, 3^2), (1, 4^2)\}$

$\text{Dom}(R_3) = \{1\}$

$\text{Range}(R_3) = \{2^2, 3^2, 4^2\}$

9) $R_4 = \{(x, y) | x \in S, y \in T \wedge x = y \in Q\}$

answer

$R_4 = \{(1, 2^2), (1, 3^2), (1, 4^2)\}$

$\text{Dom}(R_4) = \{1\}$

$\text{Range}(R_4) = \{2^2, 3^2, 4^2\}$

9) $R_5 = \{(x, y) | x \in S, y \in T \wedge y = 2x\}$

answer

$R_5 = \{(1, 2^2), (1, 3^2), (2, 3^2), (4, 3^2)\}$

$\text{Dom}(R_5) = \{1, 2, 4\}$

$\text{Range}(R_5) = \{2^2, 3^2\}$

Q.9 Find the domain and range in each case.

answer

Solution: Domain and range are given in Q.8

Miscellaneous Exercise 2

1. Encircle the correct option in the following:

1. Set builder form of $A \cap B$ is:

answer

(a) $\{x | x \in A\}$

(b) $\{x | x \in A \wedge x \in B\}$

(c) $\{x | x \in A \vee x \in B\}$

(d) $\{x | x \in B\}$

2. If $A \cup B = A$ and $A \cap B = B$, then:

answer

(a) $A \subset B$

(b) $A \not\subset B$

(c) $A \supset B$

(d) $A = B$

3. If $A \subset B$ then $A - B =$

answer

(a) A

(b) B

(c) ϕ

(d) $B - A$

4. If $A - B = B - A = \phi$, then:

answer

(a) $A = B$

(b) $B \subset A$

(c) $A \subset B$

(d) all $a, b \notin c$

5. Set of common elements of A and A' is:

answer

(a) infinite

(b) null

(c) universal

(d) singleton

6. Set of real numbers can be written in:

answer

(a) tabular form

(b) description form

(c) set builder form

(d) both b and c

7. If $R = \{(2, 1), (4, 3), (3, 2)\}$, then $\text{Dom } R =$

answer

(a) $\{2, 4, 3\}$

(b) $\{2, 4\}$

(c) $\{1, 3, 2\}$

(d) none of these

8. If $R = \{a, b\}$ $a, b \in Z$ $a + b = 0$, then:

(a) $\text{Dom } R \subset \text{Range } R$

answer

(b) $\text{Range } R \subset \text{Dom } R$

(c) $\text{Dom } R = \text{Range } R$

(d) none of these

9. If $R = \{(a, b) | a, b \in N \wedge a + b = 12\}$ then tabular form R is:

answer

(a) $\{(1, 12), (2, 6), (3, 4), (4, 3), (6, 2), (12, 1)\}$

(b) $\{(1, 12), (2, 6), (3, 4)\}$

(c) $\{(12, 1), (6, 2), (4, 3)\}$

(d) $\{(1, 12), (4, 3), (2, 6)\}$

10. If $n(A) = p$, then $n(A \times A) =$

answer

(a) p

(b) $2p$

(c) $2p^2$

(d) p^2

11. If $n(B) = t$, then number of binary relations in $B \times B$ is:

answer

(a) 1

(b) t^2

(c) 2^t

(d) 2^{t^2}



Additional MCQs

12. $A \cup B =$

answer

(a) $\{x \in A \vee x \in B\}$

(b) $\{x \in A \vee x \in B\}$

(c) $\{x \in A \wedge x \in B\}$

(d) $\{x \in A \wedge x \in B\}$

13. The set $\{x \in U \wedge x \in A\}$ is equal to:

answer

(a) A

(b) A'

(c) $A - B$

(d) B'

14. The set $\{x \in A \wedge x \in B\}$ is equal to:

answer

(a) A

(b) B

(c) $A - B$

(d) $B - A$

15. If $A = \{1, 2, 3\}$, $B = \{y, z\}$, then all the binary relations in $A \times B$ are:

answer

(a) 9

(b) 64

(c) 6

(d) 32

16. $R = \{(1, 2), (1, 3), (2, 5), (3, 10)\}$ is a binary relation. Its domain is:

answer

(a) $\{1, 2, 3\}$

(b) $\{1, 2, 3\}$

(c) $\{2, 3, 5, 10\}$

(d) $\{1, 2, 3, 4, 5, 10\}$

17. If $n(A \times B) = 6$ and $A = \{1, 3\}$ then $n(B)$ is:

answer

(a) 1

(b) 2

(c) 3

(d) 4

18. $A = \{a, b, p\}$, $B = \{2, 3\}$, $C = \{p, q, x\}$ then $n[(A \cup C) \times B]$ is:

answer

(a) 8

(b) 20

(c) 12

(d) 16

19. If there are 1024 relations from a set $A = \{1, 2, 3, 4, 5\}$ to a set B , then the number of elements in B is:

answer

(a) 3

(b) 2

(c) 4

(d) 6

20. The range of a relation $R = \{(x, x^2) | x \text{ is a prime number less than } 13\}$ is:

answer

(a) $\{2, 3, 5, 7\}$

(b) $\{2, 3, 5, 7, 11\}$

(c) $\{4, 9, 25, 49, 121\}$

(d) $\{1, 4, 9, 25, 49, 121\}$

21. If the ordered pairs $(x^2, 4)$ and $(5, 2x - b)$ are equal then (a, b) is:

answer

(a) $(2, -2)$

(b) $(5, 1)$

(c) $(2, 3)$

(d) $(3, -2)$

22. Let $n(A) = m$ and $n(B) = n$ then the total number of non-empty relations that can be defined from A to B is:

answer

(a) m^n

(b) n^n

(c) $2^{mn} - 1$

(d) 2^{mn}

23. If $B = \{x\}$, then:

answer

(a) B is the null set

(b) B is an infinite set

(c) B is a finite set

(d) B is not a set

Answer Key

1	b	2	c	3	c	4	a	5	b
6	d	7	b	8	c	9	a	10	d
11	d	12	b	13	b	14	c	15	b
16	b	17	c	18	c	19	b	20	c
21	d	22	c	23	c				

Q.2 If $R = \{a, t, h, v\}$ and $S = \{h, a, j, n\}$, then:

i) Find the number of binary relations in $R \times S$

answer

Solution

$R = \{a, t, h, v\}$ and $S = \{h, a, j, n\}$

Number of elements of $R = n(R) = 4$

Number of elements of $S = n(S) = 4$

Number of elements of $R \times S = 4 \times 4 = 16$

Number of binary relations in $R \times S = 2^{16}$

ii) Any 3 binary relations from $R \times S$

Solution

$R \times S = \{a, t, h, v\} \times \{h, a, j, n\}$



$R = S = \{(a,b), (a,c), (a,j), (a,i), (c,b), (c,i), (c,j), (c,e), (b,b), (b,i), (b,j), (b,e), (a,b), (a,c), (a,j), (a,i)\}$
Three Binary relations from $R = S$.

$R_1 = \{(a,b), (a,c), (a,j)\}$

$R_2 = \{(a,i), (c,b), (c,i)\}$

$R_3 = \{(c,j), (c,i), (b,b)\}$

iii) Write a Binary relations having domain equal set R

Solution

Binary relations having domain equal set R

$R_1 = \{(a,b), (c,b), (b,b), (a,b)\}$

$\text{Dom}(R_1) = \{a, c, b, i\} = R$

$R_2 = \{(a,b), (c,b), (b,b), (a,i)\}$

$\text{Dom}(R_2) = \{a, c, b, i\} = R$

iv) Write a Binary relation whose range is equal to set S.

Solution

Binary relation having range equal to S

$R_1 = \{(a,b), (a,c), (a,j), (a,i)\} = R$

$\text{Range}(R_1) = \{b, c, j, i\} = S$

v) Binary relation having $\text{Dom} = R$ and $\text{Range} = S$.

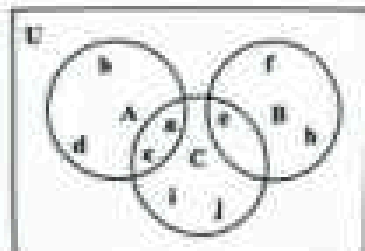
$R_1 = \{(a,b), (c,b), (b,b), (a,i)\}$

$\text{Dom}(R_1) = \{a, c, b, i\} = R$

$\text{Range}(R_1) = \{b, c, j, i\} = S$

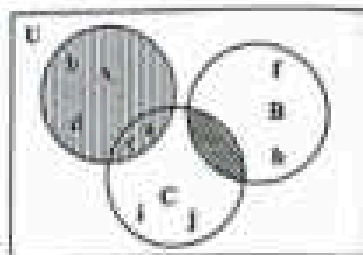
Q.3 Shade $A \cup (B \cap C)$, $A \cap (B \cup C)$.

$A \cap (B \cup C)$ and $B = (A \cup C)$, in the following Venn diagram.



Solution

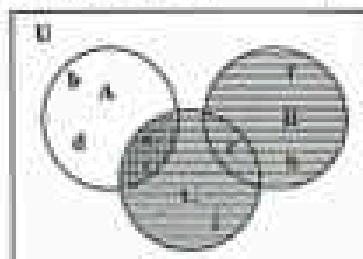
i) Shading $A \cup (B \cap C)$



$(B \cap C) = \text{Region of } \text{[diagonal lines]}$

$A \cup (B \cap C) = \text{Regions of } \text{[diagonal lines]} \text{ and } \text{[horizontal lines]}$

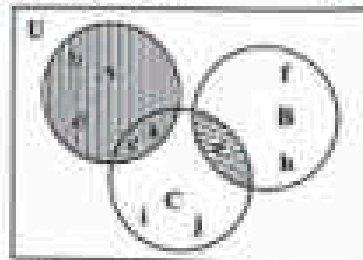
ii) Shading $A \cap (B \cup C)$



$(B \cup C) = \text{Regions of } \text{[diagonal lines]} \text{ and } \text{[horizontal lines]}$

$A \cap (B \cup C) = \text{Regions of } \text{[diagonal lines]} \text{ and } \text{[horizontal lines]}$

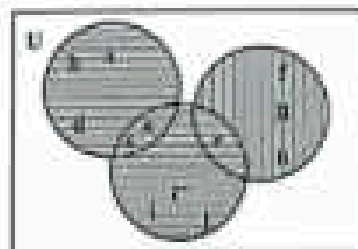
iii) Shading $A - (B \cap C)$



$B \cap C = \text{Region of } \text{[diagonal lines]}$

$A - (B \cap C) = \text{Region of } \text{[diagonal lines]}$

iv) Shading $B - (A \cup C)$



$A \cap C = \text{Region of } \text{[diagonal lines]}$

$B - (A \cup C) = \text{Region of } \text{[horizontal lines]}$

Q.4 Verify associative properties of union and intersection through Venn diagram for

$X = \{2x\} \text{ so } N \wedge x < 20$, $Y = \text{Set of first 6 natural multiples of 3}$

$Z = \{6x\} \text{ so } W \wedge x < 20$.

$X = \{2x\} \text{ so } N \wedge x < 20$.

$Y = \text{set of first 6 natural multiples of 3}$.

$Z = \{6x\} \text{ so } W \wedge x < 20$

Solution

$X = \{2x\} \text{ so } N \wedge x < 20$.

$Y = \text{set of first 6 natural multiples of 3}$.

$Z = \{6x\} \text{ so } W \wedge x < 20$

$X = \{2, 4, 6, 8, 10, \dots, 38\}$.

$Y = \{3, 6, 9, 12, 15, 18\}$

$Z = \{6, 12, 18, 24, 30, \dots, 114\}$

We observe that

X and Y are overlapping sets

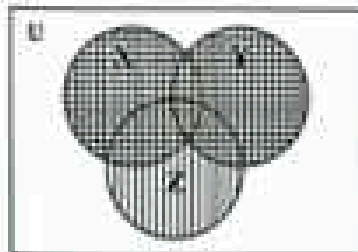
Y and Z are overlapping sets

X and Z are overlapping sets

ii) **Associative law of union**

$(X \cup Y) \cup Z = X \cup (Y \cup Z)$

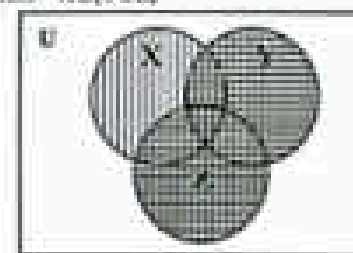
L.H.S. = $(X \cup Y) \cup Z$



$(X \cup Y) = \text{Regions of } \text{[diagonal lines]} \text{ and } \text{[horizontal lines]}$

$(X \cup Y) \cup Z = \text{Regions of } \text{[diagonal lines]} \text{ and } \text{[horizontal lines]}$

R.H.S. = $X \cup (Y \cup Z)$



$(Y \cup Z) = \text{Regions of } \text{[horizontal lines]} \text{ and } \text{[vertical lines]}$

$X \cup (Y \cup Z) = \text{Regions of } \text{[diagonal lines]} \text{ and } \text{[horizontal lines]}$

From both the figures, we observe that the regions showing $(X \cup Y) \cup Z$ and $X \cup (Y \cup Z)$ are same i.e.

$(X \cup Y) \cup Z = X \cup (Y \cup Z)$

Hence proved.

iii) **Associative property of intersection**

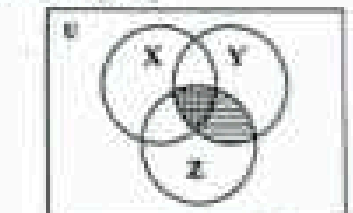
$(X \cap Y) \cap Z = X \cap (Y \cap Z)$

Hence proved.

ii) **Associative property of intersection**

$(X \cap Y) \cap Z = X \cap (Y \cap Z)$

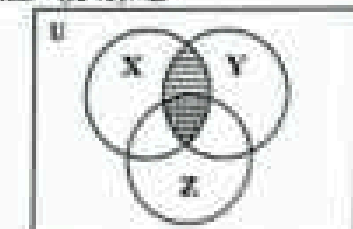
L.H.S. = $X \cap (Y \cap Z)$



$(Y \cap Z) = \text{Regions of } \text{[horizontal lines]} \text{ and } \text{[vertical lines]}$

$X \cap (Y \cap Z) = \text{Regions of } \text{[diagonal lines]} \text{ and } \text{[horizontal lines]}$

L.H.S. = $(X \cap Y) \cap Z$





$(X \cap Y) \cap Z$ = Regions of and .

$(X \cap Y) \cap Z$ = Regions of .

From both figures, we observe that the regions showing $(X \cap Y) \cap Z$ and $X \cap (Y \cap Z)$ are same i.e. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$. Hence proved.

Q.5 Verify Distributive law of union over intersection through Venn diagram for the following sets.

$A = \{1, 2, 3, \dots\}$, $B = \{-1, -2, -3, \dots\}$

$C = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

Solution

$A = \{1, 2, 3, \dots\}$

$B = \{-1, -2, -3, \dots\}$

$C = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

We observe that A and B are disjoint sets.

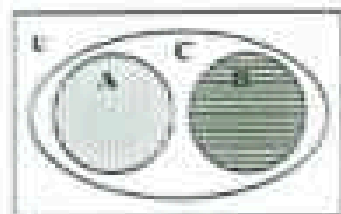
A is subset of C.

B is subset of C.

Distributive law of union over intersection

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

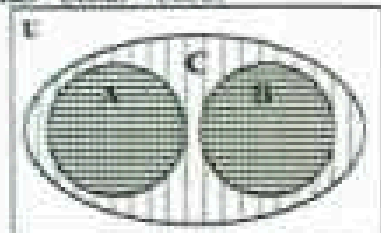
L.H.S = $A \cup (B \cap C)$



$(B \cap C) =$ Regions of .

$A \cup (B \cap C) =$ Regions of and .

R.H.S = $(A \cup B) \cap (A \cup C)$



$(A \cup B) =$ Regions of and .

$(A \cup C) =$ Regions of and .

$(A \cup B) \cap (A \cup C) =$ Regions of and .

From both the figures, we observe that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Hence proved.

Q.6 Verify distributive law of intersection over union through Venn diagram for the following sets.

$P = \{a, b, c, d, e\}$, $Q = \{c, d, e, f\}$

$R = \{g, h, i, j, k\}$

Solution

$P = \{a, b, c, d, e\}$

$Q = \{c, d, e, f\}$

$R = \{g, h, i, j, k\}$

We observe that

P and Q are overlapping sets.

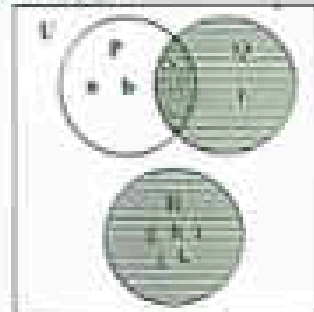
P and R are disjoint sets.

Q and R are disjoint sets.

Distributive law of intersection over union

$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$

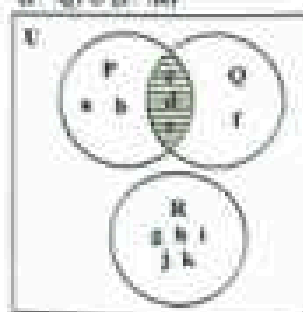
L.H.S = $P \cap (Q \cup R)$



$Q \cup R =$ Regions of and .

$P \cap (Q \cup R) =$ Regions of and .

R.H.S = $(P \cap Q) \cup (P \cap R)$



$(P \cap Q) =$ Regions of .

$(P \cap R) = \emptyset$

$(P \cap Q) \cup (P \cap R) =$ Regions of and .

From both figures, the regions showing

$P \cap (Q \cup R)$ and $(P \cap Q) \cup (P \cap R)$ are same i.e.

$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$

Q.7 Create a Venn diagram to illustrate the following information regarding the subsets

A and B in the universal set.

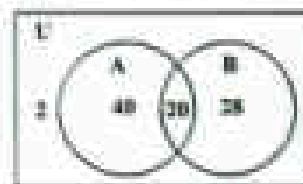
0 $n(A) = 60$, $n(B) = 48$, $n(A \cap B) = 20$, $n(U) = 90$

Solution

$n(A) = 60$, $n(B) = 48$, $n(A \cap B) = 20$, $n(U) = 90$

Venn diagram

$n(A \cap B) = 20$ shows that sets are overlapping.



$n(A) = 60 = 40 + 20 = 60$

$n(B) = 48 = 28 + 20 = 48$

$n(A \cap B) = 20$

$n(U) = 90 = 40 + 20 + 30 = 90$

Ans $n(A) = 60$, $n(B) = 48$, $n(A \cap B) = 20$, $n(U) = 90$

Solution

$n(A) = 60$, $n(B) = 48$, $n(A \cap B) = 20$, $n(U) = 90$

We know that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$60 = 40 + 20 - n(A \cap B)$

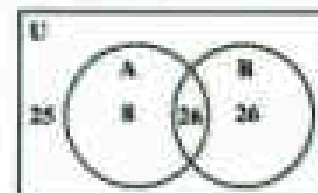
$60 = 60 - n(A \cap B)$

$\Rightarrow n(A \cap B) = 60 - 60$

$n(A \cap B) = 20$

Venn Diagram

As $n(A \cap B) = 20$ which shows that set A and set B are overlapping sets.



$n(A) = 8 + 20 = 28$

$n(B) = 20 + 20 = 40$

$n(A \cap B) = 8 + 20 + 20 = 48$

$n(A \cup B) = 28$

$n(U) = 28 + 8 + 20 + 20 = 76$

Q.8 10 boys participated in a Qirat competition. Among them Haani, Zubair and Haider recited in Naafi Qirat style.

Abdullah, Umar, Bilal and Ali recited in Al-Khulsi Qirat style while Hassan, Jaffer and Usman recited in Al-Kafi Qirat style.

Represent boys participated in Naafi, Al-Khulsi and Al-Kafi styles by sets A, B and C respectively. Find:

(i) Find tabular form of A, B and C. (ii) Draw Venn Diagram of situation

Solution

Let A represents the boys who recited the Holy Quran in Naafi style.

B represents the boys who recited the Holy Quran in Al-Khulsi style.

C represents the boys who recited the Holy Quran in Al-Kafi style.

$A = \{ \text{Haani, Zubair, Haider} \}$

$B = \{ \text{Abdullah, Umar, Bilal, Ali} \}$

$C = \{ \text{Hassan, Jaffer, Usman} \}$

(i) Draw Venn Diagram of situation

Solution

Let A represents the boys who recited the Holy Quran in Naafi style.

B represents the boys who recited the Holy Quran in Al-Khulsi style.

C represents the boys who recited the Holy Quran in Al-Kafi style.

$A = \{ \text{Haani, Zubair, Haider} \}$

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Solution

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(i) Draw Venn Diagram of situation

Solution

Let A represents the boys who recited the Holy Quran in Naafi style.

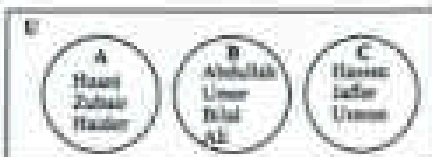
B represents the boys who recited the Holy Quran in Al-Khulsi style.

C represents the boys who recited the Holy Quran in Al-Kafi style.

$A = \{ \text{Haani, Zubair, Haider} \}$

$B = \{ \text{Abdullah, Umar, Bilal, Ali} \}$

$C = \{ \text{Hassan, Jaffer, Usman} \}$



$U = \{ \text{Haani, Zubair, Haider, Abdullah, Umar, Bilal, Ali, Hassan, Jaffer, Usman} \}$

(ii) Find $A \cap (B \cap C)^c$, $(A \cup B) \cap C^c$, $A \cap (B \cap C)$ and $A - (A \cup B)$. (100000)

Solution

(a) Finding $A \cap (B \cap C)^c$

$$(B \cap C) = \emptyset$$

$$(B \cap C)^c = U - \emptyset = U$$

$$A \cap (B \cap C)^c = A \cap U = A = \{ \text{Haani, Zubair, Haider} \}$$

(b) Finding $(A \cup B) \cap C^c$ (100000)

$$(A \cup B) = \{ \text{Haani, Zubair, Haider, Abdullah, Umar, Bilal, Ali} \}$$

$$C^c = U - C = \{ \text{Haani, Zubair, Haider, Abdullah, Umar, Bilal, Ali} \}$$

$$(A \cup B) \cap C^c = \{ \text{Haani, Zubair, Haider, Abdullah, Umar, Bilal, Ali} \}$$

(c) Finding $A \cap (B \cap C)$ (100000)

$$(B \cap C) = \emptyset$$

$$A \cap (B \cap C) = A \cap \emptyset = A$$

$$= \{ \text{Haani, Zubair, Haider} \}$$

(d) Finding $A - (A \cup B)$ (100000)

$$\text{In this case } A \subset (A \cup B)$$

$$(A \cup B) = \{ \text{Haani, Zubair, Haider, Abdullah, Umar, Bilal, Ali} \}$$

$$\text{Now, } A - (A \cup B) = \{ \text{Haani, Zubair, Haider} \} - \{ \text{Haani, Zubair, Haider, Abdullah, Umar, Bilal, Ali} \} = \{ \}$$

Q.9 100 candidates were appeared in an examination. Out of which 45 candidates passed in Mathematics, 40 in science and 30 in health. If 12 were passed in Mathematics

Health and Mathematics and 5 were passed in all three subjects. (100000)

Solution

$$\text{Total students, } n(U) = 100$$

$$\text{Candidates pass in Math, } n(M) = 45$$

$$\text{Candidates pass in Science } = n(S) = 40$$

$$\text{Candidates pass in Health } = n(H) = 30$$

$$\text{Candidates pass in Health, and Math } = n(H \cap M) = 20$$

$$\text{Candidate pass in Maths and Science } n(M \cap S) = 12$$

$$\text{Candidates pass in science and Health, } n(S \cap H) = 10$$

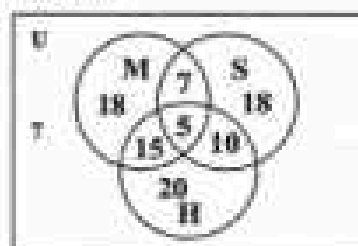
$$\text{Candidates pass in all three, } n(M \cap S \cap H) = 5$$

$$\text{Candidate pass in all three, } n(M \cap S \cap H) = 5$$

(i) Illustrate the above information by drawing a Venn diagram. (100000)

Solution

Venn Diagram



(ii) How many candidates were passed at least one subject? (100000)

Solution

$$n(M \cup S \cup H) = n(M) + n(S) + n(H) - n(M \cap S) - n(M \cap H) - n(S \cap H) + n(M \cap S \cap H)$$

$$= 45 + 40 + 30 - 12 - 20 - 10 + 5$$

$$= 140 - 47$$

$$= 93$$

(iii) How many candidates did not pass any subject? (100000)

Solution

$$n(M \cup S \cup H)^c = n(U) - n(M \cup S \cup H)$$

$$= 100 - 93$$

$$= 7$$



UNIT 04

Factorization and Algebraic Manipulation

Factorization of an Algebraic Expression

The process in which an algebraic expression can be expressed as the product of its factors is called its factorization. For example,

$$15x^2y = (5x)(3xy)$$

Here, 5x and 3xy are the factors of $15x^2y$.

Type-I

Factorizing Expression of the Form

$$(a) x^2 + a^2b^2 + b^2 \quad (b) x^2 + 4b^2$$

(a) To factorize $x^2 + a^2b^2 + b^2$ or $x^2 + 4b^2$, we shall modify it and try to make it in appropriate form to utilize the previous results for its factorization.

$$\text{Consider: } x^2 + a^2b^2 + b^2 = (x^2) + a^2b^2 + (b^2) = x^2 + b^2$$

Here first and last terms are perfect squares but middle term is not twice the product of the square root of first and last term. So, we shall add and subtract a^2b^2 to make it twice.

i.e.,

$$x^2 + a^2b^2 + b^2 = (x^2) + a^2b^2 + a^2b^2 + (b^2) - a^2b^2$$

$$= (x^2) + 2a^2b^2 + (b^2) - a^2b^2$$

$$= (x^2 + b^2) - (ab)^2$$

$$= (x^2 + b^2) + ab(a^2 + b^2) - ab^2$$

$$\text{So, } x^2 + a^2b^2 + b^2 = (x^2 + b^2) + ab(a^2 + b^2) - ab^2$$

$$(b) \text{ Similarly, } x^2 + 4b^2 = (x^2) + (2b)^2$$

$$= (x^2) + 2(x)(2b) + (2b)^2 - 2(x)(2b)$$

$$= (x^2 + 4b^2) - (4xb)$$

$$= (x^2 + 4b^2) - (4xb)$$

$$= (x^2 + 4b^2) + 4xb(x^2 + 4b^2) - 4xb$$

Example 1: Factorize the expression (100000)

$$8x^2 - 26x^2m^2 + 18m^4$$

Solution: $8x^2 - 26x^2m^2 + 18m^4$

$$= 2(4x^2 - 13x^2m^2 + 9m^4)$$

$$= 2[(2x^2)^2 - 13x^2m^2 + (3m^2)^2]$$

$$= 2[(2x^2)^2 + 2(2x^2)(3m^2) + (3m^2)^2 - 2(2x^2)(3m^2) - 13x^2m^2]$$

$$= 2[(2x^2 + 3m^2)^2 - 13x^2m^2 - 13x^2m^2]$$

$$= 2[(2x^2 + 3m^2)^2 - 26x^2m^2]$$

$$= 2[(2x^2 + 3m^2)^2 - (5xm)^2]$$

$$= 2[(2x^2 + 3m^2) + 5xm](2x^2 + 3m^2 - 5xm)$$

Type-II

Factorization of a Trinomial of the Form

$$(i) x^2 + px + q$$

$$(ii) ax^2 + bx + c$$

To factorize a trinomial of this form means to express the trinomial as the product of two binomials. To factorize such trinomials keep in mind the following steps:

(i) Make a list of all possible factors of 'product of extreme coefficients'.

(ii) Select a pair of factors among the list such that:

- Their sum is equal to middle coefficient if extremes terms have same sign.

- Their difference is equal to middle coefficient if extremes terms have opposite signs.

Example 2: Factorize the following polynomials. (100000)

$$(a) y^2 - 7y + 12 \quad (b) m^2 - 2m - 15$$



Solution: (a) $y^2 - 7y + 12$
 $= y^2 - 4y - 3y + 12$
 $= y(y - 4) - 3(y - 4)$
 $= (y - 4)(y - 3)$

So, $y^2 - 7y + 12 = (y - 3)(y - 4)$
 Product of -3 and -4
 Sum of -3 and -4

(b) $m^2 - 2m - 15$
 $= m^2 - 5m + 3m - 15$

$= m(m - 5) + 3(m - 5)$
 $= (m - 5)(m + 3)$
Example 3: Factorise $3x^2 + 22x - 16$

Solution: $3x^2 + 22x - 16$
 $= 3x^2 + 24x - 2x - 16$
 $= (3x^2 + 24x) - (2x + 16)$
 $= 3x(x + 8) - 2(x + 8)$
 $= (x + 8)(3x - 2)$

Exercise 4.1

Factorise the following polynomials

Q.1 $2x^2y^2 - 4x^2y^2 + 2xy^2$

Solution
 Taking common $2xy^2$
 $= 2xy^2(xy - 2x + 1)$

Q.2 $3xy - 3x - 3xy + 3y$

Solution
 Taking common 3
 $= 3(xy - x - xy + y)$
 $= 3(x - x) - 3(y - y)$
 $= 3(x - 1) - 3(y - 1)$
 $= 3(x - 1)(1 - y)$
 $= 3(1 - y)(x - 1)$

Q.3 $18x^2 - 108x^2y^2 + 162y^4$

Solution
 $= 18(x^2 - 6x^2y^2 + 9y^4)$
 $= 18(x^2 - 3x^2y + 3y^2x + 3y^2)$

By using formula
 $\therefore a^2 - 2ab + b^2 = (a - b)^2$
 $= 18(x^2 - 3xy + 3xy - 3y^2)$

Q.4 $(k+2)^2 - 8(k+2) + 16$

Solution
 $(k+2)^2 - 8(k+2) + 16$
 $(k+2)^2 - 2(4+2)(k+2) + (4)^2$

Using formula
 $= a^2 - 2ab + b^2 = (a - b)^2$

$= [(k+2) - 4]^2$
 $= (k-2)^2$
 $= (k-2)^2$

Q.5 $9x^2 + 4 - 108y^2 - 12x$

Solution
 $9x^2 + 4 - 108y^2 - 12x$
 $= (3x)^2 + (2)^2 - 2(3x)(2) - (12y)^2$
 $= (3x - 2)^2 - (12y)^2$
 $\therefore a^2 - b^2 = (a + b)(a - b)$
 $= (3x - 2 + 12y)(3x - 2 - 12y)$
 $= (3x + 12y - 2)(3x - 12y - 2)$

Q.6 $(x^2 - 1)(y + 1) - (y + 3)(x^2 - 1)$

Solution
 $(x^2 - 1)(y + 1) - (y + 3)(x^2 - 1)$
 $= (x^2 - 1)[(y + 1) - (y + 3)]$
 $= (x^2 - 1)(y + 1 - y - 3)$
 $= (x^2 - 1)(-2)$
 $= -2(x^2 - 1)$

Using formula
 $\therefore a^2 - b^2 = (a + b)(a - b)$
 $= -2(x + 1)(x - 1)$

Q.7 $x^2 - 4ax + 9a^2 - 10b^2$

Solution
 $= x^2 - 4ax + 9a^2 - 10b^2$
 $= (x^2 - 2(x)(3a) + (3a)^2) - (4b)^2$



Using formula
 $\therefore a^2 - 2ab + b^2 = (a - b)^2$
 $= (x - 3a)^2 - (4b)^2$

Using formula
 $\therefore a^2 - b^2 = (a + b)(a - b)$
 $= (x - 3a + 4b)(x - 3a - 4b)$

Q.8 $1 - x^2 - 2xy - y^2$

Solution
 $1 - x^2 - 2xy - y^2$
 $= 1 - (x^2 + 2xy + y^2)$
 $= (1)^2 - (x + y)^2$
 $= (1 - (x + y))(1 + (x + y))$
 $= (1 - x - y)(1 + x + y)$

Q.9 Find a polynomial whose factorisation is $(x + y - 2x)(x + 2x + y)$ by using appropriate formula.

Solution
 $(x + y - 2x)(x + 2x + y)$
 $= (x + y - 2x)(x + y + 2x) \rightarrow$ Rearranging Using the formula $= (x + y)^2 - (2x)^2$
 $(a + b + c)(a + b - c) = (a + b)^2 - (c)^2$
 $= (x + y)^2 - (2x)^2$
 $= x^2 + 2xy + y^2 - 4x^2$
 $= x^2 + y^2 + 2xy - 4x^2$

Q.10 Show the expression $x^2 + 4y^2 - a^2 + 4xy$ as the difference of two squares.

Solution
 $= x^2 + 4y^2 - a^2 + 4xy$
 $= x^2 + 4y^2 + 4xy - a^2$
 $= (x + 2y)^2 - a^2$
 $\therefore (a + b)^2 = a^2 + 2ab + b^2$
 $= [(x)^2 + 2(x)(2y) + (2y)^2] - (a)^2$
 $= (x + 2y)^2 - (a)^2$

Hence expression shows the difference of squares of $(x + 2y)$ and a .

Q.11 Find the missing factor of the following.

(a) $(2y^2 - 3y - 27) = (y + 3)(\quad)$

Solution
 $(2y^2 - 3y - 27) = (y + 3)(\quad)$
 Let missing factor be F
 $(2y^2 - 3y - 27) = (y + 3)(F)$
 $(2y^2 - 9y + 6y - 27) = (y + 3)(F)$
 $y(2y - 9) + 3(2y - 9) = (y + 3)(F)$
 $(2y - 9)(y + 3) = (y + 3)(F)$
 $\frac{(2y - 9)(y + 3)}{(y + 3)} = F$
 $2y - 9 = F$

$\therefore F = 2y - 9$
 So missing factor is $2y - 9$.

(b) $(5x^2 + 12x - 9) = (\quad)(x + 3)$

Solution
 $(5x^2 + 12x - 9) = (\quad)(x + 3)$
 Let missing factor be F
 $(5x^2 + 12x - 9) = (F)(x + 3)$
 $(5x^2 - 3x + 15x - 9) = (F)(x + 3)$
 $(5x^2 - 3x) + 3(5x - 3) = (F)(x + 3)$
 $3x(5x - 3) + 3(5x - 3) = (F)(x + 3)$
 $(3x + 3)(5x - 3) = (F)(x + 3)$
 $\frac{(3x + 3)(5x - 3)}{(x + 3)} = F$
 $3x - 3 = F$

So missing factor is $(3x - 3)$

Factorise the following expressions.

Q.12 $x^4 + 4x^2$

Solution
 Make it perfect square
 $(x^2)^2 + (2x)^2$
 We add and subtract
 $2x^2$ term $= (2x^2)$ term
 to complete a formula
 $= (x^2)^2 + (2x)^2 + 2x^2 - 2x^2 + (2x)^2$
 $= [(x^2)^2 + (2x)^2 + 2x^2](2x^2) - 4x^2x^2$



Using formula

$$a^2 + b^2 + 2ab = (a+b)^2$$

$$= (a^2 + 2ab + b^2) - 4ab + b^2$$

$$= (a^2 + 2ab + b^2) - (2ab)^2$$

Using formula

$$\therefore a^2 - b^2 = (a-b)(a+b)$$

$$= (a^2 + 2ab + b^2) - (a^2 + 2ab + b^2)$$

Q.13 $m^2 + m^2 + 1$

Solution

$$m^2 + m^2 + 1$$

Arrange the terms

$$= m^2 + 1 + m^2$$

$$= (m^2)^2 + (1)^2 + m^2$$

Add and subtract $2(m^2)(1)$

$$= (m^2)^2 + (1)^2 + 2(m^2)(1) - 2(m^2)(1) + m^2$$

$$= [(m^2)^2 + (1)^2 + 2(m^2)(1)] - 2m^2 + m^2$$

Using formula

$$\therefore a^2 + b^2 + 2ab = (a+b)^2$$

$$= (m^2 + 1)^2 - m^2$$

Using formula

$$\therefore a^2 - b^2 = (a-b)(a+b)$$

$$= (m^2 + 1) - m^2 (m^2 + 1 + m^2)$$

$$= (m^2 + m^2 + 1)(m^2 + m^2 + 1)$$

$$\text{Q.14 } -3a^2 - 21a^2 + 24a^2$$

Solution

$$-3a^2 - 21a^2 + 24a^2$$

$$= -3a^2 (a^2 + 7a + 8)$$

$$= -3a^2 (a^2 + 8a + a + 8)$$

$$= -3a^2 [a(a+8) + 1(a+8)]$$

$$= -3a^2 (a+8)(a+1)$$

$$\text{Q.15 } x^2 + x^2 + 1$$

Solution

$$x^2 + x^2 + 1$$

Arrange the terms

$$= x^2 + 1 + x^2$$

$$= (x^2)^2 + (1)^2 + x^2$$

Add and subtract $2(x^2)(1)$

$$= (x^2)^2 + (1)^2 + 2(x^2)(1) - 2(x^2)(1) + x^2$$

$$= [(x^2)^2 + (1)^2 + 2(x^2)(1)] - 2x^2 + x^2$$

Using formula

$$\therefore a^2 + b^2 + 2ab = (a+b)^2$$

$$= (a^2 + b^2) - a^2$$

$$= (a^2 + b^2) - (a^2)^2$$

Using formula

$$\therefore a^2 - b^2 = (a-b)(a+b)$$

$$= (a^2 + 1) - a^2 (a^2 + 1 + a^2)$$

$$= (a^2 + a^2 + 1)(a^2 + a^2 + 1)$$

$$\text{Q.16 } 4x^2 + 25y^2$$

Solution

Taking common 4

$$= 4(x^2 + 6y^2)$$

$$= 4(x^2 + 6y^2)$$

We add and subtract $2(x^2)(6y^2)$

$$= 4[(x^2)^2 + (6y^2)^2 + 2(x^2)(6y^2) - 2(x^2)(6y^2)]$$

Using formula

$$\therefore a^2 + b^2 + 2ab = (a+b)^2$$

$$= 4[(x^2 + 6y^2)^2 - 12(x^2)(y^2)]$$

$$= 4[(x^2 + 6y^2)^2 - 12(x^2)(y^2)]$$

Using formula

$$\therefore a^2 - b^2 = (a-b)(a+b)$$

$$= 4[(x^2 + 6y^2 - 4xy)(x^2 + 6y^2 + 4xy)]$$

$$\text{Q.17 } 12 - 7x + x^2$$

Solution

$$12 - 7x + x^2$$

Arrange the terms

$$= x^2 - 7x + 12$$

$$= x^2 - 4x - 3x + 12$$

$$= x(x-4) - 3(x-4)$$

$$= (x-4)(x-3)$$

$$\text{Q.18 } x^2 - 9x + 8$$

Solution

$$= x^2 - 9x + 8$$

$$= x^2 - 8x - x + 8$$

$$= x(x-8) - 1(x-8)$$

$$= (x-8)(x-1)$$

$$\text{Q.19 } 10x^2 - 25x + 10$$

Solution

$$10x^2 - 25x + 10$$



$$= 10x^2 - 25x + 10$$

$$= 5x(2x-5) - 5(2x-5)$$

$$= (2x-5)(5x-5)$$

$$\text{Q.20 } -3y^2 + 13y - 4$$

Solution

$$-3y^2 + 13y - 4$$

Taking common -1 as factor each term

$$= -1(3y^2 - 13y + 4)$$

$$= -1(3y^2 - 12y - y + 4)$$

$$= -1[(3y-4)(y-1) - 1(y-4)]$$

$$= -1(y-4)(3y-1)$$

$$\text{Q.21 } x^2 - 21x + 98$$

Solution

$$x^2 - 21x + 98$$

$$= x^2 - 13x - 8x + 98$$

$$= x(x-13) - 8(x-13)$$

$$= (x-13)(x-6)$$

$$\text{Q.22 } x^2 + x - 2$$

Solution

$$x^2 + x - 2$$

$$= x^2 + 2x - x - 2$$

$$= x(x+2) - 1(x+2)$$

$$= (x+2)(x-1)$$

$$\text{Q.23 } 3x^2 + 11x + 6$$

Solution

$$3x^2 + 11x + 6$$

$$= 3x^2 + 6x + 5x + 6$$

$$= 3x(x+2) + 3(x+2)$$

$$= (x+2)(3x+3)$$

$$\text{Q.24 } 2x^2 - 5xy - 3y^2$$

Solution

$$2x^2 - 5xy - 3y^2$$

$$= 2x^2 - 6xy + xy - 3y^2$$

$$= 2x(x-3y) + y(x-3y)$$

$$= (x-3y)(2x+y)$$

$$\text{Q.25 } 8 + 6x - 5x^2$$

Solution

$$8 + 6x - 5x^2$$

$$= 8 + 10x - 4x - 5x^2$$

$$= 2(4+5x) - 4(4+5x)$$

$$= (4+5x)(2-4)$$

$$\text{Q.26 } 6 - 7x - 5x^2$$

Solution

$$6 - 7x - 5x^2$$

$$= 6 - 10x + 3x - 5x^2$$

$$= 2(3-5x) + 3(3-5x)$$

$$= (3-5x)(2+3)$$

$$\text{Q.27 } 2x^2 - 4x - 6$$

Solution

Taking common 2 from each term

$$= 2(x^2 - 2x - 3)$$

$$= 2(x^2 - 3x + x - 3)$$

$$= 2[x(x-3) + 1(x-3)]$$

$$= 2(x-3)(x+1)$$

$$\text{Q.28 } a^2 - 13a^2 + 36$$

Solution

$$\text{Let } a^2 = y \dots (i)$$

Taking square

$$(a^2)^2 = y^2$$

$$a^2 = y^2 \dots (ii)$$

Now, given expression can be written as

$$= y^2 - 13y + 36$$

$$= y^2 - 9y - 4y + 36$$

$$= y(y-9) - 4(y-9)$$

$$= (y-9)(y-4)$$

Put $y = a^2$ from (i)

$$= (a^2-9)(a^2-4)$$

$$= [(a^2)^2 - 9(a^2) + (a^2)^2 - 12(a^2)]$$

Using formula

$$a^2 - b^2 = (a-b)(a+b)$$

$$= (a-3)(a+3)(a-2)(a+2)$$

$$\text{Q.29 } y^2 - 12y^2 - 64$$

Solution

$$\text{Let } y^2 = u \dots (i)$$

Taking square

$$(y^2)^2 = u^2$$

$$y^2 = u^2 \dots (ii)$$



$$= x^2 - 12x + 64$$

Now, given expression can be written as

$$= x^2 - 12x + 40 - 64$$

$$= (x-16) - 4(2-12)$$

$$= (x-16)(x-4)$$

Put $a = x^2$ then

$$= (x^2 - 16)(x^2 - 4)$$

$$= [(x^2 - 4)^2](x^2 - 4)$$

Using formula

$$\therefore x^2 - b^2 = (x-b)(x+b)$$

$$= (x-4)(x+4)(x^2-4)$$

Type-III

Factorizing Expressions of the Form

$$(a) (ax^2 + bx + c)(ax^2 + bx + d) + k$$

$$(b) (x + a)(x + b)(x + c)(x + d) + k$$

$$(c) (x + a)(x + b)(x + c)(x + d) + kx^2$$

The process of factorizing expressions of above types will be explained in the following examples.

$$(a) (ax^2 + bx + c)(ax^2 + bx + d) + k$$

Example 4 *****

Factorize

$$(x^2 + 3x + 4)(x^2 + 3x + 5) + 8$$

We observe here that first two terms inside both the parentheses are same, i.e.

$$(x^2 + 3x + 4) \quad (x^2 + 3x + 5) + 8$$

Let $x^2 + 3x = x$, then above expression will take the form

$$= (x-4)(x+5) + 8$$

$$= a(a+5) - 4(a+5) + 8$$

$$= a^2 + 5a - 4a - 20 + 8$$

$$= a^2 + a - 12$$

$$= a^2 + 4a - 3a - 12$$

$$= (a^2 + 4a) - (3a + 12)$$

$$= a(a+4) - 3(a+4)$$

$$= (a+4)(a-3)$$

Putting the value of 'x', we get

$$= (x^2 + 3x + 4)(x^2 + 3x - 3)$$

$$(b) (x + a)(x + b)(x + c)(x + d) + k$$

To factorize such expressions, consider the following example.

Example 5a: Factorize *****

$$(x + 5)(x + 3)(x + 2)(x + 6) - 88$$

Solution:

$$(x + 5)(x + 3)(x + 2)(x + 6) - 88$$

$$\text{notice here, } 5 + 3 = 2 + 6$$

$$= [(x + 5)(x + 3)][(x + 2)(x + 6)] - 88$$

$$= (x^2 + 8x + 15)(x^2 + 8x + 12) - 88$$

Let $x^2 + 8x = a$, then above expression will take the form

$$= (a + 15)(a + 12) - 88$$

$$= a^2 + 27a + 180 - 88$$

$$= a^2 + 27a + 92$$

$$= a^2 + 4a + 23a + 92$$

$$= (a^2 + 4a) + (23a + 92)$$

$$= a(a + 4) + 23(a + 4)$$

$$= (a + 4)(a + 23)$$

By back substitution

$$= (x^2 + 8x + 4)(x^2 + 8x + 23)$$

Example 5b *****

Factorize the expression

$$(x + 1)(x - 5)(x - 9)(x - 3) + 44$$

Solution:

$$(x + 1)(x - 5)(x - 9)(x - 3) + 44$$

Combine $(x + 1)$ with $(x - 9)$ and $(x - 5)$ with $(x - 3)$.

Rearranging the given expression, we have

$$= (x + 1)(x - 9)(x - 5)(x - 3) + 44$$

$$= (x^2 - 8x - 9)(x^2 - 8x + 15) + 44$$

By putting $x^2 - 8x = x$ in the above expression, it will take the form

$$= (x - 9)(x + 15) + 44$$

$$= x^2 - 9x + 15x - 135 + 44$$

$$= x^2 + 6x - 91$$

$$= x^2 + 13x - 7x - 91$$

$$= (x^2 + 13x) - (7x + 91)$$

$$= x(x + 13) - 7(x + 13)$$

$$= (x + 13)(x - 7)$$



Now replacing x by $x^2 - 8x$, we have

$$= (x^2 - 8x + 13)(x^2 - 8x - 7)$$

$$(c) (x + a)(x + b)(x + c)(x + d) + kx^2$$

Example 6 *****

Factorize the expression

$$(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$$

Solution

$$\therefore 1+6=3+2$$

$$x+6$$

$$(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$$

Rearranging the expression

$$(x + 1)(x + 6)(x + 2)(x + 3) - 3x^2$$

$$= (x^2 + 6 + 7x)(x^2 + 6 + 3x) - 3x^2$$

$$\text{let } x^2 + 6 = y$$

$$= (y + 7x)(y + 3x) - 3x^2$$

$$= y^2 + 5xy + 7xy + 21x^2 - 3x^2$$

$$= y^2 + 12xy + 18x^2$$

$$= y^2 + 4xy + 4xy + 12x^2$$

$$= y(y + 4x) + 4x(y + 3x)$$

$$= (y + 4x)(y + 3x)$$

Putting the value of y

$$= (x^2 + 6 + 8x)(x^2 + 6 + 4x)$$

$$= (x^2 + 8x + 6)(x^2 + 4x + 6)$$

Type-IV

Factorizing expressions of the form:

$$(a) x^4 + 3a^2b + 3ab^2 + b^4$$

$$(b) x^4 - 3a^2b + 3ab^2 - b^4$$

We have studied in the previous unit that

$$(i) x^4 + 3a^2b + 3ab^2 + b^4 = (a + b)^4$$

$$(ii) x^4 - 3a^2b + 3ab^2 - b^4 = (a - b)^4$$

$$(iii) x^4 - 3a^2b + 3ab^2 - b^4 = (a - b)^4$$

Example 7

Factorize the following expressions.

$$(i) 27a^3 + 1 + 27a^2 + 9a$$

$$(ii) x^4 + \frac{3}{x} - \frac{1}{x^2} - 2x$$

$$(iii) 4x^2(2x - 15) - 25(2 - 6x)$$

Solution:

$$(i) 27a^3 + 1 + 27a^2 + 9a$$

$$= (3a)^3 + (1)^3 + 3(3a)^2(1) + 3(3a)(1)^2$$

$$= (3a)^3 + 3(3a)^2(1) + 3(3a)(1)^2 + (1)^3$$

$$= (3a + 1)^3$$

$$(ii) x^4 + \frac{3}{x} - \frac{1}{x^2} - 2x$$

$$= x^4 - 2x + \frac{3}{x} - \frac{1}{x^2}$$

$$= (x^4 - 2x) + \left(\frac{3}{x} - \frac{1}{x^2}\right)$$

$$= (x^4 - 2x) + \left(\frac{3}{x} - \frac{1}{x^2}\right)$$

$$= (x^4 - 2x) + \left(\frac{3}{x} - \frac{1}{x^2}\right)$$

$$= (x^4 - 2x) + \left(\frac{3}{x} - \frac{1}{x^2}\right)$$

$$= (x^4 - 2x) + \left(\frac{3}{x} - \frac{1}{x^2}\right)$$

$$(iii) 4x^2(2x - 15) - 25(2 - 6x)$$

$$= 8x^3 - 60x^2 - 50 + 150x$$

$$= 8x^3 - 60x^2 + 150x - 50$$

$$= (2x)^3 - 3(2x)^2(5) + 3(2x)(5)^2 - (5)^3$$

$$= (2x - 5)^3$$

Type-V

Factorizing the sum and difference of two cubes

$$(a) x^3 + b^3 \quad (b) x^3 - b^3$$

We have studied in the previous unit that:

$$x^3 + b^3 = (x + b)(x^2 - xb + b^2)$$

$$x^3 - b^3 = (x - b)(x^2 + xb + b^2)$$

Example 8 *****

Factorize $x^3 + 27y^3$

Solution:

$$\text{As, } x^3 + 27y^3 = x^3 + (3y)^3 = x^3 + 27y^3$$

So,

$$x^3 + 27y^3 = x^3 + (3y)^3 = x^3 + 27y^3$$

$$= (x + 3y)(x^2 - 3xy + 9y^2)$$

$$= (x + 3y)(x^2 - 3xy + 9y^2)$$

$$= (x + 3y)(x^2 - 3xy + 9y^2)$$

$$= (x + 3y)(x^2 - 3xy + 9y^2)$$

$$= (x + 3y)(x^2 - 3xy + 9y^2)$$

Example 9 *****

Factorize $x^4 - y^4$

$$\text{Solution: } x^4 - y^4 = (x^2)^2 - (y^2)^2$$

$$= (x^2 + y^2)(x^2 - y^2)$$



$$= (x+y)(x^2-xy+y^2) \div (x+y)(x^2+xy+y^2)$$

$$= (x+y)(x-y)(x^2-xy+y^2) \div (x+y)(x^2+xy+y^2)$$

Example 10Factorize $3a^2 - 12ab^2 - 2a + 5b$ **Solution:** $3a^2 - 12ab^2 - 2a + 5b$

$$= (2ab^2 - 15b^2) - (2a - 5b)$$

$$= (2a - 5b)(2a^2 + (2a)(5b) + (5b)^2) - (2a - 5b)$$

$$= (2a - 5b)(4a^2 + 10ab + 25b^2) - (2a - 5b)$$

$$= (2a - 5b)(4a^2 + 10ab + 25b^2 - 1)$$

Exercise 4.2

Factorize the following expression completely.

Q.1 $x^2 - 125$

Solution

$x^2 - 125$

$= (x^2) - (5^3)$

$\therefore x^2 - b^3 = (x-b)(x^2+xb+b^2)$

$= (x-5)(x^2+5x+25)$

$= (x-5)(x^2+5x+25)$

Q.2 $8x^2 + 1$

Solution

$8x^2 + 1$

$= (2x^2) + (1^2)$

$\therefore a^2 + b^2 = (a+b)(a^2-ab+b^2)$

$= (2x+1)((2x)^2 - (2x)(1) + (1)^2)$

$= (2x+1)(4x^2 - 2x + 1)$

Q.3 $3p^3q^2 - 81x^2$

Solution

$3p^3q^2 - 81x^2$

Taking common 3 from each

$= 3(p^3q^2 - 27x^2)$

$= 3(pq)^2 - (3x)^2$

Using formula

$\therefore a^2 - b^2 = (a-b)(a+b)$

$= 3(pq-3x)(pq+3x)$

$= 3(pq-3x)(p^2q^2+3pqx+9x^2)$

Q.4 $27 + 512x^3$

Solution

$27 + 512x^3$

$= 3(2x^3 + 27)$

$= (3x)^3 + (3)^3$

Using formula

$\therefore a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$= (3x+3)((3x)^2 - (3x)(3) + (3)^2)$

$= (3x+3)(9x^2 - 24x + 9)$

Q.5 $t^3 - 64$

Solution

$t^3 - 64$

$= (t^3) - (4^3)$

Using formula

$\therefore a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$= (t-4)((t)^2 + t(4) + (4)^2)$

$= (t-4)(t^2 + 4t + 16)$

$\left[\left(\frac{1}{2} \right)^2 - \left(\frac{2}{3} \right)^2 \right] \left[\left(\frac{1}{2} \right)^2 + \left(\frac{1}{2} \right) \left(\frac{2}{3} \right) + \left(\frac{2}{3} \right)^2 \right] - 2 \left(\frac{1}{2} \right) \left(\frac{2}{3} \right) + 4 \left(\frac{2}{3} \right)^2$

$= (1-2)(1-2) \left[\left(\frac{1}{2} + \frac{2}{3} \right)^2 - 8 \left(\frac{1}{2} + \frac{2}{3} \right) \right]$

$= (1-2)(1-2) \left[\left(\frac{1}{2} + \frac{2}{3} \right)^2 - 4 \left(\frac{1}{2} \right) \right]$

$= (1-2)(1-2) \left[\left(\frac{1}{2} + \frac{2}{3} \right)^2 - (2)^2 \right]$

$= (1-2)(1-2) \left[\left(\frac{1}{2} + 2 \right) \left(\frac{1}{2} + 4 - 2 \right) \right]$

$= (1-2)(1-2) \left[\left(\frac{1}{2} + 2 + 4 \right) \left(\frac{1}{2} - 2 + 4 \right) \right]$

Q.6 $x^3 + y^3$

Solution

$x^3 + y^3 = (x^3) + (y^3)$

Using formula

$\therefore a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$= (x+y)((x)^2 - x(y) + (y)^2)$

$= (x+y)(x^2 - xy + y^2)$

Q.7 $Q - x^2 + (y - 2)^2$

Solution

$Q - x^2 + (y - 2)^2$



Using formula

$\therefore a^2 + b^2 = (a+b)(a^2-ab+b^2)$

$= [(2-x) + (y-2)] [(2-x)^2 - (2-x)(y-2) + (y-2)^2]$

Using formula

$\therefore (a-b)^2 = a^2 - 2ab + b^2$

$= (2-x+y-2) [(2)^2 + (x)^2 - 2(2)(x) - (2y-4-xy+2x) + (2)^2 - 2(y)(2)]$

$= (y-x) [4 + x^2 - 4x - 2y + 4xy - 2x - y^2 + 4 - 4y]$

$= (y-x) [x^2 + y^2 - 4x - 6y + 4xy + 8]$

Q.8 $64(x+y)^2 - x^2$

Solution

$64(x+y)^2 - x^2$

$= (4)^2 (x+y)^2 - (x)^2$

$= [4(x+y)]^2 - x^2$

Using formula

$\therefore a^2 - b^2 = (a-b)(a+b)$

$= [4(x+y) - x] [(4(x+y)) + x] = 4(x+y)(3x+4y)$

Using formula

$\therefore (a+b)^2 = a^2 + 2ab + b^2$

$= (4x+4y-x) [(4x)^2 + 2(4x)(y) + 4(y)^2]$

$= (3x+4y-x) [16x^2 + 16xy + 12xy + 4y^2 + 4y^2]$

$= (3x+4y-x) [16x^2 + 16xy + 12xy + 4y^2 + 4y^2]$

Q.9 $27p^3 + 144pq^2 - 108p^2q - 64q^3$

Solution

$27p^3 + 144pq^2 - 108p^2q - 64q^3$

Arrange the terms

$27p^3 - 108p^2q + 144pq^2 - 64q^3$

$= 27p^3 - 64q^3 - 108p^2q + 144pq^2$

$= (3p)^3 - (4q)^3 - 3(3p)(4q) + 3(3p)(4q)$

$= (3p)^3 - (4q)^3 - 3(3p)(4q) + 3(3p)(4q)$

Using formula

$\therefore a^3 - b^3 - 3ab(a-b) = (a-b)^3$

$= (3p-4q)^3$

Q.10 $8p^3 + q^3 + 12p^2q + 6pq^2$

Solution

$8p^3 + q^3 + 12p^2q + 6pq^2$

$= (2p)^3 + (q)^3 + 3(2p)^2(q) + 3(2p)(q)^2$

Using formula

$\therefore a^3 + b^3 + 3a^2b + 3ab^2 = (a+b)^3$

$= (2p+q)^3$

Q.11 $125x^3 - y^3 - 75x^2y + 15xy^2$

$125x^3 - y^3 - 75x^2y + 15xy^2$

Solution

$= 125x^3 - y^3 - 75x^2y + 15xy^2$

$= (5x)^3 - (y)^3 - 3(5x)^2(y) + 3(5x)(y)^2$

Using formula

$\therefore (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$= (5x-y)^3$

Q.12 $p^3 - 9p^2q + 27pq^2 - 27q^3$

Solution

$p^3 - 9p^2q + 27pq^2 - 27q^3$

$= p^3 - 27q^3 - 9p^2q + 27pq^2$

$= (p)^3 - (3q)^3 - 3(p)^2(3q) + 3(p)(3q)^2$

Using formula

$\therefore (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$= (p-3q)^3$

Q.13 $(2x^2 - 3x + 4)(2x^2 - 3x) - 55$

Solution

$(2x^2 - 3x + 4)(2x^2 - 3x) - 55$

Let $2x^2 - 3x = y$... (i)

$= (y+4)(y) - 55$

$= y^2 + 4y - 55$

$= y^2 + 11y - 55$

$= y(y+11) - 5(y+11)$

$= (y+11)(y-5)$

Put $y = 2x^2 - 3x$ from (i)

$= (2x^2 - 3x + 11)(2x^2 - 3x - 5)$

$= (2x^2 - 3x + 11)(2x^2 - 3x + 2x - 5)$

$= (2x^2 - 3x + 11)(2x^2 - 5x + 2x - 5)$

$= (2x^2 - 3x + 11)(2x^2 - 3x - 5)$

Q.14 $(y^2 + 2y - 3)(y^2 + 2y + 11) + 48$

Solution

$(y^2 + 2y - 3)(y^2 + 2y + 11) + 48$

Let $y^2 + 2y = u$... (i)

$= (u-3)(u+11) + 48$

$= u^2 + 11u - 3u - 33 + 48$

$= u^2 + 8u + 15$

$= u^2 + 5u + 3u + 15$

$= u(u+5) + 3(u+5)$

$= (u+5)(u+3)$

Put $u = y^2 + 2y$ from (i)

$= (y^2 + 2y + 5)(y^2 + 2y + 3)$

Q.15 $y(y-1)(y-2)(y-4) + 2$

Solution

$y(y-1)(y-2)(y-4) + 2$



Arrange the following:

$$= (y(y-4))[(y-1)(y-3)] + 2$$

$$= (y^2-4y)(y^2-4y+3) + 2$$

$$\text{Let } y^2-4y = u \dots (i)$$

$$= u(u+3) + 2$$

$$= u^2+3u+2$$

$$= u^2+2u+u+2$$

$$= u(u+2) + (u+2)$$

$$= (u+2)(u+1)$$

Put $u = y^2-4y$ from (i)

$$= (y^2-4y+2)(y^2-4y+1)$$

$$\text{Q.16 } (k+2)(k-3)(k+5)(k+10) + 375$$

Answer

Solution

$$(k+2)(k-3)(k+5)(k+10) + 375$$

$$\therefore 2+5 = 10 \text{ and}$$

$$-3 = -5$$

Arrange the following:

$$= [(k+2)(k+5)][(k-3)(k+10)] + 375$$

$$= (k^2+7k+10)(k^2+10k-3k-30) + 375$$

$$= (k^2+7k+10)(k^2+7k-30) + 375$$

$$\text{Let } k^2+7k = y \dots (i)$$

$$= (y+10)(y-30) + 375$$

$$= y^2-30y+10y-300+375$$

$$= y^2-20y+75$$

$$= y^2-5y-15y+75$$

$$= y(y-5) - 15(y-5)$$

$$= (y-5)(y-15)$$

Put $y = k^2+7k$ from (i)

$$= (k^2+7k-15)(k^2+7k-3)$$

$$\text{Q.17 } (x-5)(x-6)(x+3)(x+2) + 12$$

Solution

$$(x-5)(x-6)(x+3)(x+2) + 12$$

$$\therefore -5+2 = -6+3$$

$$-3 = -5$$

$$-5+2 = -6+3$$

$$-3 = -5$$

Arrange the following:

$$= [(x-5)(x+2)][(x-6)(x+3)] + 12$$

$$= (x^2+2x-5x-10)(x^2+3x-6x-18) + 12$$

$$= (x^2-3x-10)(x^2-3x-18) + 12$$

$$\text{Let } x^2-3x = y \dots (i)$$

$$= (y-10)(y-18) + 12$$

$$= y^2-18y-10y+180+12$$

$$= y^2-28y+192$$

$$= y^2-12y-16y+192$$

$$= y(y-12) - 16(y-12)$$

$$= (y-12)(y-16)$$

Put $y = x^2-3x$ from (i)

$$= (x^2-3x-12)(x^2-3x-16)$$

$$\text{Q.18 } (x+1)(x+2)(x-3)(x-6) - 21x^2$$

Answer

Solution

$$= 1+(-6) = 2+(-3)$$

$$-6 = -3$$

$$(x+1)(x+2)(x-3)(x-6) - 21x^2$$

Arrange the given expression:

$$= [(x+1)(x+2)][(x-3)(x-6)] - 21x^2$$

$$= (x^2+4x+2x+2)(x^2-3x-6x-18) - 21x^2$$

$$= (x^2+6x+2)(x^2-9x-18) - 21x^2$$

$$= (x^2+6x+2)(x^2-9x-18) - 21x^2$$

$$= (x^2+6x+2)(x^2-9x-18) - 21x^2$$

$$\text{Let } x^2+6x = y \dots (i)$$

$$= (y+2)(y-9) - 21x^2$$

$$= y^2-9y+2y-18-21x^2$$

$$= y^2-7y-18-21x^2$$

$$= y^2-7y+2y-18-21x^2$$

$$= y(y-4) + 2y(y-9)$$

$$= (y-4)(y+2)$$

$$\text{Put } y = x^2+6x \text{ from (i)}$$

$$= (x^2+6x-4)(x^2+6x-9)$$

$$= (x^2+6x-4)(x^2+6x-9)$$

$$\text{Q.19 } (x-2)(x-6)(x-3)(x-4) - 2x^2$$

Solution

$$(x-2)(x-6)(x-3)(x-4) - 2x^2$$

$$= [(x-2)(x-6)][(x-3)(x-4)] - 2x^2$$

$$= (x^2-6x-2x+12)(x^2-4x-3x+12) - 2x^2$$

$$= (x^2-8x+12)(x^2-7x+12) - 2x^2$$



$$= (x^2+12-8x)(x^2+12-7x) - 2x^2$$

$$\text{Let } x^2+12 = y \dots (i)$$

$$= (y-8x)(y-7x) - 2x^2$$

$$= y^2-7xy-8xy+56x^2-2x^2$$

$$= y^2-15xy+54x^2$$

$$= y^2-6xy-9xy+54x^2$$

$$= y(y-6x) - 9x(y-6x)$$

$$= (y-6x)(y-9x)$$

Put $y = x^2+12$ from (i)

$$= (x^2+12-6x)(x^2+12-9x)$$

$$= (x^2-6x+12)(x^2-9x+12)$$

$$\text{Q.20 } (5-x)(2+x)(10-x)(1+x) - 7x^2$$

Answer

Solution

$$(5-x)(2+x)(10-x)(1+x) - 7x^2$$

$$= [-(x-5)(x+2)][-(x-10)(x+1)] - 7x^2$$

$$= [(x-5)(x+2)][(x-10)(x+1)] - 7x^2$$

$$= (x^2+2x-10x-10)(x^2+x-10x-10) - 7x^2$$

$$= (x^2-8x-10)(x^2-9x-10) - 7x^2$$

$$= (x^2-10-8x)(x^2-10-9x) - 7x^2$$

$$\text{Let } y = x^2-10 \dots (i)$$

$$= (y-8x)(y-9x) - 7x^2$$

$$= y^2-9xy-8xy+72x^2-7x^2$$

$$= y^2-17xy+65x^2$$

$$= y^2-10xy-7xy+65x^2$$

$$= y(y-10x) - 7x(y-10x)$$

$$= (y-10x)(y-7x)$$

Put $y = x^2-10$ from (i)

$$= (x^2-10-10x)(x^2-10-7x)$$

$$= (x^2-10x-10)(x^2-7x-10)$$

$$\text{Q.21 The expression } x^4 + 729 \text{ can be written}$$

in two ways as:

(a) sum of two squares (b) sum of two cubes.

Which one will be used for factoring it and why? Also factorize the given expression.

Solution

$$(a) \text{ Given expression } = x^4+729$$

Answer

$$\text{Sum of two squares } = (x^2)^2 + (27)^2$$

Factorization is not possible.

$$(b) \text{ Given expression } = x^4+729$$

Answer

Solution

$$\text{Sum of two cubes } = (x^2)^3 + (9)^3$$

Using formula

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$

$$= (x^2+9)((x^2)^2 - (x^2)(9) + (9)^2)$$

$$= (x^2+9)(x^4-9x^2+81)$$

Factorization of sum of two cubes is possible

Which factorize $(x^2+9)(x^4-9x^2+81)$

Q.22 Express $8+12x+6x^2+x^3$ as the product

of three factors. Is each factor a binomial or

a trinomial?

Answer

Solution

Product of three factors

$$= 8+12x+6x^2+x^3$$

$$= x^3+6x^2+12x+8$$

$$= (x^3+3x^2)(2+3x)(2^2+2x)$$

Using formula

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$

$$= (x+2)^3$$

$$= (x+2)(x+2)(x+2)$$

Each factor is a binomial.

Highest Common Factor of Algebraic

Expressions

The highest common factor of two or more

than two polynomials is a highest degree

polynomial which divides the given

polynomials exactly.

HCF of Monomial Expression

(i) Determine the HCF of numerical

coefficients by prime factorization.

(ii) Determine the common variables and select

their lowest power that appears in all

monomials.

This will be the HCF of variables.

Example 11

Answer

Find the HCF of $18ab^3c$, $30a^2b^2c^2$, $24b^2c^2$

Solution: First we find the HCF of numerical

coefficients 18, 30 and 24 as

$$18 = 2 \times 3^2$$



$$30 = 2 \times 3 \times 5$$

$$24 = 2^3 \times 3$$

$$\therefore \text{HCF of 18, 30 and 24} = 2 \times 3 = 6$$

HCF of $x^2y^2z^2$ and $x^4y^4z^4$ is $x^2y^2z^2$ — least common power of x
 HCF of $x^2y^2z^2$ and $x^4y^4z^4$ is $x^2y^2z^2$ — least common power of y
 Hence, required HCF = $6x^2y^2z^2$

HCF of Compound Expressions

i) Write each expression in complete factored form. Repeated factors should be expressed as powers.

ii) Select the least power of each common factor.

iii) The highest common factor (HCF) is the product of results of step-II.

Example 12

Find the HCF of $2m^2 - 2mn$, $4m^4 - 4m^3n$ and $2m^3 - 4m^2n + 2mn^2$.

Solution

First factorize each polynomial expression completely as

$$2m^2 - 2mn = 2m(m - n)$$

$$4m^4 - 4m^3n = 4m^3(m^2 - n^2) \\ = 2^2 m^3 (m + n)(m - n)$$

$$2m^3 - 4m^2n + 2mn^2 = 2m^2(m^2 - 2mn + n^2) \\ = 2m^2(m - n)^2$$

Common factors with least power are $2, m, m - n$

$$\therefore \text{Required HCF} = 2m(m - n)$$

Example 13

Find the HCF of the following.

$$ax^2 + 7ax + 12a, ax^2 - 5ax - 24a, 2ax^2 + 5ax - 12a$$

Solution

$$ax^2 + 7ax + 12a = a(x^2 + 7x + 12)$$

$$= a(x^2 + 4x + 3x + 12)$$

$$= a\{x(x + 4) + 3(x + 4)\}$$

$$= a(x + 3)(x + 4) \dots (i)$$

$$ax^2 - 5ax - 24a = a(x^2 - 5x - 24)$$

$$= a(x^2 - 8x + 3x - 24)$$

$$= a\{x(x - 8) + 3(x - 8)\}$$

$$= a(x + 3)(x - 8) \dots (ii)$$

$$2ax^2 + 5ax - 12a = a(2x^2 + 5x - 3)$$

$$= a(2x^2 + 6x - x - 3)$$

$$= a\{2x(x + 3) - 1(x + 3)\}$$

$$= a(2x - 1)(x + 3) \dots (iii)$$

From equation (i), (ii) and (iii)

$\therefore$ Required HCF = product of all common factors with least power.
 $= a(x + 3)$

Exercise 14

Q.1 Find the H.C.F of the following monomials by completing the table.

a)

answer

Monomials	HCF of numerical coefficient	HCF of p	HCF of q	HCF of r	Required HCF
$16p^3q, 8pq^2r$					
$16p^3q^2r, 5p^2qr, 15p^3qr^2$					
$14p^3qr^2, 28p^3qr^2, 7p^3qr^2, 21p^3q^2r^2$					

b) If all common factors with least power of three unknown polynomials are $2^3, 3, pq$ and $(p+q)^2$ then what would be their HCF?

answer

c) Write any two polynomials of your choice having HCF as 1.

answer



d) The only common factor of two polynomials is $m - n$ and the only uncommon factor is $m^2 + n^2$. Can you guess the unknown polynomials?

answer

e) Can you guess HCF of two polynomials $x^2 + 5x + 1$ and $1 + 5x + x^2$ without any procedure?

answer

a) Solution

Monomials	HCF of numerical coefficient	HCF of p	HCF of q	HCF of r	Required HCF
$16p^3q, 8pq^2r$	8	p	q	r	$8pq$
$16p^3q^2r, 5p^2qr, 15p^3qr^2$	5	p^2	q	r	$5p^2qr$
$14p^3qr^2, 28p^3qr^2, 7p^3qr^2, 21p^3q^2r^2$	7	p^3	q	r^2	$7p^3qr^2$

b) Solution

Common factors of unknown polynomials are

$$2^3, 3, pq \text{ and } (p+q)^2$$

$$\text{HCF} = 2^3 \cdot 3 \cdot pq \cdot (p+q)^2$$

$$= 48pq(p+q)^2$$

$$= 12pq(p+q)^2$$

c) Solution

$$(x + y)^2, (x - y)^2$$

d) Solution

The only common factor = $m - n$

The only uncommon factor = $m^2 + n^2$

Let the polynomials be $p(x)$ and $q(x)$

$$p(x) = (m - n)(m^2 + n^2), q(x) = m - n$$

So, polynomials are $(m - n)(m^2 + n^2)$, $(m - n)$

e) Solution

The two polynomials $x^2 + 5x + 1$ and $1 + 5x + x^2$ are actually the same polynomials written in different order. So, the HCF of the two identical polynomials is the polynomial itself.

$$\text{H.C.F} = x^2 + 5x + 1$$

Find the HCF of the following by factorization

$$\text{Q.2 } (x + y)^2, x^2 - y^2$$

answer

Solution

$$(x + y)^2, x^2 - y^2$$

By factorizing we get

$$(x + y)^2 = (x + y)(x + y) \rightarrow (i)$$

$$x^2 - y^2 = (x + y)(x - y) \rightarrow (ii)$$

From equation (i) and (ii)

Common factor = $x + y$

$$\text{HCF} = x + y$$

$$\text{Q.3 } (a - b)^2, a^2 - 2ab + b^2$$

answer

Solution

$$(a - b)^2, a^2 - 2ab + b^2$$

$$(a - b)^2 = (a - b)(a - b) \rightarrow (i)$$

$$a^2 - 2ab + b^2 = (a - b)^2 \rightarrow (ii)$$

From Equation (i) and (ii)

Common factor = $(a - b)(a - b)$

$$\text{H.C.F} = (a - b)(a - b)$$

$$\text{H.C.F} = (a - b)^2$$

$$\text{H.C.F} = a^2 - 2ab + b^2$$

$$\text{Q.4 } a^2b - ab^2, a^2b^2 - a^2b^2$$

answer

Solution

$$a^2b - ab^2, a^2b^2 - a^2b^2$$

$$a^2b - ab^2 = ab(a^2 - b^2)$$

$$= ab(a + b)(a - b) \dots (i)$$

$$= a^2b^2 - a^2b^2 = a^2b^2(a^2 - b^2)$$

$$= a \cdot a \cdot b \cdot b(a - b)(a^2 + ab + b^2) \rightarrow (ii)$$

From (i) and (ii)

Common factor = $ab(a - b)$

$$\text{H.C.F} = ab(a - b)$$

$$\text{Q.5 } x^2 - 49, x^2 - 4x - 21$$

answer

Solution



$$x^2 - 49, x^2 - 4x - 21$$

$$x^2 - 49 = (x^2 - 7^2)$$

$$\therefore x^2 - 49 = (x + 7)(x - 7)$$

$$= (x + 7)(x - 7) \rightarrow (i)$$

Now

$$x^2 - 4x - 21 = x^2 - 7x + 3x - 21$$

$$= x(x - 7) + 3(x - 7)$$

$$= (x - 7)(x + 3) \rightarrow (ii)$$

From (i) and (ii)

$$\text{Common factor} = (x - 7)$$

$$\text{H.C.F} = x - 7$$

$$\text{Q.6 } 12x^2 + x - 1, 15x^2 + 8x + 1$$

Answer

Solution

$$12x^2 + x - 1, 15x^2 + 8x + 1$$

$$12x^2 + x - 1 = 12x^2 + 4x - 3x - 1$$

$$= 4x(3x + 1) - 1(3x + 1)$$

$$= (4x - 1)(3x + 1) \rightarrow (i)$$

$$15x^2 + 8x + 1 = 15x^2 + 5x + 3x + 1$$

$$= 5x(3x + 1) + 1(3x + 1)$$

$$= (5x + 1)(3x + 1) \rightarrow (ii)$$

From (i) and (ii)

$$\text{Common factor} = 3x + 1$$

$$\text{H.C.F} = 3x + 1$$

$$\text{Q.7 } x^2 - d^2, acx^2 - bcx + adx - bd$$

Answer

Solution

$$x^2 - d^2, acx^2 - bcx + adx - bd$$

$$x^2 - d^2 = (x^2 - d^2)$$

$$\therefore x^2 - d^2 = (x + d)(x - d)$$

$$= (x + d)(cx - d) \rightarrow (i)$$

$$acx^2 - bcx + adx - bd$$

$$= cx(ax - b) + d(ax - b)$$

$$= (cx + d)(ax - b) \rightarrow (ii)$$

From (i) and (ii)

$$\text{Common factor} = cx + d$$

$$\text{H.C.F} = cx + d$$

$$\text{Q.8 } m^3 - a^3, m^3 - n^3, m^3 - n^3$$

Answer

Solution

$$m^3 - a^3, m^3 - n^3, m^3 - n^3$$

$$m^3 - a^3 = (m + a)(m - a) \rightarrow (i)$$

$$\therefore a^3 - b^3 = (a + b)(a - b)$$

$$m^3 - a^3 = (m^3 - a^3)$$

$$= (m^3 + a^3)(m^2 - a^2)$$

$$= (m^3 + a^3)(m + a)(m - a) \rightarrow (ii)$$

$$m^3 - a^3 = (m^3 - a^3)$$

$$\therefore a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$= (m^3 - a^3)(m^2 + a^2)(m^2 + a^2)$$

$$= (m + a)(m - a)(m^2 + a^2)(m^2 + a^2) \rightarrow (iii)$$

From (i), (ii) and (iii)

$$\text{Common factor} = (m - a)(m + a)$$

$$\text{H.C.F} = (m - a)(m + a)$$

$$= m^2 - a^2$$

$$\text{Q.9 } ax^3 + 2ax^2 + a^2, 2ax^3 - 4ax^2 - 6a^2, 3(ax + a^2)^2$$

Answer

Solution

$$ax^3 + 2ax^2 + a^2, 2ax^3 - 4ax^2 - 6a^2, 3(ax + a^2)^2$$

$$ax^3 + 2ax^2 + a^2 = a(x^3 + 2ax + a^2)$$

$$\therefore (x + a)^2 = x^2 + 2ax + a^2$$

$$= a(x + a)^2$$

$$= a(x + a)(x + a) \rightarrow (i)$$

$$2ax^3 - 4ax^2 - 6a^2 = 2a(x^3 - 2ax - 3a^2)$$

$$= 2a(x^3 - 3ax + ax - 3a^2)$$

$$= 2a[x(x - 3a) + a(x - 3a)]$$

$$= 2a(x + a)(x - 3a) \rightarrow (ii)$$

$$3(ax + a^2)^2 = 3[a(x + a)]^2$$

$$= 3[a^2(x + a)^2]$$

$$= 3a^2(x + a)(x + a) \rightarrow (iii)$$

From (i), (ii) and (iii)

$$\text{Common factor} = a(x + a)$$

$$\text{H.C.F} = a(x + a)$$

Highest Common Factor by Division

Sometimes, it is difficult to factorize the given polynomials completely. In such cases, we adopt division method to find the HCF of the polynomials.

We will explain the procedure with the help of the following example.



Example 14 Find the HCF of two polynomials.

$$x^3 + x^2 - 5x + 3 \text{ and } x^3 + 1x$$

Solution: The following steps will be followed for finding HCF.

$$\begin{array}{r} x^3 + x^2 - 5x + 3 \\ x^3 + 1x \\ \hline x^2 + x^2 - 5x + 3 \\ -2x^2 - 5x + 3 \\ \hline 3x^2 + 6x \\ x + 3 \end{array}$$

$$\begin{array}{r} x \\ x + 3 \overline{) x^2 + 3x} \\ \underline{x^2 + 3x} \\ 0 \end{array}$$

Step-I: Arrange the given polynomials in descending order w.r.t. the variables.

In this case, polynomials are already in descending order.

Since first term of the divisor is x^2 and first term of the dividend is x^2 . Therefore, the quotient will be x .

Again here we will multiply the remainder by x .

Since $3x$ is not a factor of the polynomial, we will stop at this point.

$$\begin{array}{r} 1 \\ x^3 + 3x^2 + 13x + 14 \overline{) x^3 + 7x^2 + 15x + 12} \\ \underline{x^3 + 7x^2 + 13x + 14} \\ 2x^2 + 2x - 2 \\ \underline{2x^2 + 16x + 28} \\ -14x - 16 \\ \underline{-14x - 14} \\ -2 \end{array}$$

$$\begin{array}{r} 2x - 1 \\ x + 2 \overline{) 2x^2 + 3x - 2} \\ \underline{2x^2 + 4x} \\ -x - 2 \\ \underline{-x - 2} \\ 0 \end{array}$$



Least Common Multiple of Algebraic Expressions

The least common multiple of two or more polynomials is polynomial of lowest degree that contains the factors of each polynomial.

Least Common Multiple by Factorization

The LCM is obtained by taking product of all factors (numerical and algebraic) with highest power.

Example 16: Find LCM of $18a^3b^2c^2$, $60a^2b^3c^2$ and $42a^3b^3$.

Solution:

First we find the LCM of 18, 60 and 42 by prime factorization as

$$18 = 2 \times 3^2, 60 = 2^2 \times 3 \times 5, 42 = 2 \times 3 \times 7$$

$$\therefore \text{LCM of } 18, 60 \text{ and } 42 = \text{product of all factors with highest power} \\ = 2^2 \times 3^2 \times 5 \times 7 = 1260$$

Now we find LCM of each variable as

LCM of a^3 and $a^2 = a^3$ (highest power of a)

LCM of b^3 and $b^2 = b^3$ (highest power of b)

LCM of c^2 and $c^2 = c^2$ (highest power of c)

$$\therefore \text{Required LCM} = 1260a^3b^3c^2$$

LCM of Compound Expressions

i) Write each polynomial in completely factored form. Repeated factors should be expressed as powers.

ii) Select the highest power of every factor that appears.

iii) The least common multiple is the product of the results of step ii.

Example 17: Find LCM of $x^2y^2 - x^2y - xy^2$

Solution: First we factorize each polynomial completely.

$$x^2y^2 - x^2y = x^2y^2(x - y)$$

$$x^2y^2 - xy^2 = xy(x^2 - y^2) = xy(x - y)(x + y)$$

We choose every factor with highest power as

$$x^2y^2, x - y \text{ and } x + y.$$

Then their product is $x^2y^2(x - y)(x + y)$

$$\therefore \text{Required LCM} = x^2y^2(x - y)(x + y) \\ = x^2y^2(x^2 - y^2) \\ = x^2y^2 - y^2y^2$$

Example 18: Find the LCM of

$$2x^2 - 12xy + 18y^2, x^2 - 4xy + 4y^2 \text{ and}$$

$$1x^2 - 12y^2$$

Solution

$$2x^2 - 12xy + 18y^2 = 2(x^2 - 6xy + 9y^2)$$

$$= 2(x^2 - 2xy - 4xy + 8y^2)$$

$$= 2x(x - 2y) - 4y(x - 2y)$$

$$= 2(x - 4y)(x - 2y)$$

$$x^2 - 4xy + 4y^2 = x^2 - 4xy - 4xy + 8y^2$$

$$= x(x - 4y) - 2y(x - 4y)$$

$$= (x - 4y)(x - 2y)$$

$$1x^2 - 12y^2 = 2(x^2 - 4y^2)$$

$$= 2(x + 2y)(x - 2y)$$

All factors with highest powers are

$$2, 3, x - 4y, x - 2y, x + 2y$$

$\therefore$ Required LCM

$$= 2 \times 3 \times (x - 4y)(x - 2y)(x + 2y)$$

$$= 6(x - 4y)(x^2 - 4y^2)$$

Exercise 4.2

Q.1 Give quick answers to these questions without doing any procedure.

If HCF of two polynomials $x^2 + 5x^2 + 6x$ and $x^2 + 9x^2 + 14x$ is obtained as $x^2 + 2x$, then

i) What would be the H.C.F. of $3(x^2 + 5x^2 + 6x)$ and $x^2 + 9x^2 + 14x$?

ii) What would be the H.C.F. of $x^2 + 5x^2 + 6x$ and $x^2 + 9x^2 + 14x$?

iii) What would be the H.C.F. of $3(x^2 + 5x^2 + 6x)$ and $7(x^2 + 9x^2 + 14x)$?

iv) What would be the H.C.F. of $15(x^2 + 5x^2 + 6x)$ and $25(x^2 + 9x^2 + 14x)$?



v) Does HCF of the given polynomials will remain unchanged if both are multiplied by x ?

Solution:

i) HCF of $3(x^2 + 5x^2 + 6x)$ and $x^2 + 9x^2 + 14x$ is $x^2 + 2x$ because HCF does not effect by multiplying or dividing any non-zero number to the polynomial.

ii) HCF of $x^2 + 5x^2 + 6x$ and $2(x^2 + 9x^2 + 14x)$ is $x^2 + 2x$ because HCF any polynomial does not change by multiply or dividing the polynomial by non-zero non-zero number.

Q.2 Find the LCM of the following monomials by completing the table.

Monomials	LCM of numerical coefficients	LCM of 'x'	LCM of 'y'	LCM of 'z'	Required LCM
i) $8x^3y, 4x^2yz$					
ii) $12x^2y^2z, 24x^2z$					
iii) $18x^2z, 9xy^2z, 6x^2yz^2$					
iv) $xyz^2, x^2y^2z, 28x^2y^2z$					

Solution:

Monomials	LCM of numerical coefficients	LCM of 'x'	LCM of 'y'	LCM of 'z'	Required LCM
i) $8x^3y, 4x^2yz$	8	x^3	y	z	$8x^3yz$
ii) $12x^2y^2z, 24x^2z$	24	x^2	y^2	z	$24x^2y^2z$
iii) $18x^2z, 9xy^2z, 6x^2yz^2$	18	x^2	y^2	z^2	$18x^2y^2z^2$
iv) $xyz^2, x^2y^2z, 28x^2y^2z$	28	x^2	y^2	z^2	$28x^2y^2z^2$

Find the HCF of the following by division method

Q.3 $x^2 + x - 2, x^2 + 3x^2 + x + 2$

Solution

$$x^2 + x - 2, x^2 + 3x^2 + x + 2$$

H.C.F. by division method

$$\begin{array}{r} x^2 + x - 2 \overline{) x^2 + 3x^2 + x + 2} \\ \underline{-(x^2 + x - 2)} \\ 4x^2 + 2x + 4 \\ \underline{-(4x^2 + 4x - 8)} \\ -2x + 12 \end{array}$$



Taking 2 as common $2(x+2)$

$$\begin{array}{r} x+2 \overline{) x^2 + x - 2} \\ \underline{x+2} \\ 0 \end{array}$$

H.C.F. = $x+2$

Q.4 $x^3 + 2x^2 - 4x - 8$, $2x^3 + 7x^2 + 4x - 4$ answer

Solution

$$x^3 + 2x^2 - 4x - 8, 2x^3 + 7x^2 + 4x - 4$$

$$\begin{array}{r} x^3 + 2x^2 - 4x - 8 \overline{) 2x^3 + 7x^2 + 4x - 4} \\ \underline{2x^3 + 4x^2 - 8x - 16} \\ 3x^2 + 12x + 12 \end{array}$$

Taking 3 as common

$$3(x^2 + 4x + 4)$$

We consider

$$\begin{array}{r} x^2 + 4x + 4 \overline{) x^2 + 2x^2 - 4x - 8} \\ \underline{x^2 + 4x^2 + 4x} \\ -2x^2 - 8x - 8 \\ \underline{-2x^2 - 8x - 8} \\ 0 \end{array}$$

H.C.F. = $x^2 + 4x + 4$

Q.5 $2x^3 + x^2 - x - 2$, $3x^3 - x^2 + x - 3$ answer

Solution

$$2x^3 + x^2 - x - 2, 3x^3 - x^2 + x - 3$$

H.C.F. by division method

$$\begin{array}{r} 3 \overline{) 2x^3 + x^2 - x - 3} \\ \underline{3x^3 - x^2 + x - 3} \\ 5x^2 - 2x \\ \underline{5x^2 - 2x + 3} \\ 0 \end{array}$$

Taking -5 as common $-5(x^2 - x)$

$$\begin{array}{r} x^2 - x \overline{) 2x^3 + x^2 - x - 3} \\ \underline{2x^3 - 2x^2} \\ 3x^2 - x - 3 \\ \underline{3x^2 - 3x} \\ 2x - 3 \end{array}$$

Taking 3 as common $3(x-1)$

$$\begin{array}{r} x-1 \overline{) x^2 - x} \\ \underline{x-1} \\ 0 \end{array}$$

H.C.F. = $x-1$

Q.6 $3+2p^2+5p^3$, answer

$$5p+5p^3+3+3p^2$$

Solution

In descending order

$$2p^3 + 0p^2 + 5p^3 + 0p + 3$$

$$5p^3 + 3p^2 + 5p + 3$$

H.C.F. by division method



$$\begin{array}{r} 2p-6 \overline{) 5p^3 + 3p^2 + 3p + 3} \\ \underline{10p^3 + 0p^2 + 25p^2 + 0p + 18} \\ 10p^2 + 0p^2 + 25p^2 + 0p + 18 \\ \underline{10p^2 + 0p^2 + 10p^2 + 0p} \\ 15p^2 + 18p + 18 \\ \underline{15p^2 + 18p + 18} \\ 0 \end{array}$$

$$\begin{array}{r} 5 \overline{) 10p^2 + 0p^2 + 25p^2 + 0p + 18} \\ \underline{10p^2 + 0p^2 + 10p^2 + 0p} \\ 15p^2 + 18p + 18 \\ \underline{15p^2 + 18p + 18} \\ 0 \end{array}$$

(Taking 93 as common) $93(p^2+1)$

$$\begin{array}{r} 3p+1 \overline{) 5p^3 + 3p^2 + 3p + 3} \\ \underline{15p^3 + 15p^2} \\ 10p^2 + 3p + 3 \\ \underline{10p^2 + 10p} \\ 3p^2 + 3 \\ \underline{3p^2 + 3} \\ 0 \end{array}$$

H.C.F. = p^2+1

Q.7 $24x^4 - 2x^3 - 60x^2 - 32x$, answer

$$18x^4 - 6x^3 - 39x^2 - 18x$$

Solution

First we take common,

$$2(12x^4 - x^3 - 30x^2 - 16x),$$

$$3(6x^4 - 2x^3 - 13x^2 - 6x)$$

Since 2 and 3 are not common factors of these expressions, so we can ignore these numbers.

$$\begin{array}{r} 1 \overline{) 12x^4 - x^3 - 30x^2 - 16x} \\ \underline{12x^4 - 4x^3 - 36x^2 - 12x} \\ 7x^3 + 4x^2 - 4x \\ \underline{7x^3 + 7x^2 - 14x} \\ 3x^2 + 18x \\ \underline{3x^2 + 6x^2 + 4x} \\ 12x^2 + 14x \\ \underline{12x^2 + 12x^2 + 4x} \\ 2x^2 + 10x \\ \underline{2x^2 + 4x^2 + 4x} \\ 6x^2 + 6x \\ \underline{6x^2 + 6x} \\ 0 \end{array}$$

$$\begin{array}{r} 4x+3 \overline{) 12x^4 - x^3 - 30x^2 - 16x} \\ \underline{12x^4 - 4x^3 - 36x^2 - 12x} \\ 7x^3 + 4x^2 - 4x \\ \underline{7x^3 + 7x^2 - 14x} \\ 3x^2 + 18x \\ \underline{3x^2 + 6x^2 + 4x} \\ 12x^2 + 14x \\ \underline{12x^2 + 12x^2 + 4x} \\ 2x^2 + 10x \\ \underline{2x^2 + 4x^2 + 4x} \\ 6x^2 + 6x \\ \underline{6x^2 + 6x} \\ 0 \end{array}$$

Taking 3 as common $3(4x^2+3x)$

$$\begin{array}{r} 4x+3 \overline{) 12x^4 - x^3 - 30x^2 - 16x} \\ \underline{12x^4 - 4x^3 - 36x^2 - 12x} \\ 7x^3 + 4x^2 - 4x \\ \underline{7x^3 + 7x^2 - 14x} \\ 3x^2 + 18x \\ \underline{3x^2 + 6x^2 + 4x} \\ 12x^2 + 14x \\ \underline{12x^2 + 12x^2 + 4x} \\ 2x^2 + 10x \\ \underline{2x^2 + 4x^2 + 4x} \\ 6x^2 + 6x \\ \underline{6x^2 + 6x} \\ 0 \end{array}$$

H.C.F. = $3x^2 + 2x$

$$= x(3x+2)$$

Q.8 $2x^3 + 6x^2 + x + 3$, $3x^3 + 9x^2 - 2x - 6$, answer

Solution

$$\begin{array}{r} 2x^3 + 6x^2 + x + 3, 3x^3 + 9x^2 - 2x - 6 \\ \underline{2x^3 + 6x^2 + 2x + 6} \\ -4x - 9 \end{array}$$

$$\begin{array}{r} x^3 + 3x^2 + 2x + 6 \overline{) 3x^3 + 9x^2 - 2x - 6} \\ \underline{3x^3 + 9x^2 + 6x + 18} \\ -11x - 24 \end{array}$$

Taking -11 as common $-11(x+3)$

$$\begin{array}{r} x+3 \overline{) x^3 + 3x^2 + 2x + 6} \\ \underline{x^3 + 3x^2 + 3x + 9} \\ -x - 3 \\ \underline{-x - 3} \\ 0 \end{array}$$

H.C.F. is $x+3$

Now, we find H.C.F. of $(x+3)$ and $2x^3 + 6x^2 + x + 3$.



$$\begin{array}{r}
 2x^2 + 1 \\
 \times 3 \\
 \hline
 6x^2 + 3x + 3 \\
 6x^2 + 9x + 3 \\
 \hline
 0
 \end{array}$$

So, HCF = $x + 3$

Find the LCM of the following expressions

Q.8 $8x^2y - 4, 6x^2 + 2x$

Solution

$$8x^2y - 4 = 4x^2 \cdot 2y - 4$$

$$8x^2y - 4 = 4(2x^2y - 1)$$

$$= 4 \cdot 2(2x^2y - 1)$$

$$\therefore 8x^2y - 4 = (2x + 1)(2y - 1)$$

$$6x^2 + 2x = 2x(3x + 1) \dots (i)$$

$$8x^2y - 4 = 2x(2y - 1) \dots (ii)$$

From (i) and (ii)

$$\text{Common factor} = 2x + 1$$

$$\text{Non-common factor} = 2y(2y - 1)$$

$$\text{LCM} = 2x(2y + 1)(2y - 1)$$

$$= 2x \cdot 2y(2y^2 - 1)$$

$$= 2x \cdot 2y(4y^2 - 1)$$

$$= 2x \cdot 2y(2y^2 - 1)$$

Q.10 $p^2q - pq^2, p^2q^2 - p^2q$

Solution

$$p^2q - pq^2 = p^2q - p^2q$$

$$p^2q - pq^2 = pq(p^2 - q^2)$$

$$\therefore p^2q - pq^2 = (p + q)(p - q)$$

$$= pq(p + q)(p - q) \dots (i)$$

Now,

$$p^2q^2 - p^2q = p^2q^2 - p^2q$$

$$\therefore p^2q^2 - p^2q = (p + q)(p^2 - pq + q^2) \dots (ii)$$

$$= p \cdot p \cdot q \cdot q(p + q)(p^2 - pq + q^2) \dots (iii)$$

From equation (i) and (ii)

$$\text{Common factor} = p \cdot q \cdot (p + q)$$

$$\text{Uncommon factors} = p \cdot q \cdot (p + q)(p^2 - pq + q^2)$$

$$\text{LCM} = p \cdot q \cdot p \cdot q(p + q)(p + q)(p^2 - pq + q^2)$$

$$\text{LCM} = p^2q^2(p + q)(p^2 - q^2)$$

Q.11 $4x^2y - y, 2x^2 + x$

Solution

$$4x^2y - y = y(4x^2 - 1)$$

$$4x^2y - y = y(4x^2 - 1)$$

$$= y(2x + 1)(2x - 1)$$

$$\therefore 4x^2y - y = (2x + 1)(2x - 1)$$

$$= y(2x + 1)(2x - 1) \dots (i)$$

$$\text{Now, } 2x^2 + x = x(2x + 1) \dots (ii)$$

$$\text{Common factor} = 2x + 1$$

$$\text{Uncommon factors} = x \cdot y(2x - 1)$$

$$\text{LCM} = x \cdot y(2x + 1)(2x - 1)$$

$$\text{LCM} = xy(2x^2 - 1)$$

$$\text{LCM} = xy(4x^2 - 1)$$

Q.12 $x^2 - x - 6, x^2 + x - 2, x^2 - 4x + 3$

$$x^2 - x - 6 = (x + 3)(x - 2)$$

$$x^2 + x - 2 = (x + 2)(x - 1)$$

$$x^2 - 4x + 3 = (x - 3)(x - 1)$$

$$= (x + 3)(x - 2)(x - 1)$$

$$= (x + 3)(x - 2)(x - 1) \dots (i)$$

Now,

$$x^2 + x - 2 = (x + 2)(x - 1)$$

$$= (x + 2)(x - 1)(x - 1)$$

$$= (x + 2)(x - 1)(x - 1) \dots (ii)$$

Also,

$$x^2 - 4x + 3 = (x - 3)(x - 1)$$

$$= (x - 3)(x - 1)(x - 1)$$

$$= (x - 3)(x - 1)(x - 1) \dots (iii)$$

$$\text{Common factor} = (x - 1)(x - 2)(x - 3)$$

So,

$$\text{LCM} = (x - 1)(x - 2)(x - 3)$$

Q.13 $m^2 - 1, m^2 - 1, m^2 - 1$

Solution

$$m^2 - 1 = (m^2 - 1)(m^2 - 1)$$

$$= (m^2 - 1)(m^2 - 1)$$

$$= (m + 1)(m - 1)(m + 1)(m - 1)$$

$$= (m + 1)(m - 1)(m^2 - 1)(m^2 - 1) \dots (i)$$

Now,

$$m^2 - 1 = (m^2 - 1)(m^2 - 1)$$

$$= (m^2 - 1)(m^2 - 1)$$

$$= (m - 1)(m + 1)(m^2 - 1) \dots (ii)$$

Also,

$$m^2 - 1 = (m^2 - 1)$$

$$\therefore m^2 - 1 = (m - 1)(m + 1)$$

$$= (m - 1)(m^2 + (m + 1) + 1)$$

$$= (m - 1)(m^2 + m + 1) \dots (iii)$$

Common factors

$$= (m - 1)(m + 1)(m^2 - m + 1)$$

$$\text{Non common factors} = (m^2 - m + 1)(m^2 + 1)$$

LCM

$$= (m - 1)(m + 1)(m^2 - m + 1)(m^2 - m + 1)(m^2 + 1)$$

$$= (m - 1)(m^2 - m + 1)(m + 1)(m^2 - m + 1)(m^2 + 1)$$

$$= (m^2 - 1)(m^2 + 1)(m^2 + 1)$$

$$= [(m^2 - 1)(m^2 + 1)](m^2 + 1) = (m^2 - 1)(m^2 + 1)$$

Q.14 $x^2 + 2x^2 - x - 2, x^2 - x - 2, x^2 - 4$

$$x^2 + 2x^2 - x - 2 = x^2(x + 2) - 1(x + 2)$$

$$= (x + 2)(x^2 - 1)$$

$$\therefore x^2 - 1 = (x + 1)(x - 1)$$

$$= (x + 2)(x + 1)(x - 1) \dots (i)$$

Now,

$$x^2 - x - 2 = (x + 2)(x - 1)$$

$$= (x + 2)(x - 1)(x - 1)$$

$$= (x + 2)(x - 1)(x - 1) \dots (ii)$$

Also,

$$x^2 - 4 = (x^2 - 4)$$

$$= (x - 2)(x + 2) \dots (iii)$$

Common factor = 1

$$\text{Non common factors} = (x + 1)(x - 1)(x + 2)(x - 2)$$

$$\text{LCM} = (x + 1)(x - 1)(x + 2)(x - 2)$$

$$= (x^2 - 1)(x^2 - 4)$$

$$= (x^2 - 1)(x^2 - 4)$$

$$= x^4 - 4x^2 - x^2 + 4$$

$$= x^4 - 5x^2 + 4$$

$$\text{LCM} = x^4 - 5x^2 + 4$$



Q.15 $x^2 + x - 20, x^2 - 10x + 24, x^2 - x - 30$

Solution

$$x^2 + x - 20 = (x + 5)(x - 4)$$

$$x^2 - 10x + 24 = (x - 6)(x - 4)$$

$$x^2 - x - 30 = (x + 5)(x - 6)$$

$$= (x + 5)(x - 6) \dots (i)$$

$$= (x + 5)(x - 6) \dots (ii)$$

Now,

$$x^2 - 10x + 24 = (x - 6)(x - 4)$$

$$= (x - 6)(x - 4)$$

$$= (x - 6)(x - 4) \dots (iii)$$

Also,

$$x^2 - x - 30 = (x + 5)(x - 6)$$

$$= (x + 5)(x - 6)$$

$$= (x + 5)(x - 6) \dots (iv)$$

$$\text{So, Common factors} = (x + 5)(x - 6)(x - 4)$$

$$\text{LCM} = (x + 5)(x - 6)(x - 4)$$

Relation between HCF and LCM

Consider two polynomials $P = a^2 - b^2$ and

$Q = a^2 - 2ab + b^2$. For their LCM and HCF,

first factorize them as

we

$$P = a^2 - b^2 = (a + b)(a - b)$$

$$Q = a^2 - 2ab + b^2 = (a - b)^2$$

$$\therefore \text{HCF of } P \text{ and } Q = a - b$$

$$\text{and LCM of } P \text{ and } Q = (a + b)(a - b)^2$$

$$\text{Now } \text{HCF} \times \text{LCM} = (a - b)(a + b)(a - b)^2$$

$$= (a^2 - b^2)(a^2 - 2ab + b^2) \dots (i)$$

Also, product of P and Q

$$= (a^2 - b^2)(a^2 - 2ab + b^2) \dots (ii)$$

$$\text{Thus, } \text{LCM} \times \text{HCF} = P \times Q \dots (iii)$$

Hence, it can be generalized that:

Product of their HCF and LCM = product of

given polynomials

Example 15

Find the LCM of

$$P = 10x^4 + 3x^2 + 8 \text{ and } Q = 8x^4 + 3x + 10$$



Solution: First, we find their HCF by division method.

$$\begin{array}{r} 3 \\ 8x^2 + 3x + 10 \overline{) 10x^2 + 3x^2 + 8} \\ \underline{-8x^2} \\ 40x^2 + 12x^2 + 32 \\ \underline{+ 40x^2} \\ 12x^2 - 18 - 15x \end{array}$$

- multiplying dividend by 4

where

$$(12x^2 - 18 - 15x) \div 3 = 4x^2 - 6 - 5x = 4x^2 - 5x - 6$$

Now, we will divide

$$8x^2 + 3x + 10 \text{ by } 4x^2 - 5x - 6$$

$$\begin{array}{r} 2x \\ 4x^2 - 5x - 6 \overline{) 8x^2 + 3x + 10} \\ \underline{- 8x^2 + 13x} \\ 18x + 10 \end{array}$$

$$\text{Again, } (18x + 10) \div 2 = 9x + 5$$

$$\begin{array}{r} 2x - 3 \\ 2x^2 + 3x + 2 \overline{) 4x^2 - 5x - 6} \\ \underline{- 4x^2 + 4x} \\ - 9x - 6 \\ \underline{+ 9x + 6} \\ 0 \end{array}$$

$$\therefore \text{HCF is } 2x^2 + 3x + 2.$$

Now, we obtain the LCM using the following relation,

$$\text{LCM} = \frac{P \times Q}{\text{HCF}} = \frac{(10x^2 + 3x^2 + 8)(8x^2 + 3x + 10)}{2x^2 + 3x + 2}$$

(Since, the HCF of two polynomials divides both of them exactly. So divide any polynomial of numerator by denominator.)

$$\begin{array}{r} 5x^2 - 6x + 4 \\ 2x^2 + 3x + 2 \overline{) 10x^2 + 3x^2 + 8x + 8} \\ \underline{- 10x^2 + 15x^2 + 10x} \\ -12x^2 - 10x + 8 \\ \underline{+ 12x^2 + 18x + 12} \\ 8x^2 + 12x + 8 \\ \underline{- 8x^2 + 12x + 8} \\ 0 \end{array}$$

Hence, required LCM =

$$(5x^2 - 6x + 4)(8x^2 + 3x + 10)$$

Example 28

Find the second polynomial Q, when first polynomial $P = x^2 - 5x + 6$, $\text{HCF} = x - 3$ and

$$\text{LCM} = x^2 - 9x^2 + 26x - 24 \quad \text{[given]}$$

$$\text{Solution: } P = x^2 - 5x + 6, Q = ?$$

$$\text{LCM of P and Q} = x^2 - 9x^2 + 26x - 24$$

$$\text{HCF} = x - 3$$

$$Q = \frac{\text{HCF} \times \text{LCM}}{P}$$

$$Q = \frac{(x - 3)(x^2 - 9x^2 + 26x - 24)}{x^2 - 5x + 6}$$

$$Q = \frac{(x - 3)(x^2 - 9x^2 + 26x - 24)}{(x - 3)(x - 2)}$$

$$Q = \frac{x^2 - 9x^2 + 26x - 24}{x - 2} = x^2 - 7x + 12$$

$$\begin{array}{r} x^2 - 7x + 12 \\ x - 2 \overline{) x^2 - 9x^2 + 26x - 24} \\ \underline{- x^2 + 2x^2} \\ -7x^2 + 26x - 24 \\ \underline{+ 7x^2 - 14x} \\ 12x - 24 \\ \underline{- 12x + 24} \\ 0 \end{array}$$



Exercise 4.5

Q.1 Find the HCF and LCM of following

i) $16 - 4x^2, x^2 + x - 6$ [given]

ii) $a^2 - a^3 - a + 1, a^4 + a^2 + 1$ [given]

iii) $x^2 + 2x^2 - 3x, 2x^2 + 5x^2 - 2x$ [given]

i) $16 - 4x^2, x^2 + x - 6$

Solution

Let

$$p(x) = 16 - 4x^2$$

$$p(x) = 4(4 - x^2) \\ = 4(2 - x)(2 + x)$$

$$= 4(2 - x)(2 + x)$$

$$= -4(x - 2)(x + 2)$$

Let

$$q(x) = x^2 + x - 6$$

$$q(x) = x^2 + 3x - 2x - 6$$

$$q(x) = x(x + 3) - 2(x + 3)$$

$$q(x) = (x - 2)(x + 3)$$

$$\text{Common factor} = x - 2$$

$$\text{Non common factors} = -4(x + 2)(x + 3)$$

$$\text{H.C.F} = x - 2$$

$$\text{L.C.M} = -4(x - 2)(x + 2)(x + 3)$$

$$= -4(x^2 - 4)(x + 3)$$

ii) $a^4 - a^3 - a + 1, a^4 + a^2 + 1$

Solution

Let

$$p(x) = a^4 - a^3 - a + 1$$

$$p(x) = a^3(a - 1) - 1(a - 1)$$

$$= (a - 1)(a^3 - 1)$$

$$= (a - 1)(a - 1)(a^2 + a + 1)$$

$$q(x) = a^4 + a^2 + 1$$

$$q(x) = a^4 + 1 + a^2$$

$$= (a^2)^2 + (1)^2 + 2a^2 \cdot 1 \cdot a^2$$

$$= (a^2 + 1)^2 - a^2$$

$$= (a^2 + 1 + a)(a^2 + 1 - a)$$

$$= (a^2 + a + 1)(a^2 - a + 1)$$

$$\text{Common factor} = a^2 + a + 1$$

$$\text{Non common factor} = (a^2 - a + 1)(a - 1)(a - 1)$$

So,

$$\text{H.C.F} = a^2 + a + 1$$

$$\text{L.C.M} = (a - 1)(a^2 + a + 1)(a^2 - a + 1)(a - 1)$$

$$= (a^2 - 1)(a^2 - a + 1)(a - 1)$$

iii) $x^2 + 2x^2 - 3x, 2x^2 + 5x^2 - 2x$

Solution

Let

$$p(x) = x^2 + 2x^2 - 3x$$

$$= x(x^2 + 2x - 3)$$

$$p(x) = x(x^2 + 3x - x - 3)$$

$$p(x) = x[x(x + 3) - 1(x + 3)]$$

$$p(x) = x(x + 3)(x - 1)$$

Let

$$q(x) = 2x^2 + 5x^2 - 2x$$

$$q(x) = x(2x^2 + 5x - 2)$$

$$q(x) = x(2x^2 + 6x - x - 2)$$

$$q(x) = x[2x(x + 3) - 1(x + 3)]$$

$$q(x) = x(x + 3)(2x - 1)$$

$$\text{Common factor} = x(x + 3)$$

$$\text{Non common factor} = (x - 1)(2x - 1)$$

So,

$$\text{H.C.F} = x(x + 3)$$

$$\text{L.C.M} = x(x + 3)(x - 1)(2x - 1)$$

Q.2 If HCF and LCM of two

polynomials are $x - 7$ and $x^2 - 10x^2 + 11x + 70$

respectively. Then find product of two

polynomials. [given]

Solution

$$\text{HCF} \times \text{LCM} = \text{Product of two polynomials}$$

$$(x - 7)(x^2 - 10x^2 + 11x + 70) = \text{Product of two polynomials}$$

$$x^3 - 10x^3 + 11x^2 + 70x - 7x^2 + 70x^2$$

$$- 77x - 490 = \text{Product of two polynomials}$$

$$\text{Product of two polynomials} =$$

$$x^3 - 17x^2 + 81x^2 - 7x - 490$$



Q.3 Product of two polynomials is $x^4 + 3x^3 - 12x^2 - 20x + 48$ and their HCF is $x - 2$. Find their LCM. answer

Solution

Product of two polynomials

$$= x^4 + 3x^3 - 12x^2 - 20x + 48$$

$$\text{HCF} = x - 2$$

$$\text{LCM} = ?$$

We know that

$$\text{HCF} \times \text{LCM} = P(x) \times Q(x)$$

Putting values

$$\text{HCF} \times \text{LCM} = x^4 + 3x^3 - 12x^2 - 20x + 48$$

$$\text{LCM} = \frac{x^4 + 3x^3 - 12x^2 - 20x + 48}{x - 2}$$

$$\text{LCM} = \frac{x^4 + 3x^3 - 12x^2 - 20x + 48}{x - 2}$$

$$\begin{array}{r} x^4 + 3x^3 - 2x - 24 \\ x-2 \overline{) } \\ \underline{x^4 - 2x^3} \\ 5x^3 - 2x \\ \underline{5x^3 - 10x^2} \\ 12x^2 - 20x \\ \underline{12x^2 - 24x} \\ 4x \\ \underline{4x - 8} \\ 24x + 48 \\ \underline{24x + 48} \\ 0 \end{array}$$

$$\text{LCM} = x^3 + 5x^2 - 2x - 24$$

Q.4 The product of two polynomial is $y^4 + 6y^3 - 3y^2 - 56y - 48$ and their LCM is $y^2 + 2y^2 - 11y - 12$. Find their HCF. answer

Solution

Product two polynomials

$$= y^4 + 6y^3 - 3y^2 - 56y - 48$$

$$\text{LCM} = y^2 + 2y^2 - 11y - 12$$

$$\text{HCF} = ?$$

$\text{HCF} \times \text{LCM} = \text{Product of two polynomials}$

$$\text{HCF} = \frac{y^4 + 6y^3 - 3y^2 - 56y - 48}{y^2 + 2y^2 - 11y - 12}$$

$$\text{HCF} = \frac{y^4 + 6y^3 - 3y^2 - 56y - 48}{y^2 + 2y^2 - 11y - 12}$$

$$\begin{array}{r} y^4 + 6y^3 - 3y^2 - 56y - 48 \\ y^2 + 2y^2 - 11y - 12 \overline{) } \\ \underline{y^4 + 2y^3 - 11y^2 - 12y} \\ 4y^3 + 3y^2 - 44y - 48 \\ \underline{4y^3 + 8y^2 - 44y - 48} \\ -4y^2 - 48 \\ \underline{-4y^2 - 48} \\ 0 \end{array}$$

So,

$$\text{HCF} = y + 4$$

Q.5 Find the second polynomial when first polynomial = $x^4 + x^3 + x + 1$, HCF = $x + 1$ and LCM = $(x^2 + 1)(x^4 + x^2 - x - 1)$. answer

Solution

$$1^{\text{st}} \text{ Polynomial} = x^4 + x^3 + x + 1$$

$$= x^3(x + 1) + 1(x + 1)$$

$$= (x + 1)(x^3 + 1)$$

$$\text{HCF} = x + 1$$

$$\text{LCM} = (x^2 + 1)(x^4 + x^2 - x + 1)$$

$$\text{LCM} = [(x^2 + 1)(x^2(x + 1) - 1)(x + 1)]$$

$$\text{LCM} = (x^2 + 1)(x + 1)(x^3 - 1)$$

$$\text{HCF} \times \text{LCM} = 1^{\text{st}} \text{ polynomial} \times 2^{\text{nd}} \text{ polynomial}$$

$$(x + 1)(x^3 - 1)(x^2 + 1)(x + 1)$$

$$= (x^2 + 1)(x + 1) \times 2^{\text{nd}} \text{ polynomial}$$

$$\frac{(x + 1)(x^3 - 1)(x^2 + 1)(x + 1)}{(x^2 + 1)(x + 1)} = 2^{\text{nd}} \text{ polynomial}$$

$$(x^3 - 1)(x + 1) = 2^{\text{nd}} \text{ polynomial}$$

$$2^{\text{nd}} \text{ polynomial} = x^4 + x^2 - x - 1$$

Q.6 Find the LCM of polynomials

$$4x^2 - 10x^2 + 4x + 2 \text{ and } 3x^4 - 2x^3 - 3x + 2$$

If their HCF is $x - 1$. answer

Solution

Let



$$P(x) = 4x^2 - 10x^2 + 4x + 2$$

$$Q(x) = 3x^4 - 2x^3 - 3x + 2$$

$$= (x^2(3x - 2) - 1)(3x - 2)$$

$$= (x^2 - 1)(3x - 2)$$

$$= (x - 1)(x + 1)(3x - 2)$$

$$\text{HCF} = x - 1$$

$$\text{LCM} = ?$$

Using formula

$$\text{HCF} \times \text{LCM} = P(x) \times Q(x)$$

$$\text{LCM} = \frac{P(x) \times Q(x)}{\text{HCF}}$$

Putting values

$$\text{LCM} = \frac{(4x^2 - 10x^2 + 4x + 2)(x - 1)(x^2 + x + 1)(3x - 2)}{(x - 1)}$$

$$\text{LCM} = (4x^2 - 10x^2 + 4x + 2)(3x^2 + 3x^2 + 3x - 2)$$

$$\text{LCM} =$$

$$= (4x^2 - 10x^2 + 4x + 2)(3x^2 + x^2 + x - 2)$$

Multiplication of algebraic fractions

If $\frac{a}{b}$ and $\frac{c}{d}$ are two algebraic fractions, where $b \neq 0, d \neq 0$.

$$\text{Then, } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Example 21

Find the indicated product. answer

$$\frac{x^2 - 1}{x^2 + 4x + 4} \times \frac{x + 2}{x^2 + 2x - 3}$$

Solution

$$\begin{aligned} \frac{x^2 - 1}{x^2 + 4x + 4} \times \frac{x + 2}{x^2 + 2x - 3} &= \frac{(x + 1)(x - 1)}{(x + 2)(x + 2)} \times \frac{x + 2}{(x - 1)(x + 3)} \\ &= \frac{x + 1}{x + 2} \times \frac{1}{x + 3} \\ &= \frac{x + 1}{(x + 2)(x + 3)} = \frac{x + 1}{x^2 + 5x + 6} \end{aligned}$$

Division of algebraic fractions

If $\frac{a}{b}$ and $\frac{c}{d}$ are two algebraic fractions, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}, \text{ where } b \neq 0 \text{ and } d \neq 0$$

Example 22

Perform the indicated operation. answer

$$\frac{x^2 - x^2}{x} \div \frac{x - 1}{1}$$

Solution

$$\begin{aligned} &= \frac{x^2 - x^2}{x} \div \frac{x - 1}{1} \\ &= \frac{(x + 1)(x - 1)}{x} \div \frac{x - 1}{1} = \frac{x + 1}{x} \times \frac{1}{1} \\ &= \frac{x + 1}{x} = \frac{x + 1}{x} \end{aligned}$$

Example 23: Simplify: answer

$$\frac{3x^3 + 6x^2}{2x^2 + x - 6} \div (6x^2 - 15x)$$

$$\begin{aligned} \text{Solution: } \frac{3x^3 + 6x^2}{2x^2 + x - 6} \div (6x^2 - 15x) &= \frac{3x^2 + 6x^2}{2x^2 + x - 6} \times \frac{1}{6x^2 - 15x} \\ &= \frac{3x^2(x + 2)}{(x + 2)(2x - 3)} \times \frac{1}{3x(2x - 5)} \\ &= \frac{x}{(2x - 3)(2x - 5)} = \frac{x}{4x^2 - 16x + 15} \end{aligned}$$

Addition and subtraction of algebraic fractions

Algebraic fractions are added or subtracted just like arithmetic fractions i.e., by manipulating LCM of their denominators.

Example 24 answer

Simplify the following:

$$\frac{x^2 + 2x + 2}{2x} - \frac{(x - 2)(x - 2)}{2x}$$



Solution

$$\begin{aligned} & \frac{x^2 + 2x + 2}{2x} + \frac{(-2x - 2)}{2x} \\ &= \frac{x^2 + 2x + 2 + (-2x - 2)}{2x} \\ &= \frac{x^2 + 2x + 2 - 2x - 2}{2x} \\ &= \frac{x^2}{2x} = \frac{x}{2} \end{aligned}$$

Example 16

Perform the indicated operations. *****

$$\begin{aligned} & \frac{-2x}{x+3} + \frac{3}{1-x} - \frac{8x-12}{x^2-9} \\ \text{Solution: } & \frac{-2x}{x+3} + \frac{3}{1-x} - \frac{8x-12}{x^2-9} \\ &= \frac{-2x}{x+3} - \frac{3}{x-3} - \frac{8x-12}{(x+3)(x-3)} \\ &= \frac{-2x(x-3) + (-3)(x+3) - (8x-12)}{(x+3)(x-3)} \\ &= \frac{-2x^2 + 6x - 3x - 9 - 8x + 12}{(x+3)(x-3)} \\ &= \frac{-2x^2 - 5x + 3}{(x+3)(x-3)} \\ &= \frac{-1(2x^2 + 5x - 3)}{(x+3)(x-3)} \\ &= \frac{-1(2x-1)(x+3)}{(x+3)(x-3)} \\ &= \frac{-(2x-1)}{(x-3)} \\ &= \frac{-2x+1}{x-3} \end{aligned}$$

Algebraic Fraction with combined operations

operation

When two or more operations occur in any algebraic expression then the rule for order of operations (DMAS) must be followed.

Example 18

Simplify *****

$$\frac{3x^2 + 5xy + 2y^2}{x^2 + 5xy + 6y^2} + \frac{2x + 6y}{x^2 - 4y^2} - \frac{x^2 + 6xy + 9y^2}{x^2y - 2xy^2}$$

Solution

Division is performed before addition while simplifying an expression.

$$\begin{aligned} & \frac{3x^2 + 5xy + 2y^2}{x^2 + 5xy + 6y^2} + \frac{2x + 6y}{x^2 - 4y^2} - \frac{x^2 + 6xy + 9y^2}{x^2y - 2xy^2} \\ &= \frac{3x^2 + 5xy + 2y^2}{x^2 + 5xy + 6y^2} + \frac{2x + 6y}{x^2 - 4y^2} - \frac{x^2y - 2xy^2}{x^2 + 6xy + 9y^2} \\ &= \frac{3x^2 + 5xy + 2y^2}{x^2 + 5xy + 6y^2} + \frac{2(x+3y)}{(x+2y)(x-2y)} - \frac{xy(x-2y)}{(x+3y)^2} \\ &= \frac{3x^2 + 5xy + 2y^2}{(x+2y)(x+3y)} + \frac{2(x+3y)}{(x+2y)(x-2y)} - \frac{xy(x-2y)}{(x+3y)^2} \\ &= \frac{3x^2 + 5xy + 2y^2}{(x+2y)(x+3y)} + \frac{2xy}{(x+2y)(x+3y)} - \frac{xy(x-2y)}{(x+3y)^2} \\ &= \frac{3x^2 + 7xy + 2y^2}{(x+2y)(x+3y)} - \frac{xy(x-2y)}{(x+3y)^2} \\ &= \frac{3x^2 + 7xy + 2y^2}{(x+2y)(x+3y)} - \frac{xy^2 + 6xy + xy - 2y^2}{(x+2y)(x+3y)} \\ &= \frac{3x^2 + 7xy + 2y^2}{(x+2y)(x+3y)} - \frac{xy^2 + 5xy - 2y^2}{(x+2y)(x+3y)} \\ &= \frac{3x^2 + 7xy + 2y^2 - xy^2 - 5xy + 2y^2}{(x+2y)(x+3y)} \\ &= \frac{3x^2 + 2xy + 4y^2 - xy^2}{(x+2y)(x+3y)} \\ &= \frac{(3x+y)(x+2y) - xy^2}{(x+2y)(x+3y)} = \frac{3x+y}{x+3y} \end{aligned}$$

Exercise 4.4

Q.1 Answer these without calculations

i) Product of what algebraic fraction and $x^2 + 7x - 8$ is 1? *****

ii) Which algebraic fraction divided by

$$\frac{x^2}{x^2 + y^2} \text{ gives } 1? \quad \text{*****}$$

iii) Sum of what algebraic fraction and

$$\frac{m}{m^2 + n^2} \text{ is } \frac{m+n}{m^2 + n^2}? \quad \text{*****}$$

iv) What is product of an algebraic fraction and its reciprocal? *****

i) **Solution**

Let $P(x)$ be the algebraic expression.

Then

$$(x^2 + 7x - 8)P(x) = 1$$

$$P(x) = \frac{1}{x^2 + 7x - 8}$$

Product of $\frac{1}{x^2 + 7x - 8}$ and $x^2 + 7x - 8$ is 1.

The Algebraic expression is $\frac{1}{x^2 + 7x - 8}$

ii) **Solution**

Let $P(x)$ is algebraic expression.

$$\frac{x^2}{x^2 + y^2} \div P(x) = 1$$

$$\frac{x^2}{x^2 + y^2} \times \frac{1}{P(x)} = 1$$

$$\frac{x^2}{x^2 + y^2} = P(x)$$

$$\frac{x^2}{x^2 + y^2} \text{ is divided by } \frac{x^2}{x^2 + y^2} \text{ is } 1.$$

So, the algebraic expression is $\frac{x^2}{x^2 + y^2}$

iii) **Solution**

To find the algebraic fraction whose sum with

$$\frac{m}{m^2 + n^2} \text{ is } \frac{m+n}{m^2 + n^2}$$

Let use the equation:

$$\frac{m+n}{m^2 + n^2} = \frac{m}{m^2 + n^2} + \frac{n}{m^2 + n^2}$$



The algebraic fraction that satisfies this

$$\text{condition is } \frac{n}{m^2 + n^2}$$

iv) **Solution**

The product of an algebraic expression its reciprocal is one.

Simplify the following (where all expression in the denominator are non-zero)

$$\text{Q.2 } \frac{14x^2 - 7x}{12x^2 + 24x^2} \div \frac{x^2 + 2x}{2x - 1} \quad \text{*****}$$

Solution

$$\frac{14x^2 - 7x}{12x^2 + 24x^2} \div \frac{x^2 + 2x}{2x - 1}$$

$$= \frac{7x(2x-1)}{12x^2(2x-1)} \times \frac{x(2x-1)}{(2x-1)}$$

$$= \frac{7x^2}{12x^2}$$

$$= \frac{7}{12}$$

$$\text{Q.3 } \frac{a^2b^2 + 3ab}{4a^2 - 1} \div \frac{2a+1}{ab+3} \quad \text{*****}$$

Solution

$$\frac{a^2b^2 + 3ab}{4a^2 - 1} \div \frac{2a+1}{ab+3}$$

$$= \frac{ab(ab+3)}{(2a)^2 - 1^2} \times \frac{(2a+1)}{(ab+3)}$$

$$= \frac{ab(2a+1)}{(2a-1)(2a+1)}$$

Using formula

$$a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{ab(2a+1)}{(2a-1)(2a+1)}$$

$$= \frac{ab}{2a-1}$$

$$\text{Q.4 } \frac{a-b}{a^2 + ab} \div \frac{a^2 - b^2}{a^2 - 2ab + b^2} \div \frac{a}{a^2 + b^2} \quad \text{*****}$$



Solution

$$\begin{aligned}
 & \frac{a-b}{a^2+ab} + \frac{a^2-b^2}{a^2-2ab+b^2} + \frac{a}{a^2+b^2} \\
 &= \frac{a-b}{a(a+b)} + \frac{(a^2-b^2)}{a^2-2ab+b^2} + \frac{a}{a^2+b^2} \\
 & \quad \because a^2-b^2 = (a+b)(a-b) \\
 &= \frac{a-b}{a(a+b)} + \frac{(a^2-b^2)(a-b)}{(a-b)^2} + \frac{a}{a^2+b^2} \\
 &= \frac{a-b}{(a+b)} + \frac{(a+b)(a-b)}{(a-b)(a-b)} \\
 &= 1
 \end{aligned}$$

$$Q.5 \quad \frac{6x^2y^2}{x^2-y^2} + \frac{3xy}{x+y}$$

Solution

$$\begin{aligned}
 & \frac{6x^2y^2}{x^2-y^2} + \frac{3xy}{x+y} \\
 & \quad \because a^2-b^2 = (a+b)(a-b) \\
 & \frac{6x^2y^2}{(x-y)(x+y)} + \frac{3xy}{x+y} \\
 & \frac{6xy^2}{(x-y)(x+y)} + \frac{3xy}{x+y} \\
 &= \frac{3xy}{(x-y)}
 \end{aligned}$$

$$Q.6 \quad \frac{8x^2-1}{4x^2-2x^2} + \frac{6x^2-13x+5}{15x-25} + \frac{2x^2+x}{15x^2}$$

Solution

$$\begin{aligned}
 & \frac{8x^2-1}{4x^2-2x^2} + \frac{6x^2-13x+5}{15x-25} + \frac{2x^2+x}{15x^2} \\
 &= \frac{(2x)^2-1^2}{2x^2(2x+1)} + \frac{6x^2-10x-3x+5}{5(3x-5)} + \frac{x^2(2x+1)}{15x^2} \\
 &= \frac{(2x-1)(4x^2+2x+1)}{2x^2(2x+1)} + \frac{2x(3x-5)-1(3x-5)}{5(3x-5)} + \frac{x \cdot x^2(2x+1)}{15x^2} \\
 &= \frac{(2x-1)(4x^2+2x+1)}{2x^2(2x+1)} + \frac{(2x-1)(2x-5)}{5(2x-5)} + \frac{x(2x+1)}{15} \\
 &= \frac{(2x-1)(4x^2+2x+1)}{2x^2(2x+1)} + \frac{(2x-1)}{5} + \frac{x(2x+1)}{15} \\
 &= \frac{(2x-1)(4x^2+2x+1)}{2x(2x+1)} + \frac{1x}{(2x+1)} + \frac{x(2x+1)}{15} \\
 &= \frac{4x^2+2x+1}{6x}
 \end{aligned}$$



$$Q.7 \quad \frac{x^2-8x-9}{x^2-17x+72} + \frac{x^2-25}{x^2-1} + \frac{x^2+4x-5}{x^2-9x+8}$$

Solution

$$\begin{aligned}
 & \frac{x^2-8x-9}{x^2-17x+72} + \frac{x^2-25}{x^2-1} + \frac{x^2+4x-5}{x^2-9x+8} \\
 &= \frac{x^2-9x+x-9}{x^2-8x-8x+72} + \frac{(x^2)-(5)^2}{(x^2)-(1)^2} + \frac{x^2+5x-x-5}{x^2-8x-x+8} \\
 &= \frac{x(x-9)+1(x-9)}{x(x-9)-8(x-8)} + \frac{(x-5)(x+5)}{(x+1)(x-1)} + \frac{x(x+5)-1(x+5)}{x(x-8)-1(x-8)} \\
 &= \frac{(x-9)(x-8)}{(x-8)(x-8)} + \frac{(x-5)(x+5)}{(x+1)(x-1)} + \frac{(x+5)(x-8)}{(x-8)(x-8)} \\
 &= \frac{(x-5)(x-8)}{(x-8)(x-1)} + \frac{(x-8)}{(x-8)} \\
 &= \frac{x-5}{x-1}
 \end{aligned}$$

$$Q.8 \quad \frac{1}{2x-3y} + \frac{x+y}{4x^2-9y^2}$$

Solution

$$\begin{aligned}
 & \frac{1}{2x-3y} + \frac{x+y}{4x^2-9y^2} \\
 &= \frac{1}{2x-3y} + \frac{x+y}{(2x)^2-(3y)^2} \\
 & \quad \because a^2-b^2 = (a-b)(a+b) \\
 &= \frac{1}{2x-3y} + \frac{x+y}{(2x+3y)(2x-3y)} \\
 &= \frac{2x+3y-(x+y)}{(2x+3y)(2x-3y)} \\
 &= \frac{2x+3y-x-y}{(2x)^2-(3y)^2} \\
 &= \frac{x+2y}{4x^2-9y^2}
 \end{aligned}$$

$$Q.9 \quad \frac{1}{x(x-y)} + \frac{1}{y(x+y)}$$

Solution

$$\begin{aligned}
 & \frac{1}{x(x-y)} + \frac{1}{y(x+y)} \\
 &= \frac{y(x+y)+x(x-y)}{x(x-y)y(x+y)} \\
 &= \frac{xy(x+y)+x^2-xy}{xy(x-y)(x+y)} \\
 &= \frac{x^2+y^2}{xy(x^2-y^2)}
 \end{aligned}$$

$$Q.10 \quad \frac{5x+5}{3(2x-1)} + \frac{4-2x}{2(1-2x)}$$

Solution

$$\frac{5x+5}{3(2x-1)} + \frac{4-2x}{2(1-2x)}$$



$$= \frac{5x+3}{3(2x-1)} - \frac{8-2x}{2(2x-1)}$$

$$= \frac{10x+10-18+8x}{6(2x-1)}$$

$$= \frac{18x-8}{6(2x-1)}$$

$$= \frac{9(2x-1)}{3(2x-1)}$$

$$= \frac{3}{1}$$

Q.11 $\frac{2x}{2x-3} - \frac{5}{6x+9} + \frac{4(3x+2)}{3(4x^2-9)}$ answer

Solution

$$\frac{2x}{2x-3} - \frac{5}{6x+9} + \frac{4(3x+2)}{3(4x^2-9)}$$

$$= \frac{2x}{2x-3} - \frac{5}{3(2x+3)} + \frac{4(3x+2)}{3(2x-3)(2x+3)}$$

$$= \frac{2x(2x+3) - 5(2x-3) + 4(3x+2)}{3(2x-3)(2x+3)}$$

$$= \frac{4x(2x+3) - 5(2x-3) + 4(3x+2)}{3(2x-3)(2x+3)}$$

$$= \frac{12x^2 + 18x - 10x + 15 - 12x - 8}{3(2x-3)(2x+3)}$$

$$= \frac{12x^2 + 18x - 22x + 7}{3(2x^2 - (3x)^2)}$$

$$= \frac{12x^2 - 4x + 7}{3(4x^2 - 9)}$$

Q.12 $\frac{9}{5+x-18x^2} - \frac{2}{1+5x+2x^2}$ answer

Solution

$$\frac{9}{5+x-18x^2} - \frac{2}{1+5x+2x^2}$$

$$= \frac{9}{18x^2 - x - 5} - \frac{2}{2x^2 + 5x + 1}$$

$$= \frac{-5}{18x^2 - 10x + 9x - 5} - \frac{2}{2x^2 + 4x + x + 2}$$

$$= \frac{-5}{2x(9x-5) + 1(9x-5)} - \frac{2}{2x(x+2) + 1(x+2)}$$

$$= \frac{-5}{(9x-5)(2x+1)} - \frac{2}{(x+2)(2x+1)}$$

$$= \frac{-5(x+2) - 2(9x-5)}{(9x-5)(2x+1)(x+2)}$$

$$= \frac{-5x-10-18x+10}{(9x-5)(2x+1)(x+2)}$$

$$= \frac{-23x}{(9x-5)(2x+1)(x+2)}$$

Q.13 $\frac{1-p^2}{1+q} + \frac{1-q^2}{p+p^2} \div \left(1 + \frac{p}{1-p}\right)$ answer

Solution

$$\frac{1-p^2}{1+q} + \frac{1-q^2}{p+p^2} \div \left(1 + \frac{p}{1-p}\right)$$

$$= \frac{(1)^2 - (p)^2}{1+q} + \frac{(1)^2 - (q)^2}{p(1+p)} \div \left(\frac{1-p+p}{1-p}\right)$$

$$= \frac{(1-p)(1+p)}{1+q} + \frac{(1-q)(1+q)}{p(1+p)} \div \left(\frac{1}{1-p}\right)$$

$$= \frac{(1-p)(1+p)}{1+q} + \frac{(1-q)(1+q)}{p(1+p)} \times (1-p)$$

$$= \frac{1-p}{p}$$

Square Root of Algebraic Expressions

The square root of an algebraic expression P is another algebraic expression Q which, when squared, gives P. Thus if $P = (-Q)^2$ then Q and -Q both are square roots of P.

Example 27

Find the square root of

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right)$$

Solution: To find square root of such type of expressions, we can adopt two methods.



Method-I

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) = x^2 + 2x\left(\frac{1}{x}\right) + \frac{1}{x^2} - 4\left(x - \frac{1}{x}\right)$$

$$= x^2 + 2 + \frac{1}{x^2} - 4\left(x - \frac{1}{x}\right)$$

$$= x^2 - 2 + \frac{1}{x^2} - 4\left(x - \frac{1}{x}\right) + 2 + 2$$

$$= \left(x - \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) + 4$$

$$= \left(x - \frac{1}{x}\right)^2 - 2\left(x - \frac{1}{x}\right)(2) + (2)^2$$

$$= \left(x - \frac{1}{x} - 2\right)^2$$

$$\sqrt{\left(x - \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) + 4} = \left(x - \frac{1}{x} - 2\right)$$

Method-II

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) = x^2 + 2 + \frac{1}{x^2} - 4\left(x - \frac{1}{x}\right)$$

$$= \left(x^2 + \frac{1}{x^2}\right) + 2 - 4\left(x - \frac{1}{x}\right) \quad \dots (i)$$

Let, $x - \frac{1}{x} = z$...(ii)

Then, $\left(x - \frac{1}{x}\right)^2 = z^2$

or $x^2 - 2 + \frac{1}{x^2} = z^2$

or $x^2 + \frac{1}{x^2} = z^2 + 2$...(iii)

Substituting values from equations (i) and (ii) in equation (i)

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) = (x^2 + 2) + 2 - 4z$$

$$= x^2 - 4x + 4$$

$$= (x-2)^2 \quad \dots (iv)$$

Now replace x by $x - \frac{1}{x}$ in equation (ii)

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) = \left(x - \frac{1}{x} - 2\right)^2$$

Applying square root on both sides, we get

$$\sqrt{\left(x + \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right)} = \left(x - \frac{1}{x} - 2\right)$$

Example 18: Find the square root of

$$(2x+1)(2x+3)(2x+5)(2x+7)+16$$

answer

Solution:

$$[(2x+1)(2x+7)][(2x+3)(2x+5)] + 16$$

$$= (4x^2 + 16x + 7)(4x^2 + 16x + 15) + 16$$

$$= (4x^2 + 16x + 7)(4x^2 + 16x + 7 + 8) + 16$$

$$(2x+1)(2x+3)(2x+5)(2x+7) + 16 = (x+8) + 16$$

(put $4x^2 + 16x + 7 = x$)

$$= x^2 + 8x + 16$$

$$= x^2 + 2(x)(4) + (4)^2 = (x+4)^2$$

Now replace x by $4x^2 + 16x + 7$

$$(2x+1)(2x+3)(2x+5)(2x+7) + 16 = (4x^2 + 16x + 7 + 4)^2$$

$$= (4x^2 + 16x + 11)^2$$

Applying square root on both sides, we get

$$\sqrt{(2x+1)(2x+3)(2x+5)(2x+7) + 16} = 2(4x^2 + 16x + 11)$$

Square root by division method

To understand this method, consider the following example.

Example 19

Find the square root of $9x^2 - 42xy + 49y^2$ by division method. answer

Solution

$$\begin{array}{r} \sqrt{9x^2 - 42xy + 49y^2} \\ \underline{9x^2 - 42xy + 49y^2} \\ 0 \end{array}$$

$$\begin{array}{r} \sqrt{9x^2 - 42xy + 49y^2} \\ \underline{9x^2 - 42xy + 49y^2} \\ 0 \end{array}$$

$\therefore$ Square root = $\pm(3x - 7y)$



Example 30

1004017

Find the square root of

$$4\left(x^2 + \frac{1}{x^2}\right) + 12\left(x + \frac{1}{x}\right) + 17, x \neq 0$$

Solution: $4\left(x^2 + \frac{1}{x^2}\right) + 12\left(x + \frac{1}{x}\right) + 17$

$$= 4x^2 + \frac{4}{x^2} + 12x + \frac{12}{x} + 17$$

$$= 4x^2 + 12x + 17 + \frac{12}{x} + \frac{4}{x^2} \quad \text{arrange in descending order}$$

$$\begin{array}{r} 2x + 3 + \frac{2}{x} \\ 2x \quad \overline{4x^2 + 12x + 17 + \frac{12}{x} + \frac{4}{x^2}} \\ \quad \pm 4x^2 \\ \hline \quad \quad 12x + 17 + \frac{12}{x} + \frac{4}{x^2} \\ \quad \quad \pm 12x + 9 \\ \hline \quad \quad \quad 8 + \frac{12}{x} + \frac{4}{x^2} \\ \quad \quad \quad \pm 8 + \frac{12}{x} + \frac{4}{x^2} \\ \hline \quad \quad \quad \quad 0 \end{array}$$

$\therefore$ Required square root is $\pm\left(2x + 3 + \frac{2}{x}\right)$

Example 31

To convert $x^2 + 4x^2 + 10x + 5$ into a perfect square.

1004018

- What should be added to it?
- What should be subtracted from it?
- What should be the value of x ?

Solution: First we try to find the square root of the given expression as

$$\begin{array}{r} x^2 + 2x + 3 \\ x^2 \quad \overline{x^2 + 4x^2 + 10x + 5} \\ \quad \pm x^2 \\ \hline \quad \quad 2x^2 + 2x \\ \quad \quad \pm 2x^2 + 4x + 3 \\ \hline \quad \quad \quad 6x^2 + 10x + 5 \\ \quad \quad \quad \pm 6x^2 + 12x + 9 \\ \hline \quad \quad \quad \quad -2x - 4 \end{array}$$

For a perfect square, remainder must be zero, but here remainder is $-2x - 4$.

Hence,

i. Expression will be a perfect square if we add $-(-2x - 4) = 2x + 4$ to it.

ii. Expression will be a perfect square if we subtract $-2x - 4$ from it.

iii. For the value of x put remainder equal to zero as

$$-2x - 4 = 0$$

$$-2x = 4 \quad \text{or} \quad x = -2$$

$\therefore$ For $x = -2$, expression will be a perfect square.

Example 32: For what values of m and n , $9x^2 - 24x^2 - 14x^2 + mx + n$ is a complete square or perfect square? 1004019

Solution: First we try to find the square root of

$$\begin{array}{r} 3x^2 \quad \overline{9x^2 + 24x^2 - 14x^2 + mx + n} \\ \quad \pm 9x^2 \\ \hline \quad \quad 6x^2 - 4x \\ \quad \quad \pm 6x^2 - 4x + 3 \\ \hline \quad \quad \quad 6x^2 - 8x + 3 \\ \quad \quad \quad \pm 6x^2 - 8x + 3 \\ \hline \quad \quad \quad \quad 30x^2 + mx + n \\ \quad \quad \quad \pm 30x^2 + 40x + 25 \\ \hline \quad \quad \quad \quad \quad mx - 40x + n - 25 \\ \quad \quad \quad \quad \quad = x(m - 40) + (n - 25) \end{array}$$



The given expression will be complete square if the remainder is zero. This is only possible if $m - 40 = 0$ and $n - 25 = 0$

or $m = 40$ and $n = 25$

Hence, for $m = 40$ and $n = 25$ given expression will be a complete square.

Exercise 4.1

Find the square root of the following by factorization

Q.1 $16y^2 - 80y + 49$ 1004020

Solution

$$16y^2 - 80y + 49$$

$$(4y)^2 - 2(4y)(7) + (7)^2$$

Using formula

$$\therefore a^2 - 2ab + b^2 = (a - b)^2$$

$$= (4y - 7)^2$$

Taking square root

$$= \pm\sqrt{(4y - 7)^2}$$

$$= \pm(4y - 7)$$

Q.2 $25a^2 - 30a^2 + 9a^2$ 1004021

Solution

$$25a^2 - 30a^2 + 9a^2$$

$$= (5a)^2 - 2(5a)(3a) + (3a)^2$$

Using formula

$$\therefore a^2 - 2ab + b^2 = (a - b)^2$$

$$= (5a - 3a)^2$$

Taking square root

$$= \pm\sqrt{(5a - 3a)^2}$$

$$= \pm(5a - 3a) = \pm a(5a - 3a)$$

Q.3 $\left(x^2 - \frac{1}{x^2}\right)^2 + 4\left(x^2 - \frac{1}{x^2}\right) + 4, x \neq 0$ 1004022

Solution

$$\left(x^2 - \frac{1}{x^2}\right)^2 + 4\left(x^2 - \frac{1}{x^2}\right) + 4, x \neq 0$$

$$= \left(x^2 - \frac{1}{x^2}\right)^2 + 2\left(x^2 - \frac{1}{x^2}\right)(2) + (2)^2$$

$$\therefore a^2 + 2ab + b^2 = (a + b)^2$$

$$= \left(x^2 - \frac{1}{x^2} + 2\right)^2$$

Taking square root

$$= \pm\sqrt{\left(x^2 - \frac{1}{x^2} + 2\right)^2}$$

$$= \pm\left(x^2 - \frac{1}{x^2} + 2\right)$$

Q.4 $\left(x^2 + \frac{1}{x^2}\right) - 8\left(x - \frac{1}{x}\right) + 14, x \neq 0$ 1004023

Solution

$$\left(x^2 + \frac{1}{x^2}\right) - 8\left(x - \frac{1}{x}\right) + 14, x \neq 0 \quad (i)$$

Let

$$x - \frac{1}{x} = y \quad (ii)$$

Taking square both sides

$$\left(x - \frac{1}{x}\right)^2 = (y)^2$$

$$x^2 + \frac{1}{x^2} - 2\left(x\right)\left(\frac{1}{x}\right) = y^2$$

$$x^2 + \frac{1}{x^2} - 2 = y^2$$

$$x^2 + \frac{1}{x^2} = y^2 + 2 \quad (iii)$$

Now, expression (i) can be written as

So equation (i) becomes

$$y^2 + 2 - 8y + 14$$

$$y^2 - 8y + 16$$

$$y^2 - 2(y)(4) + (4)^2$$



$$(y+4)^2$$

Put $y = a - \frac{1}{a}$ from (i)

$$= \left(a - \frac{1}{a} + 4 \right)^2$$

Taking square root both sides

$$= \pm \sqrt{\left(a - \frac{1}{a} + 4 \right)^2}$$

$$= \pm \left(a - \frac{1}{a} + 4 \right)$$

Q.5 $(x+2)(x+4)(x+6)(x+8)+16$ Answer

Solution

$$(x+2)(x+4)(x+6)(x+8)+16$$

$$\because 2+8=4+6$$

$$= (x+2)(x+8)(x+4)(x+6)+16$$

$$= (x^2+8x+2x+16)(x^2+6x+4x+24)+16$$

$$= (x^2+10x+16)(x^2+10x+24)+16 \dots (i)$$

Let $x^2+10x=y \dots (ii)$

Expression (i) can be written as:

$$= (y+16)(y+24)+16$$

$$= y^2+24y+16y+384+16$$

$$= y^2+40y+400$$

$$= (y)^2+2(y)(20)+(20)^2$$

$$= (y+20)^2$$

Taking square root

$$= \pm \sqrt{(y+20)^2}$$

$$= \pm (y+20)$$

Put $y = x^2+10x$

$$= \pm (x^2+10x+20)$$

Find the square root of the following by division method

Q.6 $x^4+8x^3+20x^2+16x+4$ Answer

Solution

$$x^4+8x^3+20x^2+16x+4$$

Taking square root

$$\begin{array}{r} x^2+4x+2 \\ x^4+8x^3+20x^2+16x+4 \\ \underline{+x^2} \\ 8x^3+20x^2 \\ \underline{+8x^2+16x} \\ 16x^2+36x+4 \\ \underline{+16x^2+32x+8} \\ 4x+4 \\ \underline{+4x+4} \\ 0 \end{array}$$

$$\sqrt{x^4+8x^3+20x^2+16x+4} = \pm (x^2+4x+2)$$

Q.7 $x^4+10x^3+31x^2+30x+9$ Answer

Solution

$$x^4+10x^3+31x^2+30x+9$$

Taking square root

$$\begin{array}{r} x^2+5x+3 \\ x^4+10x^3+31x^2+30x+9 \\ \underline{+x^2} \\ 10x^3+31x^2 \\ \underline{+10x^2+25x} \\ 6x^2+30x+9 \\ \underline{+6x^2+30x+9} \\ 0 \end{array}$$

Thus,

$$\sqrt{x^4+10x^3+31x^2+30x+9} = \pm (x^2+5x+3)$$

Q.8 $49b^4+18ab^3+4a^2b^2+a^3+28ab^3$ Answer

Solution

$$49b^4+18ab^3+4a^2b^2+a^3+28ab^3$$

Taking square root

$$\begin{array}{r} 7b^2+2ab+a^2 \\ 49b^4+28ab^3+18a^2b^2+4a^3b^2+a^4 \\ \underline{+49b^4} \\ 28ab^3+18a^2b^2 \\ \underline{+28ab^3+4a^2b^2} \\ 14a^2b^2+4a^3b^2+a^4 \\ \underline{+14a^2b^2+4a^3b^2+a^4} \\ 0 \end{array}$$

Thus,



$$\sqrt{49b^4+18ab^3+4a^2b^2+a^3+28ab^3}$$

$$= \pm (7b^2+2ab+a^2)$$

Q.9 $4x^4-12x^3+29x^2-30x+25$ Answer

Solution

$$4x^4-12x^3+29x^2-30x+25$$

Taking square root

$$\begin{array}{r} 2x^2-3x+5 \\ 4x^4-12x^3+29x^2-30x+25 \\ \underline{+4x^4} \\ -12x^3+29x^2 \\ \underline{+12x^3-30x} \\ 19x^2-30x+25 \\ \underline{+19x^2-57x+25} \\ 37x-50 \\ \underline{+37x-50} \\ 0 \end{array}$$

Thus,

$$\sqrt{4x^4-12x^3+29x^2-30x+25} = \pm (2x^2-3x+5)$$

Q.10 $1-10x+27x^2-10x^3+x^4$ Answer

Solution

$$1-10x+27x^2-10x^3+x^4$$

In descending order

$$x^4-10x^3+27x^2-10x+1$$

$$\begin{array}{r} x^2-5x+1 \\ x^4-10x^3+27x^2-10x+1 \\ \underline{+x^2} \\ -10x^3+27x^2 \\ \underline{+10x^2-25x} \\ 17x^2-35x+1 \\ \underline{+17x^2-85x+17} \\ -68x+16 \\ \underline{+68x-68} \\ 16-67 \\ \underline{+16-67} \\ 0 \end{array}$$

Thus,

$$\sqrt{1-10x+27x^2-10x^3+x^4} = \pm (x^2-5x+1)$$

Q.11 $x^4+\frac{1}{x^2}+4x^2-\frac{4}{x^2}+2, x \neq 0$ Answer

Solution

$$x^4+\frac{1}{x^2}+4x^2-\frac{4}{x^2}+2, x \neq 0$$

Taking square root

$$\begin{array}{r} x^2+2-\frac{1}{x^2} \\ x^4+\frac{1}{x^2}+4x^2-\frac{4}{x^2}+2 \\ \underline{+x^4} \\ \frac{1}{x^2}+4x^2-\frac{4}{x^2}+2 \\ \underline{+\frac{1}{x^2}+4x^2-\frac{4}{x^2}+2} \\ 0 \end{array}$$

Thus,

$$\sqrt{x^4+\frac{1}{x^2}+4x^2-\frac{4}{x^2}+2} = \pm \left(x^2+2-\frac{1}{x^2} \right)$$

Q.12 $x^4-8x+2+\frac{56}{x}+\frac{49}{x^2}, x \neq 0$ Answer

Solution

$$x^4-8x+2+\frac{56}{x}+\frac{49}{x^2}, x \neq 0$$

Taking square root

$$\begin{array}{r} x^2-4-\frac{7}{x} \\ x^4-8x+2+\frac{56}{x}+\frac{49}{x^2} \\ \underline{+x^4} \\ -8x+2+\frac{56}{x}+\frac{49}{x^2} \\ \underline{+8x+2+\frac{56}{x}+\frac{49}{x^2}} \\ 0 \end{array}$$

Thus,

$$\sqrt{x^4-8x+2+\frac{56}{x}+\frac{49}{x^2}} = \pm \left(x^2-4-\frac{7}{x} \right)$$



Q.13 $x^2 - 2x^2 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}$ answer

Solution

$$x^2 - 2x^2 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}$$

Taking square root

$$\begin{array}{r} x^2 - 2x^2 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16} \\ \hline 2x^2 - 4x \\ \hline 2x^2 - 4x + \frac{1}{4} \\ \hline \frac{1}{4} - \frac{x}{2} + \frac{1}{16} \\ \hline \pm \frac{1}{2} \pm \frac{x}{2} \pm \frac{1}{4} \\ \hline 0 \end{array}$$

Thus,

$$\sqrt{x^2 - 2x^2 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}} = \pm \left(x^2 - x + \frac{1}{4} \right)$$

Q.14 $4x^2 + 32x^2 + 96 + \frac{64}{x^2} + \frac{128}{x^2}$ answer

Solution

$$4x^2 + 32x^2 + 96 + \frac{64}{x^2} + \frac{128}{x^2}$$

Taking square root

$$\begin{array}{r} 4x^2 + 32x^2 + 96 + \frac{64}{x^2} + \frac{128}{x^2} \\ \hline 2x^2 \\ \hline 2x^2 + 32x^2 + 96 + \frac{64}{x^2} + \frac{128}{x^2} \\ \hline 32x^2 + 96 \\ \hline \pm 32x^2 \pm 64 \\ \hline 32x^2 + 96 + \frac{64}{x^2} + \frac{128}{x^2} \\ \hline \pm 32x^2 \pm \frac{128}{x^2} \pm \frac{64}{x^2} \\ \hline 0 \end{array}$$

$$\sqrt{4x^2 + 32x^2 + 96 + \frac{64}{x^2} + \frac{128}{x^2}} = \pm \left(2x^2 + 8 + \frac{8}{x^2} \right)$$

Q.15 To make $x^2 - 10x^2 + 27x^2 - 9x + 2$ a perfect square:

i) What should be added to it? answer

ii) What should be subtracted from it? answer

iii) What will be the value of x? answer

Solution

Given expression = $x^2 - 10x^2 + 27x^2 - 9x + 2$

Taking square root

$$\begin{array}{r} x^2 - 10x^2 + 27x^2 - 9x + 2 \\ \hline 2x^2 \\ \hline 2x^2 - 10x^2 + 27x^2 - 9x + 2 \\ \hline -10x^2 + 27x^2 \\ \hline \pm 10x^2 \pm 27x^2 \\ \hline 2x^2 - 10x^2 + 27x^2 - 9x + 2 \\ \hline \pm 10x^2 \pm 27x^2 \pm 1 \\ \hline 0 \end{array}$$

i) Solution

We should be added $(- 9 - 1)$ to expression to make perfect square.

ii) Solution

We should be subtracted $9 + 1$ from expression to make perfect square.

iii) Solution

If the given expression is perfect square

Then $R = 0$ i.e.

$$a + 1 = 0$$

$$a = -1$$

Q.16 Find the values of p and q if $x^2 - 10x^2 + 27x^2 - 9x + 2$ a perfect square. answer



Solution

$$\begin{array}{r} x^2 - 10x^2 + 27x^2 - 9x + 2 \\ \hline 2x^2 \\ \hline 2x^2 - 10x^2 + 27x^2 - 9x + 2 \\ \hline -10x^2 + 27x^2 \\ \hline \pm 10x^2 \pm 27x^2 \\ \hline 2x^2 - 10x^2 + 27x^2 - 9x + 2 \\ \hline \pm 10x^2 \pm 27x^2 \pm 1 \\ \hline 0 \end{array}$$

If the given expression is perfect square then $R = 0$

$$(p - 216)x + (q - 324) = 0$$

$$(p - 216)x + (q - 324) = 0 \text{ (as } x \neq 0)$$

$$(p - 216) = 0 \text{ and } q - 324 = 0$$

$$p = 216, \quad q = 324$$

Q.17 For what value of k, the expression

$$y^2 + 4y^2 + k + \frac{8}{y} + \frac{4}{y^2}$$

square, where $y \neq 0$. answer

Solution

Given expression

$$y^2 + 4y^2 + k + \frac{8}{y} + \frac{4}{y^2}$$

Taking square root

$$\begin{array}{r} y^2 + 4y^2 + k + \frac{8}{y} + \frac{4}{y^2} \\ \hline 2y^2 \\ \hline 2y^2 + 4y^2 + k + \frac{8}{y} + \frac{4}{y^2} \\ \hline 4y^2 + k + \frac{8}{y} + \frac{4}{y^2} \\ \hline \pm 4y^2 \pm 4 \\ \hline (k - 4) + \frac{8}{y} + \frac{4}{y^2} \\ \hline \pm 4 \pm \frac{8}{y} \pm \frac{4}{y^2} \\ \hline 0 \end{array}$$

To make given expression a perfect square, the remainder should be zero i.e. $R = 0$

$$k - 8 = 0$$

$$k = 8$$

Application of factorization in daily life

Example 33

Mustafa is working on a space project to make a cuboid with the volume as the difference of 2 cubes of sides x and y respectively. Help him to find: answer

- Expression for volume of the required cuboid.
- Expression for any one side of required cuboid.
- Expression for Area of any one surface of required cuboid.

Solution

Mustafa is working with cubes of volumes:

$$\text{Volume } 2 = x^3 - y^3$$

$$\text{Difference of 2 volumes} = x^3 - y^3$$

$$a) \text{ Volume of required cuboid} = x^3 - y^3$$

$$b) x^3 - y^3 = (x - y)(x^2 + xy + y^2) \text{ (by factorizing)}$$

The volume of a cuboid is factorized in one linear and one quadratic factor. So the expression for length is " $x - y$ "

c) The quadratic factor in above factorization represents Area of a surface. i.e. Area of one surface is $x^2 + xy + y^2$.

Example 34

Volume of a cubical container is given by

$$\text{expression } (x^3 - 6x^2y + (2xy^2 - 4y^3)) \text{ m}^3. \text{ Find:}$$

answer

- Expression for length of each side
- Expression for area of the base
- Expression for total surface area
- Expense for cost of painting all outer surfaces at the rate of Rs.50/m².



Solution

Volume of cube = l^3

$$= x^3 - 6x^2y + 12xy^2 - 8y^3$$

$$= (x)^3 - 3(x)^2(2y) - 3(x)(2y)^2 - (2y)^3$$

$$= (x-2y)^3$$

a) Length of each side is $(x-2y)$ m

b) Base area = $l^2 = (x-2y)^2 \text{ m}^2$

c) Surface area = $6l^2 = 6(x-2y)^2 \text{ m}^2$

d) Cost = rate $\times$ surface area

$$= \text{Rs. } 50 \times 6 \times (x-2y)^2$$

$$= \text{Rs. } 300(x-2y)^2$$

Exercise 4.5

Q.1 In a map of an industry, expression of area of a rectangular veranda is given by $(x^2 - 2x - 3) \text{ m}^2$. Find:

(a) expression for both dimension of veranda. answ

(b) expression for perimeter of veranda. answ

(c) expression for cost of fencing veranda @ Rs.100/m. answ

d. expression for the cost of carpeting veranda floor @ Rs.250/m². answ

(a) Solution

Area of rectangular veranda

$$= (x^2 - 2x - 3) \text{ m}^2$$

Length $\times$ Width = Area of rectangle

$$\text{Length} \times \text{Width} = (x^2 - 2x - 3) \text{ m}^2$$

$$\text{Length} \times \text{Width} = (x^2 - 3x + x - 3) \text{ m}^2$$

$$L \times W = [x(x-3)(x+1)] \text{ m}^2$$

$$L \times W = (x-3)(x+1) \text{ m}^2$$

These two terms $(x-3)$ m and $(x+1)$ m represent the width and length of the rectangle.

(b) Solution

Perimeter of Veranda = 2 (l + w)

$$= 2 [(x-3) + (x+1)] \text{ m}$$

$$= 2(x-3+x+1) \text{ m}$$

$$= 3(2x-2) \text{ m}$$

$$= (4x-4) \text{ m}$$

(c) Solution

$$= \text{Rs. } 200(4x-4)$$

$$= \text{Rs. } (800x-800)$$

(d) Solution

$$\text{Cost of } 1 \text{ m}^2 = \text{Rs. } 250$$

$$\text{The cost of } (x^2 - 2x - 3) \text{ m}^2$$

$$= \text{Rs. } 250(x^2 - 2x - 3)$$

Q.2 Area of a square shaped surface of a machine is given by the expression, $(15x^2 - 30x + 9) \text{ m}^2$. Find:

(a) expression for the length of the surface answ

(b) expression for the boundary of the surface. answ

(c) expression for the cost of polishing the surface of machine @ Rs.75/m². answ

(d) expression for the cost of edging around 2 sides of the machine surface @ Rs.28/m. answ

Solution

(a) Area of square shaped

$$\text{Surface} = (15x^2 - 30x + 9) \text{ m}^2$$

$$\text{Length} \times \text{Length} = \text{Area of square}$$

$$(\text{length})^2 = \text{Area of square}$$

$$\text{Length} = \sqrt{\text{Area of square}}$$

$$\text{Length} = \sqrt{15x^2 - 30x + 9} \text{ m}^2$$

$$= \sqrt{3(5x^2 - 10x + 3)}$$

$$= \sqrt{3(5x-3)(x-1)}$$

$$\text{Length} = \sqrt{3(5x-3)} \text{ m}^2$$

$$\text{Length} = (5x-3) \text{ m}$$

(b) Solution

$$\text{Expression for boundary} = 4 \times L$$



Perimeter of square = 4 $\times$ length

$$= 4(5x-3) \text{ m} = (20x-12) \text{ m}$$

(c) Solution

$$\text{Cost of polishing surface } 1 \text{ m}^2 = \text{Rs. } 75$$

$$\text{Expression of cost of } (15x^2 - 30x + 9) \text{ m}^2$$

$$= \text{Rs. } 75(15x^2 - 30x + 9)$$

(d) Solution

$$\text{Cost of edging perimeter} = \text{Rs. } 28/\text{m}$$

$$\text{Cost of edging around 2 sides of the machine}$$

$$\text{Expression for 1 side} = (5x-3) \text{ m}$$

$$\text{Expression for 2 sides} = 2(5x-3) \text{ m}$$

$$= (10x-6)$$

$$\text{Expression for the cost of edging around 2 sides a machine surface} = 28 \times 2(5x-3) \text{ m}$$

$$= 56(5x-3) \text{ m}$$

Q.3 Volume of a cubical oil tank in an oil refinery is expressed as $(125x^3 - 150x^2 + 60x - 8) \text{ m}^3$. Find:

(a) expression for height of oil tank. answ

(b) expression for surface area of oil tank. answ

(c) expression for painting it from outside @ Rs. 12/m². answ

(a) Solution

Length of each side of cube

$$l = \sqrt[3]{\text{Volume of cube}}$$

$$l = \sqrt[3]{(125x^3 - 150x^2 + 60x - 8) \text{ m}^3}$$

$$l = \sqrt[3]{\{(5x)^3 - 3(5x)^2(2) + 3(5x)(2)^2 - (2)^3\}}$$

$$l = \sqrt[3]{(5x-2)^3 \text{ m}^3}$$

$$l = (5x-2) \text{ m}$$

$$\text{Length of each side of cube} = (5x-2) \text{ m}$$

(b) Solution

Expression for surface area of 1 face

$$= (5x-2) \text{ m} \times (5x-2) \text{ m}$$

$$= (5x-2)^2 \text{ m}^2$$

As there are six surface,

$$\text{so cost of painting } 1 \text{ m}^2 = \text{Rs. } 12 \text{ m}^2$$

Expression for surface area of cube

$$= 6(5x-2)^2 \text{ m}^2$$

Expression for painting it from outside

$$= \text{Rs. } 12 \times 6(5x-2)^2$$

$$= \text{Rs. } 192(5x-2)^2$$

(c) Solution

Expression for painting it from outside

$$= \text{Rs. } 192(5x-2)^2$$

Q.4 A mechanical engineer working on wheels of a machine, finds their areas given by $A_1 = \pi x^2 - 6\pi x + 9\pi$

$A_2 = \pi x^2 - 10\pi x + 25\pi$. Help him find radii of both wheels. answ

Solution

Let r_1 be the radius of wheel having area of A_1

$$A_1 = \pi x^2 - 6\pi x + 9\pi$$

$$\pi r_1^2 = \pi(x^2 - 6x + 9)$$

$$r_1^2 = x^2 - 6x + 9$$

$$r_1^2 = (x)^2 - 2(x)(3) + (3)^2$$

$$r_1^2 = (x-3)^2$$

Taking square root

$$\sqrt{r_1^2} = \sqrt{(x-3)^2}$$

$$r_1 = x-3$$

Let r_2 be the radius of wheel having area of A_2

$$A_2 = \pi x^2 - 10\pi x + 25\pi$$

$$\pi r_2^2 = \pi(x^2 - 10x + 25)$$

$$r_2^2 = x^2 - 10x + 25$$

$$r_2^2 = (x)^2 - 2(5)(x) + (5)^2$$

$$r_2^2 = (x-5)^2$$

Taking square root

$$\sqrt{r_2^2} = \sqrt{(x-5)^2}$$

$$r_2 = (x-5)$$

$(x-3)$ and $(x-5)$ are radii of A_1 and A_2

Q.5 A machine has two squared shape prisms with areas expressed by 25 m^2 and 16 m^2 respectively. Difference of these areas



describes a rectangular prism. Find dimension of the rectangular prism.

Solution answer

$25m^2$ and $36m^2$ are the areas of square shaped prism.

Difference of area = Area of rectangular prism

$$25m^2 - 36m^2 = \text{Length} \times \text{width}$$

$$(5m)^2 - (6m)^2 = l \times w$$

$$(5m - 6m)(5m + 6m) = l \times w$$

$5m - 6m$ and $5m + 6m$ are dimension of the rectangular prism.

Q.4 Distance covered by a missile to hit the target is given by expression $(x^2 - 5x + 6)m$.

Find

(a) the possible expression for speed of missile answer

(b) the possible expression for time to reach the target. answer

(a) Solution

Expression for distance covered =

$$(x^2 - 5x + 6)m \dots (i)$$

We know that

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$\therefore \text{Speed} \times \text{time} = \text{distance} \dots (ii)$$

From (i) and (ii)

$$\text{Speed} \times \text{time} = x^2 - 5x + 6$$

$$\text{Speed} \times \text{time} = x^2 - 2x - 3x + 6$$

$$\text{Speed} \times \text{time} = x(x - 2) - 3(x - 2)$$

$$\text{Speed} \times \text{time} = (x - 2)(x - 3)$$

By comparing we observe that

If speed $(x - 2)$, then time is $(x - 3)$

If speed is $(x - 3)$ then time is $(x - 2)$

Miscellaneous Exercise 4

Encircle the correct option in the following.

1. Factors of $-2 - a + a^2$ are: answer

(a) $(a - 2)(a - 1)$ (b) $(a - 1)(a - 2)$

(c) $(a + 2)(a - 1)$ (d) $(a - 1)(a + 2)$

2. Factorisation of $x^2 - x + \frac{1}{4}$ is: answer

(a) $\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$

(b) $\left(x - \frac{1}{2}\right)\left(x - 1\right)$

(c) $\left(x - \frac{1}{2}\right)\left(x + 1\right)$

(d) $\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$

3. $(x^2 + m^2)(x + m)(x^2 + m^2)(x - m)$ is the factored form of: answer

(a) $x^4 - m^4$ (b) $x^4 + m^4$

(c) $x^4 - m^4$ (d) $x^4 + m^4$

4. $x^2 + 64b^2$ is the product of: answer

(a) $x^2 - 4ab + 8b^2$ and $x^2 + 4ab + 8b^2$

(b) $x^2 + 8b^2 + 4ab$ and $x^2 - 8b^2 + 4ab$

(c) $(x^2 + 8b^2)^2$ (d) none of them

5. $x(2a + 3b + 7) + x(2a + 3b + 5)$ equals

(a) $2x(2a + 3b + 6)$ answer

(b) $4x(2a + 3b + 6)$

(c) $x(2a + 3b + 7)(2a + 3b + 5)$

(d) $(x + x)(2a + 3b + 12)$

6. Which is the highest common factor of $-12x^2y^2, 6xy^2, 24x^2y^2$? answer

(a) $-6xy$ (b) $6x^2y^2$

(c) $6xy^2$ (d) $-24x^2y^2$

7. What is the least common multiple of $12x^2y^2, 6xy^2, 24x^2y^2$? answer

(a) $6xy$ (b) $6x^2y^2$

(c) $6xy^2$ (d) $24x^2y^2$

8. What is the highest common factor of $7x - 6xy$ and $5xy^2 - 1x^2$? answer



(a) $(7 - 6y)(5y^2 - 3x)$

(b) $(7x - 6xy)(5y^2 - 3x^2)$

(c) x

(d) $x(7 - 6y)(5y^2 - 3x)$

9. Least common multiple of $7x - 6xy$ and $5y^2x - 3x^2$ is: answer

(a) $(7 - 6y)(5y^2 - 3x^2)$

(b) $(7x - 6xy)(5y^2 - 3x^2)$

(c) x

(d) $x(7 - 6y)(5y^2 - 3x)$

10. HCF of $7x^2 - 8y^2$ and $3x^2 - 5y^2$ is:

(a) 1 answer

(b) $(7x^2 - 8y^2)(3x^2 - 5y^2)$

(c) $7x^2 - 8y^2$

(d) $3x^2 - 5y^2$

11. LCM of $7x^2 - 8y^2$ and $3x^2 - 5y^2$ is:

(a) 1 answer

(b) $(7x^2 - 8y^2)(3x^2 - 5y^2)$

(c) $7x^2 - 8y^2$

(d) $3x^2 - 5y^2$

12. What is the product of $\frac{uv^2}{2w}$ and $\frac{8w^2}{u^2v}$? answer

(a) $\frac{uv^2}{3w} \times \frac{u^2v^2}{8w^2}$

(b) $\frac{2w}{uv}$

(c) $\frac{uv}{2w}$

(d) both a & b

13. What is the quotient of $\frac{3y^2}{10} \div \frac{y^2}{2}$? answer

(a) $\frac{3y^2}{20}$

(b) $\frac{3}{2y}$

(c) $\frac{3y^2}{10} \times \frac{2}{y^2}$

(d) $\frac{3y}{2}$

14. What is the sum of $\frac{2x}{x^2 - 1}$ and $\frac{-x}{x^2 - 1}$? answer

(a) $\frac{3x}{x^2 - 1}$

(b) $\frac{x}{x^2 - 1}$

(c) $\frac{2x - x}{(x^2 - 1)(x^2 - 1)}$ (d) $\frac{-2x}{x^2 - 1}$

15. What is the difference of $\frac{-3x}{x + y}$ and $\frac{x}{x + y}$? answer

(a) $\frac{-2x}{x + y}$

(b) $\frac{-3x^2 - x}{2x + 2y}$

(c) $\frac{3x^2}{x + y}$

(d) $\frac{-4x}{x + y}$

16. If product of two polynomials is $(a - b)^2(a^2 + ab + b^2)$ and their HCF is $a - b$, what is their LCM? answer

(a) $a^2 - b^2$

(b) $(a - b)^3(a^2 + ab + b^2)$

(c) $(a - b)^2(a^2 + ab + b^2)$

(d) $a^2 + ab + b^2$

17. If product of HCF and LCM of two polynomials is $(x^2 - y^2)(x + y)$ then what will be the product of these polynomials? answer

(a) $x^2 - y^2$

(b) $(x - y)(x^2 + y^2)$

(c) $(x^2 - y^2)(x^2 + xy + y^2)$

(d) $(x^2 - y^2)(x^2 - xy + y^2)$

18. What is the square root of $36x^2y^2$? answer

(a) $6x^2y^2$ (b) $6xy$

(c) $6x^2y^2$ (d) $16x^2y^2$

19. What is the square root of: answer

$(15x^2 - 7y^2)^2$

(a) $\pm(15x^2 - 7y^2)^2$

(b) $\pm(15x^2 - 7y^2)$

(c) $\pm(15x - 7y)^2$

(d) $\pm(15x - 7y)$

20. What is the square root of

$\left[-\left(2x + \frac{1}{x} + 1\right)\right]^2$ answer

(a) $\pm\left(2x + \frac{1}{x} + 1\right)$ (b) $\left(2x + \frac{1}{x} + 1\right)$

(c) $2x + \frac{1}{x} + 1$ (d) $\sqrt{2x + \frac{1}{x} + 1}$



Additional MCQs

21. A rectangle has an area of $x^2+3x-24$ square units. Which of the following are possible expressions for the length and the width of the rectangle?

- (a) Length: $(x-24)$ units; width: $(x+1)$ units
(b) Length: $(x-4)$ units; width: $(x+6)$ units
(c) Length: $(x-3)$ units; width: $(x+8)$ units
(d) Length: $(x-12)$ units; width: $(x+2)$ units

22. Which property of real numbers is used to transform the equation in step 1 into equation in step2?

Step 1: $4(x-5) + 8 = 88$

Step 2: $4x-20 + 8 = 88$

- (a) Distributive property
(b) Associative property of multiplication
(c) Multiplication property
(d) None of these

23. If $\frac{2x}{b} - 9 = 3$, what is value of the expression $8x-7$?

- (a) -75 (b) 61
(c) -35 (d) 141

24. Which of the following expressions is equivalent to $x^2-8x+16$?

- (a) $(x+4)^2$ (b) $(x+8)(x+2)$
(c) $(x-4)(x-4)$ (d) $(x-4)^2$

25. Which of the following shows the complete factorization of $2x^2+4x^2-6x$?

- (a) $(2x^2-2x)$ (b) $2x(x^2+2x-3)$
(c) $2x(x-1)(x+3)$ (d) $2(x^2+2x^2-3x)$

26. Which value of b would make x^2+bx-2 factorable?

- (a) -2 (b) 0
(c) -1 (d) 3

27. Factors of $8x^2-12x$ are:

- (a) $2(4x^2-6x)$ (b) $2x(4x-6)$
(c) $(2x^2-6x)$ (d) $4x(2x-3)$

28. What is the C.F of $9p^2q^2 + 12p^3q^3$?

- (a) 3 (b) $3pq$
(c) $3p^2q^2$ (d) $3p^3q^3$

29. If L.C.M of 'a' and 'b' is 'c' and L.C.M of 'b' and 'c' is 'b', then what is the L.C.M of 'c' and 'a'?

- (a) a (b) b
(c) c (d) abc

30. The H.C.F and L.C.M of two polynomials are $x+1$ and x^2-1 , respectively. If one of the polynomials is x^2+1 , then the other one is:

- (a) x^2-1 (b) $(x+1)(x^2-1)$
(c) x^2+x-1 (d) x^2-x+1

31. The H.C.F of $4p^2q^3r$, $8p^3q^3r^2$, $16p^3q^3r^2$ is:

- (a) $4p^2q^3r$ (b) $4p^3q^3r$
(c) $16p^3q^3r$ (d) $16p^3q^3r^2$

32. The product of two algebraic expressions is equal to the _____ of their H.C.F and L.C.M.

- (a) sum (b) product
(c) difference (d) quotient

Answer Key

1	d	2	d	3	c	4	a	5	a
6	c	7	d	8	c	9	d	10	a
11	b	12	b	13	b	14	b	15	d
16	a	17	c	18	c	19	a	20	c
21	c	22	a	23	d	24	d	25	c
26	c	27	b	28	c	29	d	30	b
31	b	32	b						



Factorize the following

Q.2 $(x^2-5)^2-13(x^2-5)+36$ answer

Solution

Let $x^2-5 = y$ ——— (i)

$$= y^2 - 13y + 36$$

$$= y^2 - 9y - 4y + 36$$

$$= y(y-9) - 4(y-9)$$

$$= (y-9)(y-4)$$

Put $y = x^2 - 5$ From (i)

$$= (x^2-5-9)(x^2-5-4)$$

$$= (x^2-14)(x^2-9)$$

Q.3 $6x^2+19x+15$ answer

Solution

$$6x^2+19x+15$$

$$= 6x^2+18x+9x+15$$

$$= 2x(3x+5) + 3(3x+5)$$

$$= (3x+5)(2x+3)$$

Q.4 $(x+1)(x-3)(x-5)(x-9)+44$ answer

Solution

$$(x+1)(x-3)(x-5)(x-9)+44$$

$$= 1 + (-9) = -8 + (-5)$$

$$= -8 - 5$$

Arrange the terms

$$= (x+1)(x-9)(x-3)(x-5)+44$$

$$= (x^2-9x+x-9)(x^2-5x-3x+15)+44$$

$$= (x^2-8x-9)(x^2-8x+15)+44$$

Let $x^2-8x = y$ ——— (i)

$$= (y-9)(y+15)+44$$

$$= y^2+15y-9y-135+44$$

$$= y^2+6y-91$$

$$= y^2+13y-7y-91$$

$$= y(y+13)-7(y+13)$$

$$= (y+13)(y-7)$$

Put $y = x^2 - 8x$ From (i)

$$= (x^2-8x+13)(x^2-8x-7)$$

Q.5 x^4-12x^2+16 answer

Solution

$$x^4-12x^2+16$$

$$= (x^2)^2 + (4)^2 - 12x^2$$

$$= (x^2)^2 + (4)^2 - 2(x^2)(4) + 2(x^2)(4) - 12x^2$$

$$= (x^2-4)^2 + 8x^2 - 12x^2$$

$$= (x^2-4)^2 - 4x^2$$

$$= (x^2-4)^2 - (2x)^2$$

$$= (x^2-4-2x)(x^2-4+2x)$$

$$= (x^2-2x-4)(x^2+2x-4)$$

Q.6 $(3x^2+4x-5)(3x^2-2+4x)-4$ answer

Solution

$$(3x^2+4x-5)(3x^2-2+4x)-4$$

$$(3x^2+4x-5)(3x^2+4x-2)-4$$

Let $3x^2+4x = y$ ——— (i)

$$= (y-5)(y-2)-4$$

$$= y^2-2y-5y+10-4$$

$$= y^2-7y+6$$

$$= y^2-6y-y+6$$

$$= y(y-6)-1(y-6)$$

$$= (y-6)(y-1)$$

Put $y = 3x^2 + 4x$ from (i)

$$= (3x^2+4x-6)(3x^2+4x-1)$$

Q.7 $2m^4+m^2n^2-3n^4$ answer

Solution

$$2m^4+m^2n^2-3n^4$$

$$= 2m^4-2m^2n^2+3m^2n^2-3n^4$$

$$= 2m^2(m^2-n^2)+3n^2(m^2-n^2)$$

$$= (m^2-n^2)(2m^2+3n^2)$$

$$= (m+n)(m-n)(2m^2+3n^2)$$

Q.8 $x^4+10x^2y^2-56y^4$ answer

Solution

$$x^4+10x^2y^2-56y^4$$

$$= x^4+14x^2y^2-4x^2y^2-56y^4$$

$$= x^2(x^2+14y^2)-4y^2(x^2+14y^2)$$

$$= (x^2+14y^2)(x^2-4y^2)$$

$$= (x^2+14y^2)(x^2-(2y)^2)$$

$$= (x^2+14y^2)(x-2y)(x+2y)$$



Q.9 $x^2 + 2ax - bx - 2ab$ (10001)

Solution

$$\begin{aligned} x^2 + 2ax - bx - 2ab \\ = x^2 - bx + 2ax - 2ab \\ = x(x-b) + 2a(x-b) \\ = (x-b)(x+2a) \end{aligned}$$

Q.10 $a^{12} - b^{12}$ (10002)

Solution

$$\begin{aligned} a^{12} - b^{12} \\ = (a^6)^2 - (b^6)^2 \\ = (a^6 - b^6)(a^6 + b^6) \\ = [(a^3)^2 - (b^3)^2](a^6 + b^6) \\ = (a^3 - b^3)(a^3 + b^3)(a^6 + b^6) \\ = (a-b)(a^2 + ab + b^2)(a+b)(a^2 - ab + b^2) \\ = (a^2 + b^2)(a^2)^2 - a^2b^2 + (b^2)^2 \end{aligned}$$

Arrange

$$(a+b)(a-b)(a^2 + b^2)(a^2 + ab + b^2)(a^2 - ab + b^2)(a^6 - a^2b^2 + b^6)$$

Q.11 $28x^4y + 64x^3y - 60x^2y$ (10003)

Solution

$$\begin{aligned} 28x^4y + 64x^3y - 60x^2y \\ = 4x^2y(7x^2 + 16x - 15) \\ = 4x^2y(7x^2 + 21x - 5x - 15) \\ = 4x^2y(7x(x+3) - 5(x+3)) \\ = 4x^2y(x+3)(7x-5) \end{aligned}$$

Q.12 $4(x-y)^2 - (x-y)$ (10004)

Solution

$$\begin{aligned} 4(x-y)^2 - (x-y) \\ = (x-y)[4(x-y) - 1] \\ = (x-y)[(2(x-y))^2 - (1)^2] \\ = (x-y)[2(x-y) - 1][2(x-y) + 1] \\ = (x-y)(2x-2y-1)(2x-2y+1) \end{aligned}$$

Q.13 $x^3p^3 - 8y^3p^3 - 4x^3q^3 + 32y^3q^3$ (10005)

Solution

$$\begin{aligned} x^3p^3 - 8y^3p^3 - 4x^3q^3 + 32y^3q^3 \\ = p^3(x^3 - 8y^3) - 4q^3(x^3 - 8y^3) \\ = (p^3 - 4q^3)(x^3 - 8y^3) \end{aligned}$$

$$\begin{aligned} &= [(p^3 - (2q)^3)](x^3 - (2y)^3) \\ &= (p+2q)(p-2q)(x-2y)(x^2 + (x)(2y) + (2y)^2) \\ &= (p+2q)(p-2q)(x-2y)(x^2 + 2xy + 4y^2) \end{aligned}$$

Q.14 $x^4 + y^4 - 7x^2y^2$ (10006)

Solution

$$\begin{aligned} x^4 + y^4 - 7x^2y^2 \\ = (x^2)^2 + (y^2)^2 + 2(x^2y^2) - 2(x^2)(y^2) - 7x^2y^2 \\ = (x^2 + y^2)^2 - 2x^2y^2 - 7x^2y^2 \\ = (x^2 + y^2)^2 - 9x^2y^2 \\ = (x^2 + y^2)^2 - (3xy)^2 \\ = (x^2 + y^2 - 3xy)(x^2 + y^2 + 3xy) \end{aligned}$$

Q.15 Find the highest common factor of the following.

(i) $x^2 + 3x^2 - 8x - 24, x^3 + 3x^2 - 3x - 9$ (10007)

(ii) $3x^4 - 3x^3 - 2x^2 - x - 1, 9x^4 - 3x^3 - x - 1$ (10008)

(i) **Solution**

(i) $x^2 + 3x^2 - 8x - 24, x^3 + 3x^2 - 3x - 9$

$$\begin{array}{r} x^3 + 3x^2 - 3x - 9 \\ \underline{-(x^2 + 3x^2 - 8x - 24)} \\ -5x - 15 \end{array}$$

Taking common $-5(x+3)$

$$\begin{array}{r} x^3 + 3x^2 - 3x - 9 \\ \underline{-(x^2 + 3x^2 - 8x - 24)} \\ -5x - 15 \\ \underline{+5x + 15} \\ 0 \end{array}$$

H.C.F. = $x+3$

(ii) **Solution**

H.C.F. by division method.



$$\begin{array}{r} 3x^4 - 3x^3 - 2x^2 - x - 1 \\ \underline{-(9x^4 - 3x^3 - 2x^2 - x - 1)} \\ -6x^4 + 6x^3 + 2x + 2 \\ \underline{+6x^4 + 6x^3 + 2x + 2} \\ 0 \end{array}$$

Now,

$$\begin{array}{r} 3x^4 + 3x^3 + x + 1 \\ \underline{-(3x^4 + 3x^3 + x + 1)} \\ 0 \end{array}$$

H.C.F. = $3x^2 + 1$

Q.16 Find the LCM of the following.

(i) $2x^2 + 3x + 1, 2x^2 + 5x + 2, x^2 + 3x + 2$ (10009)

(ii) $3x^2 + 11x + 6, 3x^2 + 8x + 4, x^2 + 5x + 6$ (10010)

(i) **Solution**

$$\begin{aligned} 2x^2 + 3x + 1 \\ = 2x^2 + 2x + x + 1 \\ = 2x(x+1) + 1(x+1) \\ = (x+1)(2x+1) \dots (i) \end{aligned}$$

Now,

$$\begin{aligned} 2x^2 + 5x + 2 \\ = 2x^2 + 4x + x + 2 \\ = 2x(x+2) + 1(x+2) \\ = (x+2)(2x+1) \dots (ii) \end{aligned}$$

Also,

$$\begin{aligned} x^2 + 3x + 2 \\ = x^2 + 2x + x + 2 \\ = x(x+2) + 1(x+2) \end{aligned}$$

$$\begin{aligned} &= (x+1)(x+2) \dots (iv) \\ \text{Common factor} &= (x+1)(x+2)(2x+1) \\ \text{LCM} &= (x+1)(x+2)(2x+1) \end{aligned}$$

(ii) **Solution**

$$\begin{aligned} 3x^2 + 11x + 6 \\ = 3x^2 + 9x + 2x + 6 \\ = 3x(x+3) + 2(x+3) \\ = (x+3)(3x+2) \dots (i) \end{aligned}$$

Now,

$$\begin{aligned} 3x^2 + 8x + 4 \\ = 3x^2 + 6x + 2x + 4 \\ = 3x(x+2) + 2(x+2) \\ = (x+2)(3x+2) \dots (ii) \end{aligned}$$

Also,

$$\begin{aligned} x^2 + 5x + 6 \\ = x^2 + 3x + 2x + 6 \\ = x(x+3) + 2(x+3) \\ = (x+3)(x+2) \dots (iii) \end{aligned}$$

Common factor = $(x+2)(x+3)(3x+2)$

LCM = $(x+2)(x+3)(3x+2)$

Q.17 Find the HCF and LCM of the following expressions.

$a(a+c) - b(b+c)$ (10011)

$b(b+a) - c(c+a)$

$c(c+b) - a(a+b)$

Solution

$$\begin{aligned} a(a+c) - b(b+c) \\ = a^2 + ac - b^2 - bc \\ = a^2 - b^2 + ac - bc \\ = (a-b)(a+b) + c(a-b) \\ = (a-b)(a+b+c) \dots (i) \end{aligned}$$

Now,

$$\begin{aligned} b(b+a) - c(c+a) \\ = b^2 + ab - c^2 - ac \\ = b^2 - c^2 + ab - ac \\ = (b-c)(b+c) + a(b-c) \\ = (b-c)(b+c+a) \dots (ii) \end{aligned}$$

Now,

$c(c+b) - a(a+b)$



UNIT 05

Linear Equations and Inequalities

Linear Equation in One Variable

A linear equation in one variable is an equation that can be written in the standard form as $ax + b = 0$ where a, b are real numbers and $a \neq 0$ e.g. $2x + 3 = 0, 3x + 3 = 0$

Solution of Linear Equations

The solution of a linear equation in one variable is a replacement for the variable that makes the equation true.

Equivalent Equations

Two or more equations, which have the same solutions, are called equivalent equations. e.g. $2x + 3 = 4$ and $2x = 1$ are two equivalent equations.

Example 1

Solve the following equation for x . answer

$$-7x + 24 = 3$$

Solution

$$-7x + 24 = 3 \quad \leftarrow \text{original equation}$$

$$-7x + 24 - 24 = 3 - 24 \quad \leftarrow \text{Subtract 24 from both sides}$$

$$-7x = -21 \quad \leftarrow \text{divide both sides by } -7$$

$$\frac{-7x}{-7} = \frac{-21}{-7}$$

$$x = 3$$

Check: To check this root, we replace the variable x by its value in the original equation and simplify both sides.

$$-7x + 24 = 3$$

$$-7(3) + 24 = 3$$

$$-21 + 24 = 3$$

$$3 = 3$$

Hence the root of equation is 3.

$$5.5 = (3)$$

Example 2

$$\text{Solve } \frac{3x}{5} - \frac{1}{2} = \frac{x}{4} + 1 \quad \text{answer}$$

$$\text{Solution: } \frac{3x}{5} - \frac{1}{2} = \frac{x}{4} + 1$$

$$\text{LCM of 5, 2 and 4 = 20}$$

$$20 \left(\frac{3x}{5} - \frac{1}{2} \right) = 20 \left(\frac{x}{4} + 1 \right)$$

$$12x - 10 = 5x + 20$$

$$12x - 5x = 20 + 10$$

$$7x = 30$$

$$x = \frac{30}{7}$$

Thus, $x = \frac{30}{7}$ is the required root of the given equation.

Example 3

$$\text{Solve: } \frac{2x+3}{5} = \frac{3-4x}{8} \quad \text{answer}$$

$$\text{Solution: } \frac{2x+3}{5} = \frac{3-4x}{8}$$

By cross multiplication we get:

$$8(2x+3) = 5(3-4x)$$

$$16x + 24 = 15 - 20x$$

$$16x + 20x = 15 - 24$$

$$36x = -9$$

$$x = -\frac{9}{36} = -\frac{1}{4}$$

Thus, $x = -\frac{1}{4}$ is the root of given equation.

Example 4

$$\text{Solve } 0.7(x-1) - 0.5x = 1.1 \quad \text{answer}$$

Solution:

$$0.7(x-1) - 0.5x = 1.1$$

$$0.7x - 0.7 - 0.5x = 1.1$$

$$0.2x - 0.7 = 1.1$$

$$0.2x = 1.1 + 0.7$$

$$0.2x = 1.8$$

$$2x = 18$$

$$x = 9$$

Check: Replace x by its value in the original equation.

$$0.7(9-1) - 0.5(9) = 1.1$$

$$5.6 - 4.5 = 1.1$$

$$1.1 = 1.1$$

Thus, $x = 9$ is the required solution of the given equation.

Exercise 5.1

Solve the following linear equations in one variable.

Q.1 $5x - 2 = x + 4 - 3x - 27$ answer

Solution

$$5x - 2 = x + 4 - 3x - 27 \dots (i)$$

$$4x - 2 = -3x - 23$$

$$4x + 3x = -23 + 2$$

$$7x = -21$$

$$x = \frac{-21}{7}$$

$$\boxed{x = -3}$$

Check: put $x = -3$ in eq (i)

$$5(-3) - 2 = (-3) + 4 - 3(-3) - 27$$

$$-15 - 2 = 4 + 9 - 27$$

$$-17 + 3 = 13 - 27$$

$$-14 = -14$$

It is true,

$\therefore x = -3$ is required root.

$$\text{So, } 5.5 = (-3)$$

Q.2 $4x - 3(5x - 14) = 5(7 + x) - 9$ answer

Solution

$$4x - 3(5x - 14) = 5(7 + x) - 9 \dots (i)$$

$$4x - 15x + 42 = 35 + 5x - 9$$

$$-11x + 42 = 26 + 5x$$

$$42 - 26 = 5x + 11x$$

$$16 = 16x$$

$$\frac{16}{16} = x \Rightarrow \boxed{x = 1}$$

Check: Put $x = 1$ in eq (i)

$$4(1) - 3(5(1) - 14) = 5(7 + 1) - 9$$

$$4 - 3(5 - 14) = 5(8) - 9$$

$$4 - 3(-9) = 40 - 9$$

$$4 + 27 = 31$$

$$31 = 31$$

It is true $x = 1$ is the required root.

$$5.5 = (1)$$

Q.3 $7(2 - 5x) + 27 = 18x - 3(8 - 4x)$ answer

Solution

$$7(2 - 5x) + 27 = 18x - 3(8 - 4x) \dots (i)$$

$$14 - 35x + 27 = 18x - 24 + 12x$$

$$41 - 35x = 30x - 24$$

$$41 + 24 = 30x + 35x$$

$$65 = 65x$$

$$\frac{65}{65} = x$$

$$1 = x$$

$$\boxed{x = 1}$$

Check: Put $x = 1$ in eq (i)

$$7(2 - 5(1)) + 27 = 18(1) - 3(8 - 4(1))$$

$$7(2 - 5) + 27 = 18 - 3(8 - 4)$$

$$7(-3) + 27 = 18 - 3(4)$$

$$-21 + 27 = 18 - 12$$



$8 = 8$
It is true.
So $x = 1$ is required root.
S.S. = {1}

Q.4 $\frac{5x}{4} + \frac{1}{2} = 0$

Answer

Solution
 $\frac{5x}{4} + \frac{1}{2} = 0 \dots (i)$
 $\frac{5x}{4} = -\frac{1}{2}$

By cross multiplication

$2(5x) = 4(-1)$

$10x = -4$

$x = -\frac{4}{10}$

$x = -\frac{2}{5}$

Check: Put $x = -\frac{2}{5}$ in eq. ... (i)

$\frac{5}{4}x + \frac{1}{2} = 0$
 $\left(\frac{5}{4} \times -\frac{2}{5}\right) + \frac{1}{2} = 0$

$\frac{-10}{20} + \frac{1}{2} = 0$

$-\frac{1}{2} + \frac{1}{2} = 0$

$0 = 0$

It is true.

So $x = -\frac{2}{5}$ is required root

S.S. = $\left\{-\frac{2}{5}\right\}$

Q.5 $\frac{x-2}{2} + \frac{x+10}{9} = 5$

Answer

Solution

$\frac{x-2}{2} + \frac{x+10}{9} = 5 \dots (i)$

Multiplying B.S by "18"

$18 \left(\frac{x-2}{2} + \frac{x+10}{9} \right) = 18(5)$

$9(x-2) + 2(x+10) = 90$

$9x - 18 + 2x + 20 = 90$

$11x + 2 = 90$

$11x = 90 - 2$

$11x = 88$

$x = \frac{88}{11}$

$x = 8$

Check: Put $x = 8$ in eq. (i)

$\frac{8-2}{2} + \frac{8+10}{9} = 5$

$\frac{6}{2} + \frac{18}{9} = 5$

$3 + 2 = 5$

$5 = 5$

It is true.

So $x = 8$ is required root

S.S. = {8}

Q.6 $\frac{4(x+2)}{3} - \frac{6(x-7)}{7} = 12$

Answer

Solution

$\frac{4(x+2)}{3} - \frac{6(x-7)}{7} = 12 \dots (i)$

Multiplying B.S by "21"

$21 \left(\frac{4(x+2)}{3} - \frac{6(x-7)}{7} \right) = 21 \times 12$

$28(x+2) - 18(x-7) = 252$

$28x + 56 - 18x + 126 = 252$

$46x - 70 = 252$

$46x = 252 + 70$

$46x = 322$

$x = \frac{322}{46}$

$x = 7$



Check: Put $x = 7$ in eq. (i)

$\frac{4(7+2)}{3} - \frac{6(7-7)}{7} = 12$

$\frac{4(9)}{3} - \frac{6(0)}{7} = 12$

$\frac{36}{3} - \frac{0}{7} = 12$

$12 - 0 = 12$

$12 = 12$

It is required root it is true.

So, S.S. = {7}

Q.7 $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} = 7\frac{3}{5}$

Answer

Solution

$\frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} = 7\frac{3}{5}$

$\frac{10x + 20x - 15x + 12x}{60} = \frac{47}{5}$ (Taking L.C.M)

$\frac{62x - 15x}{60} = \frac{47}{5}$

$\frac{47x}{60} = \frac{47}{5}$

$x = \frac{60}{47} \times \frac{47}{5}$

$x = 10$

Check: Put $x = 10$ in eq. (i)

$\frac{10}{2} + \frac{10}{3} - \frac{10}{4} + \frac{10}{5} = 7\frac{3}{5}$

$\frac{300 + 200 - 150 + 120}{60} = \frac{47}{5}$

$\frac{470}{60} = \frac{47}{5}$

$\frac{47}{6} = \frac{47}{6}$

It is true, so $x = 10$ is the required root.

S.S. = {10}

Q.8 $\frac{y+1}{3} + \frac{y+1}{2} = 2 - \frac{y+3}{2}$

Answer

Solution

$\frac{y+1}{3} + \frac{y+1}{2} = 2 - \frac{y+3}{2}$

Multiplying both sides by 6

$6 \left(\frac{y+1}{3} \right) + 6 \left(\frac{y+1}{2} \right) = 6(2) - 6 \left(\frac{y+3}{2} \right)$

$2(y+1) + 3(y+1) = 12 - 3(y+3)$

$2y + 2 + 3y + 3 = 12 - 3y - 9$

$5y + 5 = 3 - 3y$

$5y + 3y = 3 - 5$

$8y = -2$

$y = -\frac{2}{8}$

$y = -\frac{1}{4}$

Check: Put $y = -\frac{1}{4}$ in Eq. (i)

$\frac{-\frac{1}{4}+1}{3} + \frac{-\frac{1}{4}+1}{2} = 2 - \frac{-\frac{1}{4}+3}{2}$

$\frac{-1+4}{6} + \frac{-1+4}{4} = 2 - \frac{-1+12}{2}$

$\frac{3}{6} + \frac{3}{4} = 2 - \frac{11}{2}$

$\frac{3}{12} + \frac{9}{12} = 2 - \frac{11}{2}$

$\frac{12}{24} = \frac{18-11}{8}$

$\frac{3}{8} = \frac{7}{8}$

Which is true, so $y = -\frac{1}{4}$ is the required root.

S.S. = $\left\{-\frac{1}{4}\right\}$

Q.9 $\frac{1}{5}x(-8) + \frac{4+x}{7} = 7 - \frac{23-x}{5}$

Answer

Solution

$\frac{1}{5}x(-8) + \frac{4+x}{7} = 7 - \frac{23-x}{5}$

Multiplying both sides by 35

$35 \left(\frac{1}{5}x(-8) \right) + 35 \left(\frac{4+x}{7} \right) = 35(7) - 35 \left(\frac{23-x}{5} \right)$



$$7(x-8) + 5(4-x) = 243 - 7(23-x)$$

$$7x - 56 + 20 - 5x = 243 - 161 + 7x$$

$$12x - 36 = 84 + 7x$$

$$12x - 7x = 84 + 36$$

$$5x = 120$$

$$x = 24$$

$$x = \frac{120}{5}$$

$$x = 24$$

Check: Put $x = 24$ in Eq. (i)

$$\frac{1}{3}(24-8) + \frac{4+24}{7} = 7 - \frac{23-24}{5}$$

$$\frac{16}{3} + \frac{28}{7} = 7 - \frac{-1}{5}$$

$$\frac{16}{3} + 4 = 7 + \frac{1}{5}$$

$$\frac{16+20}{3} = \frac{35+1}{5} = \frac{36}{5} = \frac{36}{5}$$

Which is true, so $x = 24$ is the required root.

$$S.S = \{24\}$$

Q.10 $\frac{1}{2y} - \frac{1}{6} = \frac{1}{4y} - 1 - \frac{1}{y}$ $2y \neq 0$ answer

Solution

$$\frac{1}{2y} - \frac{1}{6} = \frac{1}{4y} - 1 - \frac{1}{y} \quad \dots (i)$$

Multiplying both sides by $12y$

$$12y \left(\frac{1}{2y} \right) - 12y \left(\frac{1}{6} \right) = 12y \left(\frac{1}{4y} \right) - (12y) - 12y \left(\frac{1}{y} \right)$$

$$6 - 2y = 3 - 12y - 12$$

$$6 - 2y = -12y - 9$$

$$12y - 2y = -9 - 6$$

$$10y = -15$$

$$y = \frac{-15}{10}$$

$$y = \frac{-3}{2}$$

$$y = \frac{-3}{2}$$

Check: Put $y = \frac{-3}{2}$ in (i)

$$\frac{1}{2 \left(\frac{-3}{2} \right)} - \frac{1}{6} = \frac{1}{4 \left(\frac{-3}{2} \right)} - 1 - \frac{1}{\frac{-3}{2}}$$

$$\frac{1}{3} - \frac{1}{6} = -\frac{1}{6} - 1 + \frac{2}{3}$$

$$\frac{-2+1}{6} = \frac{-1-6+4}{6}$$

$$\frac{-1}{6} = \frac{-3}{6}$$

$$\frac{-1}{2} = \frac{-1}{2}$$

Which is true, so $y = \frac{-3}{2}$ is the required root.

$$S.S = \left\{ \frac{-3}{2} \right\}$$

Q.11 $4 - 0.3(1-x) = 7$ answer

Solution

$$4 - 0.3(1-x) = 7 \rightarrow (i)$$

$$4 - 0.3 + 0.3x = 7$$

$$3.7 + 0.3x = 7$$

$$0.3x = 7 - 3.7$$

$$0.3x = 3.3$$

$$x = \frac{3.3}{0.3}$$

$$x = 11$$

Check: Put $x = 11$ in Eq. (i)

$$4 - 0.3(1-11) = 7$$

$$4 - 0.3(-10) = 7$$

$$4 + 3 = 7$$

$$7 = 7$$

Which is true, so $x = 11$ is the required root.

$$S.S = \{11\}$$



Q.12 $0.5x = 6.3 - 0.2x$ answer

Solution

$$0.5x = 6.3 - 0.2x \rightarrow (i)$$

$$0.5x + 0.2x = 6.3$$

$$0.7x = 6.3$$

$$x = \frac{6.3}{0.7}$$

$$x = 9$$

Check: Put $x = 9$ in Eq. (i)

$$0.5(9) = 6.3 - 0.2(9)$$

$$4.5 = 6.3 - 1.8$$

$$4.5 = 4.5$$

Which is true, so $x = 9$ is the required root.

$$S.S = \{9\}$$

Q.13 $1.3x - 0.2 = 6.3x - 1.5$ answer

Solution

$$1.3x - 0.2 - 6.3x = -1.5 \rightarrow (i)$$

$$1.3x - 6.3x = 0.2 - 1.5$$

$$-5.0x = -1.3$$

$$x = \frac{-1.3}{-5}$$

Check

$$1.3(-1.3) - 0.2 = 6.3(-1.3) - 1.5$$

$$-1.69 - 0.2 = -0.79 - 1.5$$

$$-1.89 = -1.89$$

Which is true, so $x = -1.3$ is the required root.

$$S.S = \{-1.3\}$$

Linear Equations Involving Radicals

Radical Equations

An equation in which the unknown letter (variable) appears under a radical sign is called a radical equation.

Examples:

$$\sqrt{x+1} = 7, \sqrt{x} = 8, \sqrt{x+2} = 3, \sqrt{2x-3} = \sqrt{x+5}$$

To solve a Radical Equation

- Arrange the terms such that a term with a radical sign is by itself on one side of the equation.
- Square both sides of the equation.

- Solve the resulting linear equation for corresponding variable.
- Check the solution in the original equation.

Example 5

Solve: $\sqrt{x+3} = 7$ answer

Solution: $\sqrt{x+3} = 7 \rightarrow (i)$

$$\sqrt{x+3} = 7$$

$$\sqrt{x+3} = 4$$

$$(\sqrt{x+3})^2 = (4)^2$$

$$x+3 = 16$$

Check: Put $x = 16$ in Eq. (i)

$$\sqrt{16+3} = 7$$

$$4+3 = 7$$

$$7 = 7$$

Which is true. So $x = 16$ is required root.

$$S.S = \{16\}$$

Hence, $x = 16$ is required solution.

Example 6

Solve: $4 + 2\sqrt{3y+1} = 3$ answer

Solution

$$4 + 2\sqrt{3y+1} = 3 \rightarrow (i)$$

$$2\sqrt{3y+1} = 3 - 4$$

$$2\sqrt{3y+1} = -1$$

$$(2\sqrt{3y+1})^2 = (-1)^2$$

$$4(3y+1) = 1$$

$$12y + 4 = 1$$

$$12y = 1 - 4$$

$$12y = -3$$

$$y = \frac{-3}{12}$$

$$y = -\frac{1}{4}$$

Check: Put $y = -\frac{1}{4}$ in equation (i)



$$4 + 2\sqrt{3\left(-\frac{1}{4} + 1\right)} = 3$$

$$4 + 2\sqrt{\frac{-3}{4} + 1} = 3$$

$$4 + 2\sqrt{\frac{-3+4}{4}} = 3$$

$$4 + 2\sqrt{\frac{1}{4}} = 3$$

$$4 + 2\left(\frac{1}{2}\right) = 3$$

$$4 + 1 = 3$$

$$5 = 3$$

Thus, $y = -\frac{1}{4}$ is an extraneous root and solution set is $\emptyset$.

Exercise 5.5

Reduce the following radical equations into simple linear equations then find their solution. In case of extraneous solutions, write $\emptyset$ for the solution set.

Q.1 $\sqrt{2x} = 4$ answer

Solution

$$\sqrt{2x} = 4 \rightarrow (1)$$

Taking square of both sides

$$(\sqrt{2x})^2 = (4)^2$$

$$2x = 16$$

$$x = \frac{16}{2}$$

$$\boxed{x = 8}$$

Check: Put $x = 8$ in Eq. (1)

$$\sqrt{2(8)} = 4$$

$$\sqrt{16} = 4$$

$$4 = 4$$

Which is true, so $x = 8$ is the required solution.

$$S.S = \{8\}$$

Q.2 $\sqrt{x-3} = 2$ answer

Solution

$$\sqrt{x-3} = 2 \rightarrow (1)$$

Taking square of both sides

$$(\sqrt{x-3})^2 = (2)^2$$

$$x - 3 = 4$$

$$x - 4 = 3$$

$$\boxed{x = 7}$$

Check: Put $x = 7$ in Eq. (1)

$$\sqrt{7-3} = 2$$

$$\sqrt{4} = 2$$

$$2 = 2$$

Which is true, so $x = 7$ is the required solution.

$$S.S = \{7\}$$

Q.3 $\sqrt{x-5} = 3$ answer

Solution

$$\sqrt{x-5} = 3 \rightarrow (1)$$

Taking square of both sides

$$(\sqrt{x-5})^2 = (3)^2$$

$$x - 5 = 9$$

$$x = 9 + 5$$

$$x = 14$$

Check: Put $x = 14$ in Eq. (1)

$$\sqrt{14-5} = 3$$

$$\sqrt{9} = 3$$

$$3 = 3$$

Which is true, so $x = 14$ is the required solution.

$$S.S = \{14\}$$

Q.4 $\sqrt{2x+1} = 9$ answer

Solution

$$\sqrt{2x+1} = 9 \rightarrow (1)$$

Taking square of both sides

$$(\sqrt{2x+1})^2 = (9)^2$$

$$2x + 1 = 81$$

$$2x = 81 - 1$$

$$2x = 80$$

$$x = \frac{80}{2}$$

$$\boxed{x = 40}$$

Check: Put $x = 40$ in Eq. (1)

$$\sqrt{2(40)+1} = 9$$

$$\sqrt{81} = 9$$

$$9 = 9$$

Which is true, so $x = 40$ is the required solution.

$$S.S = \{40\}$$

Q.5 $\sqrt{5x-4} = 14$ answer

Solution

$$\sqrt{5x-4} = 14 \rightarrow (1)$$

Taking square of both sides

$$(\sqrt{5x-4})^2 = (14)^2$$

$$5x - 4 = 196$$

$$5x = 196 + 4$$

$$5x = 200$$

$$x = \frac{200}{5}$$

$$\boxed{x = 40}$$

Check: Put $x = 40$ in Eq. (1)

$$\sqrt{5(40)-4} = 14$$

$$\sqrt{196} = 14$$

$$14 = 14$$

Which is true, so $x = 40$ is the required solution.

$$S.S = \{40\}$$

Q.6 $\sqrt{3x-5} = -10$ answer

Solution

$$\sqrt{3x-5} = -10 \rightarrow (1)$$

Taking square of both sides

$$(\sqrt{3x-5})^2 = (-10)^2$$

$$3x - 5 = 100$$

$$3x = 100 + 5$$

$$3x = 105$$

$$x = \frac{105}{3}$$

$$\boxed{x = 35}$$

Check: Put $x = 35$ in Eq. (1)

$$\sqrt{3(35)-5} = -10$$

$$\sqrt{105-5} = -10$$

$$\sqrt{100} = -10$$

$$10 = -10$$

Which is not true, so $x = 35$ is an extraneous root and $S.S = \emptyset$.

Q.7 $\sqrt{y+4} - 3 = 2$ answer

Solution

$$\sqrt{y+4} - 3 = 2 \rightarrow (1)$$

$$\sqrt{y+4} = 2 + 3$$

$$\sqrt{y+4} = 5$$

Taking square of both sides

$$(\sqrt{y+4})^2 = (5)^2$$

$$y + 4 = 25$$

$$y = 25 - 4$$

$$\boxed{y = 21}$$

Check: Put $y = 21$ in Eq. (1)

$$\sqrt{21+4} - 3 = 2$$

$$\sqrt{25} - 3 = 2$$

$$5 - 3 = 2$$

$$2 = 2$$

Which is true, so $y = 21$ is the required solution.

$$S.S = \{21\}$$



Q.8 $x - \sqrt{2x-1} = 0$

Solution

$$x - \sqrt{2x-1} = 0 \rightarrow (i)$$

Taking square of both sides

$$(x)^2 = (\sqrt{2x-1})^2$$

$$25 = 2x - 1$$

$$25 + 1 = 2x$$

$$\frac{26}{2} = x$$

$$\boxed{x = 13}$$

Check: Put $x = 13$ in Eq. (i)

$$x = \sqrt{2(13)-1}$$

$$x = \sqrt{26-1}$$

$$x = \sqrt{25}$$

$$x = 5$$

Which is true, so $x = 13$ is the required solution.

$$S.S = \{13\}$$

Q.9 $\sqrt{y+1} - 12 = -10$

Solution

$$\sqrt{y+1} - 12 = -10$$

$$\sqrt{y+1} = 12 - 10$$

$$\sqrt{y+1} = 2 \rightarrow (i)$$

Taking square of both sides

$$(\sqrt{y+1})^2 = (2)^2$$

$$y+1 = 4$$

$$y = 4 - 1$$

$$\boxed{y = 3}$$

Check: Put $x = 3$ in Eq. (i)

$$\sqrt{3+1} = 2$$

$$\sqrt{4} = 2$$

$$2 = 2$$

Which is true, so $x = 3$ is the required solution.

$$S.S = \{3\}$$

Q.10 $\sqrt{3x-2} = \sqrt{3x+4}$

Solution

$$\sqrt{3x-2} = \sqrt{3x+4} \rightarrow (i)$$

Taking square of both sides

$$(\sqrt{3x-2})^2 = (\sqrt{3x+4})^2$$

$$3x - 2 = 3x + 4$$

$$3x - 3x = 4 + 2$$

$$2x = 6$$

$$x = \frac{6}{2}$$

$$\boxed{x = 3}$$

Check: Put $x = 3$ in Eq. (i)

$$\sqrt{3(3)-2} = \sqrt{3(3)+4}$$

$$\sqrt{15-2} = \sqrt{9+4}$$

$$\sqrt{13} = \sqrt{13}$$

Which is true, so $x = 3$ is the required solution.

$$S.S = \{3\}$$

Q.11 $\sqrt{9-2x} = \sqrt{5x-12}$

Solution

$$\sqrt{9-2x} = \sqrt{5x-12} \rightarrow (i)$$

Taking square of both sides

$$(\sqrt{9-2x})^2 = (\sqrt{5x-12})^2$$

$$9 - 2x = 5x - 12$$

$$9 + 12 = 5x + 2x$$

$$21 = 7x$$

$$\frac{21}{7} = x$$

$$\boxed{x = 3}$$

Check: Put $x = 3$ in Eq. (i)

$$\sqrt{9-2(3)} = \sqrt{5(3)-12}$$

$$\sqrt{9-6} = \sqrt{15-12}$$

$$\sqrt{3} = \sqrt{3}$$

Which is true, so $x = 3$ is the required solution.

$$S.S = \{3\}$$



Q.12 $12 - \sqrt{y+1} = 14$

Solution

$$12 - \sqrt{y+1} = 14$$

$$12 - 14 = \sqrt{y+1}$$

$$\Rightarrow \sqrt{y+1} = -2$$

Taking square of both sides

$$(\sqrt{y+1})^2 = (-2)^2$$

$$y+1 = 4$$

$$y = 4 - 1$$

$$\boxed{y = 3}$$

Check: Put $y = 3$ in Eq. (i)

$$\sqrt{3+1} = -2$$

$$\sqrt{4} = -2$$

$$2 = -2$$

Which is not true, so $y = 3$ is the extraneous solution and $S.S = \emptyset$.

Q.13 $4\sqrt{x+8} = 40$

Solution

$$4\sqrt{x+8} = 40$$

$$4\sqrt{x} = 40 - 8$$

$$\sqrt{x} = \frac{32}{4}$$

$$\sqrt{x} = 8 \rightarrow (i)$$

Taking square of both sides

$$(\sqrt{x})^2 = (8)^2$$

$$x = 64$$

Check: Put $x = 64$ in Eq. (i)

$$4\sqrt{64+8} = 40$$

$$4(8) + 8 = 40$$

$$40 = 40$$

Which is true, so $x = 64$ is the required solution.

$$S.S = \{64\}$$

Q.14 $\sqrt{\frac{x+6}{x+2}} = \sqrt{\frac{x+2}{x-1}}$

Solution

$$\sqrt{\frac{x+6}{x+2}} = \sqrt{\frac{x+2}{x-1}} \rightarrow (i)$$

Taking square of both sides

$$\frac{x+6}{x+2} = \frac{x+2}{x-1}$$

$$(x+6)(x-1) = (x+2)(x+2)$$

$$x^2 + 6x - x - 6 = x^2 + 2x + 2x + 4$$

$$x^2 + 5x - 6 = x^2 + 4x + 4$$

$$x + 5x - 4x = 4 + 6$$

$$\boxed{x = 10}$$

Check: Put $x = 10$ in Eq. (i)

$$\sqrt{\frac{10+6}{10+2}} = \sqrt{\frac{10+2}{10-1}}$$

$$\sqrt{\frac{16}{12}} = \sqrt{\frac{12}{9}}$$

$$\sqrt{\frac{4}{3}} = \sqrt{\frac{4}{3}}$$

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

Which is true, so $x = 10$ is the required solution.

$$S.S = \{10\}$$

Q.15 $\sqrt{\frac{x}{x+3}} = \sqrt{\frac{x+2}{x+6}}$

Solution

$$\sqrt{\frac{x}{x+3}} = \sqrt{\frac{x+2}{x+6}} \rightarrow (i)$$

Taking square of both sides

$$\left(\sqrt{\frac{x}{x+3}}\right)^2 = \left(\sqrt{\frac{x+2}{x+6}}\right)^2$$

$$\frac{x}{x+3} = \frac{x+2}{x+6}$$

$$x(x+6) = (x+2)(x+3)$$

$$x^2 + 6x = x^2 + 2x + 3x + 6$$

$$x^2 + 6x = x^2 + 5x + 6$$



$$x^2 + 4x = x^2 + 2x + 4$$

$$x^2 + 4x - x^2 - 2x = 4 - 4$$

$$\underline{x = 4}$$

Check: Put $x = 4$ in Eq. (1)

$$\sqrt{4+3} = \sqrt{4+2}$$

$$\sqrt{7} = \sqrt{6}$$

$$\sqrt{7} \neq \sqrt{6}$$

Which is true, so $x = 4$ is the required solution.

$$\text{S.S.} = \{4\}$$

Q.10 $\sqrt{5x-4} = \sqrt{7x+2}$ www

Solution

$$\sqrt{5x-4} = \sqrt{7x+2} \Rightarrow (1)$$

Taking square of both sides

$$(\sqrt{5x-4})^2 = (\sqrt{7x+2})^2$$

$$5x-4 = 7x+2$$

$$-4-2 = 7x-5x$$

$$-6 = 2x$$

$$2x = -6$$

$$x = \frac{-6}{2}$$

$$\underline{x = -3}$$

Check: Put $x = -3$ in Eq. (1)

$$\sqrt{5(-3)-4} = \sqrt{7(-3)+2}$$

$$\sqrt{-15-4} = \sqrt{-21+2}$$

$$\sqrt{-19} = \sqrt{-19}$$

Which is true, so $x = -3$ is the required solution.

$$\text{S.S.} = \{-3\}$$

Defining an Absolute Value

The absolute value of a number is its distance from 0 on the number line. If x is any point on the number line then its distance from 0 is denoted by $|x|$. The two vertical bars are called

absolute value bars. Since distance between any two points is always a positive number or zero, then the absolute value of a number is always a positive number or zero e.g. distance from 0 to 5 or from 0 to -5 is 5 units on the number line.

Thus, $|5| = 5$ and $|-5| = 5$

The absolute value of a real number x , written as $|x|$, is defined as

$$\bullet \quad |x| = x, \text{ if } x \geq 0$$

$$\bullet \quad |x| = -x \text{ if } x < 0 \text{ e.g. } |-9| = 9 \text{ or } |-3| = -(-3) = 3$$

Solution of Absolute Value Equations

An equation that contains a variable inside the absolute value bars is called an absolute value equation.

e.g. $|x+1| = 5$, $|x-2| = 4$

Example 7

Solve the equation $|x| = 8$. www

Solution: To solve such equation we have to consider both the possible values of the number with absolute value.

Thus, if $|x| = 8$ then $x = 8$ or $x = -8$.

The solution set is $\{-8, 8\}$.

Example 8

Solve the equation $|x| = -6$. www

Solution: There is no real number x such that $|x| = -6$. So, this equation has no solution. Hence the solution set is $\emptyset$.

Conclusion

An absolute value equation of the form $|ax+b| = c$, where a, b and c are real numbers.

$a \neq 0$ and $c > 0$ is equivalent to two equations:

$$ax+b=c \quad \text{or} \quad ax+b=-c.$$

For the required solution set of the given absolute value equation, we solve both the equations separately.

Example 9

Solve the equation: www

$$|2x+5| = 11, \text{ where } x \in R$$

Solution

Step-I: Remove the absolute value bars and write two equations as:

$$2x+5=11 \quad \text{or} \quad 2x+5=-11$$

Step-II: Now solve both the equations for x .

$$2x+5=11 \quad | \quad 2x+5=-11$$

$$2x=11-5 \quad | \quad 2x=-11-5$$

$$2x=6 \quad | \quad 2x=-16$$

$$x=3 \quad | \quad x=-8$$

Step-III: Hence 3 and -8 are roots of the absolute value equation. Thus, the solution set is $\{3, -8\}$.

Note

- The equation $|x| = k$ and $k > 0$ has two solutions k and $-k$.
- The equation $|x| = 0$ has one solution, namely $x = 0$.
- The equation $|x| = k$ has no solution and the solution set in this case is $\emptyset$.

Key Fact

- $|a|b| = |a||b|$
- $-\frac{|a|}{|b|} = \frac{|a|}{|b|}$
- $-|a| = |a| = 2|a|$

Example 10

Solve the absolute value equation. www

$$\frac{|10-x|}{5} = \frac{|2x-5|}{2} \text{ where } x \in R$$

Solution: Before we remove the absolute value bars, try to isolate the absolute value expressions on either side as:

$$\frac{|10-x|}{5} = \frac{|2x-5|}{2}$$

$$\frac{|10-x|}{|2x-5|} = \frac{5}{2}$$

$$\frac{|10-x|}{|2x-5|} = \frac{5}{2}$$

$$\sqrt{\frac{|a|}{|b|}} = \sqrt{\frac{a}{b}}$$

$$\frac{|10-x|}{|2x-5|} = \frac{5}{2}$$

$$\frac{|10-x|}{|2x-5|} = \frac{5}{2}$$

$$2|10-x| = 5|2x-5|$$

$$2|10-x| = 5|2x-5|$$

$$20-2x = 10x-25$$

$$20-2x = -10x+25$$

$$-2x-10x = -25-20$$

$$-2x-10x = 25-20$$

$$-12x = -45$$

$$8x = 5$$

$$x = \frac{45}{12} = \frac{15}{4}$$

$$x = \frac{5}{8}$$

Thus, the solution set is $\left\{\frac{15}{4}, \frac{5}{8}\right\}$.

Example 11

Solve the following absolute value equation.

$$|x-1| = |2x-3|, x \in R \quad \text{www}$$

Solution:

By removing the absolute value bars we get two equations as:

$$x-1 = 2x-3 \quad \text{or} \quad x-1 = -(2x-3)$$

$$x-2x = -3+1$$

$$x-1 = -2x+3$$

$$-x = -2$$

$$x+2x = 3+1$$

$$x = 2$$

$$3x = 4$$

$$x = \frac{4}{3}$$

The solution set is $\left\{2, \frac{4}{3}\right\}$.

Exercise 3.3

Solve the following absolute value equations, where $x, y, z \in R$.

Q.1 $|x| = \frac{8}{3}$ www



Solution

$$18 = \frac{2}{3}$$

$$x = \frac{2}{3} \text{ or } x = -\frac{2}{3}$$

$$\text{So, S.S.} = \left\{ \frac{2}{3}, -\frac{2}{3} \right\}$$

 Q.2 $3x+2=6$

correct

Solution

$$3x+2=6$$

$$3x+2=6 \text{ or } 3x+2=-6$$

$$3x=6-2 \text{ or } 3x=-6-2$$

$$3x=4 \text{ or } 3x=-8$$

$$\text{So, S.S.} = \{4, -8\}$$

 Q.3 $3y-1=9$

correct

Solution

$$3y-1=9$$

$$3y-1=9 \quad \text{or} \quad 3y-1=-9$$

$$3y=9+1 \quad 3y=8+1$$

$$3y=10 \quad 3y=-8$$

$$y = \frac{10}{3} \quad y = -\frac{8}{3}$$

$$y = 2$$

$$\text{So, S.S.} = \left\{ 2, -\frac{8}{3} \right\}$$

$$\text{Q.4 } 7x+6=2$$

correct

$$\text{Solution}$$

$$7x+6=2$$

$$7x+6=2 \text{ or } 7x+6=-2$$

$$7x=2-6 \text{ or } 7x=-6-2$$

$$7x=-4 \text{ or } 7x=-8$$

$$\text{So, S.S.} = \left\{ -\frac{4}{7}, -\frac{8}{7} \right\}$$

$$\text{Q.5 } 4-3y=8$$

correct

Solution

$$4-3y=8$$

$$4-3y=8 \text{ or } 4-3y=-8$$

$$-3y=8-4 \text{ or } -3y=-8-4$$

$$-3y=4 \text{ or } -3y=-12$$

$$y = -\frac{4}{3} \text{ or } y = 4$$

$$\text{So, S.S.} = \left\{ -\frac{4}{3}, 4 \right\}$$

$$\text{Q.6 } 3x-2=4 \text{ or } -2$$

$$y = 2$$

$$\text{Then S.S.} = \{2\}$$

$$\text{Q.8 } 3x-2=4 \text{ or } -2$$

Solution

$$3x-2=4 \text{ or } -2$$

$$3x-2=4 \text{ or } -2$$

$$3x-2=4 \text{ or } -2$$

$$3x-2=4 \text{ or } -2$$

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$$3x-2=4 \text{ or } -2$$



$$\text{S.S.} = \left\{ -\frac{1}{3}, \frac{4}{3} \right\}$$

$$\text{Q.9 } \frac{4x}{3} = 12$$

correct

Solution

$$\frac{4x}{3} = 12$$

$$4x = 12 \times 3$$

$$4x = 36$$

$$4x = 36 \text{ or } 4x = -36$$

$$x = \frac{36}{4} \text{ or } x = -\frac{36}{4}$$

$$x = 9 \text{ or } x = -9$$

$$\text{S.S.} = \{9, -9\}$$

$$\text{Q.10 } 15x+10=5$$

correct

Solution

$$15x+10=5$$

$$15x=5-10$$

$$15x=-5$$

$$\text{As the value of absolute can never be negative,}$$

$$\text{so S.S.} = \{ \}$$

$$\text{Q.11 } \frac{1-2y}{4} = 3$$

correct

Solution

$$\frac{1-2y}{4} = 3$$

$$1-2y = 3 \times 4$$

$$1-2y = 12$$

$$1-2y = 12 \text{ or } 1-2y = -12$$

$$1-12 = 2y \text{ or } 1+12 = 2y$$

$$-11 = 2y \text{ or } 13 = 2y$$

$$y = -\frac{11}{2} \text{ or } y = \frac{13}{2}$$

$$\text{S.S.} = \left\{ -\frac{11}{2}, \frac{13}{2} \right\}$$

$$\text{Q.12 } \frac{x+1}{2} = \frac{2x-1}{3}$$

correct

Solution

$$\frac{x+1}{2} = \frac{2x-1}{3}$$

$$3(x+1) = 2(2x-1)$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$3x+3 = 4x-2$$

$$\frac{x+1}{2} = \frac{2x-1}{3}$$

$$\frac{x+1}{2} = \frac{2x-1}{3}$$

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$$\frac{x+1}{2} = \frac{2x-1}{3}$$

$$\frac{x+1}{2} = \frac{2x-1}{3}$$



$$3y - 12 > 0, y > 3$$

Putting $y = 3$ in inequality

$$3(3) - 12 > 0$$

$$12 - 12 > 0$$

$$0 > 0$$

It satisfies the inequality

$$\text{ii) } 4 - 2x \leq 9, x = -3$$

Solution

$$4 - 2x \leq 9, x = -3$$

Putting $x = -3$ in inequality

$$4 - 2(-3) \leq 9$$

$$4 + 6 \leq 9$$

$$10 \leq 9$$

It does not satisfy the inequality

$$\text{iii) } 5 - 2x > -4x + 6, x = 4$$

Solution

$$5 - 2x > -4x + 6, x = 4$$

Putting $x = 4$ in inequality

$$5 - 2(4) > -4(4) + 6$$

$$5 - 8 > -16 + 6$$

$$-3 > -10$$

It satisfies the inequality

$$\text{iv) } 3x - 4 \leq 8, x = -2$$

Solution

$$3x - 4 \leq 8, x = -2$$

Putting $x = -2$ in inequality

$$3(-2) - 4 \leq 8$$

$$-6 - 4 \leq 8$$

$$-10 \leq 8$$

It satisfies the inequality

$$\text{v) } 5x - 2 \geq 3(2x - 4), x = 11$$

Solution

Putting $x = 11$ in inequality

$$5(11) - 2 \geq 3(2(11) - 4)$$

$$55 - 2 \geq 3(22 - 4)$$

$$53 \geq 36 - 12$$

$$53 \geq 24$$

It satisfies the inequality

(b) In the following cases, write each solution in the set notation form.

$$\text{i) } 2 \leq x \leq 5, \text{ where } x \in \mathbb{N}$$

Solution

$$2 \leq x \leq 5, \text{ where } x \in \mathbb{N}$$

$$S = \{x | x \in \mathbb{N} \wedge 2 \leq x \leq 5\}$$

$$S = \{2, 3, 4, 5\}$$

$$\text{ii) } y < 7, \text{ where } y \in \mathbb{N}$$

Solution

$$y < 7, \text{ where } y \in \mathbb{N}$$

$$S = \{y | y \in \mathbb{N} \wedge y < 7\}$$

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$\text{iii) } x \leq -3, \text{ where } x \in \mathbb{R}$$

Solution

$$x \leq -3, \text{ where } x \in \mathbb{R}$$

$$S = \{x | x \in \mathbb{R} \wedge x \leq -3\}$$

$$\text{iv) } x \leq 4, \text{ where } x \in \mathbb{W}$$

Solution

$$x \leq 4, \text{ where } x \in \mathbb{W}$$

$$S = \{x | x \in \mathbb{W} \wedge x \leq 4\}$$

$$S = \{0, 1, 2, 3, 4\}$$

$$\text{v) } -4 < x < \frac{-3}{2}, \text{ where } x \in \mathbb{R}$$

Solution

$$-4 < x < \frac{-3}{2}, \text{ where } x \in \mathbb{R}$$

$$S = \{x | x \in \mathbb{R} \wedge -4 < x < \frac{-3}{2}\}$$

Solve the following inequalities.

$$\text{Q2 } 2x - 1 < 7, x \in \mathbb{N}$$

Solution

$$2x - 1 < 7, x \in \mathbb{N}$$

$$2x < 7 + 1$$

$$2x < 8$$

$$x < \frac{8}{2}$$

$$x < 4$$

$$S = \{x | x \in \mathbb{N} \wedge x < 4\}$$

$$S = \{1, 2, 3\}$$

$$\text{Q3 } 6x - 5 \leq 35 - 2x, x \in \mathbb{W}$$

Solution

$$6x - 5 \leq 35 - 2x, x \in \mathbb{W}$$

$$6x + 2x \leq 35 + 5$$

$$8x \leq 40$$

$$x \leq \frac{40}{8}$$

$$x \leq 5$$

$$S = \{x | x \in \mathbb{W} \wedge x \leq 5\}$$

$$S = \{0, 1, 2, 3, 4, 5\}$$

$$\text{Q4 } 1 - 6y < 4(y - 1) - 7, y \in \mathbb{R}$$

Solution

$$1 - 6y < 4(y - 1) - 7, y \in \mathbb{R}$$

$$1 - 6y < 4y - 4 - 7$$

$$1 - 6y < 4y - 11$$

$$10 < 4y + 6y$$

$$20 < 10y$$

$$\frac{20}{10} < y$$

$$2 < y \text{ or } y > 2$$

$$S = \{y | y \in \mathbb{R} \wedge y > 2\}$$

$$\text{Q5 } 10 - (7 - y) \geq 2y - 8, y \in \mathbb{R}$$

Solution

$$10 - (7 - y) \geq 2y - 8, y \in \mathbb{R}$$

$$10 - 7 + y \geq 2y - 8$$

$$3 + y \geq 2y - 8$$

$$2 + 8 \geq 2y - y$$

$$12 \geq y$$

$$\frac{12}{1} \geq y$$

$$12 \geq y \text{ or } y \leq 12$$

$$S = \{y | y \in \mathbb{R} \wedge y \leq 12\}$$

Solve the following compared inequalities

where $x \in (0 - 1)$

$$\text{Q6 } 5 - 2x < 11 \text{ or } 2x + 3 < -9$$

Solution

$$5 - 2x < 11 \quad \text{or} \quad 2x + 3 < -9$$

$$5 - 11 < 2x \quad \text{or} \quad 2x < -9 - 3$$

$$-6 < 2x \quad \text{or} \quad 2x < -12$$

$$-3 < x \quad \text{or} \quad x < -6$$

$$S = \{x | x \in \mathbb{R} \wedge -3 < x < -6\}$$



$$-6 < 2x \quad \text{or} \quad x < \frac{12}{2}$$

$$-3 < x \quad \text{or} \quad x < 6$$

$$\text{or} \quad \boxed{x > -6}$$

$$S = \{x | x \in \mathbb{R} \wedge x > -2\} \text{ or } \{x | x \in \mathbb{R} \wedge x < -6\}$$

Now, the union of both the solution sets is:

$$\{x | x \in \mathbb{R} \wedge x > -2 \text{ or } x < -6\}$$

$$\text{Q7 } 2x + 3 \leq 9 \text{ and } x - 5 > -6$$

Solution

$$2x + 3 \leq 9 \text{ and } x - 5 > -6$$

$$2x \leq 9 - 3 \text{ and } x > -6 + 5$$

$$2x \leq 6 \text{ and } x > -1$$

$$x \leq \frac{6}{2} \text{ and}$$

$$x > -1$$

$$\{x | x \in \mathbb{R} \wedge x \leq 3\} \text{ and } \{x | x \in \mathbb{R} \wedge x > -1\}$$

Now, the intersection of both the solution set:

$$\{x | x \in \mathbb{R} \wedge -1 < x \leq 3\}$$

$$\text{Q8 } 1 \leq 7 - 3x \leq 22$$

Solution

$$1 \leq 7 - 3x \quad \text{and} \quad 7 - 3x \leq 22$$

$$3x \leq 7 - 1 \quad | \quad 7 - 22 \leq 3x$$

$$3x \leq 6 \quad | \quad -15 \leq 3x$$

$$x \leq \frac{6}{3} \quad | \quad \frac{-15}{3} \leq x$$

$$x \leq 2 \quad | \quad -5 \leq x$$

$$x \leq 2 \quad | \quad x \geq -5$$

$$x \geq -5$$

$$S = \{x | x \in \mathbb{R} \wedge -5 \leq x \leq 2\}$$

$$\text{Q9 } 3x - 21 < 1 - x \text{ or } 3x + 8 > 0 - 2x$$

Solution

$$3x - 21 < 1 - x \quad \text{or} \quad 3x + 8 > 0 - 2x$$

$$3x + x < 1 + 21 \quad \text{or} \quad 3x + 2x > 0 - 8$$

$$4x < 22 \quad \text{or} \quad 5x > -8$$

$$x < \frac{22}{4} \quad \text{or} \quad x > \frac{-8}{5}$$

$$\boxed{x < \frac{11}{2}} \quad \text{or} \quad \boxed{x > -\frac{8}{5}}$$

$$\{x | x \in \mathbb{R} \wedge x < \frac{11}{2}\} \text{ or } \{x | x \in \mathbb{R} \wedge x > -\frac{8}{5}\}$$



The union of both the solution sets is:
 $\{x \in \mathbb{R} : x < -3 \text{ or } x > -1\}$

Q.10 $1 - 3x > 16$ and $3 - \frac{3x}{2} \leq 9$ answer

Solution

$$1 - 3x > 16 \text{ and } 3 - \frac{3x}{2} \leq 9$$

$$1 - 16 > 3x \text{ and } 3 - 9 \leq \frac{3x}{2}$$

$$-15 > 3x \text{ and } -6 \leq \frac{3x}{2}$$

$$-\frac{15}{3} > x \text{ and } -6 \leq \frac{3x}{3}$$

$$-5 > x \text{ and } -2 \leq x$$

$$\Rightarrow \boxed{x < -5} \text{ and } \boxed{-2 \leq x}$$

$$-4 \leq x$$

$$\Rightarrow \boxed{x \leq -2} \text{ and } \boxed{-2 \leq x}$$

$$\{x \in \mathbb{R} : x < -3\} \text{ and } \{x \in \mathbb{R} : x \geq -4\}$$

The intersection of both the solution set is:

$$\{x \in \mathbb{R} : -3 < x \leq -4\}$$

Q.11 The sum of five times a number x and 10 is less than -35 or greater than -5 . What real numbers does x represent?

answer

Solution

Sum of 5 times a number x and 10 is $5x + 10$

By given condition,

$$5x + 10 < -35 \text{ or } 5x + 10 > -5$$

$$5x < -45 - 10 \text{ or } 5x > -5 - 10$$

$$5x < -55 \text{ or } 5x > -15$$

$$x < \frac{-55}{5} \text{ or } x > \frac{-15}{5}$$

$$\boxed{x < -11}$$

$$\text{or } \boxed{x > -3}$$

$$\{x \in \mathbb{R} : x < -11 \text{ or } x > -3\}$$

Q.12 Two times a number decreased by 3 is greater than or equal to the number increased by 8. Find the possible values for the number.

answer

Solution

Let required number be ' x '.

By given condition:

Two times a number decreased by 3 is $2x - 3$

The number increased by 8 is $x + 8$.

$$2x - 3 \geq x + 8$$

$$2x - x \geq 8 + 3$$

$$x \geq 11$$

$$\text{Possible values} = \{x \in \mathbb{R} : x \geq 11\}$$

Miscellaneous Exercise 3

Select the correct option in the following.

1. Which one is the standard form of linear equation? answer

(a) $ax^2 + b = 0$

(b) $ax + b = -c$

(c) $ax + b > 0$

(d) $ax + b = 0$

2. The exponent of the variable in linear equation is: answer

(a) 1

(b) 2

(c) -1

(d) 0

3. Which one is the linear equation in one variable? answer

(a) $ax + y = 0$

(b) $xy + 3 = 0$

(c) $2x + 3 = 0$

(d) $2x^2 + 3 = 0$

4. Which one is the solution of $12x + 17 = 65$? answer

(a) 48

(b) $\frac{82}{12}$

(c) 4

(d) $\frac{65}{12}$

5. What number must be subtracted from the right side of $7x = 30$ so that 4 is a solution of the resulting equation? answer

(a) 7

(b) 2

(c) 4

(d) -2



6. Which property of equality will be applied to solve the equation $-2x = \frac{2}{3}$? answer

(a) Addition

(b) Subtraction

(c) Division

(d) Addition and subtraction

7. Which one is the solution set of $5|x| = 25$? answer

(a) $\{-25, 25\}$

(b) $\{-25\}$

(c) $\{7\}$

(d) $\{-5, 5\}$

8. Which is the solution set of $|x| + 7 = 3$? answer

(a) $\{-4\}$

(b) $\{4, -4\}$

(c) $\{1\}$

(d) $\{-7, -3\}$

9. Which one is the solution set of $\sqrt{2x} = -10$? answer

(a) $\{1\}$

(b) $\{-20\}$

(c) $\{20\}$

(d) $\{-2\}$

10. Which one is the solution set of $\sqrt{3x+1} = 5$? answer

(a) 25

(b) 8

(c) 24

(d) $\frac{25}{3}$

11. Which one is the solution of $\frac{4}{x} - \frac{2}{x} = 5$? answer

(a) -1

(b) $\frac{2}{5}$

(c) $\frac{5}{2}$

(d) zero

12. Which one is a strict inequality? answer

(a) $x + 3 = 0$

(b) $12x > 5$

(c) $2y - 3 \leq 0$

(d) $4x + 5 \geq 0$

13. What should be the value of ' k ' if $x < y$ shows $kx > ky$? answer

(a) $k = 0$

(b) $k > 0$

(c) $k < 0$

(d) $k \geq 0$

14. Which one is the compound relation? answer

(a) $1 + 2x < 4 + x$

(b) $4x + 3 > 5\frac{3}{5}$

(c) $x + y > 5\frac{1}{2}$

(d) $x \leq 0$

15. Which one is the solution of $3 - \frac{1}{2}x \geq 0$? answer

(a) $x \geq -6$

(b) $x \leq 6$

(c) $x \geq 6$

(d) $x \leq -6$

16. Which one is the solution of $2(x+6) \leq 3(x+4)$? answer

(a) $x \leq 0$

(b) $x \geq 0$

(c) $x \leq 24$

(d) $x \geq 24$

Additional MCQs

17. The standard form of a linear equation in one variable x is: answer

(a) $ax + b > 0$

(b) $ax^2 + bx + c > 0$

(c) $ax^2 + bx^2 + cx + d > 0$

(d) $ax > 0$

18. The solution of $2y + 9 = 4$ is: answer

(a) $\frac{9}{2}$

(b) $\frac{4}{9}$

(c) $-\frac{9}{2}$

(d) $-\frac{5}{2}$

19. The perimeter of a rectangle is 20cm. If the length of the rectangle is 6cm, then its breadth will be: answer

(a) 4cm

(b) 8cm

(c) 10cm

(d) 14cm

20. The age of the father is three times the age of the son. If the age of the son is 15 years old, then the age of the father is: answer

(a) 50 years

(b) 55 years

(c) 40 years

(d) 45 years



21. The difference between the two whole numbers is 66. The ratio of the two numbers is 2 : 5. The two numbers are: answer

- (a) 60 and 6 (b) 100 and 33
(c) 110 and 44 (d) 99 and 33

22. Three consecutive integers add up to 51. The integers are: answer

- (a) 16, 17, 18 (b) 18, 19, 21
(c) 17, 18, 19 (d) 18, 19, 20

23. If a number is divided by 8 it gives 6 as the value. Find the number: answer

- (a) 3 (b) 42
(c) 48 (d) 56

24. If a number is x , such that $0 < x < 30$ then what is the smaller number? answer

- (a) $x-20$ (b) $10-x$
(c) $30x$ (d) all of these

25. If $a < b$ then the sign used between

$\frac{a}{-3} \square \frac{b}{-3}$ is of: answer

- (a) less than (b) equal to
(c) greater than
(d) greater than and equal to

26. The graph of the solution on number line of the inequality $2x-2 < 2x+1$ is: answer

- (a) 
(b) 
(c) 
(d) 

27. Let $^{\circ}\text{C} = \frac{5}{9}(^{\circ}\text{F} - 32)$. If $^{\circ}\text{C}$ lies between 10

- and 20, then: answer
(a) $50 < ^{\circ}\text{F} < 70$ (b) $50 < ^{\circ}\text{F} < 68$
(c) $40 < ^{\circ}\text{F} < 68$ (d) $40 < ^{\circ}\text{F} < 70$

28. The values of $(x+y)^2 = \dots\dots\dots$ answer

- (a) x^2+y^2 (b) $(x-y)^2$
(c) $\frac{x}{y}$ (d) $(x/y)^2$

29. The equation $2x-3 = 0$ is true for $x = \dots\dots\dots$ answer

- (a) $-\frac{3}{2}$ (b) $\frac{3}{2}$
(c) $\frac{2}{3}$ (d) All of these

Answer Key

1	d	2	a	3	c	4	c
5	b	6	c	7	d	8	c
9	a	10	b	11	b	12	b
13	c	14	d	15	b	16	b
17	a	18	d	19	d	20	a
21	d	22	c	23	a	24	a
25	c	26	c	27	c	28	d
29	a						

Solve the following:

Q.2 $\frac{2x-11}{12} = \frac{2x+10}{12} \rightarrow \left(\frac{28-2x-1}{4} \right)$ answer

Solution

$$\frac{2x-11}{12} = \frac{2x+10}{12} \rightarrow \left(\frac{28-2x-1}{4} \right) \dots (i)$$

$$\frac{2x-11}{12} = \frac{2x+10}{12} \rightarrow \left(\frac{28-2x-1}{4} \right)$$

(Taking L.C.M)

$$\frac{(2x-11) - (2x+10)}{12} = \frac{-(27-2x)}{4}$$

$$\frac{2x-11-2x-10}{12} = \frac{-(27-2x)}{4}$$

$$\frac{-2x-21}{12} = \frac{-27+2x}{4}$$

$$\frac{-2x-21}{4} = \frac{2x-27}{4}$$

$$-2x-21 = 2x-27$$

$$-2x-27 = 2x$$

$$20 = 2x$$

$$\frac{20}{2} = x$$

$$10 = x$$

$$\Rightarrow \boxed{x=10}$$

Check: Put $x = 10$ in eq. (i)

$$\frac{2(10)-11}{12} = \frac{2(10)+10}{12} \rightarrow \left(\frac{28-2(10)}{4} = \frac{1}{4} \right)$$

$$\frac{9}{12} = \frac{20^{10}}{12} \rightarrow \left(\frac{8}{4} = \frac{1}{4} \right)$$

$$\frac{9-10}{4} = \frac{7}{4}$$

$$\frac{1-10}{4} = \frac{7}{4}$$

$$\frac{1}{4} = \frac{1}{4}$$

It is true so $x = 10$ is required root

Hence, S.S = {10}

Q.3 $\sqrt{x-\frac{1}{2}} = \sqrt{\frac{2x+2}{5}}$ answer

Solution

$$\sqrt{x-\frac{1}{2}} = \sqrt{\frac{2x+2}{5}} \dots (i)$$

Taking L.C.M and square of both sides.

$$\left(\sqrt{\frac{2x-1}{2}} \right)^2 = \left(\sqrt{\frac{2x+2}{5}} \right)^2$$

$$\frac{2x-1}{2} = \frac{2x+2}{5}$$

By cross multiplication

$$5(2x-1) = 2(2x+2)$$

$$10x-5 = 4x+4$$

$$10x-4x = 4+5$$

$$6x = 9$$

$$x = \frac{9}{6}$$

$$x = \frac{3}{2}$$

$$\Rightarrow \boxed{x = \frac{3}{2}}$$



Check: Put $x = \frac{3}{2}$ in eq. (i)

$$\sqrt{\frac{3}{2}-\frac{1}{2}} = \sqrt{\frac{2\left(\frac{3}{2}\right)+2}{5}}$$

$$\sqrt{\frac{3-1}{2}} = \sqrt{\frac{3+2}{5}}$$

$$\sqrt{\frac{2}{2}} = \sqrt{\frac{5}{5}}$$

$$\sqrt{1} = \sqrt{1}$$

$$1 = 1$$

It is true so required root of given equation is

$$\frac{3}{2} \text{ Hence, S.S} = \left\{ \frac{3}{2} \right\}$$

Q.4 $\frac{\sqrt{5x-4}-4}{10} = -1$ answer

Solution

$$\frac{\sqrt{5x-4}-4}{10} = -1 \dots (i)$$

$$\sqrt{5x-4}-4 = -10$$

Multiplying B.S by 10

$$\sqrt{5x-4} = -10+4$$

$$\sqrt{5x-4} = -6$$

$$\sqrt{5x-4} = -6$$

Taking square of both sides.

$$\left(\sqrt{5x-4} \right)^2 = (-6)^2$$

$$5x-4 = 36$$

$$5x = 36+4$$

$$5x = 40$$

$$x = \frac{40}{5}$$

$$x = 8$$

$$\boxed{x=8}$$

Check: Put $x = 8$ in eq. (i)

$$\frac{\sqrt{5(8)-4}-4}{10} = -1$$

$$\frac{\sqrt{40-4}-4}{10} = -1$$

$$\frac{\sqrt{36}-4}{10} = -1$$

$$\frac{6-4}{10} = -1$$

$$\frac{2}{10} = -1$$

$$\frac{\sqrt{5x-4}-4}{10} = -1$$



$$\frac{6-4}{10} = -1$$

$$\frac{2}{10} = -1$$

$$\frac{1}{5} = -1$$

It is not true. Thus $x = 8$ is an extraneous root and solution set is empty set i.e. $S.S = \{\}$.

Q.5 $5 - (5y + 1) = -9$ ****

Solution

$$5 - (5y + 1) = -9 \quad || -10$$

$$5 + 9 = (5y + 1)$$

$$14 = (5y + 1)$$

$$\Rightarrow (5y + 1) = 14$$

$$\Rightarrow \pm (5y + 1) = 14$$

$$\Rightarrow 5y + 1 = \pm 14$$

$$\Rightarrow 5y + 1 = 14 \text{ or } 5y + 1 = -14$$

$$5y = 14 - 1 \text{ or } 5y = -14 - 1$$

$$5y = 13 \text{ or } 5y = -15$$

$$\boxed{y = \frac{13}{5}} \text{ or } y = -\frac{15}{5}$$

$$\text{or } \boxed{y = -3}$$

Check: Put $y = -3$ in eq. (i)

$$5 - (5(-3) + 1) = -9$$

$$5 - (-15 + 1) = -9$$

$$5 - (-14) = -9$$

$$5 - (-14) = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

$$-9 = -9$$

Q.6 $\frac{3}{4}x - 12 \geq x + 1, x \in \mathbb{R}$ ****

Solution

$$\frac{3}{4}x - 12 \geq x + 1, x \in \mathbb{R}$$

$$\frac{3}{4}x - x \geq 1 + 1$$

$$\frac{3x - 4x}{4} \geq 2$$

$$\frac{-x}{4} \geq 2$$

$$-x \geq 8$$

$$\Rightarrow -x \geq 8$$

$$\Rightarrow x \leq -8$$

$$S.S = \{x | x \in \mathbb{R} \wedge x \leq -8\}$$

Q.7 $4(2y + 3) - (6y - 1) > 10, y \in \mathbb{R}$ ****

Solution

$$4(2y + 3) - (6y - 1) > 10, y \in \mathbb{R}$$

$$8y + 12 - 6y + 1 > 10$$

$$2y + 13 > 10$$

$$2y > 10 - 13$$

$$2y > -3$$

$$y > -\frac{3}{2}$$

$$S.S = \{y \in \mathbb{R} \wedge y > -\frac{3}{2}\}$$

Q.8 $\frac{3}{2}x \leq -3 \text{ or } \frac{2}{3}x \geq 4, x \in \mathbb{R}$ ****

Solution

$$\frac{3}{2}x \leq -3 \text{ or } \frac{2}{3}x \geq 4, x \in \mathbb{R}$$

$$3x \leq -3 \times 2 \text{ or } 2x \geq 4 \times 3$$

$$3x \leq -6 \text{ or } 2x \geq 12$$

$$x \leq -\frac{6}{3} \text{ or } x \geq \frac{12}{2}$$

$$\boxed{x \leq -2} \text{ or } \boxed{x \geq 6}$$

$$\{x | x \in \mathbb{R} \wedge x \leq -2\} \text{ or } \{x | x \in \mathbb{R} \wedge x \geq 6\}$$

$$\text{Now, the union of both the solution set is}$$



$$\{x | x \in \mathbb{R} \wedge x \leq -2 \text{ or } x \geq 6\}$$

Q.9 The difference between three times x number y and 18 is less than 12 or greater than 39. What real numbers do y represent? ****

Solution

The difference between three times x number y and 18 = $3y - 18$

By given condition

$$3y - 18 < 12 \text{ or } 3y - 18 > 39$$

$$3y < 12 + 18 \text{ or } 3y > 39 + 18$$

$$3y < 30 \text{ or } 3y > 57$$

$$y < \frac{30}{3} \text{ or } y > \frac{57}{3}$$

$$\boxed{y < 10} \text{ or } \boxed{y > 19}$$

$$\{y | y \in \mathbb{R} \wedge y < 10\} \text{ or } \{y | y \in \mathbb{R} \wedge y > 19\}$$

Thus the union of both the solution sets is:

$$\{y | y \in \mathbb{R} \wedge y < 10 \text{ or } y > 19\}$$



UNIT 06

Trigonometry and Bearing

Measurement of an angle

Union of two rays having common end point forms an angle. An angle is positive or negative if measured in anti-clockwise or clockwise direction respectively.



Angle in standard position

An angle when drawn in the cartesian plane (triangular coordinate system) is in standard position if its vertex is at the origin and its initial arm is directed along positive x-axis. Its terminal arm may lie either on any of the axis or in any of the quadrants.

- Sexagesimal System or English system
- Radian measure or Circular System
- Centesimal or French System

Sexagesimal System

Draw a circle and divide its boundary into 360 equal arcs. The central angle of each arc is of measure 1 degree (1°).

To draw an angle of 30° . We need 30 consecutive arcs of 1° . As 90° is one fourth of 360° , so 90 consecutive arcs of 1° form a quadrant of the circle, i.e., central angle of quadrant of a circle is 90° . There are 180 arcs of 1° in a half circle and all 360 arcs form a complete circle, i.e., central angle of half circle is 180° and that of full circle is 360° . To measure the central angle of an arc, whose

length is smaller than the length of arc of 1° , we divide the arc of 1° into further 60 equal arcs. The central angle of each of these small arcs is of measure 1 minute denoted as $1'$. So, there are 60 in a degree. Similarly, there are 60 seconds ($60''$) in a minute thus $1^\circ = 60'$, $1' = 60''$, $1^\circ = 3600''$

$$1^\circ = \left(\frac{1}{60}\right)', 1' = \left(\frac{1}{60}\right)'', 1'' = \left(\frac{1}{3600}\right)^\circ$$

In sexagesimal system, an angle is measured in Degrees, Minutes and Seconds.

Example 1

Convert the following measures of angles into seconds.

- 30°
- $60'$
- $45^\circ 45' 45''$

Solution:

- $30^\circ = 30 \times 60' = 1800'$
- $60' = 60 \times 60'' = 3600''$
 $3600 \times 60'' = 216000''$
- $45^\circ 45' 45''$
 $= 45^\circ + 45' + 45''$
 $= 45 \times 3600'' + 2700'' + 45''$
 $= 162745''$

Example 2

Convert the following measures of angles into minutes.

- 36°
- $45'$
- $36^\circ 45' 30''$

Solution

- $36^\circ = 36 \times 60' = 2160'$
- $45' = \left(45 \times \frac{1}{60}\right)^\circ = \left(\frac{3}{4}\right)^\circ = 0.75^\circ$

$$\text{iii) } 36^\circ 45' 30'' = 36^\circ + 45' + 30''$$

$$= 36^\circ + 60' + 45' + \left(\frac{30}{60}\right)^\circ$$

$$= 2160' + 45' + 0.5' = 2205.5'$$

Example 3

Write the following measures of angles into $D^\circ M' S''$.

- 60.5°
- 45.125°
- $3600''$
- $216000''$
- $20.555'$

Solution

- $60.5^\circ = 60^\circ + 0.5^\circ$
 $= 60^\circ + 0.5 \times 60'$
 $= 60^\circ + 30' = 60^\circ 30'$
- $45.125^\circ = 45^\circ + 0.125^\circ$
 $= 45^\circ + 0.125 \times 60'$
 $= 45^\circ + 7.5'$
 $= 45^\circ + 7' + 0.5 \times 60'' = 45^\circ 7' 30''$

$$\text{iii) } 3600'' = \left(\frac{3600}{60}\right)' = 60' = 1^\circ = 1^\circ 0' 0''$$

$$\text{iv) } 216000'' = \left(\frac{216000}{60}\right)' = 3600'$$

$$= \left(\frac{3600}{60}\right)^\circ = 60^\circ 0' 0''$$

$$\text{v) } 20.555' = 20' + 0.555'$$

$$= 20' + 0.555 \times 60''$$

$$= 20' + 33.3'' = 20^\circ 20' 33''$$

Example 4

Write the following measures of angles in degree correct to 4 decimal places.

- $60^\circ 20' 20''$
- $75^\circ 25' 15''$

Solution

$$\text{i) } 60^\circ 20' 20'' = 60^\circ + \left(\frac{20}{60}\right)' + \left(\frac{20}{3600}\right)^\circ$$

$$= 60^\circ + 0.3^\circ + 0.0055^\circ = 60.3055^\circ$$

$$\text{ii) } 75^\circ 25' 15'' = 75^\circ + \left(\frac{25}{60}\right)' + \left(\frac{15}{3600}\right)^\circ$$

$$= 75^\circ + 0.4167^\circ + 0.0042^\circ$$

$$= 75.4209^\circ$$

Circular System or Radian Measure

The ratio between length of circular arc and radius is called radian. If length of an arc is equal to radius of circle, then their ratio is 1 radian, i.e., one radian is measure of central angle of an arc whose length is equal to radius of the circle.



The ratio between circumference and radius of a circle is 2π , therefore measure of a complete angle in circular system is 2π radians.

In circular system, an angle is measured in radian.

In sexagesimal system measure of a complete angle is 360° .

$$\text{So } 2\pi \text{ radians} = 360^\circ$$

$$1 \text{ radian} = 180^\circ$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ$$

$$= 37^\circ 17' 45''$$

$$\text{Similarly, } 1^\circ = \frac{\pi}{180} \text{ Radian} = 0.0175 \text{ radian.}$$

To convert measure of an angle given in radian to degree, multiply value by $\left(\frac{180}{\pi}\right)^\circ$ and to convert measure of an angle given in degree to radian, multiply its value by $\frac{\pi}{180}$ radian.

Example 5

Convert the following measures of angles in sexagesimal system ($D^\circ M' S''$).

- 1.125 radian
- 0.5 radian
- $\frac{\pi}{5}$ radian
- $\frac{3\pi}{4}$ radian



Solution

$$(i) 2.125 \text{ radians} = 2.125 \times \left(\frac{180}{\pi}\right)^\circ$$

$$= 2.125 \times 57.2958^\circ$$

$$= 121.7335^\circ$$

$$= 121^\circ + 0.7335 \times 60'$$

$$= 121^\circ + 43.21'$$

$$= 121^\circ + 43' + 0.21 \times 60'' = 121^\circ 43' 12''$$

$$(ii) 0.5 \text{ radians} = 0.5 \times \left(\frac{180}{\pi}\right)^\circ$$

$$= 0.5 \times 57.2958^\circ$$

$$= 28.6479^\circ$$

$$= 28^\circ + 0.6479 \times 60'$$

$$= 28^\circ + 38.874'$$

$$= 28^\circ + 38' + 0.874 \times 60'' = 28^\circ 38' 52''$$

$$(iii) \frac{\pi}{3} \text{ radians} = \frac{\pi}{3} \times \left(\frac{180}{\pi}\right)^\circ$$

$$= \left(\frac{180}{3}\right)^\circ = 60^\circ$$

$$(iv) \frac{2\pi}{4} \text{ radians} = \frac{2\pi}{4} \times \left(\frac{180}{\pi}\right)^\circ$$

$$= \left(\frac{360}{4}\right)^\circ = 90^\circ$$

Example 4

Convert the following measures of angles in radians.

$$(i) 45^\circ \quad (ii) 150^\circ \quad (iii) 300^\circ \quad (iv) 30^\circ 30' 30''$$

Solution

$$(i) 45^\circ = 45 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{4} \text{ radians} = 0.7854 \text{ radians}$$

$$(ii) 150^\circ = \frac{\pi}{180} \text{ radians} \\ = \frac{5\pi}{6} \text{ radians} = 2.618 \text{ radians}$$

$$(iii) 300^\circ = 300 \times \frac{\pi}{180} \text{ radians} \\ = \frac{5\pi}{3} \text{ radians} = 5.236 \text{ radians}$$

$$(iv) 30^\circ 30' 30'' = 30.5083^\circ \\ = 30.5083 \times \frac{\pi}{180} \text{ radians} \\ = 0.5325 \text{ radians}$$

Exercise - 6.1

Q.1 Convert the following measure of angles in seconds.

Solution

$$(i) 5''$$

$$5'' = 5 \times 1'' = 5 \times 60' = 300'$$

$$= 300 \times 1'' = 300 \times 60''$$

$$= 18,000''$$

$$(ii) 30'$$

$$30' = 30 \times 1'$$

$$= 30 \times 60'' = 1800''$$

$$(iii) 10^\circ 30'$$

$$10^\circ 30' = (10 \times 60') + 30'$$

$$= 600' + 30' = 630'$$

$$= 630 \times 60'' = 37,800''$$

$$(iv) 20^\circ 20' 20''$$

$$20^\circ 20' 20'' = (20 \times 60') + (20 \times 60'') + 20''$$

$$= (1200') + (1200'') + 20''$$

$$= (1200 \times 60'') + 1220''$$

$$= (72000'') + 1220''$$

$$= 73,220''$$



Q.2 Convert the following measure of angles in minutes.

Solution

$$(i) 75'$$

$$75' = 75 \times 1'$$

$$= 75 \times 60'' = 4500''$$

$$(ii) 120''$$

$$120'' = 120 \times 1'' = \left(120 \times \frac{1}{60}\right)^\circ = 2^\circ$$

$$(iii) 50^\circ 40'$$

$$50^\circ 40' = (50 \times 1^\circ) + 40'$$

$$= (50 \times 60') + 40'$$

$$= 3000' + 40' = 3,040'$$

$$(iv) 24^\circ 30' 30''$$

$$24^\circ 30' 30'' = (24 \times 1^\circ) + (30') + (30 \times 1'')$$

$$= (24 \times 60') + (30') + \left(30 \times \left(\frac{1}{60}\right)^\circ\right)$$

$$= 1440' + 30' + 0.5'$$

$$= 1,470.5'$$

Q.3 Convert the following measure of angles in degrees and write the answer correct to 4 decimal.

Solution

$$(i) 135'$$

$$135'$$

$$= 135 \times 1'$$

$$= 135 \times \left(\frac{1}{60}\right)^\circ = 2.25^\circ$$

$$(ii) 150''$$

$$150'' = 150 \times 1''$$

$$= 150 \times \left(\frac{1}{60}\right)^\circ = 2.5^\circ$$

$$= 2.5 \times \left(\frac{1}{60}\right)^\circ = 0.0417^\circ$$

$$(iii) 60^\circ 60'$$

$$60^\circ 60'$$

$$= 60^\circ + 60'$$

$$= 60^\circ + 1^\circ = 61^\circ$$

$$(iv) 45^\circ 45' 45''$$

$$45^\circ 45' 45''$$

$$= 45^\circ + 45 \times 1' + 45 \times 1''$$

$$= 45^\circ + \left(45 \times \frac{1}{60}\right)^\circ + \left(45 \times \frac{1}{60 \times 60}\right)^\circ$$

$$= 45^\circ + (0.75)^\circ + (0.0125)^\circ$$

$$= 45.7625^\circ$$

Q.4 Write the following measure of angles in D°M'N''.

Solution

$$(i) 60.125^\circ$$

$$60.125^\circ$$

$$= 60^\circ + 0.125^\circ$$

$$= 60^\circ + (0.125 \times 60')$$

$$= 60^\circ + 7.5'$$

$$= 60^\circ + 7' + (0.5 \times 60'')$$

$$= 60^\circ + 7' + 30''$$

$$= 60^\circ 7' 30''$$

$$(ii) 135.375^\circ$$

$$135.375^\circ$$

$$= (135.375 \times \frac{1}{60} \times \frac{1}{60})'$$

$$= 0.03750'$$

$$= 0' + 0.0375'$$

$$= 0' + (0.0375 \times 60'')$$

$$= 0' + 2.250''$$

$$= 0' + 2' + 0.250''$$

$$= 0' + 2' + (0.250 \times 60'')$$

$$= 0' + 2' + 15.00''$$

$$= 0' + 2' + 15''$$

$$= 0^\circ 2' 15''$$



(iii) 60.85°

60.85°

$$= \left(60.85 \times \frac{1}{60} \right)^\circ$$

$$= 1.01416^\circ$$

$$= 1^\circ + 0.01416^\circ$$

$$= 1^\circ + (0.01416 \times 60)''$$

$$= 1^\circ + 0.8496''$$

$$= 1^\circ + 0' + 0.8496''$$

$$= 1^\circ + 0' + (0.8496 \times 60)'''$$

$$= 1^\circ + 0' + 50.976'''$$

$$= 1^\circ + 0' + 31''$$

$$= 1^\circ 31''$$

(iv) 255.45°

255.45°

$$= 255^\circ + 0.45^\circ$$

$$= 255^\circ + (0.45 \times 60)'$$

$$= 255^\circ + 27'$$

$$= 255^\circ + 27' + 00''$$

$$= 255^\circ 27' 00''$$

Q.5 Convert the following in radians. Write the answers in terms of π .

Solution

(i) 45°

45°

$$= 45 \times 1'$$

$$= 45 \times \frac{\pi}{180} \text{ rad}$$

$$= 45 \times \frac{\pi}{180} \text{ rad}$$

$$= \frac{\pi}{4} \text{ rad}$$

(ii) 150°

150°

$$= 150 \times 1'$$

$$= 150 \times \frac{\pi}{180} \text{ rad}$$

$$= 150 \times \frac{\pi}{180} \text{ rad (Dividing by 30)}$$

$$= \frac{5\pi}{6} \text{ rad}$$

(iii) $40^\circ 30'$

$40^\circ 30'$

$$= 40.5^\circ$$

$$\because 30' = 0.5^\circ$$

$$= 40.5 \times 1'$$

$$= 40.5 \times \frac{\pi}{180} \text{ rad}$$

$$= \frac{40.5}{180} \times \frac{\pi}{180} \text{ rad}$$

$$= \frac{40.5}{180} \times \frac{\pi}{180} \text{ rad (Dividing by 3)}$$

$$= \frac{121\pi}{360} \text{ rad}$$

(iv) 120°

120°

$$= 120 \times 1'$$

$$= 120 \times \frac{\pi}{180} \text{ rad}$$

$$= 120 \times \frac{\pi}{180} \text{ rad (Dividing by 60)}$$

$$= \frac{2\pi}{3} \text{ rad}$$

Q.6 Convert the following in radians.

Use $\pi = 3.1416$

Solution

(i) 170°

170°

$$= 170 \times 1'$$

$$= 170 \times \frac{\pi}{180} \text{ rad}$$

$$= 170 \times \frac{\pi}{180} \text{ rad (Dividing by 90)}$$

$$= \frac{17\pi}{9} \text{ rad}$$

$$= \frac{17 \times 3.1416}{9} \text{ rad}$$

$$= \frac{53.4072}{9} \text{ rad}$$

$$= 5.934 \text{ rad}$$

(ii) 60°

60°

$$= 60 \times 1'$$

$$= 60 \times \frac{\pi}{180} \text{ rad}$$

$$= 60 \times \frac{\pi}{180} \text{ rad}$$

$$= \frac{\pi}{3} \text{ rad (Dividing by 60)}$$

$$= \frac{3.1416}{3} \text{ rad}$$

$$= 1.05 \text{ rad}$$

(iii) $180^\circ 45'$

$180^\circ 45'$

$$= 180^\circ + 45'$$

$$= 180^\circ + \left(\frac{45}{60} \right)^\circ$$

$$= 180^\circ + 0.75^\circ$$

$$= 180.75^\circ$$

$$= 180.75 \times 1'$$

$$= 180.75 \times \frac{\pi}{180} \text{ rad}$$

$$= 180.75 \times \frac{3.1416}{180} \text{ rad}$$

$$= \frac{567.8442}{180} \text{ rad}$$

$$= 3.15 \text{ rad}$$



(iv) $75^\circ 30' 45''$

$75^\circ 30' 45''$

$$= 75^\circ + \left(\frac{30}{60} \right)^\circ + \left(\frac{45}{60 \times 60} \right)^\circ$$

$$= 75^\circ + 0.5^\circ + 0.0125^\circ$$

$$= 75.5125^\circ$$

$$= 75.5125 \times 1'$$

$$= 75.5125 \times \frac{\pi}{180} \text{ rad}$$

$$= 75.5125 \times \frac{3.1416}{180} \text{ rad}$$

$$= \frac{237.1416}{180} \text{ rad}$$

$$= 1.3179 \text{ rad}$$

$$= 1.32 \text{ rad}$$

Q.7 Write the following radian measure of angles in $D^\circ M' S''$.

Solution: First we convert radian to degree, the degree to $D^\circ M' S''$ form.

$$(i) \frac{\pi}{4} \text{ rad}$$

$$\frac{\pi}{4} \text{ rad} = \frac{1}{4} \pi \text{ rad}$$

$$= \frac{1}{4} \times 180^\circ$$

$$= 45^\circ \text{ or } 45^\circ 00' 00''$$

$$(ii) \frac{5\pi}{6} \text{ rad}$$

$$\frac{5\pi}{6} \text{ rad} = \frac{5}{6} \pi \text{ rad}$$

$$= \frac{5}{6} \times 180^\circ$$

$$= \frac{5}{6} \times 180^\circ = 150^\circ$$

$$= 5 \times 30^\circ$$

$$= 150^\circ \text{ or } 150^\circ 00' 00''$$



$$(iii) \frac{\pi}{12} \text{ rad}$$

$$\frac{\pi}{12} \text{ rad} = \frac{1}{12} \pi \text{ rad.}$$

$$= \frac{1}{12} \times 180^\circ \quad (\because \pi \text{ rad} = 180^\circ)$$

$$= \frac{1}{12} \times 180^\circ = 15^\circ$$

$$= 15^\circ \text{ or } 15^\circ 00' 00''$$

$$(iv) \frac{7\pi}{40} \text{ rad.}$$

$$\frac{7\pi}{40} \text{ rad} = \frac{7}{40} \pi \text{ rad.}$$

$$= \frac{7}{40} \times 180^\circ \quad (\because \pi \text{ rad} = 180^\circ)$$

$$= \frac{7}{40} \times 180^\circ = 31.5^\circ$$

$$= \frac{63^\circ}{2}$$

$$= 31.5^\circ$$

$$= 31^\circ + 0.5^\circ$$

$$= 31^\circ + (0.5 \times 60')^\circ$$

$$= 31^\circ + 30'$$

$$= 31^\circ 30' 00''$$

$$(v) 1 \text{ rad}$$

We know that

$$\pi \text{ rad} = 180^\circ$$

$$1 \text{ rad} = \frac{180^\circ}{\pi}$$

$$= \frac{180^\circ}{3.1416}$$

$$= 57.29563^\circ$$

$$= 57^\circ + 0.29563^\circ$$

$$= 57^\circ + (0.29563 \times 60')^\circ$$

$$= 57^\circ + 17.739'$$

$$= 57^\circ + 17' + 0.739''$$

$$= 57^\circ + 17' + (0.739 \times 60'')^\circ$$

$$= 57^\circ + 17' + 44.34''$$

$$= 57^\circ + 17' + 44''$$

$$= 57^\circ 17' 44''$$

$$(vi) 3.1416 \text{ rad}$$

$$3.1416 \text{ rad} = 3.1416 \times 1 \text{ rad.}$$

$$= 3.1416 \times \frac{180^\circ}{\pi}$$

$$= 3.1416 \times \frac{180^\circ}{3.1416} \quad (\because \pi = 3.1416)$$

$$= 180^\circ \text{ or } 9^\circ$$

$$(vii) 12\pi \text{ rad}$$

$$12\pi \text{ rad} = 12 \times \pi \text{ rad.}$$

$$= 12 \times 180^\circ$$

$$(\because \pi \text{ rad} = 180^\circ)$$

$$= 2160^\circ$$

$$= 2160^\circ \text{ or } 2160^\circ 00' 00''$$

$$(viii) 5 \text{ rad}$$

$$1 \text{ Radian} = \left(\frac{180^\circ}{\pi} \right)$$

$$= 5 \text{ Radian} = 5 \times \left(\frac{180^\circ}{\pi} \right)$$

$$= 5 \text{ Radian} = 5 \times \frac{180^\circ}{3.1416} = 289.4789^\circ$$

$$= 289^\circ + 0.4789^\circ$$

$$= 289^\circ + (0.4789 \times 60')^\circ$$

$$= 289^\circ + 28.734'$$

$$= 289^\circ + 28' + 0.734''$$

$$= 289^\circ + 28' + (0.734 \times 60'')^\circ$$

$$= 289^\circ + 28' + 44.04''$$

$$= 289^\circ 28' 44''$$

Sector of a Circle

Length of an Arc

Length of a circular arc and radian measure of its central angle are directly proportional. I.e.,



ratio between length of an arc and radian measure of its central angle remains constant. Consider a circle of radius r with centre O . Choose an arc of length l on circumference. Let θ be the radian measure of central angle of that arc. Now choose another arc whose central angle is of 2 radian. The length (l_2) of each arc is equal to the radius. I.e. $l_2 = r$.

$$l_1 : \theta :: l_2 : 2$$

$$l_1 : \theta :: r : 2$$

$$l_1 = 1; \theta = 1$$

$$l_1 = r$$

So, length of a circular arc is product of radius and radian measure of central angle.



Fig. 10

Area of Sector of a Circle

The union of a circle and its interior region is the circular region. The circular region bounded by an arc and its two corresponding radial segments is called sector of the circle. In figure (i) shaded region is a sector of the circle with radius r and θ the radian measure of central angle. Like arc length, area (A) of sector is directly proportional to the radian measure of central angle.

Area of sector : $\theta ::$ area of circle : complete angle

$$A : \theta :: \pi r^2 : 2\pi$$

$$2\pi A = \pi r^2 \theta$$

$$A = \frac{\pi r^2 \theta}{2\pi}$$

$$A = \frac{r^2 \theta}{2}$$

Do you know?

From elementary geometry:

Arc length (l)

$$= 2\pi r$$

$$= \text{Circumference of circle (C)} = \frac{\text{Area of sector of circle (A)}}{2\pi}$$

$$\begin{aligned} \frac{l}{2\pi r} &= \frac{\theta}{2\pi} = \frac{A}{\pi r^2} \\ \Rightarrow \frac{l}{2\pi r} &= \frac{\theta}{2\pi} \text{ and } \frac{\theta}{2\pi} = \frac{A}{\pi r^2} \\ \Rightarrow \text{Arc length (l)} &= \theta r \\ \text{Area of sector (A)} &= \frac{1}{2} l r \end{aligned}$$

Example 7

Find length of an arc of a circle whose central angle and radius are given below.

$$(i) \theta = \frac{\pi}{4} \text{ rad}, r = 2 \text{ cm} \quad (ii) \theta = 60^\circ, r = 30 \text{ cm}$$

Solution:

$$(i) \quad l = \theta r$$

$$l = \frac{\pi}{4} \times 2 \text{ cm}$$

$$l = \frac{3.14}{2}$$

$$l = 1.57 \text{ cm.}$$

$$(ii) \theta = 60^\circ, r = 30 \text{ cm}$$

$$\theta = 60^\circ = \left(\frac{\pi}{3} \right) \text{ rad.}$$

$$\text{Now, } l = \theta r$$

$$l = 30 \times \frac{\pi}{3}$$

$$l = 10\pi \text{ cm}$$

$$l = 10 \times 3.14$$

$$l = 31.4 \text{ cm}$$

Example 8

Find area of the sector of a circle whose central angle is 120° and radius 5 cm.

Solution:

$$r = 5 \text{ cm.}$$

$$\theta = 120^\circ = \frac{2\pi}{3} \text{ rad}$$

$$A = \frac{r^2 \theta}{2} = \frac{1}{2} (5)^2 \left(\frac{2\pi}{3} \right)$$

$$A = \frac{25\pi}{3} \text{ cm}^2 = 26.18 \text{ cm}^2$$

Example 9

The length of an arc and area of sector of a circle are 4 cm and 16 cm^2 respectively. Find radius and central angle of sector.

Solution:

$$l = 4 \text{ cm, } A = 16 \text{ cm}^2, r = ?, \theta = ?$$



$$\text{As, } l = r\theta$$

$$\Rightarrow r\theta = 4$$

$$\theta = \frac{l}{r} = \frac{4}{\frac{1}{2}} = \dots\dots\dots$$

$$\text{Now } A = \frac{1}{2}r^2\theta$$

$$16 = \frac{1}{2}r^2 \times \frac{4}{\frac{1}{2}} \text{ (substituting the value of } \theta)$$

$$16 = 2r$$

$$r = 8 \text{ cm}$$

Put $r = 8$ cm in equation (i)

$$\therefore \theta = \frac{4}{r}$$

$$\theta = \frac{4}{8}$$

$$\theta = \frac{1}{2} \text{ rad.}$$

EXERCISE 4C.2

Q.1 Find length of arc and area of sector for given radius and central angle.

$$(i) r = 5 \text{ cm, } \theta = \frac{\pi}{3} \text{ radians}$$

Arc length of sector

We know that

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$l = 5 \times \frac{\pi}{3}$$

$$l = 5 \times \frac{3.1416}{3}$$

$$l = \frac{15.708}{3}$$

$$l = 5.236 \text{ cm}$$

$$l = 5.24 \text{ cm}$$

Area of sector

We know that

$$\text{Area of sector} = \frac{1}{2}r^2\theta \quad [\theta \text{ in rad}]$$

$$A = \frac{1}{2}(5)^2 \times \frac{\pi}{3}$$

$$A = \frac{1}{2} \times 25 \times \frac{3.1416}{3}$$

$$A = \frac{78.34}{6}$$

$$A = 13.09 \text{ cm}^2$$

$$(ii) r = 12 \text{ m, } \theta = 120^\circ$$

First we convert angle in radians.

$$\theta = 120^\circ = 120 \times \frac{\pi}{180} = \frac{2\pi}{3} \text{ rad}$$

Arc length of sector

We know that

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$l = 12 \times \frac{2\pi}{3}$$

$$l = \frac{24\pi}{3}$$

$$l = 8\pi$$

$$l = 8 \times 3.1416$$

$$l = 25.1328 \text{ m}$$

$$l = 25.13 \text{ m}$$

Area of sector:

We know that

$$\text{Area of sector} = \frac{1}{2}r^2\theta \quad [\theta \text{ in rad}]$$

$$A = \frac{1}{2}(12)^2 \times \frac{2\pi}{3}$$

$$A = \frac{1}{2} \times 144 \times \frac{2\pi}{3}$$

$$A = 24 \times 2\pi$$

$$A = 48\pi$$

$$A = 48 \times 3.1416$$

$$A = 150.868 \text{ m}^2$$

$$A = 150.87 \text{ m}^2$$

$$(iii) r = 6.4 \text{ dm, } \theta = 60^\circ 45' 30''$$

First we convert angle in radians.

$$\theta = 60^\circ 45' 30''$$

$$\theta = 60^\circ + \left(\frac{45}{60}\right)^\circ + \left(\frac{30}{60 \times 60}\right)^\circ$$

$$\theta = 60^\circ + (0.75)^\circ + (0.008333)^\circ$$

$$\theta = 60.75833^\circ$$

$$\theta = 60.75833 \times \frac{\pi}{180} \text{ rad.}$$

$$\theta = 1.0604 \text{ rad.}$$

Arc length of sector

We know that

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$l = 6 \times 1.0604$$

$$l = 6.3624 \text{ dm}$$

$$l = 6.36 \text{ dm}$$

Area of sector

We know that

$$\text{Area of sector} = \frac{1}{2}r^2\theta \quad [\theta \text{ in rad}]$$

$$A = \frac{1}{2}(6)^2 \times 1.0604$$

$$A = \frac{1}{2} \times 36 \times 1.0604$$

$$A = 18 \times 1.0604$$

$$A = 19.0872 \text{ dm}^2$$

$$A = 19.09 \text{ dm}^2$$

Q.2 A central angle in a circle of radius 4 cm is 75° . Find the length of the intercepted arc and area bounded by the sector.

Solution:

$$\text{Radius} = r = 4 \text{ cm}$$

$$\text{Central angle} = \theta = 75^\circ$$

First we convert angle in radians.



$$\theta = 75^\circ = 75 \times \frac{\pi}{180} \text{ rad.} = \frac{5\pi}{12} \text{ rad}$$

Arc length of sector

We know that

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$l = 4 \times \frac{5\pi}{12}$$

$$l = \frac{5\pi}{3}$$

$$l = \frac{5 \times 3.1416}{3}$$

$$l = \frac{15.708}{3}$$

$$l = 5.24 \text{ cm}$$

Area of sector

We know that

$$\text{Area of sector} = \frac{1}{2}r^2\theta \quad [\theta \text{ in rad}]$$

$$A = \frac{1}{2}(4)^2 \times \frac{5\pi}{12}$$

$$A = 16 \times \frac{5\pi}{24}$$

$$A = 16 \times \frac{5\pi}{24}$$

$$A = \frac{10\pi}{3}$$

$$A = \frac{10 \times 3.1416}{6}$$

$$A = \frac{31.416}{3}$$

$$A = 10.472 \text{ cm}^2$$

$$A = 10.47 \text{ cm}^2$$

Q.3 Find radian measure of the central angle of a circular sector for the following data.

$$(i) l = 5 \text{ cm, } r = 4 \text{ cm}$$



We know that

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$8 = 4 \times \theta$$

$$\frac{8}{4} = \theta$$

$$2 = \theta$$

$$\Rightarrow \theta = 2 \text{ rad.}$$

$$(ii) l = 8.3 \text{ m, } r = 2.25 \text{ m}$$

We know that

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$8.3 = 2.25 \times \theta$$

$$\frac{8.3}{2.25} = \theta$$

$$3.78 = \theta$$

$$\Rightarrow \theta = 3.78 \text{ rad.}$$

$$(iii) A = 45 \text{ cm}^2, r = 10.70 \text{ cm}$$

We know that

$$\text{Area of sector} = \frac{1}{2} r^2 \theta \quad [\theta \text{ in rad}]$$

$$45 = \frac{1}{2} (10.7)^2 \times \theta$$

$$2 \times 45 = 114.49 \times \theta$$

$$\frac{90}{114.49} = \theta$$

$$0.786 = \theta$$

$$\theta = 0.78 \text{ rad.}$$

$$(iv) A = 100 \text{ cm}^2, l = 10 \text{ cm}$$

$$l = r\theta$$

$$r = \frac{l}{\theta} \quad (i)$$

$$\text{Area of sector} = \frac{1}{2} r^2 \theta \quad [\theta \text{ in rad}]$$

$$\text{Area} = \frac{1}{2} \left(\frac{l}{\theta} \right)^2 \theta \quad \left(\because \text{Putting } r = \frac{l}{\theta} \text{ from (i)} \right)$$

$$\text{Area} = \frac{1}{2} \times \frac{l^2}{\theta} = \theta$$

$$\text{Area} = \frac{l^2}{2\theta}$$

$$\Rightarrow \theta = \frac{l^2}{2 \times \text{Area}}$$

$$\theta = \frac{(10)^2}{2 \times 100}$$

$$\theta = \frac{1 \times 100}{2 \times 100}$$

$$\theta = \frac{1}{2} = 0.5$$

$$\theta = 0.5 \text{ rad}$$

Q.4 Find radius of circle for the following information.

$$(i) l = 4 \text{ cm, } \theta = \pi \text{ radians}$$

We know that

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$4 = r \times \pi$$

$$\frac{4}{\pi} = r$$

$$1.27 \text{ m} = r$$

$$\Rightarrow r = 1.27 \text{ cm}$$

$$(ii) l = 6 \text{ m, } \theta = 15^\circ$$

 First we convert θ into radians

$$\theta = 15^\circ \times \frac{\pi}{180}$$

$$\theta = \frac{\pi}{12} \text{ rad.}$$

We know that

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$6 = r \times \frac{\pi}{12}$$

$$\frac{6 \times 12}{\pi} = r$$

$$\frac{72}{\pi} = r$$

$$22.9 \text{ m} = r$$

$$\Rightarrow r = 22.9 \text{ m}$$

$$(iii) A = 200 \text{ cm}^2, \theta = \frac{\pi}{4} \text{ radians}$$

$$\text{Area of sector} = \frac{1}{2} r^2 \theta \quad [\theta \text{ in rad}]$$

$$200 = \frac{1}{2} r^2 \times \frac{\pi}{4}$$

$$200 = \frac{\pi r^2}{8}$$

$$\frac{200 \times 8}{\pi} = r^2$$

$$\frac{1600}{3.1416} = r^2$$

$$r^2 = 509.29$$

$$\Rightarrow \sqrt{r^2} = \sqrt{509.29}$$

$$r = 22.57 \text{ cm}$$

$$(iv) A = 100 \text{ dm}^2, l = 10 \text{ dm}$$

We know that

Arc length

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$\text{or } \theta = \frac{l}{r} \quad (i)$$

Area of sector

$$A = \frac{1}{2} r^2 \theta \quad (ii) \quad [\theta \text{ in rad}]$$

 Putting the value of θ in eq.(ii)

$$A = \frac{1}{2} r^2 \times \frac{l}{r}$$

$$A = \frac{1}{2} r \times l$$

Now putting the values, we get

$$100 = \frac{1}{2} r \times 10$$

$$100 = r \times 5$$

$$\frac{100}{5} = r$$

$$20 = r$$

$$\Rightarrow r = 20 \text{ dm}$$

 Q.5 A 30 inch pendulum swings through an angle of 30° . Find the length of the arc in inches through which the tip of the pendulum swings?

Solution:

 Length of pendulum = radius = $r = 30$ inch

 Angle of swing = $\theta = 30^\circ$

 Length of the arc = $l = ?$

We know that

$$\theta = 30^\circ = 30^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{6} \text{ rad.}$$

Now finding arc length

$$l = r\theta \quad [\theta \text{ in rad}]$$

$$l = 30 \times \frac{\pi}{6}$$

$$l = 5\pi \times \frac{\pi}{6}$$

$$l = 5\pi$$

$$l = 5 \times 3.1416$$

$$l = 15.71 \text{ inches}$$

Q.6 A motorcycle is travelling on a curve along a highway. The curve is an arc of a circle with radius of 10 km. If the motorcycle's speed is 42 km/h, what is the angle in degrees through which the motorcycle will turn in 21 minutes?

Solution:

 Radius of arc = $r = 10$ km

 Speed of motorcycle = $v = 42$ km/h

 Angle in degree = $\theta = ?$

 Travelling time = $t = 21$ minutes

Distance travelled in 60 min. (hour) = 42 km

 Distance travelled in t min. = $\frac{42}{60}$ km

 Distance travelled in 21 min. = $\frac{42}{60} \times 21$ km



$$= 14.7 \text{ km}$$

In this case distance travelled is equal to arc length. So $l = 14.7 \text{ km}$

We know that

$$l = r\theta \quad \boxed{\theta \text{ in rad}}$$

$$14.7 = 10 \times \theta$$

$$\frac{14.7}{10} = \theta$$

$$1.47 = \theta$$

$$\Rightarrow \theta = 1.47 \text{ rad.}$$

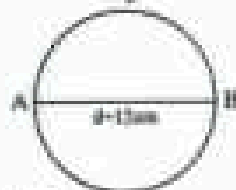
$$\theta = 1.47 \times \frac{180^\circ}{\pi} \quad \boxed{1^\circ \text{ rad} = \frac{180^\circ}{\pi}}$$

$$\theta = 1.47 \times \frac{180^\circ}{3.1416}$$

$$\theta = 84.22^\circ$$

Q.7 Find perimeter and area of a half circle in the following figure by using the formulae

$$l = r\theta, \quad A = \frac{1}{2}r^2\theta \quad \text{radians}$$



Solution

$$\text{Diameter} = d = 12 \text{ cm}$$

$$\text{Radius} = r = \frac{d}{2} = \frac{12 \text{ cm}}{2} = 6 \text{ cm}$$

Perimeter of half circle

In a half circle central angle $\theta = \pi \text{ rad}$.

We know that

$$l = r\theta \quad \boxed{\theta \text{ in rad}}$$

$$l = 6 \times \pi \text{ rad.}$$

$$l = 6 \times 3.1416$$

$$l = 6 \times 3.1416$$

$$l = 18.85 \text{ cm}$$

We know that

Perimeter of half circle

= Arc length of semicircle + diameter

$$\text{Perimeter of half circle} = (18.85 + 12) \text{ cm}$$

$$= 30.85 \text{ cm}$$

Area of half circle

In a half circle central angle $\theta = \pi \text{ rad}$.

We know that

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(6 \text{ cm})^2 \times \pi \text{ rad.}$$

$$A = \frac{1}{2} \times 36 \times \pi$$

$$A = 18\pi$$

$$A = 18 \times 3.1416$$

$$A = 56.55 \text{ cm}^2$$

Q.8 Find circular measure of the angle between the hour and minutes hand of a clock at:

(i) 9 o'clock

(ii) 02:30

(iii) 06:45

Solution (i):

Total central angle in 1 rotation = $2\pi \text{ rad}$.

$$\boxed{1 \text{ rotation} = 12 \text{ hours}}$$

Angle subtended by hour hand during 12 hours = $2\pi \text{ rad}$.

Angle subtended by hour hand during 1-hour

$$= \frac{2\pi}{12} \text{ rad.} = \frac{\pi}{6} \text{ rad.}$$

Similarly,

Angle subtended by minute hand during 60 minutes = $2\pi \text{ rad}$.

Angle subtended by minute hand during 1 minute

$$= \frac{2\pi}{60} \text{ rad.} = \frac{\pi}{30} \text{ rad.}$$



The angle between the hour hand and minute hand at any given time equals the absolute value of the difference of angles subtended by the hour hand and minute hand.

ii) 9 o'clock

Time in hours = 9 hours

Time in minutes = 60 minutes

The angle subtended by hour hand during 1

$$\text{hour} = \frac{\pi}{6} \text{ rad.}$$

The angle subtended by hour hand during 9

$$\text{hours} = 9 \times \frac{\pi}{6} \text{ rad.} = \frac{3\pi}{2} \text{ rad.}$$

The angle subtended by minute hand during

$$1 \text{ minute} = \frac{\pi}{30} \text{ rad.}$$

The angle subtended by minute hand during 60

$$\text{minutes} = 60 \times \frac{\pi}{30} \text{ rad.} = 2\pi \text{ rad.}$$

Now the angle between the hands of clock

$$= \left| 2\pi - \frac{3\pi}{2} \right| \text{ rad} = \left| \frac{4\pi - 3\pi}{2} \right| \text{ rad} = \frac{\pi}{2} \text{ rad}$$

Solution (ii): 02:30

Time in hours =

$$\left(2 + \frac{30}{60} \right) \text{ hours} = \left(2 + \frac{1}{2} \right) \text{ hours} = \frac{5}{2} \text{ hours}$$

The angle subtended by hour hand during 1

$$\text{hour} = \frac{\pi}{6} \text{ rad.}$$

The angle subtended by hour hand during

$$\frac{5}{2} \text{ hours} = \frac{5}{2} \times \frac{\pi}{6} \text{ rad.} = \frac{5\pi}{12} \text{ rad.}$$

Similarly,

Time in Minutes = 30 minutes

The angle subtended by minute hand during 1

$$\text{minute} = \frac{\pi}{30} \text{ rad.}$$

The angle subtended by minute hand during 30

$$\text{minutes} = 30 \times \frac{\pi}{30} \text{ rad.} = \pi \text{ rad.}$$

Now,

The angle between the hour hand and minute hand =

$$= \left| \pi - \frac{5\pi}{12} \right| \text{ rad}$$

$$= \left| \frac{12\pi - 5\pi}{12} \right| \text{ rad}$$

$$= \left| \frac{7\pi}{12} \right| \text{ rad}$$

$$= \frac{7\pi}{12} \text{ rad}$$

Solution (iii): 06:45

$$\text{Time in hours} = \left(6 \text{ hrs} + \frac{45}{60} \text{ hrs} \right)$$

$$= \left(6 + \frac{3}{4} \right) \text{ hrs} = \left(\frac{27}{4} \right) \text{ hrs}$$

The angle subtended by hour hand during 1

$$\text{hour} = \frac{\pi}{6} \text{ rad.}$$

The angle subtended by hour hand during

$$\frac{27}{4} \text{ hrs}$$

$$= \frac{27}{4} \times \frac{\pi}{6} \text{ rad.} = \frac{27\pi}{24} \text{ rad.} = \frac{9\pi}{8} \text{ rad}$$

Time in minutes = 45 minutes

The angle subtended by minute hand during

$$1 \text{ minute} = \frac{\pi}{30} \text{ rad.}$$

The angle subtended by minute hand during 45

$$\text{minutes} = 45 \times \frac{\pi}{30} \text{ rad.} = \frac{3\pi}{2} \text{ rad.}$$



The angle between hour hand and minutes hand

$$= \left| \frac{9\pi}{8} - \frac{3\pi}{2} \right| \text{ rad}$$

$$= \left| \frac{9\pi - 12\pi}{8} \right| \text{ rad}$$

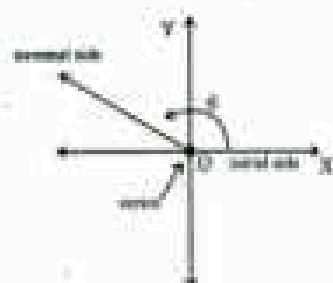
$$= \left| \frac{-3\pi}{8} \right| \text{ rad}$$

$$= \frac{3\pi}{8} \text{ rad}$$

$$| -\theta | = \theta$$

General Angle

In coordinate plane, the two axes divide the plane in four equivalent parts called quadrants. If we place an angle in the coordinate plane such that its vertex coincides with origin and one of its rays lies along $\overrightarrow{OX}$ (i.e. x-axis), then this angle is said to be in standard position. The ray along $\overrightarrow{OX}$ is called initial ray and second ray in the plane is called terminal ray. Such an angle is called general angle.



Measure of an angle is positive in anticlockwise direction and is negative in clockwise direction.

In the adjoining figure two angles in standard position are drawn. Both have same terminal ray. One of them is 150° (in clock wise direction) and the other is -210° (in clock wise direction).

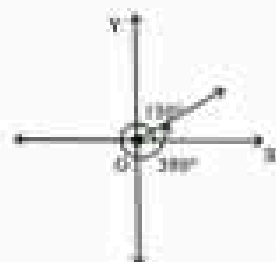
Two angles having same terminal ray in standard position are called co-terminal angles.

The measure of a general angle can be greater than 360° .

To draw a general angle whose measure is greater than 360° , we subtract any multiple of 360° .

Terminal ray of 150° is that of any of its multiple coincides with 0° , i.e. $\overrightarrow{OX}$.

In general, two angles are co-terminal if their measures differ by some multiple of 360° . e.g., to draw an angle of 390° , first find $390^\circ - 360^\circ = 30^\circ$, then draw 30° in anticlockwise direction.



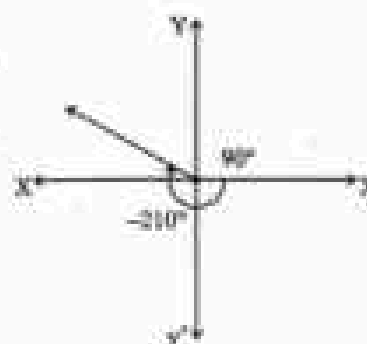
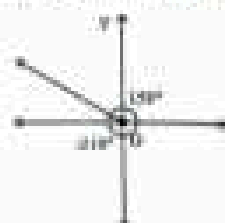
390° and 30° have same terminal ray as they are co-terminal angles.

Quadrantal angles

If terminal ray of an angle in standard position coincides with any of the axes, then it is called quadrantal angle. The measure of quadrantal angle is multiple of 90° .

e.g. $0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ, 450^\circ, \dots$ are the quadrantal angles.

The measure of a quadrantal angle is a multiple of $\frac{\pi}{2}$.



e.g., $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \dots$ are quadrantal angles.

Unit Circle

A circle whose radius is of 1 unit is called a unit circle. In standard position. The centre of circle is on on O (origin). The equation of a unit circle centered at O is $x^2 + y^2 = 1$.



Trigonometric Ratios with the Help of Unit Circle

Consider a unit circle centered at O. Mark a point P(x, y) on the circle. Join P with O. Draw PA perpendicular to x-axis.

$\overline{OA} = x$, $\overline{PA} = y$ and $\overline{OP} = 1$ (radius of circle).

If $\angle AOP = \theta$, then $\sin \theta = \frac{y}{1} = y$ and $\cos \theta =$

$\frac{x}{1} = x$. So, coordinates of P are $(\cos \theta, \sin \theta)$.

i.e., $x = \cos \theta$ and $y = \sin \theta$. Also $\tan \theta = \frac{y}{x}$.

Reciprocal of y is $\csc \theta$, reciprocal of x is $\sec \theta$ and reciprocal of $\frac{y}{x}$ is $\cot \theta$. If P is a point on a unit circle centered at O and $\angle AOP = \theta$, then coordinates of P are $(\cos \theta, \sin \theta)$.

Trigonometric ratios of 30°

To find trigonometric ratios of 30° , choose a point P(x, y) on unit circle in standard position such that $\angle O = \theta = 30^\circ$.



Draw PA perpendicular to x-axis. As in a right-angled triangle length of side opposite to 30° is half of hypotenuse, so in right angled triangle

OAP, $y_1 = \frac{1}{2}$. Equation of unit circle is $x^2 + y^2 = 1$.

Put $x = x_1$ and $y = y_1 = \frac{1}{2}$ in above equation to get

$$x^2 + \left(\frac{1}{2}\right)^2 = 1 \quad \Rightarrow x^2 + \frac{1}{4} = 1$$

$$x^2 = 1 - \frac{1}{4}$$

$$x_1^2 = \frac{3}{4} \text{ or } x_1 = \frac{\sqrt{3}}{2} \quad (\because P \text{ lies in quadrant I})$$

$$\text{Coordinates of P are } \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\text{So, } \cos 30^\circ = x_1 = \frac{\sqrt{3}}{2}$$

$$\sin 30^\circ = y_1 = \frac{1}{2}, \quad \tan 30^\circ = \frac{y_1}{x_1} = \frac{1}{\sqrt{3}}$$



$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}$$

$$\csc 30^\circ = \frac{1}{\sin 30^\circ} = 2, \text{ and } \cot 30^\circ = \frac{\cos 30^\circ}{\sin 30^\circ} = \sqrt{3}$$



Trigonometric Ratios of 45°

To find trigonometric ratios of 45° , choose a point $P(x_1, y_1)$ on a unit circle in standard position such that $\angle O = \theta = 45^\circ$. Draw PA perpendicular to x -axis. Triangle OAP is right isosceles.

Therefore, $\overline{AP} = \overline{OA}$ or $x_1 = y_1$.

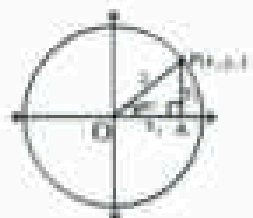
Equation of unit circle is $x^2 + y^2 = 1$.

Put $x = y_1$ and $y = y_1 = x_1$ in above equation to get

$$x_1^2 + x_1^2 = 1$$

$$2x_1^2 = 1$$

$$x_1^2 = \frac{1}{2} \Rightarrow x_1 = \frac{1}{\sqrt{2}}, \text{ also } y_1 = \frac{1}{\sqrt{2}}$$



Coordinates of P are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\text{So, } \cos 45^\circ = x_1 = \frac{1}{\sqrt{2}}$$

$$\sin 45^\circ = y_1 = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{y_1}{x_1} = 1$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2}$$

$$\csc 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}$$

$$\cot 45^\circ = \frac{\cos 45^\circ}{\sin 45^\circ} = 1$$

Trigonometric Ratios of 60°

To find trigonometric ratios of 60° , choose point $P(x_1, y_1)$ on unit circle in standard position such that $\angle O = \theta = 60^\circ$.



Draw PA perpendicular to x -axis. As it is right-angled triangle two acute angles are complementary so $\angle OPA = 30^\circ$. As in a right-angled triangle length of side opposite to 30° is half of hypotenuse, so $x_1 = \frac{1}{2}$.

Equation of unit circle is $x^2 + y^2 = 1$.

$$\left(\frac{1}{2}\right)^2 + y_1^2 = 1 \Rightarrow \frac{1}{4} + y_1^2 = 1$$

$$y_1^2 = 1 - \frac{1}{4} \Rightarrow y_1^2 = \frac{3}{4} \Rightarrow y_1 = \frac{\sqrt{3}}{2}, \text{ also } x_1 = \frac{1}{2}$$

Coordinates of P are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$\text{So, } \cos 60^\circ = x_1 = \frac{1}{2}, \sin 60^\circ = y_1 = \frac{\sqrt{3}}{2}$$

$$\tan 60^\circ = \frac{y_1}{x_1} = \sqrt{3}, \sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$$

$$\csc 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}}, \text{ and } \cot 60^\circ = \frac{\cos 60^\circ}{\sin 60^\circ} = \frac{1}{\sqrt{3}}$$

Signs of Trigonometric Ratios in the Fourth Quadrant

If (x_1, y_1) is a point on a unit circle such that $\angle XOP = \theta$, then $x_1 = \cos \theta$ and $y_1 = \sin \theta$.

First Quadrant

If P lies in first quadrant, then θ lies between 0° and 90° . Both coordinates of P are positive i.e., $x_1 > 0$ and $y_1 > 0$. So, in first quadrant $\cos \theta > 0$ and $\sin \theta > 0$. As the quotient and reciprocals of positive numbers are also positive so in first quadrant all trigonometric ratios are positive.



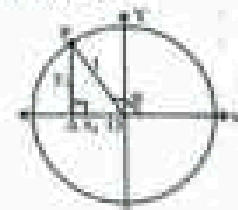
Second Quadrant

If P lies in second quadrant, then θ lies between 90° and 180° . In second quadrant abscissa of a point is negative and ordinate is positive i.e., $x_1 < 0$ and $y_1 > 0$.

So, in second quadrant $\sin \theta > 0$ and $\csc \theta > 0$.

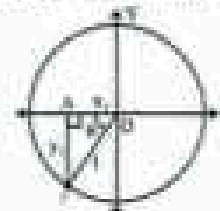
As the quotient of a negative and a positive number is negative so that $\theta < 0$ and $\cot \theta < 0$.

As reciprocal of a negative number is negative and that of positive number is positive so $\sec \theta < 0$ and $\cos \theta < 0$.



Third Quadrant

If P lies in third quadrant, then θ lies between 180° and 270° . In third quadrant both abscissa and ordinate of a point are negative i.e., $x_1 < 0$ and $y_1 < 0$. So, in third quadrant $\cos \theta < 0$ and $\sin \theta < 0$. As the quotient of two negative numbers is positive so $\tan \theta > 0$ and $\cot \theta > 0$. As reciprocals of negative numbers are negative, so $\sec \theta < 0$ and $\csc \theta < 0$.

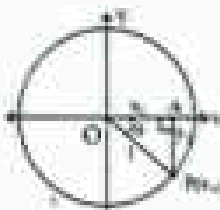


Fourth Quadrant

If P lies in fourth quadrant, then θ lies between 270° and 360° . In fourth quadrant abscissa of a point is positive and ordinate is negative i.e., $x_1 > 0$ and $y_1 < 0$.

So in fourth quadrant $\cos \theta > 0$ and $\sec \theta > 0$.

As the quotient of a negative and a positive number is negative and that of positive number is positive, so $\tan \theta < 0$ and $\cot \theta < 0$.



Example 10

If $\sin \theta = \frac{\sqrt{3}}{2}$ and terminal ray of θ is in first quadrant, then find remaining trigonometric ratios for same value of θ .



Solution

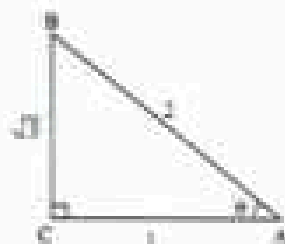
Method-I

Consider a right-angled triangle ABC, in which $\angle C = 90^\circ$, $\angle A = \theta$

By definition $\sin \theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{BC}{AB}$

But $\sin \theta = \frac{\sqrt{3}}{2}$ (given)

So, $BC = \sqrt{3}$ and $AB = 2$



By Pythagoras theorem $AB^2 = AC^2 + BC^2$

$$2^2 = AC^2 + (\sqrt{3})^2$$

$$4 = AC^2 + 3$$

$$4 - 3 = AC^2$$

$$AC^2 = 1 \Rightarrow AC = 1$$

$$\cos \theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{1}{2} \Rightarrow \cos \theta = \frac{1}{2}$$

$$\tan \theta = \frac{\text{perpendicular}}{\text{base}} = \frac{BC}{AC} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{1}{2}} \Rightarrow \sec \theta = \frac{1}{\frac{1}{2}} = 2$$

Method-II

Consider a unit circle at O. Mark a point P on the circle in first quadrant whose y coordinate is $\frac{\sqrt{3}}{2}$.

is $\frac{\sqrt{3}}{2}$.

The equation of unit circle is $x^2 + y^2 = 1$.



$$\text{Put } y = \frac{\sqrt{3}}{2}$$

$$x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$$x^2 + \frac{3}{4} = 1$$

$$x^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$$

Since, P is in first quadrant so $x > 0$, i.e.,

$$x = \frac{1}{2}$$

$$\tan \theta = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \Rightarrow \tan \theta = \frac{1}{\sin \theta} = \frac{2}{\sqrt{3}}$$

$$\sec \theta = \frac{1}{\cos \theta} = 2 \Rightarrow \sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{1}{2}}$$

Trigonometric Ratios of Quadrantal Angles
Trigonometric Ratios of 0°

To find trigonometric ratios of 0° , choose a point P(x, y) on unit circle in standard position such that $\angle XOP = \theta = 0^\circ$.

As P lies on x-axis so its coordinates are $x_1 = 1$ and $y_1 = 0$.

$$\text{So, } \sin 0^\circ = y_1 = 0$$

$$\cos 0^\circ = x_1 = 1$$

$$\tan 0^\circ = \frac{y_1}{x_1} = \frac{0}{1} = 0$$

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$$

$$\csc 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty \text{ (undefined)}$$

$$\cot 0^\circ = \frac{\cos 0^\circ}{\sin 0^\circ} = \frac{1}{0} = \infty \text{ (undefined)}$$



Trigonometric Ratios of 90°

To find trigonometric ratios of 90° , choose a point P(x, y) on unit circle in standard position such that $\angle XOP = \theta = 90^\circ$.

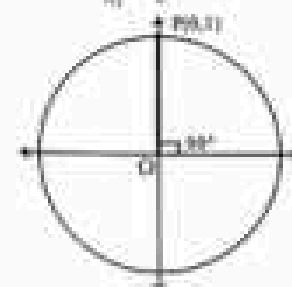
As P lies on y-axis so its coordinates are $x_1 = 0$ and $y_1 = 1$.

$$\text{So, } \sin 90^\circ = y_1 = 1$$

$$\cos 90^\circ = x_1 = 0$$

$$\tan 90^\circ = \frac{y_1}{x_1} = \frac{1}{0} = \infty \text{ (undefined)}$$

$$\sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} = \infty \text{ (undefined)}$$



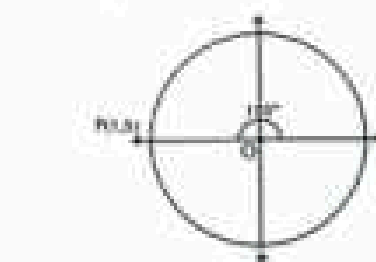
$$\csc 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = 1$$

$$\cot 90^\circ = \frac{\cos 90^\circ}{\sin 90^\circ} = \frac{0}{1} = 0$$

Trigonometric Ratios of 180°

To find trigonometric ratios of 180° , choose a point P(x, y) on unit circle in standard position such that $\angle XOP = \theta = 180^\circ$.

As P lies on x-axis at left side of O so its coordinates are $x_1 = -1$ and $y_1 = 0$.



$$\text{So, } \sin 180^\circ = y_1 = 0$$

$$\cos 180^\circ = x_1 = -1$$

$$\tan 180^\circ = \frac{y_1}{x_1} = \frac{0}{-1} = 0$$

$$\sec 180^\circ = \frac{1}{\cos 180^\circ} = \frac{1}{-1} = -1$$

$$\csc 180^\circ = \frac{1}{\sin 180^\circ} = \frac{1}{0} = \infty \text{ (undefined)}$$

$$\cot 180^\circ = \frac{\cos 180^\circ}{\sin 180^\circ} = \frac{-1}{0} = \infty \text{ (undefined)}$$

Trigonometric Ratios of 270°

To find trigonometric ratios of 270° , choose a point P(x, y) on unit circle in standard position such that $\angle XOP = \theta = 270^\circ$.

As P lies on y-axis below origin, therefore, its coordinates are $x_1 = 0$ and $y_1 = -1$.

$$\text{So, } \sin 270^\circ = y_1 = -1$$

$$\cos 270^\circ = x_1 = 0$$

$$\tan 270^\circ = \frac{y_1}{x_1} = \frac{-1}{0} = \infty \text{ (undefined)}$$

$$\sec 270^\circ = \frac{1}{\cos 270^\circ} = \frac{1}{0} = \infty \text{ (undefined)}$$

$$\csc 270^\circ = \frac{1}{\sin 270^\circ} = \frac{1}{-1} = -1$$

$$\cot 270^\circ = \frac{\cos 270^\circ}{\sin 270^\circ} = \frac{0}{-1} = 0$$





Trigonometric Ratios of 360°

To find trigonometric ratios of 360° , choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 360^\circ$. It coincides with 0° .

As P lies on x -axis so its coordinates are $x_1 = 1$ and $y_1 = 0$.



$$\text{So, } \cos 360^\circ = x_1 = 1$$

$$\sin 360^\circ = y_1 = 0$$

$$\tan 360^\circ = \frac{y_1}{x_1} = \frac{0}{1} = 0$$

$$\sec 360^\circ = \frac{1}{x_1} = \frac{1}{1} = 1$$

$$\csc 360^\circ = \frac{1}{y_1} = \frac{1}{0} = \infty \text{ (undefined)}$$

$$\cot 360^\circ = \frac{y_1}{x_1} = \frac{0}{1} = 0$$

Exercise 6.3

Q.1 Solution:

Angle $\rightarrow$ Ratio $\downarrow$	0° or 0 rad.	30° or $\frac{\pi}{6} \text{ rad.}$	45° or $\frac{\pi}{4} \text{ rad.}$	60° or $\frac{\pi}{3} \text{ rad.}$	90° or $\frac{\pi}{2} \text{ rad.}$	180° or $\pi \text{ rad.}$	270° or $\frac{3\pi}{2} \text{ rad.}$	360° or $2\pi \text{ rad.}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
$\csc \theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞	1
$\cot \theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞

Q.2 Evaluate the following.

(i) $\sin 60^\circ - \cos 30^\circ$

$$\begin{aligned} \sin 60^\circ - \cos 30^\circ \\ = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = 0 \end{aligned}$$

(ii) $\sin 45^\circ \cos 45^\circ - \sin 30^\circ \cos 60^\circ$

Solution

$$\sin 45^\circ \cos 45^\circ - \sin 30^\circ \cos 60^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$= \frac{2}{4} - \frac{1}{4}$$

$$= \frac{2-1}{4}$$

$$= \frac{1}{4}$$

(iii) $\frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{6}}$

Solution

$$\frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{6}}$$

$$= \left(\tan \frac{\pi}{3} - \tan \frac{\pi}{6}\right) \div \left(1 + \tan \frac{\pi}{3} \tan \frac{\pi}{6}\right)$$

$$= \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) \div \left(1 + \sqrt{3} \times \frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{(\sqrt{3})^2 - 1}{\sqrt{3}}\right) \div \left(1 + \sqrt{3} \times \frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{3-1}{\sqrt{3}}\right) \div (1+1)$$

$$= \left(\frac{2}{\sqrt{3}}\right) \div (2)$$

$$= \left(\frac{2}{\sqrt{3}}\right) \times \left(\frac{1}{2}\right) \leftarrow \text{Reciprocal}$$

$$= \frac{1}{\sqrt{3}}$$

(iv) $\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$

Solution

$$\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$$

$$= \left(\cos \frac{\pi}{3}\right)^2 - \left(\sin \frac{\pi}{3}\right)^2$$

$$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} - \frac{3}{4}$$

$$= \frac{1-3}{4}$$

$$= \frac{-2}{4} = \frac{-1}{2}$$

$$= \frac{-1}{2}$$

(v) $\sin \frac{\pi}{2} \cos \frac{\pi}{3} - \cos \frac{\pi}{2} \sin \frac{\pi}{3}$

Solution

$$\sin \frac{\pi}{2} \cos \frac{\pi}{3} - \cos \frac{\pi}{2} \sin \frac{\pi}{3}$$

$$= (1)\left(\frac{1}{2}\right) - (0)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

Q.3 For $\theta = \frac{\pi}{6}$, verify the following statements.

(i) $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

Solution

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

We know that $\theta = \frac{\pi}{6} = \frac{180^\circ}{6} = 30^\circ$



$$\begin{aligned} \text{L.H.S} &= \tan 2\left(\frac{\pi}{6}\right) \\ &= \tan\left(\frac{\pi}{3}\right) \\ &= \sqrt{3} \quad \text{--- (i)} \end{aligned}$$

$$\begin{aligned} \text{R.H.S} &= \frac{2 \tan \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}} \\ &= \left(2 \tan \frac{\pi}{6}\right) \div \left(1 - \tan^2 \frac{\pi}{6}\right) \\ &= 2\left(\frac{1}{\sqrt{3}}\right) \div \left(1 - \left(\frac{1}{\sqrt{3}}\right)^2\right) \\ &= \frac{2}{\sqrt{3}} \div \left(1 - \frac{1}{3}\right) \\ &= \frac{2}{\sqrt{3}} \div \frac{2}{3} \\ &= \frac{2}{\sqrt{3}} \times \frac{3}{2} \\ &= \frac{3}{\sqrt{3}} \quad \left(\because \frac{a}{\frac{a}{b}} = b\right) \\ &= \sqrt{3} \quad \text{--- (ii)} \end{aligned}$$

From (i) and (ii) L.H.S = R.H.S
Hence proved

$$(ii) \sin 2\theta = 2 \sin \theta \cos \theta$$

Solution

$$\text{L.H.S} = \sin 2\theta$$

$$\text{L.H.S} = \sin\left(2 \times \frac{\pi}{6}\right)$$

$$= \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \text{--- (i)}$$

$$\text{R.H.S} = 2 \sin \theta \cos \theta$$

$$= 2 \sin \frac{\pi}{6} \cos \left(\frac{\pi}{6}\right)$$

$$\begin{aligned} &= 2\left(\frac{1}{2}\right) \times \left(\frac{\sqrt{3}}{2}\right) \\ &= \left(\frac{1}{1}\right) \times \left(\frac{\sqrt{3}}{1}\right) \\ &= \frac{1}{1} \times \frac{\sqrt{3}}{1} \\ &= \frac{1 \times \sqrt{3}}{1 \times 1} \\ &= \frac{\sqrt{3}}{1} = \sqrt{3} \quad \text{--- (ii)} \end{aligned}$$

From (i) and (ii) L.H.S = R.H.S
Hence proved

$$(iii) \cos 2\theta = 1 - 2 \sin^2 \theta$$

Solution

$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

$$\text{L.H.S} = \cos 2\theta$$

$$= \cos\left(2 \times \frac{\pi}{6}\right) \quad \left(\because \theta = \frac{\pi}{6}\right)$$

$$= \cos \frac{\pi}{3}$$

$$= \frac{1}{2} \quad \text{--- (i)} \quad \left(\because \cos \frac{\pi}{3} = \frac{1}{2}\right)$$

$$\text{R.H.S} = 1 - 2 \sin^2 \theta$$

$$= 1 - 2\left(\sin \theta\right)^2$$

$$= 1 - 2\left(\sin \frac{\pi}{6}\right)^2 \quad \left(\because \theta = \frac{\pi}{6}\right)$$

$$= 1 - 2\left(\frac{1}{2}\right)^2 \quad \left(\because \sin \frac{\pi}{6} = \frac{1}{2}\right)$$

$$= 1 - 2\left(\frac{1}{4}\right)$$

$$= 1 - \frac{1}{2}$$

$$= \frac{2-1}{2}$$

$$= \frac{1}{2} \quad \text{--- (ii)}$$

From (i) and (ii) L.H.S = R.H.S

Hence proved

$$(iv) \sec^2 \theta - \tan^2 \theta = 1$$

Solution

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$\text{L.H.S} = \sec^2 \theta - \tan^2 \theta$$

$$= \sec^2 \theta - \tan^2 \theta$$

$$= \left(\sec \frac{\pi}{6}\right)^2 - \left(\tan \frac{\pi}{6}\right)^2 \quad \left(\because \theta = \frac{\pi}{6}\right)$$

$$= \left(\frac{2}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2 \quad \left(\because \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}\right)$$

$$= \frac{4}{3} - \frac{1}{3}$$

$$= \frac{4-1}{3}$$

$$= \frac{3}{3} = 1 = \text{R.H.S}$$

$$\text{L.H.S} = \text{R.H.S}$$

$$(v) \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$$

Solution

$$\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$$

$$\text{L.H.S} = \operatorname{cosec}^2 \theta$$

$$= \operatorname{cosec}^2 \theta$$

$$= \left(\operatorname{cosec} \frac{\pi}{6}\right)^2$$

$$= (2)^2$$

$$= 4 \quad \text{--- (i)}$$

$$\text{R.H.S} = 1 + \cot^2 \theta$$

$$= 1 + \left(\cot \theta\right)^2$$

$$= 1 + \left(\cot \frac{\pi}{6}\right)^2$$

$$= 1 + \left(\sqrt{3}\right)^2$$

$$= 1 + 3$$

$$= 4 \quad \text{--- (ii)}$$

From (i) and (ii) L.H.S = R.H.S

Hence proved

Q.4 Write the value of θ for the following ratios, ($0 < \theta < 2\pi$)

$$(i) \sin \theta = \frac{\sqrt{2}}{2}$$

Solution

$$\sin \theta = \frac{\sqrt{2}}{2}$$

$$\sin \theta = \frac{1}{\sqrt{2}} \quad \text{--- (i)} \quad \left(\because \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\right)$$

We know that

$$\sin 45^\circ = \frac{1}{\sqrt{2}} \quad \text{--- (ii)}$$

From (i) and (ii) we get

$$\sin \theta = \sin 45^\circ$$

$$\Rightarrow \theta = 45^\circ$$

$$(ii) \operatorname{cosec} \theta = -1$$

Solution

$$\operatorname{cosec} \theta = -1 \quad \text{--- (i)}$$

We know that

$$\operatorname{cosec} 270^\circ = -1 \quad \text{--- (ii)}$$

From (i) and (ii) we get, $\operatorname{cosec} \theta = \operatorname{cosec} 270^\circ$

$$\Rightarrow \theta = 270^\circ$$

$$(iii) \tan \theta = 1$$

Solution

$$\tan \theta = 1 \quad \text{--- (i)}$$

We know that

$$\tan 45^\circ = 1 \quad \text{--- (ii)}$$

From (i) and (ii) we get

$$\tan \theta = \tan 45^\circ$$

$$\Rightarrow \theta = 45^\circ$$

$$(iv) \sec \theta = \frac{\sqrt{2}}{2}$$



Solution

$$\sec \theta = \frac{\sqrt{13}}{3}$$

$$\sec \theta = \frac{\sqrt{4+3}}{3}$$

$$\sec \theta = \frac{2\sqrt{3}}{3} \quad \left(\because \frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}} \right)$$

$$\sec \theta = \frac{2 \times 1}{\sqrt{3}}$$

$$\sec \theta = \frac{2}{\sqrt{3}} \quad \text{--- (i)}$$

We know that

$$\sec 90^\circ = \frac{2}{\sqrt{3}} \quad \text{--- (ii)}$$

From equation (i) and (ii)

$$\sec \theta = \sec 90^\circ$$

$$\Rightarrow \theta = 90^\circ$$

$$\therefore \sec \theta = 0$$

Answer

Solution

$$\sec \theta = 0 \quad \text{--- (i)}$$

We know that

$$\sec 90^\circ = 0 \quad \text{--- (ii)}$$

$$\sec \theta = \sec 90^\circ$$

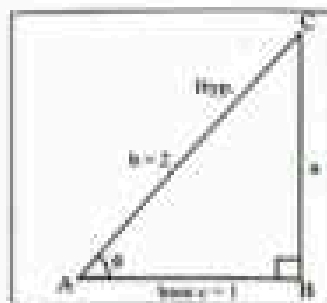
$$\Rightarrow \theta = 90^\circ$$

Q.5 If terminal ray of θ is in first quadrant

and $\cos \theta = \frac{1}{2}$, then find remaining trigonometric ratios of θ . Answer

Solution

Consider a right-angled $\triangle ABC$ in which $\angle A = \theta$ and $\angle B = 90^\circ$.



$$\cos \theta = \frac{1}{2} \quad \text{--- (i)} \quad \text{(Given)}$$

In right-angled $\triangle ABC$

$$\cos \theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{c}{b} \quad \text{--- (ii)} \quad \text{(From } \triangle ABC)$$

From (i) and (ii) we get

$$\frac{c}{b} = \frac{1}{2}$$

$$\Rightarrow \boxed{c=1, b=2}$$

In right-angled $\triangle ABC$ by Pythagoras' theorem

$$b^2 = c^2 + a^2$$

$$(2)^2 = (1)^2 + a^2$$

$$4 - 1 = a^2$$

$$4 - 1 = a^2$$

$$3 = a^2$$

$$\sqrt{a^2} = \sqrt{3}$$

$$\boxed{a = \sqrt{3}} \quad (\theta \text{ is in Q.I})$$

Remaining Ratios

$$\sin \theta = \frac{\text{Perp.}}{\text{Hyp.}} = \frac{a}{b} = \frac{\sqrt{3}}{2}$$

$$\tan \theta = \frac{\text{Perp.}}{\text{Base}} = \frac{a}{c} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\csc \theta = \frac{\text{Hyp.}}{\text{Perp.}} = \frac{b}{a} = \frac{2}{\sqrt{3}}$$

$$\sec \theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{b}{c} = \frac{2}{1} = 2$$

$$\sec \theta = \frac{\text{Base}}{\text{Perp.}} = \frac{c}{a} = \frac{1}{\sqrt{3}}$$

As θ is in first quadrant so all ratios are positive.

Q.6 If terminal ray of θ is in second quadrant and $\csc \theta = 2$, then find the

$$\text{value of } \csc \theta \left(\frac{\sin^2 \theta - \cos^2 \theta}{\sin^2 \theta + \cos^2 \theta} \right) \quad \text{Answer}$$

Solution

$$\csc \theta \left(\frac{\sin^2 \theta - \cos^2 \theta}{\sin^2 \theta + \cos^2 \theta} \right) \quad \text{--- (i)}$$

Given that

$$\csc \theta = 2$$

$$\sin \theta = \frac{1}{2} \quad \text{--- (ii)} \quad (\because \text{Reciprocal of } \csc \theta)$$

We know that

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\sqrt{\cos^2 \theta} = \pm \sqrt{1 - \sin^2 \theta}$$

$$\cos \theta = \pm \sqrt{1 - \left(\frac{1}{2}\right)^2} \quad (\cos \theta \text{ is } -ve \text{ in Q.II})$$

$$\cos \theta = -\sqrt{1 - \frac{1}{4}} = -\sqrt{\frac{4-1}{4}} = -\frac{\sqrt{3}}{2} \quad \text{--- (iii)}$$

From (ii) and (iii) putting the values of $\sin \theta$ and $\cos \theta$ in (i)

$$\begin{aligned} & \csc \theta \left(\frac{\sin^2 \theta - \cos^2 \theta}{\sin^2 \theta + \cos^2 \theta} \right) \\ &= \frac{\sqrt{3}}{2} \left[\frac{\left(\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} \right] \\ &= \frac{\sqrt{3}}{2} \left[\frac{\left(\frac{1}{4} - \frac{3}{4}\right)}{\left(\frac{1}{4} + \frac{3}{4}\right)} \right] \\ &= \frac{\sqrt{3}}{2} \left[\frac{\left(\frac{1-3}{4}\right)}{\left(\frac{1+3}{4}\right)} \right] \\ &= \frac{\sqrt{3}}{2} \left[\frac{\left(\frac{-2}{4}\right)}{\left(\frac{4}{4}\right)} \right] \end{aligned}$$

$$= \frac{\sqrt{3}}{2} \left[\left(\frac{-2}{4}\right) \times \left(\frac{4}{4}\right) \right]$$

$$= \frac{\sqrt{3}}{2} \left[\frac{-1}{2} \times 1 \right]$$

$$= \frac{\sqrt{3}}{2} \times \left(\frac{-1}{2}\right)$$

$$= \frac{\sqrt{3}}{4}$$

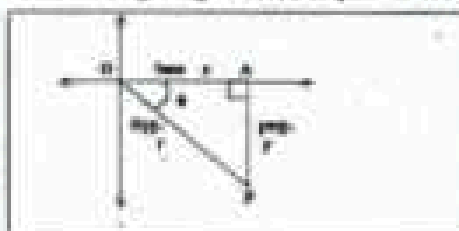
Q.7 If $\tan \theta = -1$ and terminal ray of θ is not in second quadrant, then find the remaining trigonometric ratios of θ . Answer

Solution

As $\tan \theta < 0$ in quadrant II and quadrant IV.

It is given that θ is not in Q.II. It means θ is in quadrant IV.

Consider a right-angled $\triangle PAO$ in quadrant IV.



$$\tan \theta = -1 \quad \text{--- (i)} \quad \text{(Given)}$$

In right-angled $\triangle PAO$

$$\tan \theta = \frac{\text{Perp.}}{\text{Base}} = \frac{a}{b} \quad \text{--- (ii)} \quad \text{(From } \triangle PAO)$$

From (i) and (ii) we get

$$\frac{a}{b} = -1$$

$$\frac{a}{b} = \frac{1}{-1}$$

$$\Rightarrow \boxed{a=1, b=-1} \quad (\because y < 0 \text{ in Q.IV})$$

In right-angled $\triangle PAO$ by Pythagoras' theorem

$$c^2 = a^2 + b^2$$

$$c^2 = (1)^2 + (-1)^2$$



$$x^2 = 1 + 1$$

$$x^2 = 2$$

$$\sqrt{x^2} = \sqrt{2}$$

$$x = \sqrt{2}$$

Remaining Ratios

$$\sin \theta = \frac{\text{Opp.}}{\text{Hyp.}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos \theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\csc \theta = \frac{\text{Hyp.}}{\text{Opp.}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\sec \theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\cot \theta = \frac{\text{Base}}{\text{Opp.}} = \frac{1}{1} = 1$$

Trigonometric Identities

Identity is an equation which remains true for all possible replacements of variable involved in it. e.g. $(x+1)(x-1) = x^2-1$ is an identity.

Trigonometric identity is an identity involving different trigonometric ratios in it. Trigonometric identities are of many types. Some of them are given below.

- Reciprocal identities
- Quotient identities
- Pythagorean identities

Reciprocal Identities

Cosec θ , sec θ and cot θ are reciprocals of sin θ , cos θ and tan θ respectively. i.e. $\csc \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$ and $\cot \theta = \frac{1}{\tan \theta}$

Quotient Identities

$$\text{i) } \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \text{ii) } \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Proof

Consider a right-angled triangle ABC, in which $\angle C = 90^\circ$, $\angle A = \theta$, $\overline{AB} = c$, $\overline{BC} = a$ and $\overline{AC} = b$

By definition

$$\sin \theta = \frac{a}{c}$$

$$\cos \theta = \frac{b}{c}$$

$$\tan \theta = \frac{a}{b}$$

$$\tan \theta = \frac{\frac{a}{c}}{\frac{b}{c}} \quad (\text{dividing numerator and denominator by } c)$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

Similarly, we can prove the second quotient identity.



Pythagorean Identities

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\sec^2 \theta - \tan^2 \theta = 1$
- $\csc^2 \theta - \cot^2 \theta = 1$

Proof

Consider a right-angled triangle ABC, in which $\angle C = 90^\circ$, $\angle A = \theta$, $\overline{AB} = c$, $\overline{BC} = a$ and $\overline{AC} = b$

$$\text{i) } \sin^2 \theta + \cos^2 \theta = 1$$

$$\text{ii) } \sec^2 \theta - \tan^2 \theta = 1$$

$$\text{iii) } \csc^2 \theta - \cot^2 \theta = 1$$



Proof

Consider a right-angled triangle ABC,

in which $\angle C = 90^\circ$, $\angle A = \theta$, $\overline{AB} = c$, $\overline{BC} = a$ and $\overline{AC} = b$.

$$\text{i) } \sin^2 \theta + \cos^2 \theta = 1$$

By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by c^2 we get



$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$(\sin \theta)^2 + (\cos \theta)^2 = 1 \text{ or } \sin^2 \theta + \cos^2 \theta = 1$$

$$\text{ii) } \sec^2 \theta - \tan^2 \theta = 1$$

$$\text{iii) } \csc^2 \theta - \cot^2 \theta = 1$$

By definition, $\sec \theta = \frac{c}{b}$ and $\tan \theta = \frac{a}{b}$

By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by b^2 we get

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\tan^2 \theta + 1 = \sec^2 \theta \text{ or } \sec^2 \theta - \tan^2 \theta = 1$$

$$\text{iii) } \csc^2 \theta - \cot^2 \theta = 1$$

By definition, $\csc \theta = \frac{c}{a}$ and $\cot \theta = \frac{b}{a}$

By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by a^2 we get

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\text{or } \csc^2 \theta - \cot^2 \theta = 1$$

Example 11

Use basic trigonometric identities to prove the following relations.

- $\sec \alpha = \sin \alpha \tan \alpha + \csc \alpha$ Exercise 11.1
- $\frac{\tan \alpha}{\sec \alpha - 1} = \frac{\sec \alpha + 1}{\tan \alpha}$ Exercise 11.2
- $\frac{\sin^2 \alpha + 2 \cos \alpha - 1}{\sin^2 \alpha + 3 \cos \alpha - 2} = \frac{1}{1 - \cos \alpha}$ Exercise 11.3

Solution

$$\text{i) L.H.S.} = \sec \alpha = \sin \alpha \tan \alpha + \csc \alpha$$

$$= \sin \alpha + \sin \alpha \frac{\sin \alpha}{\cos \alpha}$$

$$= \sin \alpha + \frac{\sin^2 \alpha}{\cos \alpha}$$

$$= \frac{\sin^2 \alpha + \cos^2 \alpha}{\cos \alpha}$$

$$= \frac{1}{\cos \alpha} = \sec \alpha = \text{R.H.S.}$$

Hence, $\sec \alpha = \sin \alpha \tan \alpha + \csc \alpha$

$$\text{ii) L.H.S.} = \frac{\tan \alpha}{\sec \alpha - 1}$$

$$= \frac{\tan \alpha}{\sec \alpha - 1} \cdot \frac{\sec \alpha + 1}{\sec \alpha + 1}$$



$$= \frac{\tan \alpha (\sec \alpha + 1)}{\sec^2 \alpha - 1}$$

$$= \frac{\tan \alpha (\sec \alpha + 1)}{\sec^2 \alpha - 1}$$

$$= \frac{\tan \alpha (\sec \alpha)}{\tan^2 \alpha}$$

$$= \frac{\sec \alpha (\sec \alpha + 1)}{1 + \sec \alpha}$$

$$= \frac{\sec \alpha + 1}{\tan \alpha} = \text{R.H.S.}$$

Hence, $\frac{\tan \alpha}{\sec \alpha - 1} = \frac{\sec \alpha + 1}{\tan \alpha}$

(ii) L.H.S. = $\frac{\sin^2 \alpha + 2 \cos \alpha - 1}{\sin^2 \alpha + 3 \cos \alpha - 2}$

$$= \frac{1 - \cos^2 \alpha + 2 \cos \alpha - 1}{1 - \cos^2 \alpha + 3 \cos \alpha - 2}$$

$$= \frac{-\cos^2 \alpha + 2 \cos \alpha}{-\cos^2 \alpha + 3 \cos \alpha - 2}$$

$$= \frac{-\cos \alpha (\cos \alpha - 2)}{-(\cos^2 \alpha - 3 \cos \alpha + 2)}$$

$$= \frac{\cos \alpha (\cos \alpha - 2)}{\cos^2 \alpha - 2 \cos \alpha - \cos \alpha + 2}$$

$$= \frac{\cos \alpha (\cos \alpha - 2)}{\cos \alpha (\cos \alpha - 2) - \cos \alpha (\cos \alpha - 2)}$$

$$= \frac{\cos \alpha (\cos \alpha - 2)}{(\cos \alpha - 1)(\cos \alpha - 2)} = \frac{\cos \alpha}{\cos \alpha - 1}$$

Dividing numerator and denominator by $\cos \alpha$.

$$= \frac{\cos \alpha}{\cos \alpha}$$

$$= \frac{\cos \alpha - 1}{\cos \alpha}$$

$$= \frac{1}{\frac{\cos \alpha}{\cos \alpha} - 1} = \frac{1}{1 - \sec \alpha} = \text{R.H.S.}$$

Hence $\frac{\sin^2 \alpha + \cos \alpha - 1}{\sin^2 \alpha + 3 \cos \alpha - 2} = \frac{1}{1 - \sec \alpha}$

Example 12

Solve the following triangles. (Find unknown elements of the following triangles) **Given**

(i) $\triangle ABC$, $\angle C = 90^\circ$, $\angle A = 30^\circ$, $a = 6$ cm

(ii) $\triangle ABC$, $\angle C = 90^\circ$, $c = 12$ cm, $b = 6\sqrt{2}$ cm

1. Given Elements

$\triangle ABC$, $\angle C = 90^\circ$, $\angle A = 30^\circ$, $a = 6$ cm

Required Elements

$\angle B$, b , c .

Solution

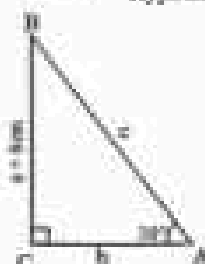
In $\triangle ABC$, $m\angle A + m\angle B + m\angle C = 180^\circ$

$$30^\circ + m\angle B + 90^\circ = 180^\circ$$

$$m\angle B = 180^\circ - 90^\circ - 30^\circ$$

$$m\angle B = 60^\circ$$

$$\sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$$



$$\sin 30^\circ = \frac{a}{c}$$

$$\frac{1}{2} = \frac{6}{c}$$

$$c = 12 \text{ cm}$$

In $\triangle ABC$, by Pythagoras theorem,

$$c^2 = a^2 + b^2$$

$$12^2 = 6^2 + b^2$$

$$144 = 36 + b^2$$

$$b^2 = 144 - 36$$

$$b^2 = 108$$

$$b = \sqrt{108} = 6\sqrt{3} \text{ cm}$$

Therefore, required elements are $m\angle B = 60^\circ$,

$$b = 6\sqrt{3} \text{ cm}, c = 12 \text{ cm}$$

(ii) Given Elements

$\triangle ABC$, $m\angle C = 90^\circ$, $c = 12$ cm, $b = 6\sqrt{2}$ cm

Required Elements

$\angle A$, $\angle B$, a

Solution

In $\triangle ABC$,

by Pythagoras theorem,

$$c^2 = a^2 + b^2$$

$$12^2 = a^2 + (6\sqrt{2})^2$$

$$a^2 = 144 - 72$$

$$a^2 = 72$$



$$a = \sqrt{72} = 6\sqrt{2} \text{ cm}$$

$$\cos \theta = \frac{\text{Base}}{\text{Hypotenuse}}$$

$$\cos m\angle A = \frac{b}{c}$$

$$\cos m\angle A = \frac{6\sqrt{2}}{12} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$m\angle A = 45^\circ$$

In $\triangle ABC$, $m\angle A + m\angle B + m\angle C = 180^\circ$

$$45^\circ + m\angle B + 90^\circ = 180^\circ$$

$$m\angle B = 180^\circ - 90^\circ - 45^\circ \text{ so } m\angle B = 45^\circ$$

Therefore, required elements are $m\angle A = 45^\circ$,

$$m\angle B = 45^\circ, a = 6\sqrt{2} \text{ cm}$$

Exercise - 6.4

Q.1 Prove the following relations by using basic trigonometric identities.

(i) $(1 - \sin^2 \theta) \sec^2 \theta = 1$ **Given**

Solution

$$(1 - \sin^2 \theta) \sec^2 \theta = 1$$

$$\text{L.H.S.} = (1 - \sin^2 \theta) \sec^2 \theta \quad \because (1 - \sin^2 \theta = \cos^2 \theta)$$

$$= (\cos^2 \theta) \sec^2 \theta$$

$$= (\cos \theta \sec \theta)^2 \quad \because \sec \theta = \frac{1}{\cos \theta}$$

$$= (1)^2 \quad \because \cos \theta \sec \theta = 1$$

$$= 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

(ii) $\frac{1 - \sin \theta}{1 + \sin \theta} = (\sec \theta - \tan \theta)^2$ **Given**

Solution:

$$\frac{1 - \sin \theta}{1 + \sin \theta} = (\sec \theta - \tan \theta)^2$$

$$\text{L.H.S.} = \frac{1 - \sin \theta}{1 + \sin \theta}$$

$$\text{L.H.S.} = (1 - \sin \theta)(1 + \sin \theta)$$

$$= \left(\frac{1 - \sin \theta}{\cos \theta} \right) \times \left(\frac{1 + \sin \theta}{\cos \theta} \right) \text{ (Dividing by } \cos \theta)$$

$$= \left(\frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} \right) \times \left(\frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta} \right)$$

$$= (\sec \theta - \tan \theta) \times (\sec \theta + \tan \theta)$$

$$= \frac{(\sec \theta - \tan \theta)}{(\sec \theta + \tan \theta)} \quad \therefore \tan \theta = \frac{\sin \theta}{\cos \theta}$$

Multiplying & dividing by $(\sec \theta - \tan \theta)$

$$= \frac{(\sec \theta - \tan \theta)}{(\sec \theta + \tan \theta)} \times \frac{(\sec \theta - \tan \theta)}{(\sec \theta - \tan \theta)}$$

$$= \frac{(\sec \theta - \tan \theta)^2}{\sec^2 \theta - \tan^2 \theta}$$



$$= \frac{(\sec \theta - \tan \theta)^2}{1} \quad \boxed{\sec^2 \theta - \tan^2 \theta = 1}$$

$$= (\sec \theta - \tan \theta)^2$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$(iii) \frac{\sqrt{1-\cos \theta}}{\sqrt{1+\cos \theta}} = \frac{\sin \theta}{1+\cos \theta}$$

Solution:

$$\frac{\sqrt{1-\cos \theta}}{\sqrt{1+\cos \theta}} = \frac{\sin \theta}{1+\cos \theta}$$

$$\text{L.H.S.} = \frac{\sqrt{1-\cos \theta}}{\sqrt{1+\cos \theta}}$$

Multiplying & dividing by $(1+\cos \theta)$

$$\text{L.H.S.} = \frac{\sqrt{1-\cos \theta} \cdot \sqrt{1+\cos \theta}}{\sqrt{1+\cos \theta} \cdot \sqrt{1+\cos \theta}}$$

$$= \frac{\sqrt{(1)^2 - (\cos \theta)^2}}{\sqrt{(1+\cos \theta)^2}}$$

$$= \frac{\sqrt{1-\cos^2 \theta}}{\sqrt{(1+\cos \theta)^2}}$$

$$= \frac{\sqrt{\sin^2 \theta}}{\sqrt{(1+\cos \theta)^2}}$$

$$= \frac{\sqrt{\sin^2 \theta}}{\sqrt{(1+\cos \theta)^2}}$$

$$\boxed{\therefore 1 - \cos^2 \theta = \sin^2 \theta}$$

$$= \frac{\sin \theta}{1+\cos \theta}$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$(iv) \frac{\sin \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - \cos \theta} = \tan \theta$$

Solution:

$$\frac{\sin \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - \cos \theta} = \tan \theta$$

$$\text{L.H.S.} = \frac{\sin \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - \cos \theta}$$

$$= \frac{\sin \theta (1 - 2 \sin^2 \theta)}{\cos \theta (2 \cos^2 \theta - 1)} \quad (\text{Taking common})$$

$$= \frac{\sin \theta (1 - 2(1 - \cos^2 \theta))}{\cos \theta (2 \cos^2 \theta - 1)} \quad \boxed{\sin^2 \theta = 1 - \cos^2 \theta}$$

$$= \frac{\sin \theta (1 - 2 + 2 \cos^2 \theta)}{\cos \theta (2 \cos^2 \theta - 1)}$$

$$= \frac{\sin \theta (2 \cos^2 \theta - 1)}{\cos \theta (2 \cos^2 \theta - 1)}$$

$$= \frac{\sin \theta}{\cos \theta}$$

$$\text{L.H.S.} = \tan \theta$$

$$\boxed{\frac{\sin \theta}{\cos \theta} = \tan \theta}$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$(v) \sqrt{\sec^2 \theta + \csc^2 \theta} = \tan \theta + \cot \theta$$

Solution:

$$\sqrt{\sec^2 \theta + \csc^2 \theta} = \tan \theta + \cot \theta$$

$$\boxed{\csc^2 \theta = \frac{1}{\sin^2 \theta}}$$

$$\text{L.H.S.} = \sqrt{\sec^2 \theta + \csc^2 \theta}$$

$$\boxed{\therefore 1 - \tan^2 \theta = \sec^2 \theta}$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$= \sqrt{1 + \tan^2 \theta + 1 + \cot^2 \theta}$$

$$\boxed{\therefore 1 - \tan^2 \theta = \sec^2 \theta}$$

$$= \sqrt{\tan^2 \theta + \cot^2 \theta + 2}$$

$$= \sqrt{\tan^2 \theta + \cot^2 \theta + 2 \times 1}$$

$$\boxed{\therefore 1 - \tan^2 \theta = \sec^2 \theta}$$

$$= \sqrt{\tan^2 \theta + \cot^2 \theta + 2 \tan \theta \cot \theta}$$

$$\boxed{(\therefore a^2 + b^2 + 2ab = (a+b)^2)}$$

$$= \sqrt{(\tan \theta + \cot \theta)^2} = \tan \theta + \cot \theta$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$(vi) \frac{1}{\sec \theta - \tan \theta} = \sec \theta + \tan \theta$$

Solution:

$$\text{L.H.S.} = \frac{1}{\sec \theta - \tan \theta}$$

Multiplying & dividing by $(\sec \theta + \tan \theta)$

$$\text{L.H.S.} = \frac{1}{\sec \theta - \tan \theta} \cdot \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta}$$

$$= \frac{1}{(\sec \theta - \tan \theta)(\sec \theta + \tan \theta)}$$

$$= \frac{\sec \theta + \tan \theta}{(\sec \theta)^2 - (\tan \theta)^2} \quad \boxed{\sec^2 \theta - \tan^2 \theta = 1}$$

$$= \frac{\sec \theta + \tan \theta}{\sec^2 \theta - \tan^2 \theta}$$

$$= \frac{\sec \theta + \tan \theta}{1} \quad (\sec^2 \theta - \tan^2 \theta = 1)$$

$$= \sec \theta + \tan \theta$$

$$(vii) \sin^2 \theta + \cos^2 \theta = \sin^2 \theta + \cos^2 \theta \quad \text{Answer}$$

Solution:

$$\sin^2 \theta + \cos^2 \theta = \sin^2 \theta + \cos^2 \theta$$

$$\text{L.H.S.} = \sin^2 \theta + \cos^2 \theta$$

$$= \sin^2 \theta + \cos^2 \theta \cos^2 \theta$$

$$= \sin^2 \theta + \cos^2 \theta (1 - \sin^2 \theta) \quad \boxed{\cos^2 \theta = 1 - \sin^2 \theta}$$

$$= \sin^2 \theta + \cos^2 \theta - \cos^2 \theta \sin^2 \theta$$

$$= 1 - \cos^2 \theta \sin^2 \theta$$

$$\boxed{\therefore \sin^2 \theta + \cos^2 \theta = 1}$$

$$= 1 - \sin^2 \theta (1 - \sin^2 \theta)$$

$$= 1 - \sin^2 \theta + \sin^4 \theta$$

$$= \cos^2 \theta + \sin^2 \theta$$

$$\boxed{1 - \sin^2 \theta = \cos^2 \theta}$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$(viii) \frac{\tan \theta + \sec \theta}{\tan \theta - \sec \theta} = \frac{\sec \theta + 1}{\sec \theta - 1}$$

Solution:

$$\frac{\tan \theta + \sec \theta}{\tan \theta - \sec \theta} = \frac{\sec \theta + 1}{\sec \theta - 1}$$

$$\text{L.H.S.} = \frac{\tan \theta + \sec \theta}{\tan \theta - \sec \theta}$$

$$= (\tan \theta + \sec \theta) + (\tan \theta - \sec \theta)$$



$$= \left(\frac{\sin \theta}{\cos \theta} + \sec \theta \right) + \left(\frac{\sin \theta}{\cos \theta} - \sec \theta \right) \quad \boxed{\therefore \tan \theta = \frac{\sin \theta}{\cos \theta}}$$

$$= \left(\frac{\sin \theta + \sec \theta \cos \theta}{\cos \theta} \right) + \left(\frac{\sin \theta - \sec \theta \cos \theta}{\cos \theta} \right) \quad (\text{L.C.M.})$$

$$= \left(\frac{\sin \theta + \sec \theta \cos \theta}{\cos \theta} \right) + \left(\frac{\sin \theta - \sec \theta \cos \theta}{\cos \theta} \right)$$

$$= \left(\frac{\sin \theta + \sec \theta \cos \theta}{\cos \theta} \right) + \left(\frac{\sin \theta - \sec \theta \cos \theta}{\cos \theta} \right)$$

$$= \left(\frac{\sin \theta + \sec \theta \cos \theta}{\cos \theta} \right) + \left(\frac{\sin \theta - \sec \theta \cos \theta}{\cos \theta} \right)$$

$$\text{Dividing numerator & denominator by } (\sec \theta \cos \theta)$$

we get

$$= \left(\frac{\sec \theta}{\sec \theta \cos \theta} + \frac{\sec \theta \cos \theta}{\sec \theta \cos \theta} \right) + \left(\frac{\sec \theta}{\sec \theta \cos \theta} - \frac{\sec \theta \cos \theta}{\sec \theta \cos \theta} \right)$$

$$= \left(\frac{1}{\cos \theta} + 1 \right) + \left(\frac{1}{\cos \theta} - 1 \right)$$

$$= (\sec \theta + 1) + (\sec \theta - 1)$$

$$= \frac{\sec \theta + 1}{\sec \theta - 1}$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$(ix) \sec^2 \theta + \csc^2 \theta = \sec^2 \theta \csc^2 \theta \quad \text{Answer}$$

Solution:

$$\sec^2 \theta + \csc^2 \theta = \sec^2 \theta \csc^2 \theta$$

$$\text{L.H.S.} = \sec^2 \theta + \csc^2 \theta$$

$$= \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta}$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin^2 \theta} \quad \boxed{\therefore \sin^2 \theta + \cos^2 \theta = 1}$$

$$= \frac{1}{\cos^2 \theta \sin^2 \theta}$$

$$= \frac{1}{\cos^2 \theta \sin^2 \theta} \quad \therefore \frac{1}{\cos \theta} = \sec \theta$$

$$= \sec^2 \theta \csc^2 \theta \quad \therefore \frac{1}{\sin \theta} = \csc \theta$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$(x) \sin^2 \theta + \cos^2 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta \quad \text{Answer}$$

Solution:

$$\sin^2 \theta + \cos^2 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta$$

$$\text{L.H.S.} = \sin^2 \theta + \cos^2 \theta$$



$$= (\sin^2 \theta) + (\cos^2 \theta)$$

$$\boxed{x^2 + y^2 = (a+b)(a^2 + b^2 - ab)}$$

$$= (\sin^2 \theta + \cos^2 \theta) \left[(\sin^2 \theta) + (\cos^2 \theta) - \sin^2 \theta \cos^2 \theta \right]$$

Adding & subtracting $(2\sin^2 \theta \cos^2 \theta)$ to complete the square:

$$= (1) \left[(\sin^2 \theta) + (\cos^2 \theta) + 2\sin^2 \theta \cos^2 \theta - 2\sin^2 \theta \cos^2 \theta - \sin^2 \theta \cos^2 \theta \right]$$

$$= (\sin^2 \theta + \cos^2 \theta)^2 - 3\sin^2 \theta \cos^2 \theta$$

$$\because \sin^2 \theta + \cos^2 \theta = 1$$

$$= (1)^2 - 3\sin^2 \theta \cos^2 \theta$$

$$= 1 - 3\sin^2 \theta \cos^2 \theta$$

L.H.S = R.H.S

Q.2 If $x \cos \theta + y \sin \theta = m$ and

$x \sin \theta - y \cos \theta = n$, Prove that

$$x^2 + y^2 = m^2 + n^2$$

Solution:

$$x \cos \theta + y \sin \theta = m \quad \text{--- (i)}$$

$$x \sin \theta - y \cos \theta = n \quad \text{--- (ii)}$$

Squaring (i) we get

$$(x \cos \theta + y \sin \theta)^2 = m^2$$

$$(x \cos \theta)^2 + (y \sin \theta)^2 + 2(x \cos \theta)(y \sin \theta) = m^2$$

$$x^2 \cos^2 \theta + y^2 \sin^2 \theta + 2xy \cos \theta \sin \theta = m^2 \quad \text{--- (iii)}$$

Now squaring eq. (ii) we get

$$(x \sin \theta - y \cos \theta)^2 = n^2$$

$$(x \sin \theta)^2 + (y \cos \theta)^2 - 2(x \sin \theta)(y \cos \theta) = n^2$$

$$x^2 \sin^2 \theta + y^2 \cos^2 \theta - 2xy \sin \theta \cos \theta = n^2 \quad \text{--- (iv)}$$

Adding eq. (iii) and (iv) we get

$$x^2 \cos^2 \theta + y^2 \sin^2 \theta + 2xy \cos \theta \sin \theta = m^2$$

$$x^2 \sin^2 \theta + y^2 \cos^2 \theta - 2xy \sin \theta \cos \theta = n^2$$

$$x^2 \cos^2 \theta + x^2 \sin^2 \theta + y^2 \sin^2 \theta + y^2 \cos^2 \theta = m^2 + n^2$$

$$x^2 (\cos^2 \theta + \sin^2 \theta) + y^2 (\sin^2 \theta + \cos^2 \theta) = m^2 + n^2$$

$$x^2 (1) + y^2 (1) = m^2 + n^2 \quad \therefore \sin^2 \theta + \cos^2 \theta = 1$$

$$x^2 + y^2 = m^2 + n^2$$

Hence proved

Q.3 Find the value of θ for the following

equations in the interval $0 < \theta < \frac{\pi}{2}$

Solution:

$$(i) \quad 2\sin^2 \theta = \frac{1}{2}$$

Solution

$$2\sin^2 \theta = \frac{1}{2}$$

$$\sin^2 \theta = \frac{1}{4}$$

$$(\sin \theta)^2 = \left(\frac{1}{2}\right)^2$$

$$\therefore \sin \theta = \frac{1}{2} \quad \text{--- (i)}$$

We know that

$$\sin 30^\circ = \frac{1}{2} \quad \text{--- (ii)}$$

From (i) & (ii)

$$\sin \theta = \sin 30^\circ$$

$$\therefore \theta = 30^\circ$$

$$(ii) \quad \sin \theta = \cos \theta$$

Solution

$$\sin \theta = \cos \theta$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}} \quad \text{--- (i)}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} \quad \text{--- (ii)}$$

From (i) and (ii)

$$\sin 45^\circ = \cos 45^\circ$$

$$\sin \theta = \cos \theta$$

$$\therefore \theta = 45^\circ$$

$$(iii) \quad \sec^2 \theta - 2\tan^2 \theta = 0$$

Solution

$$\sec^2 \theta - 2\tan^2 \theta = 0$$

$$\sec^2 \theta = 2\tan^2 \theta$$

$$\sec^2 \theta = (\sqrt{2})^2 \tan^2 \theta$$

$$\sec^2 \theta = (\sqrt{2} \tan \theta)^2$$

$$\therefore \sec \theta = \sqrt{2} \tan \theta$$

$$\frac{1}{\sqrt{2}} = \frac{\tan \theta}{\sec \theta}$$

$$\frac{\tan \theta = 1}{\sec \theta} = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{\sin \theta}{\cos \theta} = \cos \theta = \frac{1}{\sqrt{2}}$$

$$\sin \theta = \frac{1}{\sqrt{2}} \quad \text{--- (i)}$$

We know that

$$\sin 45^\circ = \frac{1}{\sqrt{2}} \quad \text{--- (ii)}$$

From (i) & (ii)

$$\sin \theta = \sin 45^\circ$$

$$\therefore \theta = 45^\circ$$

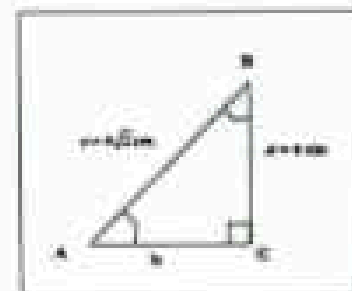
Q.4 Solve the following right angled triangles.

$$(i) \quad \triangle ABC, m\angle C = 90^\circ, a = 4 \text{ cm}, c = 4\sqrt{2} \text{ cm}$$

Solution:

Let $\triangle ABC$ be a right-angled triangle in which

$$m\angle C = 90^\circ, a = 4 \text{ cm}, c = 4\sqrt{2} \text{ cm}$$



In right-angled triangle ABC by Pythagoras' theorem,

$$c^2 = a^2 + b^2$$

$$(4\sqrt{2})^2 = (4)^2 + b^2$$

$$4^2 (\sqrt{2})^2 = 16 + b^2$$

$$16(2) = 16 + b^2$$

$$32 - 16 = b^2$$

$$16 = b^2$$

$$\sqrt{b^2} = \sqrt{16}$$

$$\boxed{b = 4 \text{ cm}}$$

In right-angled $\triangle ABC$

$$\tan m\angle A = \frac{a}{b}$$

$$\tan m\angle A = \frac{4 \text{ cm}}{4 \text{ cm}}$$

$$\tan m\angle A = 1$$

$$\Rightarrow \boxed{m\angle A = 45^\circ} \quad (\because \tan 45^\circ = 1)$$

(ii) In right-angled $\triangle ABC$ with $m\angle C = 90^\circ$

$$m\angle A + m\angle B = 90^\circ$$

$$45^\circ + m\angle B = 90^\circ$$

$$m\angle B = 90^\circ - 45^\circ$$

$$\boxed{m\angle B = 45^\circ}$$

(iii)

$$\triangle PQR, m\angle Q = 90^\circ, m\angle P = 60^\circ, m\overline{PQ} = 4\sqrt{3} \text{ cm}$$

Solution:



Solution

Let $\triangle PQR$ be a right-angled triangle in which $m\angle Q = 90^\circ$, $m\angle P = 60^\circ$, $m\overline{PQ} = 4\sqrt{3}$ cm



(i) In right-angled $\triangle PQR$ with $m\angle Q = 90^\circ$

$$m\angle P + m\angle R = 90^\circ$$

$$60^\circ + m\angle R = 90^\circ$$

$$m\angle R = 90^\circ - 60^\circ$$

$$\boxed{m\angle R = 30^\circ}$$

(ii) In right-angled triangle ABC:

$$\cos m\angle P = \frac{m\overline{PQ}}{m\overline{PR}}$$

$$\cos 60^\circ = \frac{4\sqrt{3}}{m\overline{PR}}$$

$$\frac{1}{2} = \frac{4\sqrt{3}}{m\overline{PR}}$$

$$\boxed{m\overline{PR} = 8\sqrt{3} \text{ cm}}$$

(iii) In right-angled triangle ABC:

$$\tan m\angle P = \frac{m\overline{QR}}{m\overline{PQ}}$$

$$\tan 60^\circ = \frac{m\overline{QR}}{4\sqrt{3}}$$

$$\sqrt{3} = \frac{m\overline{QR}}{4\sqrt{3}}$$

$$4\sqrt{3} \times \sqrt{3} = m\overline{QR}$$

$$4(\sqrt{3})^2 = m\overline{QR}$$

$$4(3) = m\overline{QR}$$

$$\boxed{m\overline{QR} = 12 \text{ cm}}$$

(iii)

$\triangle PQR$, $m\angle R = 90^\circ$, $m\overline{PR} = 8$ m, $m\overline{RQ} = 3$ m

theorem

Solution

Let $\triangle PQR$ be a right-angled triangle in which

$m\angle R = 90^\circ$, $m\overline{PR} = 8$ m, $m\overline{RQ} = 3$ m



(i) In right-angled $\triangle PQR$ by Pythagoras' theorem,

$$(m\overline{PQ})^2 = (m\overline{PR})^2 + (m\overline{RQ})^2$$

$$\sqrt{(m\overline{PQ})^2} = \sqrt{(8 \text{ m})^2 + (3 \text{ m})^2}$$

$$m\overline{PQ} = \sqrt{64 \text{ m}^2 + 9 \text{ m}^2}$$

$$m\overline{PQ} = \sqrt{73 \text{ m}^2}$$

$$m\overline{PQ} = \sqrt{73} \text{ m}$$

$$m\overline{PQ} = \sqrt{73} \text{ m}$$

(ii) In right-angled $\triangle PQR$,

$$\tan m\angle P = \frac{m\overline{RQ}}{m\overline{PR}}$$

$$\tan m\angle P = \frac{3 \text{ m}}{8 \text{ m}}$$

$$\tan m\angle P = 1$$

$$\Rightarrow \boxed{m\angle P = 45^\circ} \quad (\because \tan 45^\circ = 1)$$

(iii) In right-angled $\triangle PQR$ with $m\angle R = 90^\circ$



$$m\angle P + m\angle Q = 90^\circ$$

$$45^\circ + m\angle Q = 90^\circ$$

$$m\angle Q = 90^\circ - 45^\circ$$

$$\boxed{m\angle Q = 45^\circ}$$

(iv)

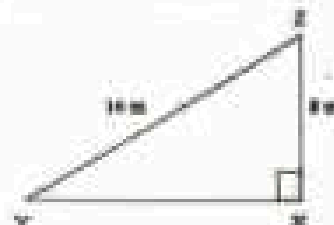
$\triangle XYZ$, $m\angle X = 90^\circ$, $m\overline{YZ} = 16$ m, $m\overline{XZ} = 8$ m

theorem

Solution

Let $\triangle XYZ$ be a right-angled triangle in which

$m\angle X = 90^\circ$, $m\overline{YZ} = 16$ m, $m\overline{XZ} = 8$ m



(i) In right-angled $\triangle XYZ$ by Pythagoras' theorem,

$$(m\overline{YZ})^2 = (m\overline{XY})^2 + (m\overline{XZ})^2$$

$$(16 \text{ m})^2 = (m\overline{XY})^2 + (8 \text{ m})^2$$

$$256 \text{ m}^2 = (m\overline{XY})^2 + 64 \text{ m}^2$$

$$256 \text{ m}^2 - 64 \text{ m}^2 = (m\overline{XY})^2$$

$$192 \text{ m}^2 = (m\overline{XY})^2$$

$$\sqrt{(m\overline{XY})^2} = \sqrt{192 \text{ m}^2}$$

$$m\overline{XY} = \sqrt{64 \times 3 \text{ m}^2}$$

$$\boxed{m\overline{XY} = 8\sqrt{3} \text{ m}}$$

(ii) In right-angled $\triangle XYZ$

$$\sin m\angle Y = \frac{m\overline{XZ}}{m\overline{YZ}}$$

$$\sin m\angle Y = \frac{8 \text{ m}}{16 \text{ m}}$$

$$\sin m\angle Y = \frac{1}{2}$$

$$\Rightarrow \boxed{m\angle Y = 30^\circ} \quad \left(\because \sin 30^\circ = \frac{1}{2} \right)$$

(iii) In right-angled $\triangle XYZ$ with $m\angle X = 90^\circ$

$$m\angle Y + m\angle Z = 90^\circ$$

$$30^\circ + m\angle Z = 90^\circ$$

$$m\angle Z = 90^\circ - 30^\circ$$

$$\boxed{m\angle Z = 60^\circ}$$

(iv)

$\triangle LMN$, $m\overline{LM} = 10$ cm, $m\overline{MN} = 8$ cm, $m\overline{LN} = 6$ cm

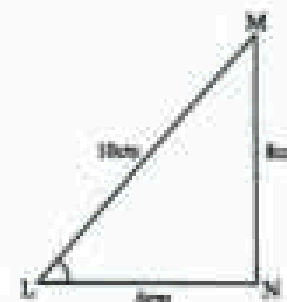
theorem

Solution

Given that $\triangle LMN$ is right-angled triangle.

Being longest side $m\overline{LM}$ is hypotenuse.

Being opposite to hypotenuse $m\angle N = 90^\circ$.



(i) In right-angled $\triangle LMN$

$$\tan m\angle L = \frac{m\overline{MN}}{m\overline{LN}}$$

$$\tan m\angle L = \frac{8 \text{ cm}}{6 \text{ cm}}$$



$$\tan m\angle L = \frac{4}{3}$$

$$m\angle L = \tan^{-1}\left(\frac{4}{3}\right)$$

$$m\angle L = 53.13^\circ$$

$$\boxed{m\angle L = 53.13^\circ}$$

100 In right-angled $\triangle LMN$ with $m\angle N = 90^\circ$

$$m\angle L + m\angle M = 90^\circ$$

$$53.13^\circ + m\angle M = 90^\circ$$

$$m\angle M = 90^\circ - 53.13^\circ$$

$$m\angle M = 36.87^\circ$$

$$\boxed{m\angle M = 36.87^\circ}$$

Q.5 If $x = r \cos \alpha \sin \beta$, $y = r \cos \alpha \cos \beta$ and $z = r \sin \alpha$, then find the value of $x^2 + y^2 + z^2$

Solution:

$$x = r \cos \alpha \sin \beta$$

$$y = r \cos \alpha \cos \beta$$

$$z = r \sin \alpha$$

Squaring and adding the equations

$$x^2 + y^2 + z^2 = (r \cos \alpha \sin \beta)^2 + (r \cos \alpha \cos \beta)^2 + (r \sin \alpha)^2$$

$$= r^2 [\cos^2 \alpha \sin^2 \beta + \cos^2 \alpha \cos^2 \beta + \sin^2 \alpha]$$

$$= r^2 [\cos^2 \alpha (\sin^2 \beta + \cos^2 \beta) + \sin^2 \alpha]$$

$$= r^2 [\cos^2 \alpha (1) + \sin^2 \alpha] \quad \because \sin^2 \beta + \cos^2 \beta = 1$$

$$= r^2 (\cos^2 \alpha + \sin^2 \alpha)$$

$$= r^2 (1) \quad \because \cos^2 \alpha + \sin^2 \alpha = 1$$

$$x^2 + y^2 + z^2 = r^2$$

Line of sight

If an object P is observed from a point of

observation O, then the line OP is called line of sight.

Angle of Elevation

If O is point of observation, OX is a horizontal line (ray) and an object P lies above the line OX, then the angle between the horizontal line and the line of sight to an object above the horizontal line is called angle of elevation. In the adjoining figure $\angle XOP$ is angle of elevation.



Angle of Depression

If O is point of observation, OX is a horizontal line (ray) and an object P lies below the line OX, then angle between the horizontal line and the line of sight to an object below the horizontal line is called angle of depression. In the adjoining figure $\angle XOP$ is angle of depression.



Example 13

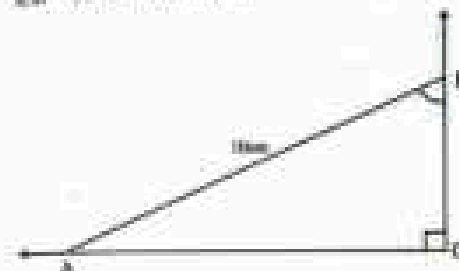
A ladder 10 m long leaning against a vertical wall makes an angle of 60° with wall. Find height of ladder along the wall.

Solution

Let AB is a ladder. BC is height of ladder along wall.

AC is along horizontal ground. In triangle ABC, $\angle C = 90^\circ$,

$\angle B = 60^\circ$ and AB = 10m.



$$\cos 60^\circ = \frac{BC}{AB}$$

$$\frac{1}{2} = \frac{BC}{10}$$

$$BC = 5m$$

So, height of ladder along wall = 5m.

Example 14

An aeroplane at an altitude of 900m finds that two ships are sailing towards it in the same direction. The angles of depression of ships as observed from plane are 30° and 60° . Find distance between ships.

Solution

In the figure O is position of plane. A and B are two ships. $\angle XOA$ and $\angle XOB$ are angles of

depression. As alternate angles of parallel lines are equal.

So, $m\angle A = 30^\circ$ and $m\angle B = 60^\circ$

Let AC = x and BC = y.

In right angled triangle ACO,



$$\tan 30^\circ = \frac{900}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{900}{x}$$

$$x = 900\sqrt{3} = 1558.85m$$

In right angled triangle BCO,

$$\tan 60^\circ = \frac{900}{y}$$

$$\sqrt{3} = \frac{900}{y} \Rightarrow y = \frac{900}{\sqrt{3}} = 519.62m$$

Distance between ships = x - y

$$= 1558.85 - 519.62$$

$$= 1039.23m$$

Exercise - 6.5

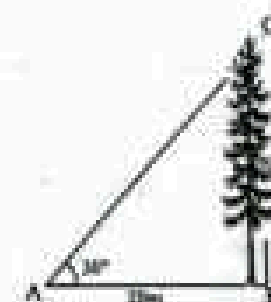
Q.1 From a point at a distance of 20 m from a tree, the angle of elevation of the top of a tree is 36° . Find the height of the tree.

Solution

Distance of point from tree = 20 m

Angle of elevation of tree = 36°

Height of tree = h = ?



In right-angled triangle ABC,



$$\tan \theta = \frac{\text{Per.}}{\text{base}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\tan 30^\circ = \frac{h}{20 \text{ m}}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{20 \text{ m}}$$

$$\Rightarrow h = \frac{20 \text{ m}}{\sqrt{3}}$$

$$h = 11.55 \text{ m}$$

Thus the height of tree is 11.55 m.

Q.3 The length of the shadow of a 10 m high pole is $10\sqrt{3}$ m. Find the angle of elevation of the sun.

Solution

Height of pole = $h = 10$ m

Length of the shadow = $10\sqrt{3}$ m

The angle of elevation of sun = $\theta = ?$



In right-angled triangle ABC,

$$\tan \theta = \frac{\text{Per.}}{\text{base}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\tan \theta = \frac{10 \text{ m}}{10\sqrt{3} \text{ m}}$$

$$\tan \theta = \frac{1}{\sqrt{3}} \quad \text{--- (i)}$$

We know that

$$\tan 30^\circ = \frac{1}{\sqrt{3}} \quad \text{--- (ii)}$$

$$\tan \theta = \tan 30^\circ$$

$$\Rightarrow \theta = 30^\circ$$

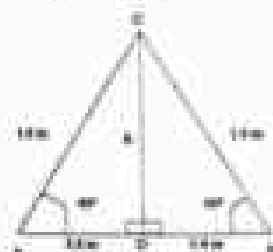
$$\therefore \angle A = 30^\circ$$

Q.3 The window of a room is in the shape of an equilateral triangle. The length of each side is 1.6 m. Find the height of the window.

Solution

The length of each side = 1.6 m.

The height of the window = $h = ?$



Let $\triangle ABC$ be an equilateral triangle.

An equilateral triangle is also an equilateral triangle having each angle of 60° i.e.

$$\angle A = \angle B = \angle C = 60^\circ$$

$AD \perp BC$ and bisects it. $\therefore AD = 0.8 \text{ m}$

In right-angled triangle ADC,

$$\tan \theta = \frac{\text{Per.}}{\text{base}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\tan 60^\circ = \frac{h}{0.8 \text{ m}}$$

$$\sqrt{3} = \frac{h}{0.8 \text{ m}} \quad \left(\because \tan 60^\circ = \sqrt{3} \right)$$

$$0.8 \text{ m} \times \sqrt{3} = h$$

$$h = 1.39 \text{ m}$$

Thus the height of window is 1.39 m.

Q.4 The height of a slide in a children park is 5 m. A girl standing at the top of the slide observes that the angle of depression of its bottom is 30° . Find the length of the slide.

Solution



From the figure,

$AB \parallel CD$ and being alternate angles

$$\angle ACD = \angle BAC$$

$$\angle BAC = 30^\circ$$

Height of slide = $AB = 5 \text{ m}$

Length of slide = $AC = x = ?$

In right-angled triangle ABC,

$$\sin \theta = \frac{\text{Per.}}{\text{Hyp.}} \quad \left(\because \sin \theta = \frac{\text{Per.}}{\text{Hyp.}} \right)$$

$$\sin 30^\circ = \frac{5 \text{ m}}{x}$$

$$\frac{1}{2} = \frac{5 \text{ m}}{x} \quad \left(\because \sin 30^\circ = \frac{1}{2} \right)$$

$$x = 2 \times 5 \text{ m}$$

$$x = 10 \text{ m}$$

Thus the length of the slide is 10 m.

Q.5 Two pillars of equal height stand on either side of a roadway which is 120 m wide. At a point on the road between pillars, the angles of elevation of the pillars are 60° and 30° . Find the height of each pillar and the position of this point.

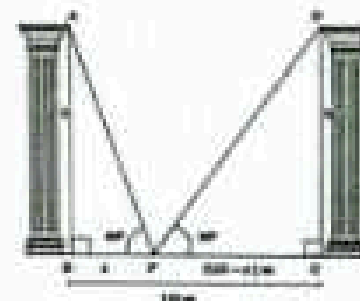
Solution

Two pillars of equal height = $AB = CD = h$

Distance between the pillars = $BC = 120 \text{ m}$

If the distance of point P from pillar AB is x then the distance of point P from pillar CD is $(120 - x) \text{ m}$

From point P angles of elevation of pillar AB and CD are 60° and 30° respectively.



In right-angled $\triangle ABP$,

$$\tan 60^\circ = \frac{\text{Per.}}{\text{base}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\sqrt{3} = \frac{h}{x} \quad \left(\because \tan 60^\circ = \sqrt{3} \right)$$

$$\sqrt{3} x = h \quad \text{--- (i)}$$

Now in right-angled $\triangle DCP$,

$$\tan 30^\circ = \frac{\text{Per.}}{\text{base}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\frac{1}{\sqrt{3}} = \frac{h}{120 - x} \quad \left(\because \tan 30^\circ = \frac{1}{\sqrt{3}} \right)$$

$$\frac{120 - x}{\sqrt{3}} = h \quad \text{--- (ii)}$$

From (i) and (ii) we get

$$h = h$$

$$\sqrt{3} x = \frac{120 - x}{\sqrt{3}}$$

$$\sqrt{3} \cdot \sqrt{3} x = 120 - x$$



$$(\sqrt{3})^2 x = 120 - x$$

$$3x = 120 - x$$

$$3x + x = 120$$

$$4x = 120$$

$$x = \frac{120}{4}$$

$$x = 30 \text{ m}$$

Put $x = 30$ m in (i)

$$\sqrt{3}x = h$$

$$h = \sqrt{3} \times 30 \text{ m}$$

$$h = 51.96 \text{ m}$$

$$h = 52 \text{ m}$$

Distance of point P from pillar AB = $x = 30$ m

Distance of point P from pillar CD = $120 - 30$
= 90 m

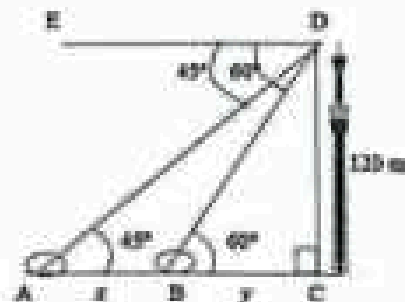
The height of each pillar is 52 m

Q.6 From the top of a tower of the height 120 m, the angles of depression of two boats on the same side of the tower at water level are 60° and 45° . Find the distance between the boats.

Answer

Solution:

Let A and B be two boats.



Height of tower = $CD = 120$ m

$AB = x$, $BC = y$ and $AC = x + y$

As $DE \parallel AC$ and alternate angles are equal
 $\angle A = 45^\circ$ and $\angle B = 60^\circ$. In right-angled triangle BCD,

$$\tan \angle C = \frac{\text{Per.}}{\text{base}} \quad (\because \tan \theta = \frac{\text{Per.}}{\text{base}})$$

$$\tan 60^\circ = \frac{120}{y}$$

$$\sqrt{3} = \frac{120}{y} \quad (\because \tan 60^\circ = \sqrt{3})$$

$$y = \frac{120}{\sqrt{3}}$$

$$y = 69.3 \text{ m}$$

In right-angled $\triangle ACD$,

$$\tan \angle A = \frac{\text{Per.}}{\text{base}} \quad (\because \tan \theta = \frac{\text{Per.}}{\text{base}})$$

$$\tan 45^\circ = \frac{120}{x+y}$$

$$1 = \frac{120}{x+y} \quad (\because \tan 45^\circ = 1)$$

$$x + y = 120 \text{ m}$$

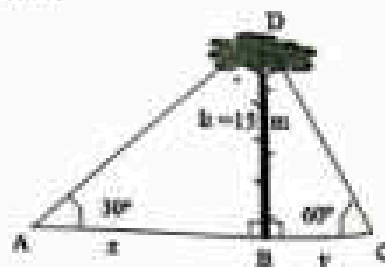
$$x = 120 \text{ m} - y$$

$$x = 120 \text{ m} - 69.3 \text{ m}$$

$$x = 50.7 \text{ m}$$

Q.7 Two men on the opposite side of a tree are in line with it. They observe that the angles of elevation of the top of the tree are 30° and 60° . The height of the tree is 15 m. Find the distance between them.

Answer



Height of tree = $BD = 15$ m.

Let distance of man A and C from the tree BD be x and y respectively.

The distance between two men = $x + y$

In right-angled $\triangle ABD$,

$$\tan 30^\circ = \frac{\text{Per.}}{\text{base}} \quad (\because \tan \theta = \frac{\text{Per.}}{\text{base}})$$

$$\frac{1}{\sqrt{3}} = \frac{15}{x} \quad (\because \tan 30^\circ = \frac{1}{\sqrt{3}})$$

$$x = 15\sqrt{3} \text{ m} \quad \text{--- (i)}$$

In right-angled $\triangle CBD$

$$\tan 60^\circ = \frac{\text{Per.}}{\text{base}} \quad (\because \tan \theta = \frac{\text{Per.}}{\text{base}})$$

$$\sqrt{3} = \frac{15}{y} \quad (\because \tan 60^\circ = \sqrt{3})$$

$$y\sqrt{3} = 15 \text{ m}$$

$$y = \frac{15}{\sqrt{3}} \text{ m}$$

$$y = 5\sqrt{3} \text{ m} \quad \text{--- (ii)}$$

Adding (i) and (ii) we get

$$x + y = 15\sqrt{3} \text{ m} + 5\sqrt{3} \text{ m}$$

$$x + y = 20\sqrt{3} \text{ m}$$

$$x + y = 34.64 \text{ m}$$

Thus the distance between two men is 34.64 m.

Q.8 From the top of a tree, the angle of elevation and depression of the top and bottom of a building are 30° and 45° respectively. If the height of the tree is 12 m, find the height of the building and the distance between the tree and the building.

Answer

Solution:

The angle of elevation = 30°

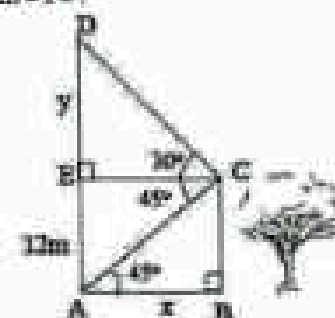
The angle of depression = 45°

Height of tree = $BD = 12$ m

The height of the building = $AD = ?$

The distance between the tree and the building

= $AB = x = ?$



In right-angled $\triangle ABC$

$$\tan \angle A = \frac{\text{Per.}}{\text{base}} \quad (\because \tan \theta = \frac{\text{Per.}}{\text{base}})$$

$$\tan 45^\circ = \frac{12}{x} \quad (\because \tan 45^\circ = 1)$$

$$1 = \frac{12}{x}$$

$$x = 12 \text{ m}$$

$$AB = 12 \text{ m}$$

The distance between the tree and the building

= $AB = 12$ m. (Given)

$\therefore AD = BC$ (opp. sides of rectangle ABCE)

$\therefore AD = 12$ m. --- (i)

In right-angled $\triangle DEC$,

$$\tan 30^\circ = \frac{\text{Per.}}{\text{base}} \quad (\because \tan \theta = \frac{\text{Per.}}{\text{base}})$$

$$\frac{1}{\sqrt{3}} = \frac{12}{y} \quad (\because \tan 30^\circ = \frac{1}{\sqrt{3}})$$

$$\frac{1}{\sqrt{3}} = \frac{12}{y}$$

$$y = \frac{12}{\sqrt{3}}$$

$$y = 6.93 \text{ m}$$



$$m\overline{ED} = 6.93 \text{ m} \quad \text{--- (i)}$$

The height of the building

$$m\overline{AD} = m\overline{AE} + m\overline{ED}$$

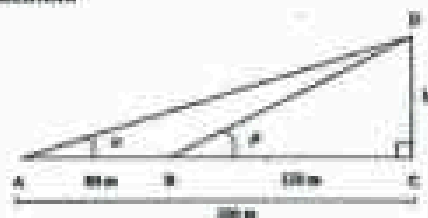
Putting value from (i) and (ii)

$$m\overline{AD} = 12 \text{ m} + 6.93 \text{ m}$$

$$m\overline{AD} = 18.93 \text{ m}$$

Q.9 From a point on the ground at a distance of 200 m, the angle of elevation of the top of a tower is α . There is another point B 80 m nearer to the tower. The angle of elevation of the top of a tower from B is β . If $\tan \alpha = \frac{2}{5}$, find the height of the tower and value of β .

Solution



Distance of point A from tower $m\overline{AC} = 200 \text{ m}$

The distance of point A from B, $m\overline{AB} = 80 \text{ m}$

The distance of point B from C, $m\overline{BC} = 120 \text{ m}$

The angle of elevation from point A = α

The angle of elevation from point B = β

Height of tower = $m\overline{CD} = h = ?$

In right-angled $\triangle ACD$

$$\tan \alpha = \frac{m\overline{CD}}{m\overline{AC}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\frac{2}{5} = \frac{h}{200} \quad \left(\because \tan \alpha = \frac{2}{5} \text{ (Given)} \right)$$

$$\frac{2 \times 200}{5} = h$$

$$\frac{400}{5} = h$$

$$80 = h$$

$$h = 80 \text{ m}$$

In right-angled $\triangle BCD$

$$\tan \beta = \frac{m\overline{CD}}{m\overline{BC}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\tan \beta = \frac{80}{120} \quad \left(m\overline{BC} = (200 - 80) \text{ m} = 120 \text{ m} \right)$$

$$\beta = \tan^{-1} \left(\frac{80}{120} \right)$$

$$\beta = 33.69^\circ$$

Q.10 A boat moving away from the lighthouse 206.6 m high, takes 120 seconds to change the angles of elevation of the top of the lighthouse from 60° to 45° . Find the speed of the boat.

Solution



Time = $T = 120 \text{ s}$

Height of lighthouse = $m\overline{CD} = h = 206.6 \text{ m}$

The angle of elevation from point B = 60°

The angle of elevation from point A = 45°

The distance of boat A from B, $m\overline{AB} = x \text{ m}$

The distance of boat B from light house,

$$m\overline{BC} = y \text{ m}$$

The distance of boat A from light house,

$$m\overline{AC} = (x + y) \text{ m}$$

In right-angled $\triangle BCD$,

$$\tan \alpha = \frac{m\overline{CD}}{m\overline{BC}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\tan 60^\circ = \frac{206.6 \text{ m}}{y}$$

$$\sqrt{3} = \frac{206.6 \text{ m}}{y} \quad \left(\because \tan 60^\circ = \sqrt{3} \right)$$

$$y = \frac{206.6 \text{ m}}{\sqrt{3}}$$

$$y = 119.28 \text{ m} \quad \text{--- (i)}$$

In right-angled $\triangle ACD$,

$$\tan \alpha = \frac{m\overline{CD}}{m\overline{AC}} \quad \left(\because \tan \theta = \frac{\text{Per.}}{\text{base}} \right)$$

$$\tan 45^\circ = \frac{206.6 \text{ m}}{x + y}$$

$$1 = \frac{206.6 \text{ m}}{x + y} \quad \left(\because \tan 45^\circ = 1 \right)$$

$$x + y = 206.6 \text{ m}$$

$$x = 206.6 \text{ m} - y$$

$$x = 206.6 \text{ m} - 119.28 \text{ m}$$

$$x = 87.32 \text{ m} \quad \text{--- (i)}$$

Distance covered from B to A while changing angle of elevation from 60° to 45° is 87.32 m.

We know that

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Speed} = \frac{87.32 \text{ m}}{120 \text{ s}}$$

$$\text{Speed} = 0.73 \text{ m/s}$$

Bearing

When we want to move from one position to another, we need to know not only the distance to be traveled but also the direction of movement. Direction sense is the ability to know a person's current location and then find the correct way towards the destination. It helps us in getting clear picture of bearing.



The four cardinal directions are north, east, south and west. These are marked by the initials N, E, S and W. Starting from north with an angle of zero degree, east is at right angle to north and intermediate direction north-east is at an angle of 45 degrees to north. Similarly south-east is at angle of 135 degrees to north and so on.

Bearings are the angles measured clockwise from due north and are expressed as three digit numbers. Bearings are mostly used by sailors and pilots to describe the direction they are travelling and to navigate from one point to another.



Points to ponder!

- Bearing is an angle expressed with three-digit numbers.

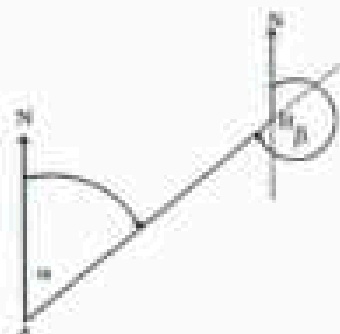
Measured in clockwise direction from north.

- If α is acute, then $\beta = 180^\circ + \alpha$
(corresponding angles property)

$$\Rightarrow \alpha = \beta - 180^\circ$$

This means that bearing of A from B = $180^\circ + \text{bearing of B from A}$

- Exact north has a bearing of 0° .



Example 15

Find the bearing of boat from light house in the given figure.

Solution

For finding the bearing of boat from light house, first draw the north line at light house, then draw the line connecting the light house and the boat. Place your protractor with zero point north and measure the angle from north line to the line connecting light house and the boat. Thus, the bearing of boat from light house = 70° .

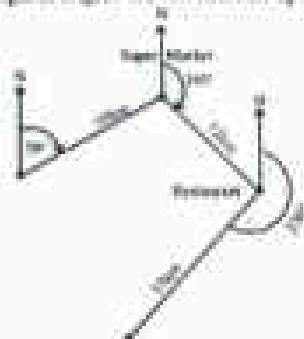
(The figure is not drawn accurately)

Example 16

Anand travelled at a bearing of 70° for 10 km to reach super market. Then, he travelled at a bearing of 145° for 12 km to reach a restaurant. Finally, he travelled at a bearing of 230° for 15 km to reach his friend's house. Make a labeled illustration to show his journey.

Solution

In the figure, angles are not marked up to scale.



Example 17

A ship sails from port A to port B at a bearing of 040° for 35 km. Then, it sails at a bearing of 130° for 50 km from port B to port C. Find:

- the distance between port A and port C
- the bearing of C from A

Solution

In the figure:

a) By drawing the path according to given information, we observe that $\theta = 130^\circ - 40^\circ - 90^\circ$ (since two north at A and B are parallel).

Therefore, $\beta = 180^\circ - 90^\circ - 40^\circ$.

Thus, triangle ABC is right with right angle at B. In right $\triangle ABC$ by using Pythagoras theorem,

$$\begin{aligned} m\overline{AC} &= \sqrt{(m\overline{AB})^2 + (m\overline{BC})^2} \\ &= \sqrt{(35)^2 + (50)^2} = \sqrt{3725} = 61.03 \text{ km} \end{aligned}$$

b) Bearing of C from A = $040^\circ + \alpha$

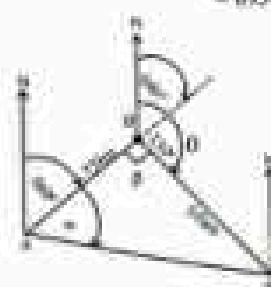
First we find α in right $\triangle ABC$

$$\tan \alpha = \frac{m\overline{BC}}{m\overline{AB}}$$

$$\tan \alpha = \frac{50}{35} = 1.4286$$

$$\Rightarrow \alpha = \tan^{-1}(1.4286) = 55^\circ$$

$$\text{Thus, bearing of C from A} = 040^\circ + 055^\circ = 095^\circ$$



Exercise 6.6

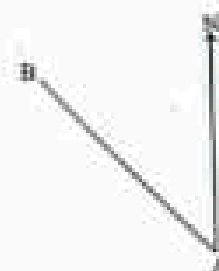
Q.1 Measure three figure bearing of B from A with the help of protractor in each the following.



ii)

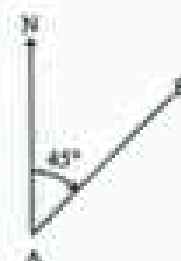


iii)



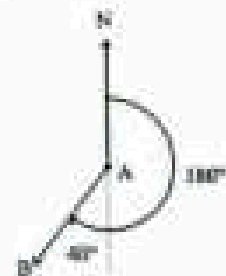
Solution:

i)



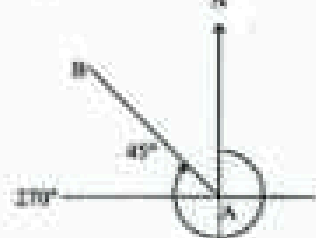
The bearing of B from A is

ii)



The bearing of B from A = $180^\circ - 40^\circ = 140^\circ$

iii)



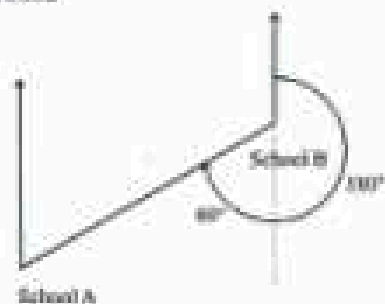
The bearing of B from A = $270^\circ - 45^\circ = 225^\circ$

Q.2 School A and school B are marked, find the bearing of school A from school B





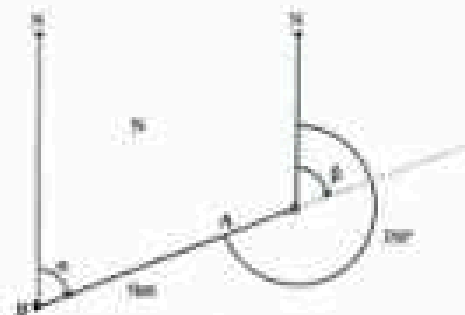
Solution



The bearing of school A from school B = $180^\circ - 60^\circ = 120^\circ$.

Q.3 Two boats A and B are 5km apart and the bearing of boat B from boat A is 250° . Draw the diagram showing relative positions of both boats, then find the bearing of boat A from boat B.

Solution



From diagram,

$$\beta = 250^\circ - 180^\circ$$

$$\beta = 70^\circ$$

Bearing of boat A from B is α

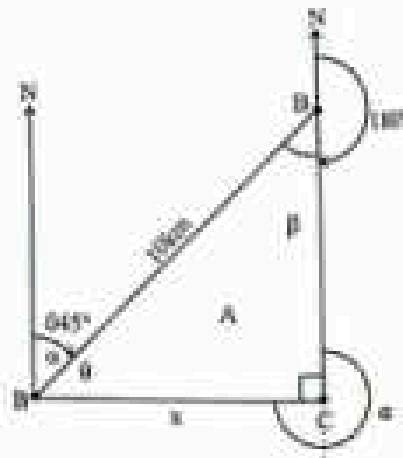
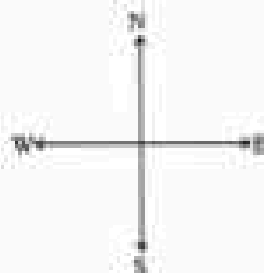
$$\alpha = \beta \text{ (corresponding angles)}$$

$$\alpha = 70^\circ$$

Hence, bearing of boat A from boat B is 070° .

Q.4 The positions of three ships A, B and C are such that the bearing of B from A is 045° and ship C is to the east of A, bearing of C from B is 180° and $m\angle A = 10^\circ$, what is the bearing of A from C? Also find the distance between ship A and ship C.

Solution



$$\text{Bearing of ship A from C} = \alpha = 180^\circ - 090^\circ = 270^\circ$$

In right angled $\triangle ABC$,

$$\beta = 45^\circ \text{ (alternate angles)}$$

$$\theta = 90^\circ - \beta$$

$$\theta = 90^\circ - 45^\circ$$

$$\theta = 45^\circ$$

Now, in $\triangle ABC$

$$\cos \theta = \frac{m\overline{AC}}{m\overline{AB}} \quad \left(\because \cos \theta = \frac{\text{base}}{\text{Hyp}} \right)$$

$$\cos 45^\circ = \frac{x}{10}$$

$$\frac{1}{\sqrt{2}} = \frac{x}{10} \quad \left(\because \cos 45^\circ = \frac{1}{\sqrt{2}} \right)$$

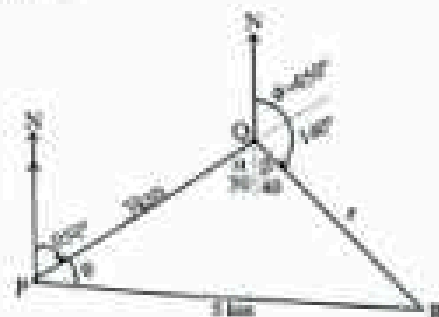
$$\Rightarrow x = \frac{10}{\sqrt{2}} \text{ km}$$

$$\Rightarrow x = 7.07 \text{ km}$$

The distance between the ship A and ship B is 7.07 km.

Q.5 Babar Azam walked from point P to point Q at a bearing of 050° for 3km. Then, he walked from point Q to point R at a bearing of 140° . Find the distance QR, if the distance between P and R is 5km. Also find the bearing of R from P.

Solution



From Diagram

$$\theta = 050^\circ \text{ (corresponding angles)}$$

$$\alpha = \phi = 50^\circ \text{ (vertical angles)}$$

$$\beta = 180^\circ - 140^\circ = 40^\circ$$

$$\text{So, } \alpha + \beta = 50^\circ + 40^\circ = 90^\circ$$

This shows that $\triangle PQR$ is right-angled triangle in which $m\angle Q = 90^\circ$

By Pythagoras theorem

$$\{m\overline{PR}\}^2 = \{m\overline{PQ}\}^2 + \{m\overline{QR}\}^2$$

$$(5\text{km})^2 = (3\text{km})^2 + x^2$$



$$25\text{km}^2 = 9\text{km}^2 + x^2$$

$$25\text{km}^2 - 9\text{km}^2 = x^2$$

$$16\text{km}^2 = x^2$$

$$\Rightarrow \sqrt{x^2} = \sqrt{16\text{km}^2} \Rightarrow x = 4\text{km}$$

Thus the distance $\overline{QR}$ is 4 km.

ii) Bearing of R from P = $050^\circ + \theta$

First we find θ in right $\triangle PQR$

$$\cos \theta = \frac{m\overline{PQ}}{m\overline{PR}} \quad \left(\because \cos \theta = \frac{\text{base}}{\text{Hyp}} \right)$$

$$\cos \theta = \frac{3\text{km}}{5\text{km}}$$

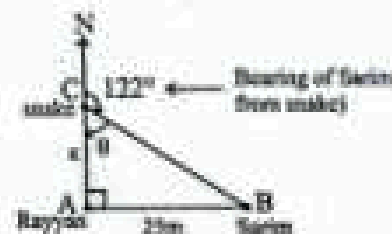
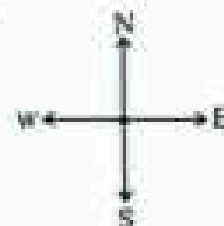
$$\theta = \cos^{-1} \left(\frac{3}{5} \right)$$

$$\theta = 53^\circ$$

Thus bearing of R from P = $050^\circ + 053^\circ = 103^\circ$

Q.6 Rayyan spots a snake directly north from his location. Sarim is 25m east of Rayyan and spots the same snake. If the bearing of Sarim from snake is 122° , how far is Rayyan from the snake?

Solution



Let distance of Rayyan from snake = x m



From figure,

$$\theta = 180^\circ - 122^\circ = 58^\circ$$

Angle between North and East is 90°

Thus triangle ABC is right triangle with right angle at A

$$\tan \theta = \frac{mAB}{mAC}$$

$$\tan 58^\circ = \frac{25m}{x}$$

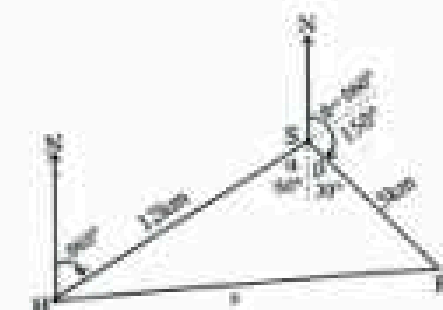
$$\Rightarrow x = \frac{25m}{\tan 58^\circ}$$

$$x = 15.6 \text{ m}$$

Thus distance of Rayyan from school is 15.6m.

Q.7 Bilal traveled 12km on his bike to reach school at a bearing of 46° from his home. Then, he went to football ground from school at a bearing of 150° . If the football ground is at a distance of 5km from school, find the distance he has to cover to reach home from football ground?

Solution



Let distance between home and football ground = x

From figure,

$$\theta = 96^\circ \text{ (corresponding angles)}$$

$$\alpha = \theta = 96^\circ \text{ (vertical angle)}$$

$$\beta = 180^\circ - 150^\circ = 30^\circ$$

$$\alpha = \beta = 96^\circ + 30^\circ = 126^\circ$$

This shows that $\triangle HSF$ is right with right angle at S.

By Pythagoras theorem

$$(mHF)^2 = (mHS)^2 + (mSF)^2$$

$$x^2 = (12\text{km})^2 + (5\text{km})^2$$

$$x = \sqrt{144\text{km}^2 + 25\text{km}^2}$$

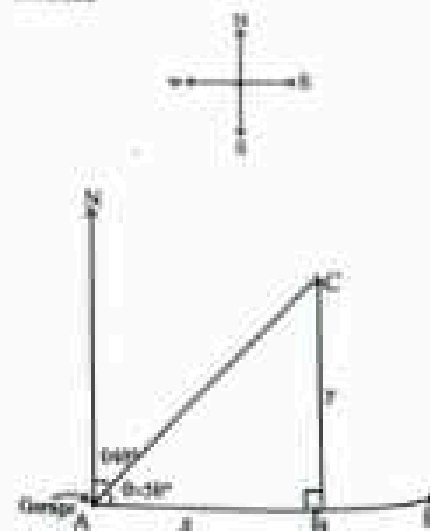
$$x = \sqrt{169\text{km}^2}$$

$$x = 13 \text{ km}$$

Thus distance from football ground to home is 13km.

Q.8 A car leaves the garage at a bearing of 90° and travels in a straight line to 12km. How many kilometers north and how many kilometers east has the car traveled from the garage?

Solution



Angle between East and North = 90°

$$\theta = 90^\circ - 40^\circ = 50^\circ$$

i) In right-angled $\triangle ABC$,



$$\cos \theta = \frac{mAB}{mAC} \quad \left(\because \cos \theta = \frac{\text{base}}{\text{hyp}} \right)$$

$$\cos 50^\circ = \frac{x}{12}$$

$$12 \cos 50^\circ = x$$

$$\Rightarrow x = 7.36 \text{ km}$$

Thus distance covered along east is 7.36 km.

ii) In right-angled $\triangle ABC$,

$$\sin \theta = \frac{mBC}{mAC}$$

$$\sin 50^\circ = \frac{y}{12}$$

$$12 \sin 50^\circ = y$$

$$\Rightarrow y = 9.36 \text{ km}$$

Thus distance covered along the North is 9.36 km.

Miscellaneous Exercise 6

Q.1 Encircle the correct option in the following:

1. If $\sin x = \frac{1}{4}$, what is the value of $\cos x$?

(a) $\frac{\sqrt{15}}{4}$

(b) $\frac{3}{4}$ correct

(c) $\frac{\sqrt{17}}{4}$

(d) $\frac{15}{4}$

2. $\cos \frac{2\pi}{3} = ?$

correct

(a) $\frac{1}{2}$

(b) $-\frac{1}{2}$

(c) $-\frac{\sqrt{3}}{2}$

(d) $\frac{\sqrt{3}}{2}$

3. $\frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = ?$

correct

(a) $\csc^2 \theta$

(b) $2 \sec^2 \theta$

(c) $\sec^2 \theta$

(d) $2 \csc^2 \theta$

4. $-2 - 3 \tan^2 \theta = ?$

correct

(a) $2 \csc^2 \theta$

(b) $-3 \csc^2 \theta$

(c) $-3 \sec^2 \theta$

(d) $3 \sec^2 \theta$

5. If $\cos x = \sin x$, then the value of x is:

correct

(a) 30°

(b) 45°

(c) 60°

(d) 90°

6. $4 \csc^2 \theta - 4 \cot^2 \theta - 4 \tan^2 \theta = ?$

correct

(a) 0

(b) 1

(c) -1

(d) 4

7. If $\tan x = \frac{a}{b}$, then $\sin x = ?$

correct

(a) $\frac{a}{\sqrt{a^2 + b^2}}$

(b) $\frac{b}{\sqrt{a^2 + b^2}}$

(c) $\frac{a}{\sqrt{a^2 + b^2}}$

(d) $\frac{b}{\sqrt{a^2 + b^2}}$

8. $50^\circ 30' = ?$

correct

(a) 50.2°

(b) 50.3°

(c) 50.4°

(d) 50.5°

9. $(1 - \sin^2 \theta) \sec^2 \theta = ?$

correct

(a) $\csc^2 \theta$

(b) $-3 \cot^2 \theta$

(c) $\sec^2 \theta$

(d) $\tan^2 \theta$

10. In which quadrant do the angles between 90° and 180° lie?

correct

(a) 1°

(b) 2°

(c) 3°

(d) 4°

11. The system in which the angles are measured in radians is called:

correct

(a) CGS

(b) arcangular

(c) circular

(d) centesimal

12. $270^\circ = ?$

correct

(a) $\frac{3\pi}{2}$

(b) $\frac{\pi}{2}$

(c) $-\frac{\pi}{2}$

(d) $-\frac{3\pi}{2}$

13. What is the bearing of south-west direction line?

correct

(a) 135°

(b) 225°

(c) 315°

(d) 45°



- (a) 45° (b) 135°
 (c) 225° (d) 270°
14. If an automobile travels at the bearing of 070° from city A to city B, then the bearing from city B to city A is: reverse
 (a) 070° (b) 110°
 (c) 230° (d) 290°

Additional MCQs

16. The union of two rays, which have common end-point is called: reverse
 (a) an angle (b) a degree
 (c) a minute (d) a radian
17. The system of measurement in which the angle is measured in $D^\circ M' S''$ is called... System. reverse
 (a) CGS (b) rectangular
 (c) MKS (d) circular
18. $1^\circ =$ reverse
 (a) 180° radian (b) π radian
 (c) $\frac{\pi}{180}$ radian (d) $\frac{180}{\pi}$ radian
19. $1 \text{ rad} =$ reverse
 (a) 60° (b) 180°
 (c) 57.30° (d) 67.30°
20. $1^\circ =$ reverse
 (a) $360'$ (b) $60'$
 (c) $330'$ (d) $720'$
21. $\frac{3\pi}{4}$ radians = reverse
 (a) 135° (b) 125°
 (c) 150° (d) 30°
22. If $\tan \theta = \sqrt{3}$, then θ is equal to: reverse
 (a) 90° (b) 45°
 (c) 60° (d) 30°
23. $\sec^2 \theta - \tan^2 \theta =$ reverse
 (a) $\sin^2 \theta$ (b) 1

24. A boat sail 4km north from point L, to M, then it sails 3km west from M to P. What's the bearing of P from L? Provide your answer to the nearest degree. reverse
 (a) 53.7° (b) 65.3°
 (c) 24.7° (d) 32.3°

- (c) $\cos^2 \theta$ (d) -1
24. $\frac{1}{1+\sin \theta} + \frac{1}{1-\sin \theta} =$ reverse
 (a) $2\sec^2 \theta$ (b) $2\cos^2 \theta$
 (c) $\sec^2 \theta$ (d) $2\cos^2 \theta$
25. $\frac{1}{\frac{1}{\tan \theta} + \tan \theta} =$ reverse
 (a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$
 (c) $\sqrt{2}$ (d) $\frac{\sqrt{2}}{2}$
26. $\cot \theta \sec \theta =$ reverse
 (a) $\sin \theta$ (b) $\sec \theta$
 (c) $\cos \sec \theta$ (d) $\tan \theta$
27. $\cot^2 \theta - \csc^2 \theta =$ reverse
 (a) -1 (b) 1
 (c) 0 (d) $\tan \theta$
28. $\sin^2 \theta + \cos^2 \theta =$ reverse
 (a) $\sec^2 \theta$ (b) 1
 (c) $\csc^2 \theta$ (d) -1
29. In which quadrants all trigonometric ratios are positive? reverse
 (a) I (b) II
 (c) III (d) IV
30. Fundamental trigonometric ratios are: reverse
 (a) 3 (b) 4
 (c) 5 (d) 6
31. Which one is a quadrantal angle? reverse
 (a) 30° (b) 45°



- (c) 60° (d) 90°
32. $\sin \theta \csc \theta =$ reverse
 (a) 1 (b) 0
 (c) $\sin \theta$ (d) $\cos \theta$
33. $\cos \theta \sec \theta =$ reverse
 (a) 1 (b) $\tan \theta$
 (c) 0 (d) $\cot \theta$
34. $\tan \theta \cot \theta =$ reverse
 (a) $\sin \theta$ (b) $\sec \theta$
 (c) 1 (d) 0
35. $\sin 45^\circ + \cos 45^\circ =$ reverse
 (a) 1 (b) $\sqrt{2}$
 (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{2}}{2}$
36. $1 + \tan^2 \theta =$ reverse
 (a) $\frac{\sqrt{2}}{2}$ (b) $\sqrt{2}$
 (c) $\frac{1}{\sqrt{2}}$ (d) 2

37. $\sec 60^\circ =$ reverse
 (a) $\frac{1}{2}$ (b) $\frac{2}{\sqrt{3}}$
 (c) $\sqrt{3}$ (d) 0
38. $\sec 30^\circ =$ reverse
 (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$
 (c) 2 (d) $\frac{2}{\sqrt{3}}$
39. $\csc 60^\circ =$ reverse
 (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$
 (c) 2 (d) $\frac{2}{\sqrt{3}}$
40. $\sin \theta \csc \theta =$ reverse
 (a) $\cot + \sec$ (b) $\cot \csc \theta$
 (c) $\tan \theta + \cot \theta$ (d) $\cot \sec \theta$
41. Formula for arc length of a sector is $l = r\theta$ and θ is in: reverse
 (a) degree (b) radians
 (c) Minutes (d) seconds

Answer Key

1	a	2	b	3	d	4	c	5	b
6	a	7	c	8	d	9	d	10	b
11	c	12	a	13	c	14	a	15	d
16	a	17	b	18	c	19	c	20	d
21	b	22	c	23	b	24	d	25	b
26	c	27	a	28	b	29	a	30	d
31	d	32	a	33	a	34	c	35	b
36	d	37	d	38	d	39	d	40	b
41	b								

Q.2 Find $\sin \alpha$, $\tan \alpha$ and $\sec \alpha$, if $\cos \alpha = \frac{-5}{13}$.

Where $\frac{\pi}{2} < \alpha < \pi$.

Solution

$\cos \alpha = \frac{-5}{13}$ where $\frac{\pi}{2} < \alpha < \pi$

Angle α lies in Q-II

(i) If $\cos \alpha = \frac{-5}{13}$.



$$\text{then } \sec \alpha = \frac{13}{-5} \text{ (reciprocal)}$$

(ii) We know that

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\sin^2 \alpha = 1 - \cos^2 \alpha$$

$$\sqrt{\sin^2 \alpha} = \sqrt{1 - \cos^2 \alpha}$$

$$\sin \alpha = \sqrt{1 - \left(\frac{-5}{13}\right)^2}$$

$$\sin \alpha = \sqrt{1 - \frac{25}{169}}$$

$$\sin \alpha = \sqrt{\frac{169 - 25}{169}}$$

$$\sin \alpha = \sqrt{\frac{144}{169}}$$

$$\sin \alpha = \frac{12}{13} \dots\dots (\sin \alpha > 0 \text{ in Q-II})$$

(iii) We know that

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$$

$$\tan \alpha = \frac{\frac{12}{13}}{\frac{-5}{13}}$$

$$\tan \alpha = \frac{12}{-5} = \frac{12}{5}$$

$$\tan \alpha = -\frac{12}{5} \text{ Ans.}$$

Q.3 The angle of elevation of a hot air balloon 500 m away, climbing up vertically, changes from 30° to 60° in 100 seconds. What is the upward speed in meters per second?

Solution

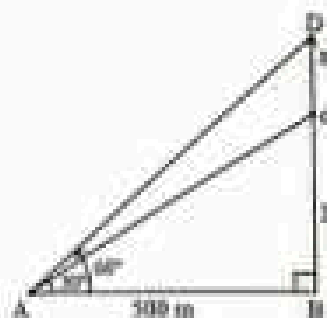
$$\text{Time} = t = 100 \text{ s}$$

$$\text{Distance} = m\overline{AB} = 500 \text{ m}$$

$$\text{Let } m\overline{BC} = y$$

$$\text{Vertical distance covered} = CD = x$$

$$m\overline{BD} = x + y$$



In right $\triangle ABC$

$$\tan m\angle A = \frac{m\overline{BC}}{m\overline{AB}} \quad \left(\because \tan \theta = \frac{\text{opp}}{\text{base}} \right)$$

$$\tan 30^\circ = \frac{y}{500 \text{ m}}$$

$$500 \text{ m} = \tan 30^\circ \cdot y$$

$$\Rightarrow y = 500 \text{ m} = \frac{1}{\sqrt{3}} \quad \left(\because \tan 30^\circ = \frac{1}{\sqrt{3}} \right)$$

$$y = 288.68 \text{ m}$$

In right $\triangle ABD$,

$$\tan m\angle A = \frac{m\overline{BD}}{m\overline{AB}} \quad \left(\because \tan 60^\circ = \sqrt{3} \right)$$

$$\tan 60^\circ = \frac{x + y}{500}$$

$$500 \times \tan 60^\circ = x + y$$

$$\Rightarrow x + y = 866 \text{ m}$$

$$x = 866 \text{ m} - y$$

$$x = 866 \text{ m} - 288.68 \text{ m}$$

$$x = 577.32$$

$$x = 577 \text{ m}$$

Vertical distance covered while changing 30° of elevation from 30° to 60° is 577 m.

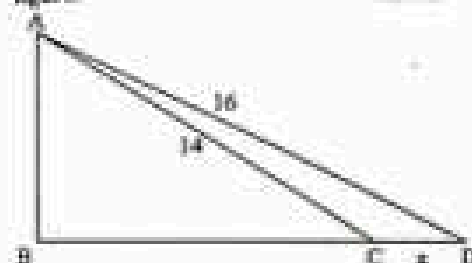
$$\text{Speed} = \frac{\text{dis. trav.}}{\text{time}}$$

$$\text{Speed} = \frac{577 \text{ m}}{100 \text{ s}}$$

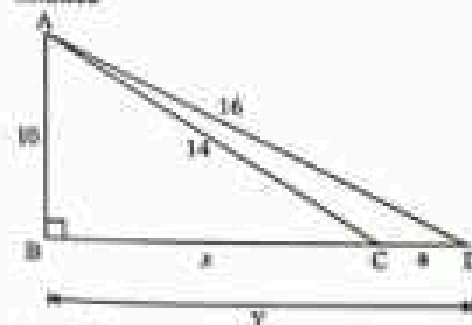
$$\text{Speed} = 5.77 \text{ m/s}$$



Q.4 Find the value of x in the following figure.



Solution



$$\text{Let } m\overline{BC} = x$$

$$m\overline{BD} = y$$

We observe that

$$m\overline{BD} = m\overline{BC} + m\overline{CD}$$

$$y = x + x \dots\dots (1)$$

First we find x and y .

In right $\triangle ABC$, by Pythagoras theorem

$$(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2$$

$$(14)^2 = (16)^2 + (x)^2$$

$$196 = 256 + x^2$$

$$96 = x^2$$

$$96 = x^2$$

$$\Rightarrow \sqrt{x^2} = \sqrt{96}$$

$$x = 9.8$$

Now, in right $\triangle ABD$, by Pythagoras theorem

$$(m\overline{AD})^2 = (m\overline{AB})^2 + (m\overline{BD})^2$$

$$(16)^2 = (10)^2 + (y)^2$$

$$256 = 100 + y^2$$

$$256 - 100 = y^2$$

$$156 = y^2$$

$$\Rightarrow \sqrt{y^2} = \sqrt{156}$$

$$y = 12.5$$

Putting the values of x and y in eq. (1)

$$y = x + x$$

$$y = x + x$$

$$12.5 = 9.8 + x$$

$$2.7 = x$$

$$\Rightarrow x = 2.7$$

Thus required value of x is 2.7.

Q.5 The area of a right triangle is 50 cm^2 . One of its angle is 45° . Find the lengths of the sides and hypotenuse of the triangle.

Solution



$$\text{Area} = A = 50 \text{ cm}^2$$

In right $\triangle ABC$,

$$\therefore \tan m\angle A = \frac{m\overline{BC}}{m\overline{AB}} \quad \left(\because \tan \theta = \frac{\text{opp}}{\text{base}} \right)$$

$$\tan 45^\circ = \frac{h}{b}$$

$$1 = \frac{h}{b}$$

$$[\because \tan 45^\circ = 1]$$



$$\Rightarrow \boxed{h = b} \dots\dots (1)$$

We know that Area of $\Delta = \frac{1}{2} (b \times h)$

$$50 = \frac{1}{2} (b \times b)$$

$$50 \times 2 = b \times b$$

Put $b = h$ from (1)

$$100 = b \times b$$

$$100 = b^2$$

$$\Rightarrow \sqrt{b^2} = \sqrt{100}$$

$$b = 10 \text{ cm}$$

$$\Rightarrow h = 10 \text{ cm}$$

$$\boxed{(\therefore h = b)}$$

In right angled ΔABC ,

By Pythagoras theorem

$$\{m\overline{AC}\}^2 = \{m\overline{AB}\}^2 + \{m\overline{BC}\}^2$$

$$x^2 = (10 \text{ cm})^2 + (10 \text{ cm})^2$$

$$x^2 = 100 \text{ cm}^2 + 100 \text{ cm}^2$$

$$x^2 = 200 \text{ cm}^2$$

$$\sqrt{x^2} = \sqrt{2 \times 100 \text{ cm}^2}$$

$$x = 10\sqrt{2} \text{ cm}$$

Thus length of hypotenuse is $10\sqrt{2}$ cm.

Base = 10 cm

Height = 10 cm

Hypotenuse = $10\sqrt{2}$ cm

Q.6 If the shadow of a building increases by 12 metres, when angle of elevation of the sun rays decreases from 60° to 45° what is the height of building.



Increase in length of shadow = 12m

Angle of elevation decreases from 60° to 45°

$$m\angle B = 60^\circ$$

$$m\angle H = 45^\circ$$

Let $m\overline{BC} = x$, $m\overline{AC} = (12+x)m$

Height of building = $m\overline{CD} = h$

In right angled ΔACD

$$\tan m\angle A = \frac{m\overline{CD}}{m\overline{AC}} \quad \left(\because \tan \theta = \frac{\text{opp}}{\text{adj}} \right)$$

$$\tan 45^\circ = \frac{h}{12+x}$$

$$1 = \frac{h}{12+x} \quad \left(\because \tan 45^\circ = 1 \right)$$

$$\Rightarrow 12 + x = h$$

$$x = h - 12 \dots\dots (1)$$

In right ΔBCD ,

$$\tan m\angle B = \frac{m\overline{CD}}{m\overline{BC}}$$

$$\tan 60^\circ = \frac{h}{x}$$

$$\sqrt{3}x = h \dots\dots (2)$$

Putting the value of x from (1)

$$\sqrt{3}(h - 12) = h$$

$$\sqrt{3}h - 12\sqrt{3} = h$$

$$\sqrt{3}h - h = 12\sqrt{3}$$

$$h(\sqrt{3} - 1) = 12\sqrt{3}$$

$$h = \frac{12\sqrt{3}}{\sqrt{3} - 1}$$

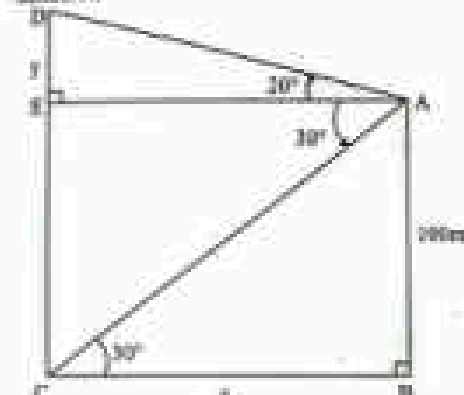
$$h = 28.39m$$

Thus the height of building is 28.39.

Q.7 From the top of a 100 metres high building, the angle of depression to the bottom of a second building is 30° . From the same point, the angle of elevation to the top of the second building is 20° . Calculate the height of the second building.



Solution



Height of 1st building = $m\overline{AB} = 100m$

Height of 2nd building = $m\overline{CD} = ?$

$m\overline{EC} = m\overline{AB}$ (Opposite sides of rectangle)

$$m\overline{EC} = 100m$$

Let $m\overline{BC} = x$, $m\overline{ED} = y$

In right ΔABC ,

$$\tan m\angle C = \frac{m\overline{AB}}{m\overline{BC}}$$

$$\tan 30^\circ = \frac{100}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{100}{x}$$

$$\Rightarrow x = 100\sqrt{3}m$$

$$\boxed{x = 173.2m}$$

Now, $m\overline{AE} = m\overline{CD}$ (Opp. Sides of rectangle)

$$m\overline{AE} = 173.2m \dots\dots (1)$$

In right ΔAED ,

$$\tan m\angle A = \frac{m\overline{ED}}{m\overline{AE}}$$

$$\tan 20^\circ = \frac{y}{173.2}$$

$$173.2 \times \tan 20^\circ = y$$

$$63m = y$$

$$\boxed{y = 63m}$$

Now, height of 2nd building = $m\overline{CD}$

$$= m\overline{BC} + m\overline{ED}$$

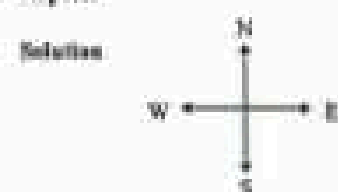
$$m\overline{CD} = 100m + 63m$$

$$\boxed{\therefore m\overline{CD} = 163m}$$

$$m\overline{CD} = 163m$$

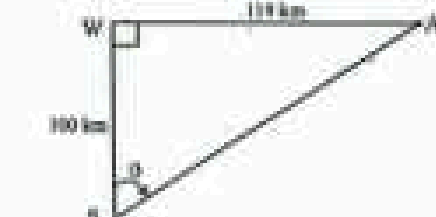
Thus, height of 2nd building is 163m.

Q.8 A plane is 119km west and 100km south of an airport. At what bearing, the pilot wants to fly directly back to the airport?



Toward west = 119 km

Toward south = 100 km



Since angle between West and South is 90° .

Bearing of pilot directly back to airport = $\theta = ?$

ΔAWS is right with $m\angle W = 90^\circ$.

$$\tan \theta = \frac{m\overline{AW}}{m\overline{WS}}$$

$$\tan \theta = \frac{119km}{100km}$$

$$\theta = \tan^{-1} \left(\frac{119}{100} \right)$$

$$\theta = \tan^{-1}(1.19)$$

$$\theta = 49.95^\circ \approx 50^\circ$$

Thus bearing of pilot directly back to airport is 050° .



Coordinate Geometry

Coordinate Geometry

Distance Between Points on a Coordinate Line

Let a and b be the coordinates of two points A and B respectively, on a coordinate line. The distance between A and B , denoted by $d(A, B)$ is defined by

$$d(A, B) = AB = |b - a|$$

The number $d(A, B)$ is the length of the line AB , and

$$d(A, B) = d(B, A) \\ |b - a| = |a - b|$$

So, we have

$$AB = d(A, B) = d(B, A) = BA$$

Hence, the formula

$d(A, B) = |b - a|$ is true regardless the sign of a and b .

Example 1

A, B, C, D, E and F have coordinates $-6, -4, -2, 0, 2$ and 5 respectively on a coordinate line as shown below.



Find

- (i) $d(A, B)$ (ii) $d(A, C)$ (iii) $d(B, F)$
(iv) $d(D, F)$ (v) $d(A, F)$ (vi) $d(A, E)$

Solution

By definition we know that

- (i) $d(A, B) = |b - a|$, a and b are the coordinates of points A and B respectively.
 $a = -6, b = -4$

$$d(A, B) = AB = |-4 - (-6)| = |-4 + 6| = 2$$

Similarly,

$$\text{So } d(A, C) = AC = |-2 - (-6)| = |-2 + 6| = 4$$

$$(i) \quad d(B, F) = BF = |-2 - 5| = |-7| = 7$$

$$(ii) \quad d(D, F) = DF = |5 - 0| = |5| = 5$$

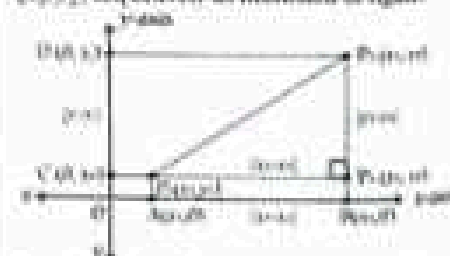
$$(iii) \quad d(A, F) = AF = |5 - (-6)| = |11| = 11$$

The Distance Formula

Given the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the distance between these points is given by the formula

$$d(P_1, P_2) = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof: If $x_1 \neq x_2$ and $y_1 \neq y_2$, then plot two points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2) respectively as illustrated in figure.



In the above figure P_1A and P_2B are perpendiculars drawn from two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on x -axis, and P_1C and P_2D are perpendiculars on y -axis. The coordinates of the points A, B, C and D are also shown in figure accordingly.

Let the point where P_1C and P_2D be P_3 with coordinate (x_2, y_1) so that $\Delta P_1P_2P_3$ is a right triangle with right angle at P_3 .

From figure, we have

$$d(A, B) = AB = P_1P_3 = |x_2 - x_1|$$

$$\text{and } d(C, D) = CD = d(P_2, P_3) = P_2P_3 = |y_2 - y_1|$$

We next use the Pythagoras Theorem in right triangle $P_1P_2P_3$

$$[d(P_1, P_2)]^2 = [d(P_1, P_3)]^2 + [d(P_3, P_2)]^2$$

Or

$$P_1P_2^2 = P_1P_3^2 + P_3P_2^2$$

$$P_1P_2^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{So } d(P_1, P_2) = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Example 2

Find the distance between the two points $A(-1, 3)$ and $B(4, 15)$.

Solution

By distance formula distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let $(-1, 3)$ be denoted by (x_1, y_1) i.e.

$x_1 = -1, y_1 = 3$ and $(4, 15)$ be denoted by (x_2, y_2) i.e. $x_2 = 4, y_2 = 15$

Now put them into the distance formula

$$AB = d = \sqrt{(4 - (-1))^2 + (15 - 3)^2}$$

$$d = \sqrt{(4 + 1)^2 + (12)^2}$$

$$d = \sqrt{5^2 + 144}$$

$$d = \sqrt{25 + 144}$$

$$d = \sqrt{169}$$

$$d = 13$$

Collinear and Non-Collinear Point

A set of points that lie on a same straight line are said to be collinear. If three points A, B and C are collinear then, by distance formula, we have

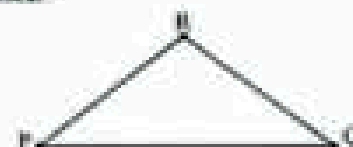
$$d(A, C) = d(A, B) + d(B, C)$$

$$\text{Or } AC = AB + BC$$



When three or more points do not lie on the same straight line, they are said to be non-collinear points.

In the figure, points P, Q and R are non-collinear.

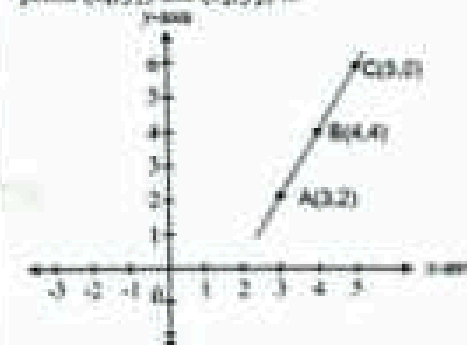


Example 3

Prove that $A(3, 2), B(4, 4)$ and $C(5, 6)$ are collinear points.

Solution

In first step we find the distance between them. By Distance Formula, distance between two points (x_1, y_1) and (x_2, y_2) is



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d(A, B) = AB = \sqrt{(4 - 3)^2 + (4 - 2)^2}$$

$$AB = \sqrt{1^2 + 2^2} = \sqrt{1 + 4}$$

$$AB = \sqrt{5} \dots (i)$$

$$d(B, C) = BC = \sqrt{(5 - 4)^2 + (6 - 4)^2}$$

$$BC = \sqrt{1^2 + 2^2} = \sqrt{1 + 4}$$

$$BC = \sqrt{5} \dots (ii)$$

$$d(A, C) = AC = \sqrt{(5 - 3)^2 + (6 - 2)^2}$$

$$AC = \sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20}$$

$$= 2\sqrt{5} \dots (iii)$$



From (i), (ii) and (iii), we find that

$$AC = AB + BC$$

$$2\sqrt{5} = \sqrt{5} + \sqrt{5}$$

$$2\sqrt{5} = 2\sqrt{5}$$

Hence, the given points are collinear.

Example 4

Prove that the points $(3,4)$, $(3,1)$ and $(8,4)$ are the vertices of a right triangle.

Solution

Suppose $A(3,4)$, $B(3,1)$ and $C(8,4)$ are the vertices of a triangle.

In first step we find the distance between them:

$$d(A,B) = AB = \sqrt{(3-3)^2 + (4-1)^2}$$

$$AB = \sqrt{(-3)^2 + 9}$$

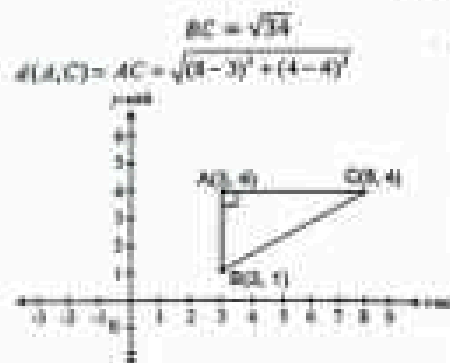
$$AB = \sqrt{9+9}$$

$$AB = 3$$

$$d(B,C) = BC = \sqrt{(8-1)^2 + (4-3)^2}$$

$$= \sqrt{3^2 + 1^2}$$

$$= \sqrt{9+1}$$



$$AC = \sqrt{25+0} = 5$$

In right triangle we know that the greatest length is always the hypotenuse. So, BC is the hypotenuse. By Pythagoras Theorem,

$$BC^2 = AB^2 + AC^2$$

$$(\sqrt{34})^2 = 3^2 + 5^2$$

$$34 = 25 + 9$$

$$34 = 34$$

Hence, A, B and C form a right triangle.

Exercise 7.1

Q.1. Find the distance between each pair of points.

i. $(-2,1)$ and $(1,5)$ answer

ii. $(1,1)$ and $(7,9)$ answer

iii. $(0,-5)$ and $(-2,1)$ answer

iv. $(0,-5)$ and $(10,5)$ answer

v. $(4,-1)$ and $(2,0)$ answer

vi. $(-3,1)$ and $(-2,-3)$ answer

vii. $\left(\frac{1}{2}, 2\right)$ and $(-2,1)$ answer

viii. $\left(\frac{7}{2}, 11\right)$ and $\left(13, \frac{17}{2}\right)$ answer

ix. $\left(-\frac{9}{4}, -\frac{7}{3}\right)$ and $\left(-\frac{3}{2}, -\frac{5}{4}\right)$ answer

x. $\left(2\frac{1}{2}, 3\frac{1}{4}\right)$ and $(3,-1)$ answer

Solution

i. Let $A(-2,1)$, $B(1,5)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{[1 - (-2)]^2 + [5 - 1]^2}$$

$$|AB| = \sqrt{(1+2)^2 + (4)^2}$$

$$|AB| = \sqrt{3^2 + 4^2}$$

$$|AB| = \sqrt{9+16}$$

$$|AB| = \sqrt{25}$$

$$|AB| = 5 \text{ Units}$$

ii. Let $A(1,1)$, $B(7,9)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(7-1)^2 + (9-1)^2}$$

$$|AB| = \sqrt{(6)^2 + (8)^2}$$

$$|AB| = \sqrt{36+64} = \sqrt{100} = 10 \text{ Units}$$

iii. Let $A(0,-5)$, $B(-2,1)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(-2-0)^2 + [1 - (-5)]^2}$$

$$|AB| = \sqrt{(-2)^2 + (1+5)^2}$$

$$|AB| = \sqrt{4 + (6)^2}$$

$$|AB| = \sqrt{4+36}$$

$$|AB| = \sqrt{40}$$

$$|AB| = \sqrt{4 \times 10}$$

$$|AB| = 2\sqrt{10} \text{ Units}$$

iv. Let $A(0,-5)$, $B(10,5)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(10-0)^2 + [-5 - (-5)]^2}$$

$$|AB| = \sqrt{(10)^2 + (-5+5)^2}$$

$$|AB| = \sqrt{(10)^2 + (-(-10))^2}$$

$$|AB| = \sqrt{100+100}$$

$$|AB| = \sqrt{200}$$

$$|AB| = \sqrt{100 \times 2} = 10\sqrt{2} \text{ Units}$$

v. Let $A(4,-1)$, $B(2,0)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(2-4)^2 + [0 - (-1)]^2}$$

$$|AB| = \sqrt{(-2)^2 + (0+1)^2}$$

$$|AB| = \sqrt{4 + (1)^2}$$

$$|AB| = \sqrt{4+1}$$

$$|AB| = \sqrt{5} \text{ Units}$$

vi. Let $A(-3,1)$, $B(-2,-3)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{[-2 - (-3)]^2 + (-3-1)^2}$$

$$|AB| = \sqrt{(-2+3)^2 + (-4)^2}$$

$$|AB| = \sqrt{(1)^2 + (-4)^2}$$

$$|AB| = \sqrt{1+16}$$

$$|AB| = \sqrt{17} \text{ Units}$$

vii. Let $A\left(\frac{1}{2}, 2\right)$, $B(-2,1)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{\left[-2 - \frac{1}{2}\right]^2 + (1-2)^2}$$

$$|AB| = \sqrt{\left[-\frac{4+1}{2}\right]^2 + (-1)^2}$$

$$|AB| = \sqrt{\left[-\frac{5}{2}\right]^2 + (1)^2}$$

$$|AB| = \sqrt{\frac{25}{4} + 1}$$

$$|AB| = \sqrt{\frac{25+4}{4}}$$

$$|AB| = \sqrt{\frac{29}{4}}$$

$$|AB| = \frac{\sqrt{29}}{2} \text{ Units}$$

viii. Let $A\left(\frac{7}{2}, 11\right)$, $B\left(13, \frac{17}{2}\right)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{\left[13 - \frac{7}{2}\right]^2 + \left[\frac{17}{2} - 11\right]^2}$$

$$|AB| = \sqrt{\left[\frac{26-7}{2}\right]^2 + \left[\frac{17-22}{2}\right]^2}$$

$$|AB| = \sqrt{\left[\frac{19}{2}\right]^2 + \left[\frac{-5}{2}\right]^2}$$

$$|AB| = \sqrt{\frac{361}{4} + \frac{25}{4}}$$

$$|AB| = \sqrt{\frac{386+25}{4}}$$

$$|AB| = \sqrt{\frac{411}{4}} = \frac{\sqrt{411}}{2} \text{ Units}$$



$$10. \text{ Let } A\left(-\frac{3}{4}, -\frac{7}{3}\right), B\left(\frac{3}{2}, -\frac{5}{4}\right)$$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{\left(-\frac{3}{2} - \left(-\frac{3}{4}\right)\right)^2 + \left(-\frac{5}{4} - \left(-\frac{7}{3}\right)\right)^2}$$

$$|AB| = \sqrt{\left(-\frac{3}{2} + \frac{3}{4}\right)^2 + \left(-\frac{5}{4} + \frac{7}{3}\right)^2}$$

$$|AB| = \sqrt{\left(-\frac{6+3}{4}\right)^2 + \left(\frac{-15+28}{12}\right)^2}$$

$$|AB| = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{13}{12}\right)^2}$$

$$|AB| = \sqrt{\frac{9}{16} + \frac{169}{144}}$$

$$|AB| = \sqrt{\frac{81+169}{144}}$$

$$|AB| = \frac{\sqrt{250}}{\sqrt{144}} = \frac{\sqrt{25} \times \sqrt{10}}{\sqrt{144}}$$

$$|AB| = \frac{5\sqrt{10}}{12} \text{ Units}$$

$$11. \text{ Let } A\left(2\frac{1}{2}, 3\frac{1}{4}\right), B(5, -1)$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{\left(5 - 2\frac{1}{2}\right)^2 + \left(-1 - 3\frac{1}{4}\right)^2}$$

$$|AB| = \sqrt{\left(\frac{10-5}{2}\right)^2 + \left(\frac{-4-21}{4}\right)^2}$$

$$|AB| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{-25}{4}\right)^2}$$

$$|AB| = \sqrt{\frac{25}{4} + \frac{625}{16}}$$

$$|AB| = \sqrt{\frac{100+625}{16}}$$

$$|AB| = \sqrt{\frac{725}{16}}$$

$$|AB| = \sqrt{\frac{25 \times 29}{16}}$$

$$|AB| = \frac{\sqrt{25} \times \sqrt{29}}{\sqrt{16}}$$

$$|AB| = \frac{5\sqrt{29}}{4} \text{ Units}$$

Q.2. Check by distance formula whether these points are collinear or not.

i. (0,1), (2,3) and (3,4) yes

ii. (-5,-4), (1,0) and (6,7) yes

iii. (8,-5), (3,7) and (5,15) yes

iv. (6,3), (14,7) and (-6,-3) yes

v. (0,15), (2,9) and (7,-4) yes

vi. (1,-2), (7,8) and (-2,-7) yes

Solution

i. (0,1), (2,3) and (3,4)

Let A(0,1), B(2,3), C(3,4)

First we find distance between them & know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(2-0)^2 + (3-1)^2}$$

$$|AB| = \sqrt{(2)^2 + (2)^2} = \sqrt{4+4}$$

$$|AB| = \sqrt{8}$$

$$|AB| = \sqrt{4 \times 2}$$

$$|AB| = 2\sqrt{2} \dots (i)$$

$$|BC| = \sqrt{(3-2)^2 + (4-3)^2}$$

$$|BC| = \sqrt{(1)^2 + (1)^2}$$

$$|BC| = \sqrt{1+1}$$

$$|BC| = \sqrt{2} \dots (ii)$$

$$|AC| = \sqrt{(3-0)^2 + (4-1)^2}$$

$$|AC| = \sqrt{(3)^2 + (3)^2}$$

$$|AC| = \sqrt{9+9}$$

$$|AC| = \sqrt{18 \times 2} = 3\sqrt{2} \dots (iii)$$

$$|AB| + |BC| = |AC|$$

$$2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$$

$$3\sqrt{2} = 3\sqrt{2}$$

$$3\sqrt{2} = 3\sqrt{2}$$

$$3\sqrt{2} = 3\sqrt{2}$$

$$|AC| = \sqrt{(3)^2 + (3)^2}$$

$$|AC| = \sqrt{9+9}$$

$$|AC| = \sqrt{18}$$

$$|AC| = \sqrt{9 \times 2}$$

$$|AC| = 3\sqrt{2} \dots (iii)$$

$$|AB| + |BC| = |AC|$$

$$2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$$

$$3\sqrt{2} = 3\sqrt{2}$$

$$3\sqrt{2} = 3\sqrt{2}$$

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First we find distance between them

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(3-0)^2 + (7-(-5))^2}$$

$$|AB| = \sqrt{(3)^2 + (12)^2}$$

$$|AB| = \sqrt{9 + 144}$$

$$|AB| = \sqrt{153} = \sqrt{9 \times 17} = 3\sqrt{17} \dots (i)$$

$$|BC| = \sqrt{(5-3)^2 + (15-7)^2}$$

$$|BC| = \sqrt{(2)^2 + (8)^2}$$

$$|BC| = \sqrt{4 + 64} = \sqrt{68}$$

$$|BC| = \sqrt{4 \times 17} = 2\sqrt{17} \dots (ii)$$

$$|AC| = \sqrt{(5-0)^2 + (15-(-5))^2}$$

$$|AC| = \sqrt{(5)^2 + (20)^2}$$

$$|AC| = \sqrt{25 + 400}$$

$$|AC| = \sqrt{425} = \sqrt{25 \times 17} = 5\sqrt{17} \dots (iii)$$

$$|AB| + |BC| = |AC|$$

$$3\sqrt{17} + 2\sqrt{17} = 5\sqrt{17}$$

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Now,

$$|AC| = \sqrt{(-6-6)^2 + (-3-3)^2}$$

$$|AC| = \sqrt{(-12)^2 + (-6)^2}$$

$$|AC| = \sqrt{144 + 36}$$

$$|AC| = \sqrt{180} = \sqrt{36 \times 5}$$

$$|AC| = 6\sqrt{5} \dots\dots\dots (ii)$$

From (i), (ii) and (iii)

$$4\sqrt{5} + 6\sqrt{5} = 10\sqrt{5}$$

$$|AB| + |AC| = |BC|$$

Hence, these points are collinear points.

(9,15), (2,9), (7, -4)

Let $A(9,15), B(2,9), C(7, -4)$

First we find distances

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(2-9)^2 + (9-15)^2}$$

$$|AB| = \sqrt{(-7)^2 + (-6)^2}$$

$$|AB| = \sqrt{49 + 36}$$

$$|AB| = \sqrt{85} = \sqrt{5 \times 17}$$

$$|AB| = 2\sqrt{17} \dots\dots\dots (i)$$

Now

$$|BC| = \sqrt{(7-2)^2 + (-6-9)^2}$$

$$|BC| = \sqrt{(5)^2 + (-15)^2}$$

$$|BC| = \sqrt{25 + 225}$$

$$|BC| = \sqrt{250}$$

$$|BC| = \sqrt{25 \times 10}$$

$$|BC| = 5\sqrt{10} \dots\dots\dots (ii)$$

Now

$$|AC| = \sqrt{(7-9)^2 + (-6-15)^2}$$

$$|AC| = \sqrt{(-2)^2 + (-21)^2}$$

$$|AC| = \sqrt{4 + 441}$$

$$|AC| = \sqrt{445}$$

$$|AC| = \sqrt{49 \times 9}$$

$$|AC| = 7\sqrt{9} \dots\dots\dots (iii)$$

From (i), (ii) and (iii)

$$2\sqrt{17} + 5\sqrt{10} = 7\sqrt{9}$$

$$7\sqrt{9} = 7\sqrt{9}$$

$$|AB| + |BC| = |AC|$$

Hence, these points are collinear points.

(1, -2), (7,8), (-2, -7)

Let $A(1, -2), B(7,8), C(-2, -7)$

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(7-1)^2 + (8-(-2))^2}$$

$$|AB| = \sqrt{(6)^2 + (10)^2}$$

$$|AB| = \sqrt{36 + 100}$$

$$|AB| = \sqrt{136 + 100}$$

$$|AB| = \sqrt{236} = \sqrt{4 \times 59}$$

$$|AB| = 2\sqrt{59} \dots\dots\dots (i)$$

Now,

$$|BC| = \sqrt{(-2-7)^2 + (-7-8)^2}$$

$$|BC| = \sqrt{(-9)^2 + (-15)^2}$$

$$|BC| = \sqrt{81 + 225}$$

$$|BC| = \sqrt{306} = \sqrt{9 \times 34}$$

$$|BC| = 3\sqrt{34} \dots\dots\dots (ii)$$

Now,

$$|AC| = \sqrt{(-2-1)^2 + (-7-(-2))^2}$$

$$|AC| = \sqrt{(-3)^2 + (-7+2)^2}$$

$$|AC| = \sqrt{9 + 25}$$

$$|AC| = \sqrt{34} \dots\dots\dots (iii)$$

$$|AC| = \sqrt{9 + 25}$$

$$|AC| = \sqrt{34} \dots\dots\dots (iii)$$

$$2\sqrt{34} + \sqrt{34} = 3\sqrt{34}$$

$$3\sqrt{34} = 3\sqrt{34}$$

$$|AB| + |AC| = |BC|$$

Hence, these points are collinear points.

Q.3. Check whether the following points are the vertices of:

(a) an equilateral triangle,

(b) an isosceles triangle,

(c) a right triangle,

(d) a scalene triangle.

i. (2,3), (5,3) and (2,1)

same

ii. (0,0), (1, $\sqrt{3}$) and (2,0)

same

iii. (0,1), (0,4) and (0,8)

same

iv. (-1, -5), (1, -3) and (9,6)

same

v. (-1, -1), (-1,5) and (4,2)

same

vi. (1,2), (7,2) and (7,8)

same

Also find the perimeter of triangle in vi.

Ans.

i. (2,3), (5,3), (2,1)

Suppose $A(2,3), B(5,3)$ and $C(2,1)$ are vertices of a triangle.

First we find the sides lengths of the triangle.

We know that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(5-2)^2 + (3-3)^2}$$

$$|AB| = \sqrt{(3)^2 + (0)^2}$$

$$|AB| = \sqrt{9 + 0}$$

$$|AB| = \sqrt{9}$$

$$|AB| = 3 \dots\dots\dots (i)$$

$$|BC| = \sqrt{(2-5)^2 + (1-3)^2}$$

$$|BC| = \sqrt{(-3)^2 + (-2)^2}$$

$$|BC| = \sqrt{9 + 4}$$

$$|BC| = \sqrt{13} \dots\dots\dots (ii)$$

Now

$$|AC| = \sqrt{(2-2)^2 + (1-3)^2}$$

$$|AC| = \sqrt{(0)^2 + (-2)^2}$$

$$|AC| = \sqrt{0 + 4}$$

$$|AC| = \sqrt{4}$$

$$|AC| = 2 \dots\dots\dots (iii)$$

From (i), (ii), (iii) we observe that three sides are different in length. So triangle with given vertices is a scalene triangle.

By applying converse Pythagoras theorem.

Now

$$|BC|^2 = (\sqrt{13})^2 = 13 \dots\dots\dots (iv)$$

$$\text{and } |AB|^2 + |AC|^2 = (3)^2 + (2)^2$$

$$|AB|^2 + |AC|^2 = 9 + 4 = 13 \dots\dots\dots (v)$$

From (iv) and (v)

$$|BC|^2 = |AB|^2 + |AC|^2$$

This relation shows that $\triangle ABC$ is also a right angled triangle in which

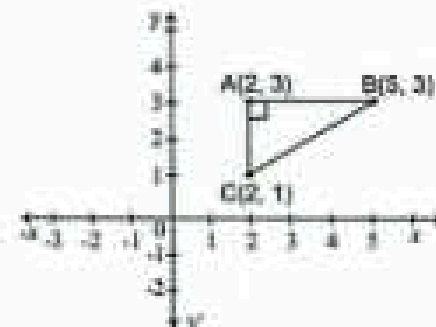
$$\angle C = 90^\circ$$

Perimeter

Perimeter = sum of three sides

$$P = |AB| + |BC| + |CA|$$

$$P = 3 + \sqrt{13} + 2 = (5 + \sqrt{13}) \text{ units}$$



ii. (0,0), (1, $\sqrt{3}$), (2,0)
Suppose $A(0,0), B(1, \sqrt{3})$ and $C(2,0)$ are vertices of a triangle.

First we find lengths of all sides

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(1-0)^2 + (\sqrt{3}-0)^2}$$

$$|AB| = \sqrt{(1)^2 + (\sqrt{3})^2}$$

$$|AB| = \sqrt{1 + 3}$$

$$|AB| = \sqrt{4} = 2 \dots\dots\dots (i)$$

$$|BC| = \sqrt{(2-1)^2 + (0-\sqrt{3})^2}$$

$$|BC| = \sqrt{(1)^2 + (-\sqrt{3})^2}$$

$$|BC| = \sqrt{1 + 3} = \sqrt{4} = 2 \dots\dots\dots (ii)$$

$$|CA| = \sqrt{(2-0)^2 + (0-0)^2}$$

$$|CA| = \sqrt{(2)^2 + (0)^2}$$

$$|CA| = \sqrt{4 + 0}$$

$$|CA| = \sqrt{4} = 2 \dots\dots\dots (iii)$$

$$|BC| = \sqrt{1 + 3} = \sqrt{4} = 2 \dots\dots\dots (ii)$$

$$|CA| = \sqrt{(2-0)^2 + (0-0)^2}$$

$$|CA| = \sqrt{(2)^2 + (0)^2}$$

$$|CA| = \sqrt{4 + 0}$$

$$|CA| = \sqrt{4} = 2 \dots\dots\dots (iii)$$

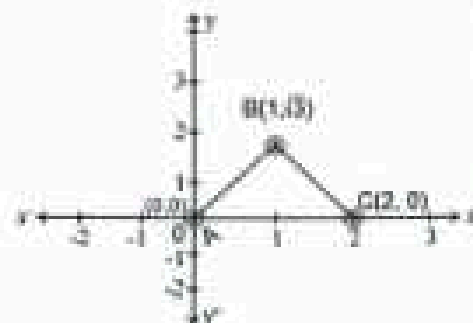
From (i), (ii) and (iii) we observe that lengths of three sides are equal. So triangle with given vertices is an equilateral triangle.

Perimeter

Perimeter = sum of three sides

$$P = |AB| + |BC| + |CA|$$

$$P = 2 + 2 + 2 = 6 \text{ units}$$



- iii. $(0,1), (1,4), (2,0)$

Suppose $A(0,1), B(1,4), C(2,0)$ are vertices of a triangle.

First we find the length of three sides of $\triangle ABC$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(1-0)^2 + (4-1)^2}$$

$$|AB| = \sqrt{(1)^2 + (3)^2}$$

$$|AB| = \sqrt{64 + 9} = \sqrt{73} \dots (i)$$

Now,

$$|BC| = \sqrt{(2-1)^2 + (0-4)^2}$$

$$|BC| = \sqrt{(-1)^2 + (4)^2}$$

$$|BC| = \sqrt{64 + 16} = \sqrt{80} \dots (ii)$$

Now,

$$|AC| = \sqrt{(2-0)^2 + (0-1)^2}$$

$$|AC| = \sqrt{(2)^2 + (1)^2}$$

$$|AC| = \sqrt{5 + 4}$$

$$|AC| = \sqrt{49} = 7 \dots (iii)$$

From (i), (ii) and (iii), we observe that lengths of three sides are different. So the triangle with given vertices is scalene triangle.

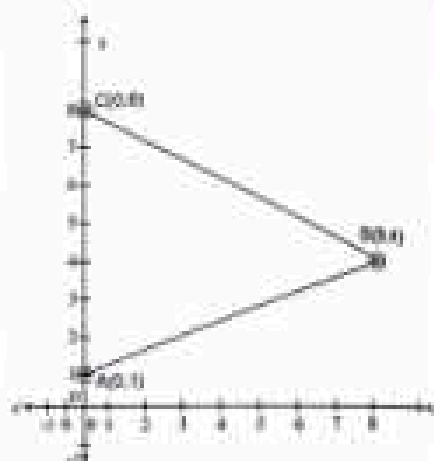
Perimeter

P = Sum of three sides

$$P = |AB| + |BC| + |AC|$$

$$P = \sqrt{73} + \sqrt{80} + 7$$

$$P = 24.4908 = 24.49 \text{ units}$$



- iv. $(-3,-5), (3,-3), (0,6)$

Suppose $A(-3,-5), B(3,-3)$ and $C(0,6)$ are vertices of a triangle.

First we find lengths of three sides.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(3 - (-3))^2 + (-3 - (-5))^2}$$

$$|AB| = \sqrt{(3+3)^2 + (-3+5)^2}$$

$$|AB| = \sqrt{(6)^2 + (2)^2}$$

$$|AB| = \sqrt{36 + 4} = \sqrt{40} \dots (i)$$

Now,

$$|BC| = \sqrt{(0-3)^2 + (6-(-3))^2}$$

$$|BC| = \sqrt{(-3)^2 + (6+3)^2}$$

$$|BC| = \sqrt{9 + (9)^2}$$

$$|BC| = \sqrt{9 + 81} = \sqrt{90} \dots (ii)$$

Now,

$$|AC| = \sqrt{(0 - (-3))^2 + (6 - (-5))^2}$$

$$|AC| = \sqrt{(3)^2 + (6+5)^2}$$

$$|AC| = \sqrt{9 + (11)^2} = \sqrt{9 + 121}$$

$$|AC| = \sqrt{130} \dots (iii)$$

From (i), (ii) and (iii) we observe that three sides are not equal in length. So $\triangle ABC$ is a scalene triangle.

Now, applying converse Pythagoras theorem

$$|AC|^2 = (\sqrt{130})^2 = 130 \dots (iv)$$

Now,

$$|AB|^2 + |BC|^2 = (\sqrt{40})^2 + (\sqrt{90})^2$$

$$|AB|^2 + |BC|^2 = 40 + 90$$

$$|AB|^2 + |BC|^2 = 130 \dots (v)$$

From (iv) and (v)

$$|AC|^2 = |AB|^2 + |BC|^2$$

This relation shows that $\triangle ABC$ is also a right triangle in which $\angle C = 90^\circ$.

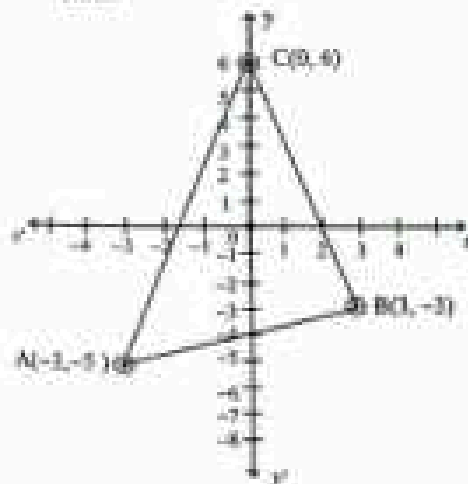
Perimeter

P = sum of three sides

$$P = |AB| + |BC| + |AC|$$

$$P = \sqrt{40} + \sqrt{90} + \sqrt{130} = 27.21$$

units.



- v. $(-1,-1), (-1,5), (4,2)$

Suppose $A(-1,-1), B(-1,5), C(4,2)$ are vertices of a triangle.

First we find lengths of three sides of $\triangle ABC$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(-1 - (-1))^2 + (5 - (-1))^2}$$

$$|AB| = \sqrt{(-1+1)^2 + (5+1)^2}$$

$$|AB| = \sqrt{(0)^2 + (6)^2} = \sqrt{0 + 36}$$

$$|AB| = \sqrt{36} = 6 \dots (i)$$

Now,

$$|BC| = \sqrt{(4 - (-1))^2 + (2 - 5)^2}$$

$$|BC| = \sqrt{(4+1)^2 + (-3)^2}$$

$$|BC| = \sqrt{(5)^2 + (9)}$$

$$|BC| = \sqrt{25 + 9} = \sqrt{34} \dots (ii)$$

Now,

$$|AC| = \sqrt{(4 - (-1))^2 + (2 - (-1))^2}$$

$$|AC| = \sqrt{(4+1)^2 + (2+1)^2}$$

$$|AC| = \sqrt{(5)^2 + (3)^2}$$

$$|AC| = \sqrt{25 + 9} = \sqrt{34} \dots (iii)$$

From (i), (ii) and (iii) we observe that length of two sides are equal while third side has different length. So $\triangle ABC$ is an isosceles triangle.

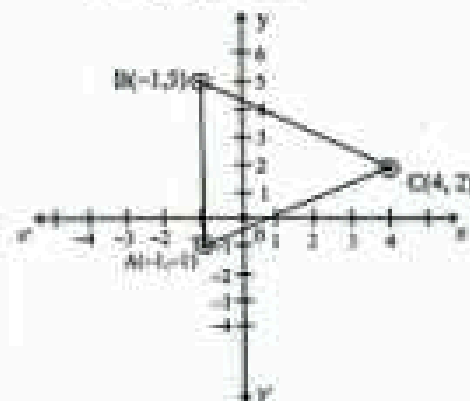
Perimeter

P = sum of three sides of $\triangle ABC$

$$P = |AB| + |BC| + |AC|$$

$$P = 6 + \sqrt{34} + \sqrt{34}$$

$$P = (6 + 2\sqrt{34}) \text{ Units}$$



- vi. $(1,2), (7,2), (7,8)$

Suppose $A(1,2), B(7,2), C(7,8)$ are vertices of a triangle.

First we find lengths of three sides of $\triangle ABC$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(7 - 1)^2 + (2 - 2)^2}$$

$$|AB| = \sqrt{(6)^2 + (0)^2}$$

$$|AB| = \sqrt{36 + 0} = \sqrt{36}$$

$$|AB| = 6 \dots (i)$$

Now,

$$|BC| = \sqrt{(7 - 7)^2 + (8 - 2)^2}$$



$$|BC| = \sqrt{(0)^2 + (6)^2}$$

$$|BC| = \sqrt{0 + 36} = \sqrt{36}$$

$$|BC| = 6 \dots\dots (ii)$$

Now,

$$|AC| = \sqrt{(7-1)^2 + (0-2)^2}$$

$$|AC| = \sqrt{(6)^2 + (4)^2}$$

$$|AC| = \sqrt{36 + 16}$$

$$|AC| = \sqrt{52} \dots\dots (iii)$$

From (i), (ii) and (iii) we observe that two sides are equal in length. So $\triangle ABC$ is an isosceles triangle.

Now, apply converse of Pythagoras theorem

$$|AC|^2 = (\sqrt{52})^2 = 52 \dots\dots (iv)$$

Now,

$$|AB|^2 + |BC|^2 = (4)^2 + (6)^2$$

$$|AB|^2 + |BC|^2 = 16 + 36$$

$$|AB|^2 + |BC|^2 = 52 \dots\dots (v)$$

From (iv) and (v)

$$|AC|^2 = |AB|^2 + |BC|^2$$

This relation shows that $\triangle ABC$ is also a right angled triangle.

Thus $\triangle ABC$ is right angled isosceles triangle with $m\angle ABC = 90^\circ$

Perimeter

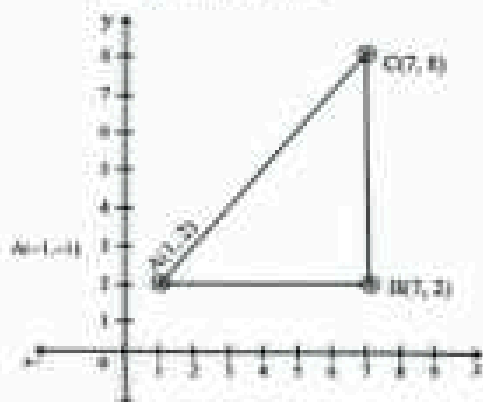
$P =$ Sum of three sides

$$P = |AB| + |BC| + |AC|$$

$$P = 6 + 6 + \sqrt{52}$$

$$P = 12 + \sqrt{36 \times 2}$$

$$P = (12 + 4\sqrt{2}) \text{ units.}$$

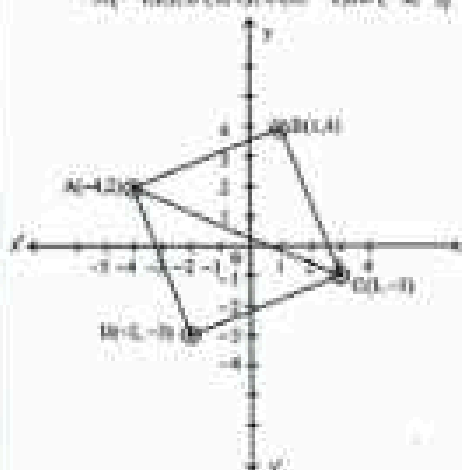


Q.4. Show that

$A(-4, 2), B(1, 4), C(3, -1)$ and $D(-2, -3)$ are the vertices of a square. short

Solution

$$A(-4, 2), B(1, 4), C(3, -1), D(-2, -3)$$



First we find lengths of all sides

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(1 - (-4))^2 + (4 - 2)^2}$$

$$|AB| = \sqrt{(5)^2 + (2)^2}$$

$$|AB| = \sqrt{25 + 4}$$

$$|AB| = \sqrt{29} \dots\dots (i)$$

$$|AB| = \sqrt{29} \dots\dots (i)$$

Now,

$$|BC| = \sqrt{(3 - 1)^2 + (-1 - 4)^2}$$

$$|BC| = \sqrt{(2)^2 + (-5)^2}$$

$$|BC| = \sqrt{4 + 25}$$

$$|BC| = \sqrt{29} \dots\dots (ii)$$

Now

$$|CD| = \sqrt{(-3 - 3)^2 + (-1 - (-1))^2}$$

$$|CD| = \sqrt{(-6)^2 + (0)^2}$$

$$|CD| = \sqrt{36 + 0}$$

$$|CD| = \sqrt{36} + 0$$

$$|CD| = \sqrt{36} \dots\dots (iii)$$

Now,

$$|AD| = \sqrt{(-2 - (-4))^2 + (-3 - 2)^2}$$

$$|AD| = \sqrt{(-2 + 4)^2 + (-5)^2}$$

$$|AD| = \sqrt{(2)^2 + (-5)^2}$$

$$|AD| = \sqrt{4 + 25}$$

$$|AD| = \sqrt{29} \dots\dots (iv)$$

From (i), (ii), (iii) and (iv)

$$|AB| = |BC| = |CD| = |AD|$$

(All sides are equal)

To check angles at each vertex, we find diagonal $|AC|$ and then apply converse of Pythagoras theorem.

$$|AC| = \sqrt{(3 - (-4))^2 + (-1 - 2)^2}$$

$$|AC| = \sqrt{(7)^2 + (-3)^2}$$

$$|AC| = \sqrt{49 + 9}$$

$$|AC| = \sqrt{58} \dots\dots (v)$$

Now,

$$|AC|^2 = (\sqrt{58})^2 = 58 \dots\dots (vi)$$

Now,

$$|AB|^2 + |BC|^2 = (\sqrt{29})^2 + (\sqrt{29})^2$$

$$|AB|^2 + |BC|^2 = 29 + 29 = 58 \dots\dots (vii)$$

From (vi) and (vii)

$$|AC|^2 = |AB|^2 + |BC|^2$$

This result shows that $\triangle ABC$ is right $\angle$ in which $m\angle B = 90^\circ$.

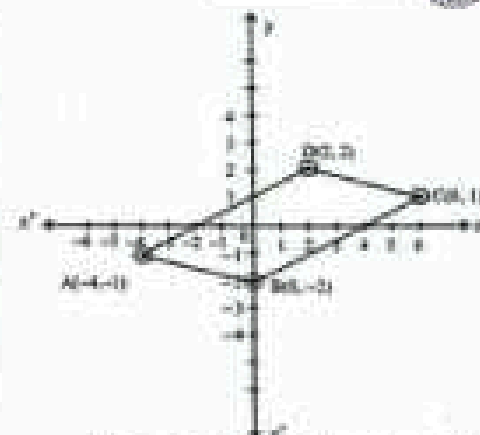
Thus angle at each vertex of quadrilateral is 90° .

Hence, given points are vertices of a square.

Q.5. Show that $A(-4, -1), B(0, -2), C(6, 1)$ and $D(2, 2)$ are the vertices of a parallelogram. short

Solution

$$A(-4, -1), B(0, -2), C(6, 1), D(2, 2)$$



First we find the lengths of all sides using distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(0 - (-4))^2 + (-2 - (-1))^2}$$

$$|AB| = \sqrt{(4)^2 + (-1)^2}$$

$$|AB| = \sqrt{16 + 1}$$

$$|AB| = \sqrt{17} \dots\dots (i)$$

$$|AB| = \sqrt{17} \dots\dots (i)$$

$$|BC| = \sqrt{(6 - 0)^2 + (1 - (-2))^2}$$

$$|BC| = \sqrt{(6)^2 + (3)^2}$$

$$|BC| = \sqrt{36 + 9}$$

$$|BC| = \sqrt{45} \dots\dots (ii)$$

$$|BC| = \sqrt{45} \dots\dots (ii)$$

$$|CD| = \sqrt{(2 - 6)^2 + (2 - 1)^2}$$

$$|CD| = \sqrt{(-4)^2 + (1)^2}$$

$$|CD| = \sqrt{16 + 1}$$

$$|CD| = \sqrt{17} \dots\dots (iii)$$

$$|CD| = \sqrt{17} \dots\dots (iii)$$

Now,

$$|AD| = \sqrt{(2 - (-4))^2 + (2 - (-1))^2}$$

$$|AD| = \sqrt{(6)^2 + (3)^2}$$

$$|AD| = \sqrt{36 + 9}$$

$$|AD| = \sqrt{45} \dots\dots (iv)$$

From (i), (ii), (iii) and (iv) we observe that



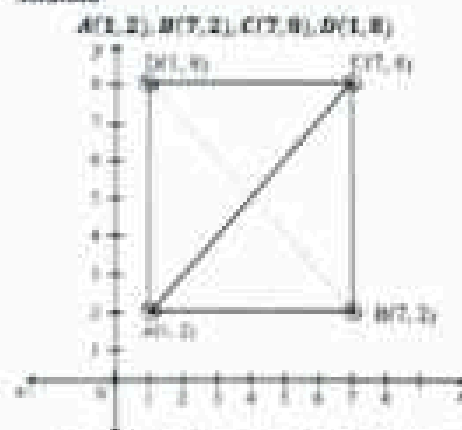
$$|AB| = |CD| = \sqrt{13}$$

$$\text{and } |BC| = |AD| = \sqrt{5}$$

opposite sides are equal which shows that ABCD is a parallelogram.

Q.6. Show that A(1, 2), B(7, 2), C(7, 8), and D(1, 8) are the vertices of a rectangle. Prove that its diagonals are also equal in length.

Solution



First we find lengths of all sides by using distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(7 - 1)^2 + (2 - 2)^2}$$

$$|AB| = \sqrt{(6)^2 + (0)^2}$$

$$|AB| = \sqrt{36 + 0} = \sqrt{36}$$

$$|AB| = 6 \dots\dots\dots (i)$$

$$|BC| = \sqrt{(7 - 7)^2 + (8 - 2)^2}$$

$$|BC| = \sqrt{(0)^2 + (6)^2}$$

$$|BC| = \sqrt{0 + 36} = \sqrt{36}$$

$$|BC| = 6 \dots\dots\dots (ii)$$

$$|CD| = \sqrt{(7 - 1)^2 + (8 - 8)^2}$$

$$|CD| = \sqrt{(6)^2 + (0)^2}$$

$$|CD| = \sqrt{36 + 0} = \sqrt{36}$$

$$|CD| = 6 \dots\dots\dots (iii)$$

$$|AD| = \sqrt{(1 - 1)^2 + (8 - 2)^2}$$

$$|AD| = \sqrt{(0)^2 + (6)^2}$$

$$|AD| = \sqrt{0 + 36} = \sqrt{36}$$

$$|AD| = 6 \dots\dots\dots (iv)$$

From (i), (ii), (iii) and (iv) we show:

$$|AB| = |CD| \text{ and } |BC| = |AD|$$

Opposite sides are equal (All sides are equal). To check the angle at each vertex we find diagonals |AC| and |BD|, then apply converse of Pythagoras theorem.

$$|AC| = \sqrt{(7 - 1)^2 + (8 - 2)^2}$$

$$= \sqrt{(6)^2 + (6)^2}$$

$$= \sqrt{36}$$

$$= \sqrt{72} \dots\dots\dots (v)$$

$$|BD| = \sqrt{(1 - 7)^2 + (8 - 2)^2}$$

$$= \sqrt{(-6)^2 + (6)^2}$$

$$= \sqrt{36 + 36}$$

$$= \sqrt{72} \dots\dots\dots (vi)$$

$$|AC|^2 = (\sqrt{72})^2 = 72 \dots\dots\dots (vii)$$

$$\text{Now, } |AB|^2 + |BC|^2 = (6)^2 + (6)^2 = 36 + 36 = 72 \dots\dots\dots (viii)$$

From (vii) and (viii)

$$|AC|^2 = |AB|^2 + |BC|^2$$

This result shows that $\triangle ABC$ is right triangle with $m\angle B = 90^\circ$.

So angle at each vertex of quadrilateral is 90° . Hence quadrilateral ABCD is a rectangle (square).

Now, From (v) and (vi) $|AC| = |BD|$

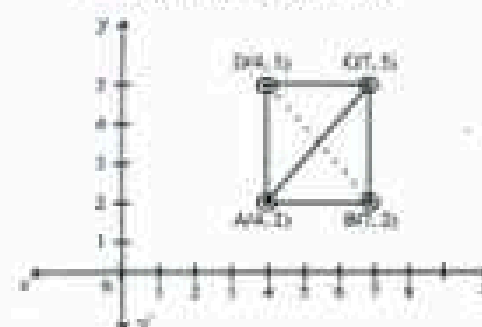
Both diagonals are equal in length.

This confirms that ABCD is a rectangle and its diagonals are equal.

Q.7. Show that A(4, 2), B(7, 2), C(7, 5) and D(4, 5) are the vertices of a square. Also show that the length of its diagonals is $\sqrt{18}$.

Solution

$$A(4, 2), B(7, 2), C(7, 5), D(4, 5)$$



We find the lengths of all sides by using distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(7 - 4)^2 + (2 - 2)^2}$$

$$|AB| = \sqrt{(3)^2 + (0)^2}$$

$$|AB| = \sqrt{9 + 0}$$

$$|AB| = \sqrt{9}$$

$$|AB| = 3 \dots\dots\dots (i)$$

Now,

$$|BC| = \sqrt{(7 - 7)^2 + (5 - 2)^2}$$

$$|BC| = \sqrt{(0)^2 + (3)^2}$$

$$|BC| = \sqrt{0 + 9}$$

$$|BC| = \sqrt{9}$$

$$|BC| = 3 \dots\dots\dots (ii)$$

Now,

$$|CD| = \sqrt{(4 - 7)^2 + (5 - 5)^2}$$

$$|CD| = \sqrt{(-3)^2 + (0)^2}$$

$$|CD| = \sqrt{9 + 0}$$

$$|CD| = \sqrt{9}$$

$$|CD| = 3 \dots\dots\dots (iii)$$

Now,

$$|AD| = \sqrt{(4 - 4)^2 + (5 - 2)^2}$$

$$|AD| = \sqrt{(0)^2 + (3)^2}$$

$$|AD| = \sqrt{0 + 9}$$

$$|AD| = \sqrt{9}$$

$$|AD| = 3 \dots\dots\dots (iv)$$

From (i), (ii), (iii) and (iv)

$$|AB| = |BC| = |CD| = |AD|$$

All sides are equal in length.

To check the angle at each vertex we find diagonals |AC| and |BD| then apply converse of Pythagoras theorem.

Diagonal AC:

$$|AC| = \sqrt{(7 - 4)^2 + (5 - 2)^2}$$

$$|AC| = \sqrt{(3)^2 + (3)^2}$$

$$|AC| = \sqrt{9 + 9}$$

$$|AC| = \sqrt{18} \dots\dots\dots (v)$$

Diagonal BD:

$$|BD| = \sqrt{(4 - 7)^2 + (5 - 2)^2}$$

$$|BD| = \sqrt{(-3)^2 + (3)^2}$$

$$|BD| = \sqrt{9 + 9}$$

$$|BD| = \sqrt{18} \dots\dots\dots (vi)$$

From (v) and (vi) $|AC| = |BD|$ both diagonals are equal.

Now,

$$|AC|^2 = (\sqrt{18})^2 = 18 \dots\dots\dots (vii)$$

$$|AB|^2 + |BC|^2 = (3)^2 + (3)^2$$

$$|AB|^2 + |BC|^2 = 9 + 9$$

$$|AB|^2 + |BC|^2 = 18 \dots\dots\dots (viii)$$

From (vii) and (viii)

$$|AC|^2 = |AB|^2 + |BC|^2$$

This relation shows that $\triangle ABC$ is right triangle with $m\angle B = 90^\circ$.

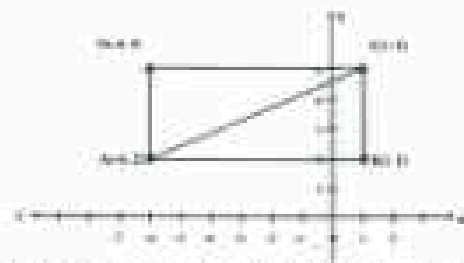
Thus angle at each vertex of quadrilateral is 90° .

Hence, quadrilateral ABCD is a square and its diagonals are same in length.

Q.8. Are A(-6, 2), B(1, 2), C(1, 5) and D(-6, 5) the vertices of a rectangle? Also plot them.

Solution

$$A(-6, 2), B(1, 2), C(1, 5), D(-6, 5)$$



We find the lengths of all sides by using distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{[1 - (-6)]^2 + (2 - 2)^2}$$

$$|AB| = \sqrt{(7)^2 + (0)^2}$$

$$|AB| = \sqrt{49 + 0}$$

$$|AB| = \sqrt{49}$$

$$|AB| = 7 \dots\dots\dots (i)$$

Now,

$$|BC| = \sqrt{(1 - 1)^2 + (5 - 2)^2}$$

$$|BC| = \sqrt{(0)^2 + (3)^2}$$

$$|BC| = \sqrt{0 + 9}$$

$$|BC| = \sqrt{9}$$

$$|BC| = 3 \dots\dots\dots (ii)$$

Now,

$$|CD| = \sqrt{(-8 - 1)^2 + (3 - 5)^2}$$

$$|CD| = \sqrt{(-7)^2 + (8)^2}$$

$$|CD| = \sqrt{49 + 64}$$

$$|CD| = \sqrt{113}$$

$$|CD| = 7 \dots\dots\dots (iii)$$

Now,

$$|AD| = \sqrt{[-6 - (-6)]^2 + (5 - 2)^2}$$

$$|AD| = \sqrt{(-6 + 6)^2 + (3)^2}$$

$$|AD| = \sqrt{0 + 9}$$

$$|AD| = \sqrt{9}$$

$$|AD| = 3 \dots\dots\dots (iv)$$

From (i), (ii) and (iv)

$$|AB| = |CD| \text{ and } |BC| = |AD|$$

i.e Opposite sides are equal.

To check the angles at each vertex we find diagonal $|AC|$ then apply converse of Pythagoras theorem.

$$|AC| = \sqrt{[1 - (-6)]^2 + (5 - 2)^2}$$

$$|AC| = \sqrt{(7)^2 + (3)^2}$$

$$|AC| = \sqrt{49 + 9}$$

$$|AC| = \sqrt{58}$$

$$|AC| = \sqrt{58} \dots\dots\dots (v)$$

Now,

$$|AC|^2 = (\sqrt{58})^2$$

$$|AC|^2 = 58 \dots\dots\dots (vi)$$

$$|AB|^2 + |BC|^2 = (7)^2 + (3)^2$$

$$|AB|^2 + |BC|^2 = 49 + 9$$

$$|AB|^2 + |BC|^2 = 58 \dots\dots\dots (vii)$$

From (vi) and (vii)

$$|AC|^2 = |AB|^2 + |BC|^2$$

This relation shows that $\triangle ABC$ is right triangle with $\angle B = 90^\circ$.

Then angle at each vertex is 90° .

Hence, quadrilateral formed by joining the given points is a rectangle.

Q.9. Prove that the points $A(-3, 0)$, $B(3, 0)$, $C(6, 4)$ and $D(0, 4)$ are the vertices of a parallelogram.

Solution



First we find the lengths of all sides by using distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{[3 - (-3)]^2 + (0 - 0)^2}$$



$$|AB| = \sqrt{(3 + 3)^2 + (0)^2}$$

$$|AB| = \sqrt{(6)^2 + 0}$$

$$|AB| = \sqrt{36}$$

$$|AB| = 6 \dots\dots\dots (i)$$

Now,

$$|BC| = \sqrt{(6 - 3)^2 + (4 - 0)^2}$$

$$|BC| = \sqrt{(3)^2 + (4)^2}$$

$$|BC| = \sqrt{9 + 16}$$

$$|BC| = \sqrt{25}$$

$$|BC| = 5 \dots\dots\dots (ii)$$

Now,

$$|CD| = \sqrt{(0 - 6)^2 + (4 - 4)^2}$$

$$|CD| = \sqrt{(-6)^2 + (0)^2}$$

$$|CD| = \sqrt{36 + 0}$$

$$|CD| = 6 \dots\dots\dots (iii)$$

Now,

$$|AD| = \sqrt{[0 - (-3)]^2 + (4 - 0)^2}$$

$$|AD| = \sqrt{(3)^2 + (4)^2}$$

$$|AD| = \sqrt{9 + 16}$$

$$|AD| = \sqrt{25}$$

$$|AD| = 5 \dots\dots\dots (iv)$$

From (i), (ii), (iii) and (iv) we observe that

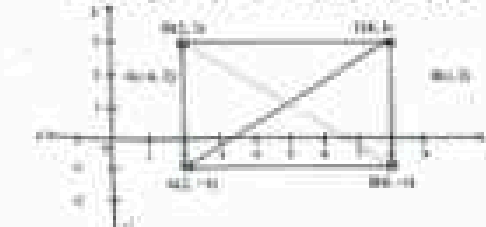
$$|AB| = |CD| = 6 \text{ and } |BC| = |AD| = 5$$

i.e opposite sides are equal in length.

Hence, quadrilateral ABCD is a parallelogram.

Q.10. Show that $A(2, -1)$, $B(8, -1)$, $C(8, 3)$, and $D(2, 3)$ are the vertices of a rectangle and prove that $\triangle ABC$ and $\triangle ABD$ form right triangles.

Solution: $A(2, -1)$, $B(8, -1)$, $C(8, 3)$, $D(2, 3)$



First we find the lengths of all sides by using distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(8 - 2)^2 + [-1 - (-1)]^2}$$

$$|AB| = \sqrt{(6)^2 + (-1 + 1)^2}$$

$$|AB| = \sqrt{36 + (0)^2}$$

$$|AB| = \sqrt{36 + 0}$$

$$|AB| = \sqrt{36}$$

$$|AB| = 6 \dots\dots\dots (i)$$

Now,

$$|BC| = \sqrt{(8 - 8)^2 + [3 - (-1)]^2}$$

$$|BC| = \sqrt{(0)^2 + (3 + 1)^2}$$

$$|BC| = \sqrt{0 + (4)^2}$$

$$|BC| = \sqrt{0 + 16}$$

$$|BC| = \sqrt{16}$$

$$|BC| = 4 \dots\dots\dots (ii)$$

Now,

$$|CD| = \sqrt{(2 - 8)^2 + (3 - 3)^2}$$

$$|CD| = \sqrt{(-6)^2 + (0)^2}$$

$$|CD| = \sqrt{36 + 0}$$

$$|CD| = \sqrt{36}$$

$$|CD| = 6 \dots\dots\dots (iii)$$

Now,

$$|AD| = \sqrt{(2 - 2)^2 + [3 - (-1)]^2}$$

$$|AD| = \sqrt{(0)^2 + (3 + 1)^2}$$

$$|AD| = \sqrt{0 + (4)^2}$$

$$|AD| = \sqrt{0 + 16}$$

$$|AD| = \sqrt{16}$$

$$|AD| = 4 \dots\dots\dots (iv)$$

From (i), (ii), (iii) and (iv)

$$|AB| = |CD| \text{ and } |BC| = |AD|$$

Opposite sides are equal in lengths. To check right angle at each vertex we find diagonals $|AC|$ and $|BD|$, then apply converse of Pythagoras theorem.

$$|AC| = \sqrt{(8 - 2)^2 + [3 - (-1)]^2}$$

$$|AC| = \sqrt{(6)^2 + (3 + 1)^2}$$



$$|AC| = \sqrt{36 + (4)^2}$$

$$|AC| = \sqrt{36 + 16}$$

$$|AC| = \sqrt{52} \dots\dots (v)$$

Now,

$$|BD| = \sqrt{(2-8)^2 + [3-(-1)]^2}$$

$$|AC| = \sqrt{(-6)^2 + [3+1]^2}$$

$$|AC| = \sqrt{(-6)^2 + (4)^2}$$

$$|AC| = \sqrt{36 + 16}$$

$$|AC| = \sqrt{52} \dots\dots (vi)$$

Now,

$$|AC|^2 = (\sqrt{52})^2$$

$$|AC|^2 = 52 \dots\dots (vii)$$

$$|AB|^2 + |BC|^2 = (6)^2 + (4)^2$$

$$|AB|^2 + |BC|^2 = 36 + 16$$

$$|AB|^2 + |BC|^2 = 52 \dots\dots (viii)$$

From (vii) and (viii)

$$|AC|^2 = |AB|^2 + |BC|^2$$

This relation shows that $\triangle ABC$ is right triangle with $m\angle B = 90^\circ$.

Similarly,

$$|BD|^2 = (\sqrt{52})^2$$

$$|BD|^2 = 52 \dots\dots (ix)$$

Now,

$$|AB|^2 + |AD|^2 = (6)^2 + (4)^2$$

$$|AB|^2 + |AD|^2 = 36 + 16$$

$$|AB|^2 + |AD|^2 = 52 \dots\dots (x)$$

From (ix) and (x)

$$|BD|^2 = |AB|^2 + |AD|^2$$

This relation shows that $\triangle ABD$ is right triangle with $m\angle A = 90^\circ$.

Hence, quad. ABCD is a rectangle.

Mid-Point Formula

The mid-point P of the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Example 5

By using mid point formula, prove that the diagonals of a parallelogram with vertices (1, 1), (2, 4), (6, 1) and (7, 4) intersect each other.

Solution

First we plot these vertices on a Cartesian plane as shown in fig. In parallelogram ABCD, AC and BD are its diagonals.

Let both the diagonals intersect each other at point E (x, y). By mid point formula, mid point of AC has the coordinates as



$$\left(\frac{1+7}{2}, \frac{1+4}{2} \right) \\ = (4, 2.5) \dots\dots (i)$$

And the coordinates of the mid point of

$$\text{diagonal BD is } \left(\frac{2+6}{2}, \frac{4+1}{2} \right)$$

$$= (4, 2.5) \dots\dots (ii)$$

From (i) and (ii) both the diagonals have the same mid point, so they intersect each other at point E(4, 2.5).

Example 6

If (1, 4) is the mid point of the line segment joining (6, b) and (a, 2). Find a and b.

Solution

By mid point formula, mid point of the line segment with end points (x_1, y_1) and (x_2, y_2)

$$\text{is } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



is, mid point of the line segment joining the end points (6, b) and (a, 2) is

$$M \left(\frac{6+a}{2}, \frac{b+2}{2} \right)$$

By using the given condition, we have

$$\left(\frac{6+a}{2}, \frac{b+2}{2} \right) = (1, 4)$$

By the definition of equal ordered pairs

$$\frac{6+a}{2} = 1 \text{ and } \frac{b+2}{2} = 4$$

$$6+a = 2 \text{ and } b+2 = 8$$

$$a = 2-6 \text{ and } b = 8-2$$

$$a = -4 \text{ and } b = 6$$

Hence, $a = -4$ and $b = 6$

Example 7

Use $\triangle XYZ$ in the adjoining figure to find the following.

i. Midpoint L and M of XY and YZ respectively.

ii. $d(L, M)$ and $d(X, Z)$

iii. The relationship between LM and XZ.



Solution

i. By mid point formula the coordinates of L

$$\text{are } \left(\frac{2+6}{2}, \frac{3+9}{2} \right) = \left(\frac{8}{2}, \frac{12}{2} \right)$$

Exercise 7.2

Q.1. Find the mid point of the line segment joining the given points.

i. (2, 5), (6, 9)

ii. (-1, 0), (2, 2)

iii. (1, 4, -1.5), (2, 6, 3, 5)

Coordinates of M are

$$\left(\frac{2+7}{2}, \frac{3+4}{2} \right) = \left(\frac{9}{2}, \frac{7}{2} \right)$$

$$\begin{aligned} \text{ii. } d(L, M) &= \sqrt{\left(\frac{7}{2} - \frac{9}{2} \right)^2 + \left(\frac{15}{2} - \frac{7}{2} \right)^2} \\ &= \sqrt{\left(\frac{7-9}{2} \right)^2 + \left(\frac{15-7}{2} \right)^2} \\ &= \sqrt{\left(\frac{-2}{2} \right)^2 + \left(\frac{8}{2} \right)^2} = \sqrt{(-1)^2 + (4)^2} \\ &= \sqrt{1+16} \\ &= \sqrt{17} \end{aligned}$$

$$\text{Now, } d(X, Z) = \sqrt{(7-2)^2 + (4-12)^2}$$

$$= \sqrt{(5)^2 + (-8)^2}$$

$$= \sqrt{4+64}$$

$$= \sqrt{68} = \sqrt{4 \times 17}$$

$$= 2\sqrt{17}$$

iii. From part (ii), we observe that

$$d(X, Z) = 2 \text{ times } d(L, M)$$

$$\text{Or } XZ = 2 LM$$

$$\text{Or } LM = \frac{XZ}{2}$$

Hence,

Length of the line which is obtained by joining the mid points of any two sides of a triangle is half of the length of third side.



$$= M\left(\frac{8+14}{2}, \frac{14}{2}\right)$$

$$= M(4, 7)$$

ii. $(-1, 0), (2, 2)$

Let $A(-1, 0), B(2, 2)$

By midpoint formula, midpoint of $\overline{AB}$

$$= M\left(\frac{-1+2}{2}, \frac{0+2}{2}\right)$$

$$= M\left(\frac{1}{2}, \frac{2}{2}\right)$$

$$= M\left(\frac{1}{2}, 1\right)$$

iii. $(1, 4, -1, 5), (2, 6, 3, 5)$

Let $A(1, 4), B(2, 6)$

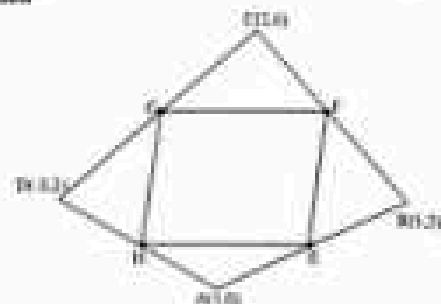
By midpoint formula, midpoint of $\overline{AB}$

$$= M\left(\frac{1+2}{2}, \frac{4+6}{2}\right)$$

$$= M\left(\frac{3}{2}, \frac{10}{2}\right)$$

$$= M(1.5, 5)$$

Q.2. Look at the quadrilateral ABCD in the adjoining figure. If E, F, G and H are the mid points of its sides as shown in the figure, then



i. Find the coordinates of E, F, G and H

ii. Draw line segments EF, FG, GH and HE

iii. Find the distance of EF, FG, GH and HE

iv. Give the type of quadrilateral EFGH

Solution:

i. Coordinates of E, F, G and H

a) E is midpoint of $\overline{AB}$

$$\text{Coordinates of E} = M\left(\frac{1+3}{2}, \frac{6+2}{2}\right)$$

$$= E\left(\frac{4}{2}, \frac{8}{2}\right)$$

$$= E(2, 4)$$

b) F is midpoint of $\overline{BC}$

$$\text{Coordinates of F} = M\left(\frac{3+5}{2}, \frac{2+6}{2}\right)$$

$$= F\left(\frac{8}{2}, \frac{8}{2}\right)$$

$$= F(4, 4)$$

c) G is midpoint of $\overline{CD}$

Coordinates of G

$$= G\left(\frac{5+3}{2}, \frac{6+10}{2}\right)$$

$$= G\left(\frac{8}{2}, \frac{16}{2}\right)$$

$$= G(4, 8)$$

d) H is midpoint of $\overline{DA}$

$$\text{Coordinates of H} = M\left(\frac{3+1}{2}, \frac{10+6}{2}\right)$$

$$= H\left(\frac{4}{2}, \frac{16}{2}\right)$$

$$= H(2, 8)$$

i. Joining E to F, F to G, G to H and H to E.

ii. Find distances of EF, FG, GH and HE

By distance formula

a) Distance of $\overline{EF}$: $E(2, 4), F(4, 4)$

$$|\overline{EF}| = \sqrt{\left(\frac{4-2}{2}\right)^2 + (4-4)^2}$$

$$|\overline{EF}| = \sqrt{\left(\frac{2-4}{2}\right)^2 + (0)^2}$$

$$|\overline{EF}| = \sqrt{\left(\frac{1}{2}\right)^2 + (0)}$$

$$|\overline{EF}| = \sqrt{\frac{1}{4} + \frac{0}{1}}$$

$$|\overline{EF}| = \sqrt{\frac{1+0}{4}}$$

$$|\overline{EF}| = \sqrt{\frac{1}{4}}$$

$$|\overline{EF}| = \frac{\sqrt{1}}{2} \text{ units} \dots\dots (i)$$

b) Distance of $\overline{FG}$: $F(4, 4), G(4, 8)$

$$|\overline{FG}| = \sqrt{\left(\frac{4-4}{2}\right)^2 + (4-8)^2}$$

$$|\overline{FG}| = \sqrt{\left(\frac{-1-7}{2}\right)^2 + (0)^2}$$

$$|\overline{FG}| = \sqrt{\left(\frac{-8}{2}\right)^2 + 0}$$

$$|\overline{FG}| = \sqrt{(-4)^2}$$

$$|\overline{FG}| = \sqrt{16}$$

$$|\overline{FG}| = 4 \text{ units} \dots\dots (ii)$$

c) Distance of

$$\overline{GH}: G(4, 8), H(2, 8)$$

$$|\overline{GH}| = \sqrt{\left(-1 - \left(-\frac{1}{2}\right)\right)^2 + (8-8)^2}$$

$$|\overline{GH}| = \sqrt{\left(-1 + \frac{1}{2}\right)^2 + (0)^2}$$

$$|\overline{GH}| = \sqrt{\left(\frac{-2+1}{2}\right)^2 + 0}$$

$$|\overline{GH}| = \sqrt{\left(\frac{-1}{2}\right)^2 + 0}$$

$$|\overline{GH}| = \sqrt{\frac{1}{4} + 0}$$

$$|\overline{GH}| = \sqrt{\frac{1+0}{4}}$$

$$|\overline{GH}| = \sqrt{\frac{1}{4}}$$

$$|\overline{GH}| = \frac{\sqrt{1}}{2} \text{ units} \dots\dots (iii)$$

Distance of $\overline{HE}$: $H(2, 8), E(2, 4)$

$$|\overline{HE}| = \sqrt{(2-2)^2 + (8-4)^2}$$

$$|\overline{HE}| = \sqrt{(0)^2 + (4)^2}$$

$$|\overline{HE}| = \sqrt{(4)^2 + 0}$$

$$|\overline{HE}| = \sqrt{16}$$

$$|\overline{HE}| = 4 \text{ units} \dots\dots (iv)$$

From (i), (ii), (iii) and (iv)

$$|\overline{EF}| = |\overline{GH}| \text{ and } |\overline{FG}| = |\overline{HE}|$$

Opposite sides of quadrilateral EFGH are equal in length.



Hence, quadrilateral EFGH is a parallelogram.

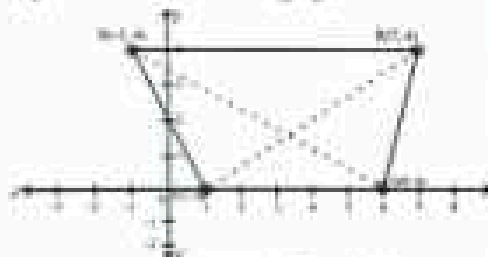
Q.3. Check the using the mid point formula whether the diagonals of trapezium PQRS with vertices at $P(1,0)$, $Q(6,0)$, $R(7,4)$, $S(-1,4)$ bisect each other or not.

Answer

Solution

$P(1,0)$, $Q(6,0)$, $R(7,4)$, $S(-1,4)$

First we plot these vertices on a Cartesian plane as shown in following figure



In trapezium PQRS, $\overline{PR}$ and $\overline{QS}$ are its diagonals.

Coordinates of midpoint of $\overline{PR}$:

$$= M_1\left(\frac{1+7}{2}, \frac{0+4}{2}\right)$$

$$= M_1\left(\frac{8}{2}, \frac{4}{2}\right)$$

$$= M_1(4,2) \dots\dots (i)$$

Coordinates of midpoint of $\overline{QS}$:

$$= M_2\left(\frac{-1+6}{2}, \frac{4+0}{2}\right)$$

$$= M_2\left(\frac{5}{2}, \frac{4}{2}\right)$$

$$= M_2(2.5, 2) \dots\dots (ii)$$

From (i) and (ii) we observe that both diagonals have different midpoints so they do not bisect each other.

Q.4. If vertices of a rhombus are $A(-3,-1)$, $B(0,0)$, $C(1,3)$, $D(-2,2)$ show

that its diagonals bisect each other. Also find the length of each diagonal.

Answer

Solution

$A(-3,-1)$, $B(0,0)$, $C(1,3)$, $D(-2,2)$

First we plot vertices on a Cartesian plane as shown in figure.



In rhombus ABCD, $\overline{AC}$ and $\overline{BD}$ are its diagonals.

Coordinates of midpoint of diagonal $\overline{AC}$:

$$= M_1\left(\frac{-3+1}{2}, \frac{-1+3}{2}\right)$$

$$= M_1\left(\frac{-2}{2}, \frac{2}{2}\right)$$

$$= M_1(-1,1) \dots\dots (i)$$

Coordinates of midpoint of diagonal $\overline{BD}$:

$$= M_2\left(\frac{-2+0}{2}, \frac{2+0}{2}\right)$$

$$= M_2\left(\frac{-2}{2}, \frac{2}{2}\right)$$

$$= M_2(-1,1) \dots\dots (ii)$$

From (i) and (ii) we observe that both diagonals have same midpoint $(-1,1)$. $\therefore$ they bisect each other at point $(-1,1)$.

Now, we find the length of each diagonal by using distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Length of diagonal $\overline{AC}$:

$$|AC| = \sqrt{[1 - (-3)]^2 + [3 - (-1)]^2}$$

$$|AC| = \sqrt{(1+3)^2 + (3+1)^2}$$

$$|AC| = \sqrt{4^2 + 4^2}$$

$$|AC| = \sqrt{16+16}$$

$$|AC| = \sqrt{32}$$

$$|AC| = \sqrt{16 \times 2}$$

$$|AC| = 4\sqrt{2} \text{ units} \dots\dots (iii)$$

Length of diagonal $\overline{BD}$:

$$|BD| = \sqrt{[-2-0]^2 + [2-0]^2}$$

$$|BD| = \sqrt{(-2)^2 + (2)^2}$$

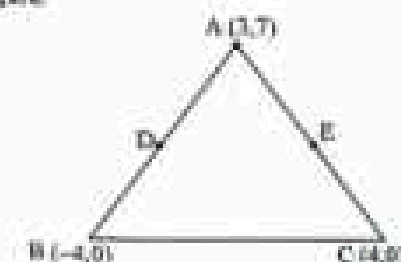
$$|BD| = \sqrt{4+4} = \sqrt{8}$$

$$|BD| = \sqrt{4 \times 2}$$

$$|BD| = 2\sqrt{2} \text{ units} \dots\dots (iv)$$

From (iii) and (iv) we observe that the lengths of diagonals of a rhombus ABCD are not equal.

Q.5. Look at the $\triangle ABC$ in the adjoining figure.



- Find the coordinates of the mid points D and E of AB and AC respectively. *Answer*
- Find $d(D, E)$ and $d(B, C)$. *Answer*
- Find half of $d(B, C)$. *Answer*
- Compare the length of DE and BC and draw conclusion. *Answer*

Solution

- Given that D is a midpoint of $\overline{AB}$.

Coordinates of midpoint D

$$= D\left(\frac{-4+3}{2}, \frac{0+7}{2}\right)$$

$$= D\left(\frac{-1}{2}, \frac{7}{2}\right) = D(-0.5, 3.5)$$

Given that E is midpoint of $\overline{AC}$.

Coordinates of midpoint E

$$= E\left(\frac{3+4}{2}, \frac{7+0}{2}\right)$$

$$= E\left(\frac{7}{2}, \frac{7}{2}\right) = E(3.5, 3.5)$$

Now, we find $d(D, E)$, using distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d(D, E) = \sqrt{\left[\frac{7}{2} - \left(-\frac{1}{2}\right)\right]^2 + \left[\frac{7}{2} - \frac{7}{2}\right]^2}$$

$$d(D, E) = \sqrt{\left[\frac{7}{2} + \frac{1}{2}\right]^2 + (0)^2}$$

$$d(D, E) = \sqrt{\left[\frac{7+1}{2}\right]^2 + 0}$$

$$d(D, E) = \sqrt{\left[\frac{8}{2}\right]^2}$$

$$d(D, E) = \sqrt{4^2}$$

$$d(D, E) = 4 \text{ units} \dots\dots (i)$$

Now,

$$d(B, C) = \sqrt{[4 - (-4)]^2 + [0 - 0]^2}$$

$$d(B, C) = \sqrt{(4+4)^2 + (0)^2}$$

$$d(B, C) = \sqrt{8^2 + 0}$$

$$d(B, C) = \sqrt{64}$$

$$d(B, C) = 8 \text{ units} \dots\dots (ii)$$

- Finding half of $d(B, C)$



$$\text{Half of } d(B, C) = \frac{1}{2} \text{ of } d(B, C)$$

$$d(B, C) = \frac{1}{2} \times 8$$

$$d(B, C) = 4 \text{ units. } \dots (iv)$$

iv. By comparing Eq (i) and (iv)

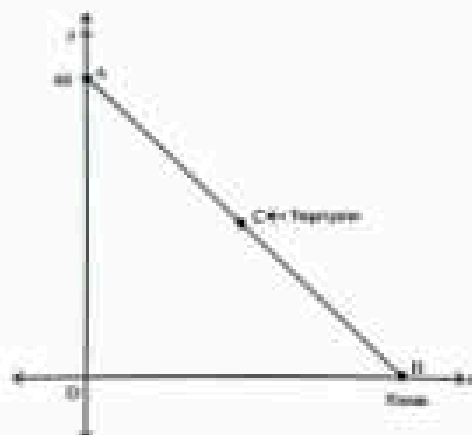
$$\text{Half of } d(B, C) = d(D, E)$$

Or

$$d(D, E) = \text{Half of } d(B, C)$$

Conclusion: The length of line segment obtained by joining the midpoints of two sides of a triangle is half of length of third side.

Q.4. Hassan and Ali participated in a race competition. They both have to start running from different points on coordinate axes at a distance of 6 units from origin but they must reach at the same target point C, after covering an equal distance. If Hassan starting point is on x -axis and Ali's starting point is on y -axis as shown in the figure, then find the coordinates of the:



(i) starting points of Hassan and Ali

answer

(ii) target point C

answer

Solution

- i. We observe that Hassan is at 6 units on the x -axis. The x -coordinate of B is 6 and y -coordinate of any point on x -axis is 0. So coordinates of starting point of Hassan on x -axis are (6,0) i.e. B(6,0).

Ali is at 6 units on the y -axis. The y -coordinate of A is 6 and x -coordinate of any point on y -axis is 0. So coordinates of starting point of Ali on y -axis are (0,6) i.e. A(0,6).

- ii. It is given that Hassan and Ali are at an equal distance from target point C. It means C is midpoint of B and A. Coordinates of target point C

$$= C \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$= C \left(\frac{6+0}{2}, \frac{0+6}{2} \right)$$

$$= C \left(\frac{6}{2}, \frac{6}{2} \right)$$

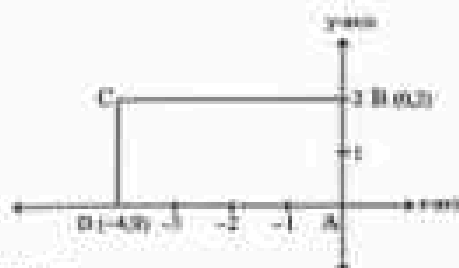
$$= C (3, 3)$$

Q.7. Label the missing coordinates of the vertices of the following figures. answer

- i. Square OLMN



- ii. Rectangle ABCD



Solution

- i. Square OLMN

All sides of a square are equal in length.

Miscellaneous Exercise 2

- Q.1. Encircle the correct option in the following:

1. If the coordinates of four points are

$$A \left(\frac{3}{2}, -\frac{7}{4} \right), B \left(3, \frac{3}{4} \right), C (3, -2) \text{ and}$$

$$D \left(1, \frac{1}{2} \right), \text{ the relationship between the}$$

distance AB and CD is: answer

$$(a) AB > CD \quad (b) AB < CD$$

$$(c) AB = CD \quad (d) AB = \frac{1}{2} CD$$

2. If a and b are any two points on a coordinate line (number line), then its mid point will be: answer

$$(a) \frac{a+b}{2}$$

$$(b) \left(\frac{a}{2}, \frac{b}{2} \right)$$

$$(c) \left(\frac{a}{2}, b \right)$$

$$(d) \left(a, \frac{b}{2} \right)$$

3. (5,2), (7,0), (1,-2) and (-1,0) are the vertices of a: answer

(a) parallelogram

(b) square

(c) trapezium

(d) kite

4. The triangle with vertices (1,-3), (3,2) and (-2,4) is: answer

$$m\overline{OL} = 3 \text{ units}$$

Coordinates of point L = L(3,0) given

Coordinates of point M = M(3,3)

Coordinates of point N = N(0,3)

Coordinates of point O = O(0,0) origin

- ii. Rectangle ABCD

Opposite sides of a rectangle are equal.

Point A is on origin,

Coordinates of point A = A(0,0) origin

Coordinates of point B = B(2,2) Given

Coordinates of point C = C(-4,2)

Coordinates of point D = D(-4,0) Given

(a) a right triangle

(b) an isosceles triangle

(c) a right isosceles triangle

(d) an equilateral triangle

3. Mid-point of $(x, -d)$ and $(-d, c)$ is: answer

$$(a) \left(\frac{c-d}{2}, \frac{c-d}{2} \right)$$

$$(b) \left(\frac{-c-d}{2}, \frac{-d-c}{2} \right)$$

$$(c) \left(\frac{-d+c}{2}, \frac{c-d}{2} \right)$$

$$(d) (c-d, -d+c)$$

4. If P(2,3) is the mid point of A(x,4) and B(-3,y), then the coordinates of (x,y) are: answer

$$(a) \left(\frac{-1}{2}, \frac{3+b}{2} \right)$$

$$(b) (3, -1)$$

$$(c) \left(\frac{x-3}{2}, \frac{x+4}{2} \right)$$

$$(d) (7, 2)$$



7. If two ordered pairs $(4,3)$ and $\left(\frac{x+1}{2}, 3-3\right)$ are equal then, the value

of x is: answer

- (a) 4 (b) 8
(c) 7 (d) 3

8. In which quadrant, the point $(-4, -(-6))$ lies? answer

- (a) first (b) second
(c) third (d) fourth

9. In the adjoining figure of a line: answer



$$d(A, C) = d(B, C) =$$

- (a) $d(B, A)$ (b) $d(C, B)$
(c) $d(A, B)$ (d) both a & b

10. The distance on y -axis is: answer

- (a) 1 (b) x
(c) zero (d) y

11. Minimum number of sides in a polygon is: answer

- (a) 2 (b) 3
(c) 4 (d) 5

12. If $x < 0$ and $y > 0$, then $(-x, y)$ will lie in quadrant: answer

- (a) I (b) II
(c) III (d) IV

13. The point $(3,5)$ is at minimum distance from: answer

- (a) x -axis (b) y -axis
(c) origin (d) both a & b

14. If N is the mid point of the line segment LN , then $LN : MN$ is: answer

- (a) 2 : 1 (b) 1 : 2
(c) 1 : 1 (d) 1 : 3

15. All the bisectors of a line segment always pass through: answer

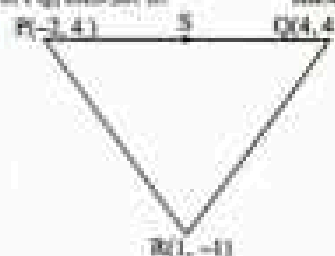
- (a) x -axis (b) origin
(c) mid-point (d) y -axis

16. The below figure is kite, what will be the coordinate of the point of intersection of both diagonals? answer



- (a) $(0,2)$ (b) $\left(2\frac{1}{2}, 0\right)$
(c) $(0,0)$ (d) $(2,0)$

17. In the given figure, if S is the mid point of PQ , then SR is: answer



- (a) $\sqrt{2}$ (b) ± 5
(c) 6 (d) 5

18. If distance between two points $A(x, 0)$ and $B(2, -1)$ is 20 units, then the value of x is: answer

- (a) 279 (b) $3\sqrt{21}$
(c) 3 (d) 40

19. The length of the hypotenuse of $\triangle OPQ$ whose vertices are $(7, 4)$, $(7, 1)$ and $(-3, 1)$ is: answer

- (a) 3 (b) 10
(c) $\sqrt{109}$ (d) $\sqrt{92}$

20. The mid point of the hypotenuse of a right triangle with vertices $(4, 0)$, $(2, 3)$ and $(-1, -5)$ is: answer

- (a) $\left(2\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, -2\right)$



- (a) $\left(\frac{3}{2}, -\frac{5}{2}\right)$ (b) $\left(\frac{3}{2}, \frac{5}{2}\right)$

21. If end points of a diameter of the circle are $(4, 5)$ and $(6, 9)$, then coordinates of its center will be: answer

- (a) $(7, 5)$ (b) 1, 2
(c) $(5, 7)$ (d) $(0, 0)$

22. If two points $(2, 3)$ and $(4, 5)$ are at equal distance from $(1, 1)$ then, the value of t is: answer

- (a) 5 (b) 4
(c) 3 (d) 7

Additional MCQs

23. Point $(0, -2)$ lies in / on: answer

- (a) the negative direction of x -axis
(b) the negative direction of y -axis
(c) 1st quadrant
(d) 2nd quadrant

24. Abscissa of the all the points on x -axis is: answer

- (a) 0 (b) 1
(c) -1 (d) any number

25. Ordinate of the all the points on y -axis is: answer

- (a) 0 (b) 1
(c) -1 (d) any number

26. Ordinate of the all the points on y -axis is: answer

- (a) 0 (b) 1
(c) -1 (d) any number

27. A point both of whose coordinates are negative will lie in: answer

- (a) 1st quadrant (b) 2nd quadrant
(c) 3rd quadrant (d) y -axis

28. A point both of whose coordinates are positive will lie in: answer

- (a) x -axis (b) 2nd quadrant
(c) 1st quadrant (d) y -axis

29. If y -coordinate of a point is zero, then this point always lies in/on: answer

- (a) 1st quadrant (b) 2nd quadrant

25. A point $P(x, y)$ is at equal distance from both the axes and lies in the third quadrant. If it is at a distance of 4 units from y -axis, then its coordinates are: answer

- (a) $(-4, 4)$ (b) $(4, -4)$
(c) $(4, 4)$ (d) $(-4, -4)$

26. A line AB intersects x -axis at C with a distance of 3 units from origin and y -axis at D with a distance of 6 units from origin, then $d(C, D)$ is: answer

- (a) $3\sqrt{5}$ (b) $3\sqrt{2}$
(c) 3 (d) 45

- (a) x -axis (b) y -axis

32. If x -coordinate of a point is zero, then this point always lies in/on: answer

- (a) 1st quadrant (b) 2nd quadrant
(c) x -axis (d) y -axis

33. The points $(-5, 2)$ and $(2, -3)$ lie in: answer

- (a) same quadrant
(b) 2nd and 4th quadrants, respectively
(c) 2nd and 3rd quadrants, respectively
(d) 4th and 2nd quadrants, respectively

34. The perpendicular distance of the point $(3, 4)$ from the y -axis is: answer

- (a) 3 (b) 4
(c) 5 (d) 7

35. The points on x -axis which are at a distance of $\sqrt{13}$ units from $(-2, 3)$ is: answer

- (a) $(0,0)$, $(-2, -3)$
(b) $(0,0)$, $(-4, 0)$
(c) $(0,0)$, $(2, 3)$
(d) None of these

36. The point P lying in the fourth quadrant which is at a distance of 4 units from x -axis and 3 units from y -axis is: answer

- (a) $(4, -3)$ (b) $(4, 3)$
(c) $(3, -4)$ (d) $(-3, 4)$



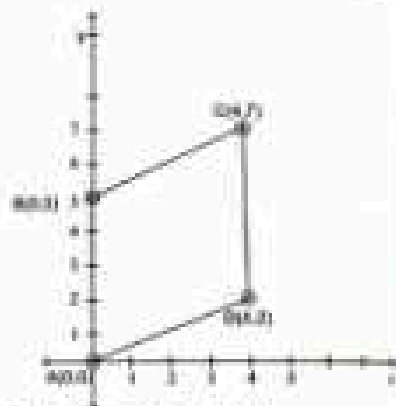
35. The lines, $x = 2$ and $y = 3$ are: correct
 (a) parallel to each other
 (b) perpendicular to each other
 (c) neither parallel nor perpendicular to each other
 (d) none of above
36. The lines, $x = -2$ and $y = 3$ intersect at the point: correct
 (a) $(-2, 3)$ (b) $(2, -3)$
 (c) $(3, -2)$ (d) $(-3, 2)$
39. If the end points of the diameter of a circle are $A(3, 4)$ and $B(-9, 8)$, then the center of the circle is: correct
 (a) $(1, -11)$ (b) $(-2, 4)$
 (c) $(1, 8)$ (d) None of these
40. The distance between the points $(0, 0)$ and $(1, 2)$ is: correct
 (a) 0 (b) 2
 (c) $\sqrt{5}$ (d) $\sqrt{3}$

Answer Key

1	b	2	a	3	a	4	c
5	c	6	d	7	c	8	b
9	d	10	c	11	b	12	b
13	b	14	a	15	c	16	d
17	d	18	b	19	c	20	c
21	c	22	b	23	d	24	a
25	b	26	d	27	a	28	d
29	a	30	c	31	c	32	d
33	b	34	a	35	b	36	c
37	b	38	a	39	b	40	d

Q.3. The consecutive vertices of a quadrilateral are $A(0, 0)$, $B(0, 5)$, $C(4, 7)$ and $D(4, 2)$. Show that quadrilateral is a parallelogram. correct

Solution:



$A(0, 0)$, $B(0, 5)$, $C(4, 7)$, $D(4, 2)$

We find the lengths of sides, using distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$|AB| = \sqrt{(0 - 0)^2 + (5 - 0)^2}$$

$$|AB| = \sqrt{(0)^2 + (5)^2}$$

$$|AB| = \sqrt{0 + 25}$$

$$|AB| = \sqrt{25}$$

$$|AB| = 5 \text{ units} \dots\dots\dots (i)$$

$$|BC| = \sqrt{(4 - 0)^2 + (7 - 5)^2}$$

$$|BC| = \sqrt{(4)^2 + (2)^2}$$

$$|BC| = \sqrt{16 + 4}$$

$$|BC| = \sqrt{20}$$

$$|BC| = \sqrt{4 \times 5}$$

$$|BC| = 2\sqrt{5} \text{ units} \dots\dots\dots (ii)$$

$$|CD| = \sqrt{(4 - 4)^2 + (2 - 7)^2}$$

$$|CD| = \sqrt{(0)^2 + (-5)^2}$$

$$|CD| = \sqrt{0 + 25}$$

$$|CD| = \sqrt{25}$$

$$|CD| = 5 \text{ units} \dots\dots\dots (iii)$$

$$|AD| = \sqrt{(4 - 0)^2 + (2 - 0)^2}$$

$$|AD| = \sqrt{(4)^2 + (2)^2}$$

$$|AD| = \sqrt{16 + 4}$$

$$|AD| = \sqrt{20}$$



$$|AD| = \sqrt{4 \times 5}$$

$$|AD| = 2\sqrt{5} \text{ units} \dots\dots\dots (iv)$$

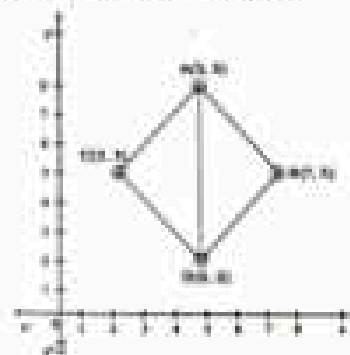
From (i), (ii), (iii) and (iv) we conclude that $|AB| = |CD|$ and $|BC| = |AD|$ opposite sides are equal.

This shows that quadrilateral having given vertices is a parallelogram.

Q.3. Show that the four points $(5, 8)$, $(7, 5)$, $(3, 5)$ and $(5, 2)$ are the vertices of a rhombus. correct

Solution:

Let $A(5, 8)$, $B(7, 5)$, $C(3, 5)$, $D(5, 2)$



Now, we find the lengths of all sides using distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(7 - 5)^2 + (5 - 8)^2}$$

$$|AB| = \sqrt{(2)^2 + (-3)^2}$$

$$|AB| = \sqrt{4 + 9}$$

$$|AB| = \sqrt{13} \text{ units} \dots\dots\dots (i)$$

Now,

$$|BD| = \sqrt{(5 - 7)^2 + (2 - 5)^2}$$

$$|BD| = \sqrt{(-2)^2 + (-3)^2}$$

$$|BD| = \sqrt{4 + 9}$$

$$|BD| = \sqrt{13} \text{ units} \dots\dots\dots (ii)$$

Now,

$$|CD| = \sqrt{(5 - 3)^2 + (2 - 5)^2}$$

$$|CD| = \sqrt{(2)^2 + (-3)^2}$$

$$|CD| = \sqrt{4 + 9}$$

$$|CD| = \sqrt{13} \text{ units} \dots\dots\dots (iii)$$

Now,

$$|AC| = \sqrt{(3 - 5)^2 + (5 - 8)^2}$$

$$|AC| = \sqrt{(-2)^2 + (-3)^2}$$

$$|AC| = \sqrt{4 + 9}$$

$$|AC| = \sqrt{13} \text{ units} \dots\dots\dots (iv)$$

From (i), (ii), (iii) and (iv)

$$|AB| = |BD| = |DC| = |CA|$$

This result shows that all sides are equal in lengths.

Now, we find diagonal AD .

$$|AD| = \sqrt{(5 - 5)^2 + (2 - 8)^2}$$

$$|AD| = \sqrt{(0)^2 + (-6)^2}$$

$$|AD| = \sqrt{0 + 36}$$

$$|AD| = \sqrt{36}$$

$$|AD| = 6 \text{ units} \dots\dots\dots (v)$$

Now, using converse Pythagoras theorem

$$|AD|^2 = (6)^2 = 36 \dots\dots\dots (vi)$$

Now

$$|AB|^2 + |BD|^2 = (\sqrt{13})^2 + (\sqrt{13})^2$$

$$|AB|^2 + |BD|^2 = 13 + 13$$

$$|AB|^2 + |BD|^2 = 26 \dots\dots\dots (vii)$$

From (vi) and (vii)

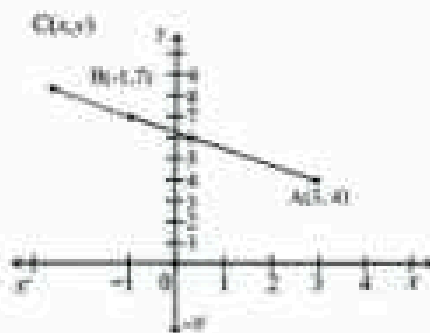
$$|AD|^2 = |AB|^2 + |BD|^2$$

This result shows that $\triangle ABD$ is not a right.

So, $m\angle B \neq 90^\circ$.

The above discussion shows that given points are vertices of a rhombus.

Q.4. $A(3, 4)$, $B(-1, 7)$ and $C(x, y)$ are three collinear points such that B is midpoint of AC . Find values of x and y . correct


Solution

Given that B is midpoint of $\overline{AC}$. Whereas $A(x_1, y_1)$ and $C(x_2, y_2)$ are end points. We

know that $M(x, y) = M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$$M(-1, 7) = M\left(\frac{3 + x_2}{2}, \frac{4 + y_2}{2}\right)$$

By comparing equal ordered pairs, we get

$$\begin{aligned} -1 &= \frac{3 + x_2}{2} & 7 &= \frac{4 + y_2}{2} \\ -2 &= 3 + x_2 & 14 &= 4 + y_2 \\ -2 - 3 &= x_2 & 14 - 4 &= y_2 \\ -5 &= x_2 & 10 &= y_2 \\ \Rightarrow \boxed{x_2 = -5} & & \Rightarrow \boxed{y_2 = 10} \end{aligned}$$

Thus coordinates of point C are $C(-5, 10)$.


UNIT 08

Geometry of Straight Lines

Inclination of Straight Line

Key fact

Gradient

Gradient is ratio of the vertical distance to the horizontal distance.

$$\begin{aligned} \text{Gradient} &= \frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{\text{rise}}{\text{run}} \\ &= \frac{2m}{50m} = \frac{1}{25} = \tan \theta \end{aligned}$$



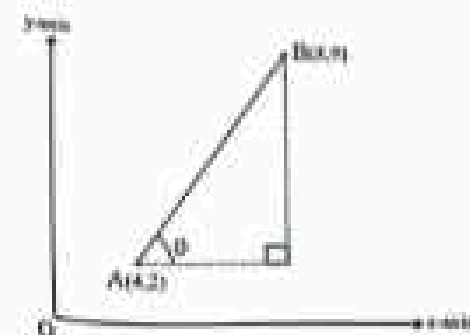
Where the angle θ is inclination.

The gradient of a straight line is the tangent of the angle which the line makes with the positive direction of the x-axis.

Gradient of Straight Line

The gradient of straight line is defined as:

$$\tan \theta = \frac{\text{rise}}{\text{run}}$$



In the figure line AB is inclined at an angle θ with x-axis.

$$\begin{aligned} \text{Gradient of line AB} &= \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\text{difference in } y\text{-coordinates of AB}}{\text{difference in } x\text{-coordinates of AB}} \\ &= \frac{4 - 2}{6 - 4} = \frac{2}{2} \end{aligned}$$

$$\text{Alternatively, gradient of AB} = \frac{2 - 0}{4 - 0} = \frac{2}{4} = \frac{1}{2}$$

Generally, gradient of a straight line is denoted by m . That is $m = \tan \theta$

Example 1

- Find the gradient (slope) of line whose inclination is 45° .
- Find inclination of lines whose gradient is 1.732.

Solution

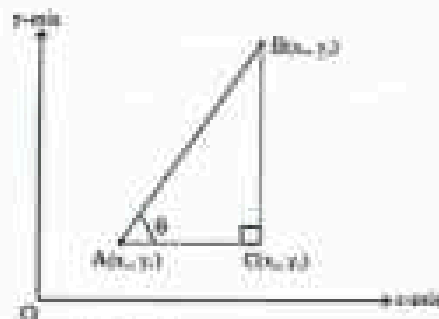
- Here $\theta = 45^\circ$
 $\therefore \text{Gradient} = \tan 45^\circ = 1$
- Gradient = $\tan \theta = 1.732$
 $\therefore \text{Inclination} = \theta = \tan^{-1}(1.732) = 60^\circ$

Formula of Gradient

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line AB, then gradient of AB is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Gradient of AB cannot be } \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$



Sign of Gradient



• In the figure 1, line AB makes an angle α ($0^\circ < \alpha < 90^\circ$) with the positive x-axis. As the value of $\tan \alpha$ is positive in first quadrant, therefore gradient of line AB is positive for $0^\circ < \alpha < 90^\circ$.

• In the figure 2, line LM makes an angle β ($90^\circ < \beta < 180^\circ$) with the positive x-axis. As the value of $\tan \beta$ is negative in second quadrant, therefore gradient of line LM is negative for $90^\circ < \beta < 180^\circ$.

Gradient of parallel and perpendicular lines



Figure 3 shows two parallel lines l_1 and l_2 making angle α with positive x-axis.

If gradients of l_1 and l_2 are m_1 and m_2 respectively then:

$$m_1 = m_2$$

Figure 2 shows two perpendicular lines p_1 and p_2 making angles α and β with positive x-axis respectively. If gradients of p_1 and p_2 are m_1 and m_2 respectively then:

$$m_1 = \frac{1}{m_2} \text{ or } m_2 = \frac{1}{m_1}$$

$$m_1 \times m_2 = -1$$

$$m_1 + m_2 = 1 \text{ or } 0$$

Example 2

Find gradients of lines joining

- (i) A(1, 4), B(3, 4) (ii) C(-1, 5), D(0, 3)

Solution

$$(i) \text{ Gradient of AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 4}{3 - 1} = 0$$

$$(ii) \text{ Gradient of CD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 5}{0 - (-1)} = -2$$

Example 3

Prove that lines AB and CD are parallel when

- A(3, 5), B(4, 6), C(7, 11), D(9, 13)

Solution

$$\text{Gradient of AB} = m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 5}{4 - 3} = 1$$

$$\text{Gradient of CD} = m_2 = \frac{13 - 11}{9 - 7} = \frac{2}{2} = 1$$

$$\text{As, } m_1 = m_2$$

Therefore, AB and CD are parallel lines.

Example 4

Lines AB and CD with coordinates A(1, 8), B(2, 3), C(3, 10), D(3, k) are perpendicular. Find k.

Solution

$$\text{Gradient of AB} = m_1 = \frac{8 - 3}{2 - 1} = \frac{5}{1} = 5$$

$$\text{Gradient of CD} = m_2 = \frac{k - 10}{3 - 3} = \frac{k - 10}{0}$$

As, AB and CD are perpendicular,

Therefore, $m_1 \times m_2 = -1$

$$5 \times \frac{k - 10}{0} = -1$$

$$5k - 50 = -1$$

$$\Rightarrow 5k - 50 = -1$$

$$\Rightarrow 5k = -1 + 50$$

$$\Rightarrow 5k = 49$$

$$\Rightarrow k = \frac{49}{5}$$

Exercise 8.1

Q.1. Find the gradient (slope) of line whose inclination is:

- (i) 0° (ii) 30° (iii) 45°

- (iv) 60° (v) 90° (vi) 120°

- (vii) 135° (viii) 150° (ix) 180°

Solution

(i) 0°

Line $\theta = 0^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 0^\circ = 0$$

(ii) 30°

Line $\theta = 30^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 30^\circ = \frac{1}{\sqrt{3}} = 0.577$$

(iii) 45°

Line $\theta = 45^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 45^\circ = 1$$

(iv) 60°

Line $\theta = 60^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 60^\circ = \sqrt{3} = 1.732$$

(v) 90°

Line $\theta = 90^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 90^\circ = \text{undefined}$$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 120^\circ = -1.732$$

(vi) 150°

Line $\theta = 150^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 150^\circ = -0.577$$

(vii) 180°

Line $\theta = 180^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 180^\circ = 0$$

(viii) 135°

Line $\theta = 135^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 135^\circ = -1$$

(ix) 45°

Line $\theta = 45^\circ$

$\therefore$ Gradient

$$= \tan \theta$$

$$= \tan 45^\circ = 1$$

Q.2. Find inclination of the line whose slope is:

- (i) 0 (ii) 0.577 (iii) 1

- (iv) -1.732 (v) -0.577 (vi) -1

Solution

$\therefore$ Gradient (Slope) = $\tan \theta = 0$

$$\theta = \tan^{-1}(0)$$

$$\theta = 0^\circ$$

(ii) 0.577

$\therefore$ Slope = $\tan \theta = 0.577$

$$\theta = \tan^{-1}(0.577)$$

$$\theta = 30^\circ$$





(iii) Solution

$$= -1.732$$

$$\therefore \text{Slope} = \tan \theta = -1.732$$

$$\theta = \tan^{-1}(-1.732) \approx 20^\circ$$

$$\theta = 120^\circ$$

(iv) Solution

$$= -0.564$$

$$\therefore \text{Slope} = \tan \theta = -0.564$$

$$\theta = \tan^{-1}(-0.564)$$

Making angle positive

$$\theta = 1 - 20^\circ + 180^\circ = 160^\circ$$

Q.3. Find gradient and inclination of lines joining:

(i) A(2, 6), B(5, 8) answer

(ii) C(-2, 4), D(1, -3) answer

(iii) E(5, -2), F(-2, -3) answer

(b) Solution

A(2, 6), B(5, 8)

$$\therefore \text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Gradient of } \overline{AB} = \frac{8 - 6}{5 - 2} = \frac{2}{3}$$

$$\therefore \text{Gradient} = \tan \theta = \frac{2}{3}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\theta = 33.69^\circ$$

(ii) Solution

C(-2, 4), D(1, -3)

$$\therefore \text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}, \theta = 1 - 20^\circ + 180^\circ = 160^\circ$$

$$\text{Gradient of } \overline{CD} = \frac{-3 - 4}{1 - (-2)} = \frac{-7}{3}$$

$$\therefore \text{Gradient} = \tan \theta = \frac{-7}{3}$$

$$\theta = \tan^{-1}\left(\frac{-7}{3}\right) = -66.80^\circ$$

Making angle positive

$$\theta = -66.80^\circ + 180^\circ = 113.20^\circ$$

(iii) Solution

E(5, -2), F(-2, -3)

$$\therefore \text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Gradient of } \overline{EF} = \frac{-3 - (-2)}{-2 - 5} = \frac{-1}{-7} = \frac{1}{7}$$

$$\therefore \text{Gradient} = \tan \theta = \frac{1}{7}$$

$$\theta = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\theta = 8.11^\circ$$

Q.4. If A(-2, 6) and B(7, -3), find the slope of line.

(i) parallel to AB answer

(ii) perpendicular to AB answer

Solution

$$\therefore \text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } \overline{AB} = \frac{-3 - 6}{7 - (-2)} = \frac{-9}{9} = -1$$

(i) Slope of line parallel to AB

Slope of line parallel to $\overline{AB}$ = Slope of $\overline{AB}$

Slope of line parallel to $\overline{AB}$ = -1

(ii) Slope of line perpendicular to AB

If m_1 and m_2 are slopes of perpendicular lines then $m_1 \times m_2 = -1$

Slope of line perpendicular to $\overline{AB}$ = 1

Q.5. Find x if the slope of line passing through A(3, x), B(5, 8) is 4. answer

Solution

A(3, x), B(5, 8) is 4

$$\text{Slope of line } \overline{AB} = 4$$

$$\therefore \text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$4 = \frac{8 - x}{5 - 3}$$

$$4 = \frac{8 - x}{2}$$

$$4 \times 2 = 8 - x$$

$$8 = 8 - x$$

$$= x - 8 - 8$$

$$= x - 16$$

Q.6. Find k if lines passing through A(3, 2), B(3, 5) and C(5, -1), D(8, 7) are parallel. answer

Solution

$$\therefore \text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } \overline{AB} = \frac{5 - 2}{3 - 3} = \frac{3}{0}$$

$$\text{Slope of } \overline{CD} = \frac{7 - (-1)}{8 - 5} = \frac{8}{3}$$

$\therefore$ Slopes are parallel so

$$\text{Slope of } \overline{AB} = \text{Slope of } \overline{CD}$$

$$\frac{3}{0} = \frac{8}{3}$$

$$3(3) = 8(0 - k)$$

$$9 = 24 - 8k$$

$$= 8k = 24 - 9$$

$$8k = 15$$

$$= k = \frac{15}{8}$$

Q.7. Find k if lines passing through P(-1, 2), Q(4, 7) and R(2, k), S(7, 10) are perpendicular. answer

Solution

P(-1, 2), Q(4, 7) and R(2, k), S(7, 10)

$$\text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } \overline{PQ} = \frac{7 - 2}{4 - (-1)} = \frac{5}{5} = 1$$

$$\text{Slope of } \overline{RS} = \frac{10 - k}{7 - 2} = \frac{10 - k}{5}$$

$$\therefore \text{Slope of } \overline{PQ} \times \text{Slope of } \overline{RS} = -1$$

$$(1) \left(\frac{10 - k}{5} \right) = -1$$

$$10 - k = -1 \times 5$$

$$10 - k = -5$$



$$10 - k = k$$

$$\Rightarrow k = 10$$

Q.8. Using slopes, prove that points X(0, -3), Y(4, 7) and Z(8, 12) are collinear. answer

Solution

X(0, -3), Y(4, 7) and Z(8, 12)

$$\text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } \overline{XY} = \frac{7 - (-3)}{4 - 0} = \frac{10}{4} = \frac{5}{2}$$

$$\text{Slope of } \overline{YZ} = \frac{12 - 7}{8 - 4} = \frac{5}{4}$$

$$\text{Slope of } \overline{XZ} = \frac{12 - (-3)}{8 - 0} = \frac{15}{8}$$

$$\therefore \text{Slope of } \overline{XY} = \text{Slope of } \overline{YZ} = \text{Slope of } \overline{XZ}$$

Hence points X(0, -3), Y(4, 7) and Z(8, 12) are collinear.

Q.9. Find value of y if points P(4, y), Q(5, 2) and R(6, 2y+1) are collinear. answer

Solution

P(4, y), Q(5, 2) and R(6, 2y+1)

$$\therefore \text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } \overline{PQ} = \frac{2 - y}{5 - 4} = \frac{2 - y}{1} = 2 - y$$

$$\text{Slope of } \overline{QR} = \frac{2y + 1 - 2}{6 - 5} = \frac{2y - 1}{1} = 2y - 1$$

$$\therefore \overline{PQ} \text{ and } \overline{QR} \text{ are collinear so}$$

$$\text{Slope of } \overline{PQ} = \text{Slope of } \overline{QR}$$

$$2 - y = 2y - 1$$

$$2 + 1 = 2y + y$$

$$3 = 3y$$

$$\Rightarrow y = \frac{3}{3} \Rightarrow y = 1$$

Q.10. Prove by using slopes that points A(3, -1), B(-5, -5) and C(1, 3) are vertices of a right angled triangle. answer



Solution

$A(3, -1)$, $B(-5, -3)$ and $C(1, 1)$

$$\therefore \text{Slope of line } = \frac{y_2 - y_1}{x_2 - x_1}$$

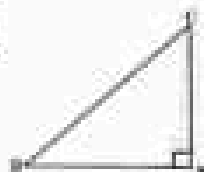
$$\text{Slope of } \overline{AB} = m_1 = \frac{-5 - 3}{-3 - 1} = \frac{-8}{-4} = 2$$

$$\text{Slope of } \overline{BC} = m_2 = \frac{1 - (-3)}{1 - (-5)} = \frac{4}{6} = \frac{2}{3}$$

$$\text{Slope of } \overline{AC} = m_3 = \frac{1 - (-1)}{1 - 3} = \frac{2}{-2} = -1$$

$$m_1 m_2 = \frac{2}{3} \times (-1) = -\frac{2}{3}$$

Hence $\overline{AB}$ and $\overline{AC}$ are perpendicular as given vertices are of right angled triangle with $\angle A = 90^\circ$



Q.11. Using slope, prove that $A(-2, 1)$, $B(6, 3)$, $C(10, 5)$ and $D(2, 3)$ are vertices of parallelogram.

Solution

$A(-2, 1)$, $B(6, 3)$, $C(10, 5)$ and $D(2, 3)$

$$\text{Slope of line } = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of line } \overline{AB} = \frac{3 - 1}{6 - (-2)} = \frac{2}{8} = \frac{1}{4}$$

$$\text{Slope of line } \overline{BC} = \frac{5 - 3}{10 - 6} = \frac{2}{4} = \frac{1}{2}$$

$$\text{Slope of line } \overline{CD} = \frac{3 - 5}{2 - 10} = \frac{-2}{-8} = \frac{1}{4}$$

$$\text{Slope of line } \overline{AD} = \frac{3 - 1}{2 - (-2)} = \frac{2}{4} = \frac{1}{2}$$

$$\text{Slope of } \overline{AB} = \text{Slope of } \overline{CD} \Rightarrow \overline{AB} \parallel \overline{CD}$$

$$\text{Slope of } \overline{BC} = \text{Slope of } \overline{AD} \Rightarrow \overline{BC} \parallel \overline{AD}$$



Thus given points are vertices of a parallelogram.

Q.12. $P(x, y)$, $Q(-2, 3)$, $R(1, 4)$ and $S(10, 1)$ are vertices of a parallelogram. Find $P(x, y)$.

Solution

$P(x, y)$, $Q(-2, 3)$, $R(1, 4)$ and $S(10, 1)$

$$\text{Slope of line } = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of line } \overline{PQ} = \frac{3 - y}{-2 - x} \quad \text{--- (i)}$$

$$\text{Slope of line } \overline{QR} = \frac{4 - 3}{1 - (-2)} = \frac{1}{3} \quad \text{--- (ii)}$$

$$\text{Slope of line } \overline{RS} = \frac{1 - 4}{10 - 1} = \frac{-3}{9} = -\frac{1}{3} \quad \text{--- (iii)}$$

$$\text{Slope of line } \overline{PS} = \frac{1 - y}{10 - x} \quad \text{--- (iv)}$$

$$\therefore \overline{PQ} \parallel \overline{RS}$$

$$\text{Slope of line } \overline{PQ} = \text{Slope of } \overline{RS} \text{ (from i and iii)}$$

$$\frac{3 - y}{-2 - x} = -\frac{1}{3}$$

$$3(3 - y) = -1(-2 - x)$$

$$9 - 3y = 2 + x$$

$$9 - 2 = x + 3y$$

$$\Rightarrow x + 3y = 7 \quad \text{--- (i)}$$

$$\therefore \overline{QR} \parallel \overline{PS}$$

$$\text{Slope of } \overline{QR} = \text{Slope of } \overline{PS} \text{ (from ii and iv)}$$

$$\frac{1}{3} = \frac{1 - y}{10 - x}$$

$$2(10 - x) = 3(1 - y)$$

$$20 - 2x = 3 - 3y$$

$$20 - 3 = -3y + 2x$$

$$\Rightarrow 2x - 3y = 17 \quad \text{--- (ii)}$$

Adding equation (i) and (ii)

$$x + 3y = 7$$

$$2x - 3y = 17$$

$$3x = 24$$

$$x = \frac{24}{3}$$

$$x = 8$$

By putting $x = 8$ in equation (i)

$$8 + 3y = 7$$

$$3y = 7 - 8$$

$$3y = -1$$

$$y = -\frac{1}{3}$$

$$y = -1$$

Hence, $P(x, y) = P(8, -1)$

Q.13. Three vertices of a rhombus are $A(2, -1)$, $B(3, 4)$ and $C(-2, 3)$. Find fourth vertex.

Solution

$A(2, -1)$, $B(3, 4)$ and $C(-2, 3)$

Let $D(x, y)$



Midpoint formula

$$M(x, y) = M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\text{Midpoint of } \overline{AC} = \left(\frac{2 + (-2)}{2}, \frac{-1 + 3}{2}\right)$$

$$= \left(\frac{2 - 2}{2}, \frac{2}{2}\right) = (0, 1)$$

$$\text{Midpoint of } \overline{BD} = \left(\frac{3 + x}{2}, \frac{4 + y}{2}\right)$$

Diagonal of rhombus bisect each other so

$$\Rightarrow \left(\frac{3 + x}{2}, \frac{4 + y}{2}\right) = (0, 1)$$

By comparing, we get

$$\Rightarrow \frac{3 + x}{2} = 0 \quad \left| \quad \frac{4 + y}{2} = 1 \right.$$

$$\Rightarrow 3 + x = 2 \times 0 \quad \left| \quad 4 + y = 2 \times 1 \right.$$

$$x + 3 = 0 \quad \left| \quad y = 2 - 4 \right.$$

$$x = -3 \quad \left| \quad y = -2 \right.$$

So, $D(x, y) = D(-3, -2)$ is required vertex.

Q.14. If $(3, 4)$, $(8, 5)$ and $(8, 8)$ are vertices of a triangle, find:

Solution

(i) slopes of sides

(ii) slopes of medians

(iii) slopes of altitudes

i) Solution

Slopes of sides

Let $A(3, 4)$, $B(8, 5)$, $C(8, 8)$

$$\therefore \text{Slope of line } = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } \overline{AB} = \frac{5 - 4}{8 - 3} = \frac{1}{5}$$

$$\therefore \text{Slope of } \overline{BC} = \frac{8 - 5}{8 - 8} = \frac{3}{0}$$

$$\text{Slope of } \overline{AC} = \frac{8 - 4}{8 - 3} = \frac{4}{5}$$

Slope of Medians

ii) Solution

First we find midpoints of sides

M_1 is the midpoint of

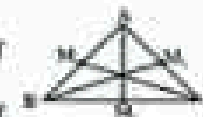
$\overline{AB}$

Midpoint formula

$$\therefore M(x, y) = M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M(x, y) = M_1\left(\frac{3 + 8}{2}, \frac{4 + 5}{2}\right)$$

$$= M_1\left(\frac{11}{2}, \frac{9}{2}\right)$$





$$= M_1(2.5, 2.5)$$

M_2 is the midpoint of $\overline{BC}$

$$M_2(x, y) = M_1\left(\frac{8+0}{2}, \frac{3+4}{2}\right)$$

$$= M_1\left(\frac{8-0}{2}, \frac{3+4}{2}\right)$$

$$= M_1(4, 3.5)$$

M_3 is the midpoint of $\overline{AC}$

$$M_3(x, y) = M_1\left(\frac{3+8}{2}, \frac{0+4}{2}\right)$$

$$= M_1\left(\frac{11}{2}, \frac{4}{2}\right)$$

$$= M_1(5.5, 2)$$

$$\text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of median } \overline{M_1C} = \frac{4 - 2.5}{8 - 2.5} = \frac{1.5}{5.5} = \frac{3}{11}$$

$$\text{Slope of median } \overline{M_1A} = \frac{0 - 6.5}{3 - 4} = \frac{-6.5}{-1} = 6.5$$

$$\text{Slope of median } \overline{M_1B} = \frac{3 - 4}{0 - 6.5} = \frac{-1}{-6.5} = \frac{2}{13}$$

(iii) Slope of altitudes

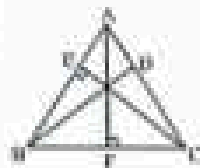
$$\text{Slopes of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

As we know

$$\text{Slope of side } \overline{AB} = -1$$

$$\text{Slope of side } \overline{BC} = \frac{1}{8}$$

$$\text{Slope of side } \overline{AC} = \frac{8}{1}$$



Altitudes are perpendicular from the vertices on the opposite sides.

Let $\overline{EC}$ be the altitude from C on $\overline{AB}$

$$\therefore \text{Slope of } \overline{AB} \times \text{Slope of } \overline{EC} = -1$$

$$-1 \times \text{Slope of } \overline{EC} = -1$$

$$\text{Slope of } \overline{EC} = \frac{-1}{-1}$$

$$\text{Slope of } \overline{EC} = -1$$

Let $\overline{AD}$ be altitude from A on $\overline{BC}$

$$\text{Slope of } \overline{BC} \times \text{Slope of } \overline{AD} = -1$$

$$\frac{1}{8} \times \text{Slope of } \overline{AD} = -1$$

$$\therefore \text{Slope of } \overline{AD} = \frac{-8}{1}$$

Let $\overline{BE}$ be altitude from B on $\overline{AC}$

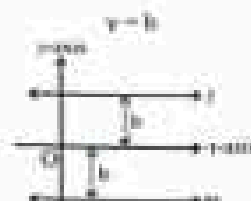
$$\therefore \text{Slope of } \overline{AC} \times \text{Slope of } \overline{BE} = -1$$

$$\frac{8}{1} \times \text{Slope of } \overline{BE} = -1$$

$$\therefore \text{Slope of } \overline{BE} = \frac{-1}{8}$$

Equation of Straight Line Parallel to x-axis
In the figure, all the points on the line parallel to x-axis remain at a constant distance b from x-axis.

Therefore, all points on the line satisfy the equation:

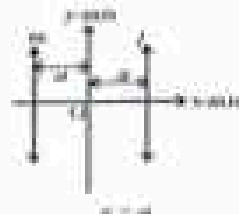


- If $b > 0$, then the line is above the x-axis (line l).
- If $b < 0$, then the line is below the x-axis (line m).
- If $b = 0$, then the line becomes x-axis and equation of x-axis becomes $y = 0$.

Equation of Straight Line Parallel to y-axis
In the figure, all the points on the line parallel to y-axis remain at a constant distance a from y-axis.



Therefore, all points on the line satisfy the equation:



- If $a > 0$, then the line is on the right of the y-axis (line l).
- If $a < 0$, then the line is on the left of the y-axis (line m).
- If $a = 0$, then the line becomes y-axis and equation of y-axis becomes $x = 0$.

Intercepts

If a line AB intersects x-axis at $(a, 0)$, then a is called x-intercept of the line AB .

If a line AB intersects y-axis at $(0, b)$, then b is called y-intercept of the line AB .



Derivations of Standard Forms of Equation of Straight Lines

1. Slope of Intercept Form

Theorem: Equation of straight line (non-vertical) with slope (gradient) m and y-intercept c is: $y = mx + c$.

Proof:

Let $P(x, y)$ be any point on the straight line l with slope m and y-intercept c .

y-intercept c means the line intersects



y-axis at point $A(0, c)$. As $P(x, y)$ and $A(0, c)$ lie on the line l , therefore slope of line is:

$$m = \frac{y - c}{x - 0}$$

$$\therefore y - c = mx$$

$$\therefore y = mx + c$$

Which is equation of line l in slope intercept form. If $c = 0$, then the line $y = mx$ passes through origin.

2. Point Slope Form

Theorem: Equation of non-vertical straight line passing through point $B(x_1, y_1)$ with slope m is:

$$y - y_1 = m(x - x_1)$$

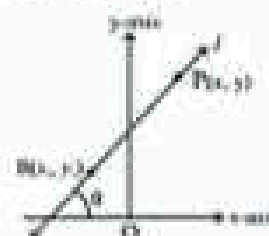
Proof

Let $P(x, y)$ be any point on the straight line l with slope m and passing through point $B(x_1, y_1)$.

As $P(x, y)$ and $B(x_1, y_1)$ lie on the line l , therefore slope m of the line is:

$$m = \frac{y - y_1}{x - x_1}$$

$$\therefore y - y_1 = m(x - x_1)$$



Which is equation of line l in point slope form.

3. Two-Point Form

Theorem: Equation of non-vertical straight line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$



Proof

Let $P(x, y)$ be any point on the straight line l passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$

As $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the line l , therefore slope m of the line is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



Now using point slope form of line, we have:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Which is equation of line l in two-point form.

4. Two-Intercept Form

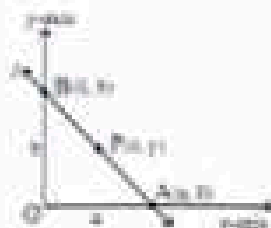
Theorem: Equation of straight line having x -intercept a and y -intercept b is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Proof

Let $P(x, y)$ be any point on the straight line l with x -intercept a and y -intercept b .

Obviously, $A(a, 0)$ and $B(0, b)$ lie on the line l . Therefore slope m of line is:



$$m = \frac{b - 0}{0 - a} = -\frac{b}{a}$$

Now using point slope form of line, we have:

$$y - b = \frac{b}{-a}(x - 0)$$

$$\Rightarrow -ay = ab + bx$$

$$\Rightarrow bx + ay = ab$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \quad (\text{dividing both sides by } ab)$$

Which is equation of line l in two intercept form.

5. Symmetric Form



If α is inclination of a line l , then slope of line is: $m = \tan \alpha$

$$\Rightarrow \tan \alpha = \frac{y - 0}{x - a}$$

$$\Rightarrow \frac{\sin \alpha}{\cos \alpha} = \frac{y - 0}{x - a}$$

$$\Rightarrow \frac{y - 0}{\cos \alpha} = \frac{y - 0}{\sin \alpha}$$

Which is symmetric form of equation of line.

6. Normal Form

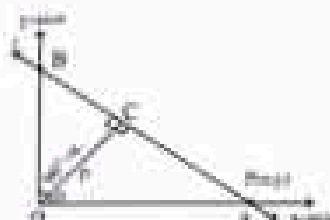
Theorem: If p is perpendicular from line l to the origin and α is the inclination of the perpendicular then:

$$x \cos \alpha + y \sin \alpha = p$$

Proof

Let the line l intersects x -axis and y -axis at points A and B respectively. Let $P(x, y)$ be any point on the line l and OC be perpendicular to line from origin, then $OC = p$.

From right angled triangle OCA :



$$\cos \alpha = \frac{OC}{OA} = \frac{p}{OA}$$

$$\Rightarrow OA = \frac{p}{\cos \alpha}$$

From right angled triangle OCP :

$$\cos(90^\circ - \alpha) = \frac{OC}{OP} = \frac{p}{OP}$$

$$\sin \alpha = \frac{p}{OP} \text{ or } OP = \frac{p}{\sin \alpha}$$

As OA and OB are x -intercept and y -intercept of AB , therefore using two-intercept form, we have:

$$\frac{x}{OA} + \frac{y}{OB} = 1$$

$$\Rightarrow \frac{x}{\frac{p}{\cos \alpha}} + \frac{y}{\frac{p}{\sin \alpha}} = 1$$

$$\frac{x \cos \alpha}{p} + \frac{y \sin \alpha}{p} = 1$$

$$\Rightarrow x \cos \alpha + y \sin \alpha = p$$

Which is normal form of equation of line l .

Example 4(a)

Find the equation of straight line with the following information:

(i) slope = 5, y -intercept = 3 Answer

(ii) through $(5, 0)$ with slope -3 Answer

(iii) through $(2, 4)$ and $(-1, -4)$ Answer

(iv) x -intercept = 6, y -intercept = -4

Solution

(i) Slope = $m = 5$, y -intercept = $c = 3$

Equation of line in slope intercept is:

$$y = mx + c$$

$$\Rightarrow y = 5x + 3$$

or $5x - y + 3 = 0$

(ii) $(x_1, y_1) = (5, 0)$, $m = -3$

Equation of line in point slope form is:

$$y - y_1 = m(x - x_1)$$

Substituting for (x_1, y_1) and m , we have:

$$y - 0 = -3(x - 5)$$

$$\Rightarrow y = -3x + 15 \text{ or } 3x + y - 15 = 0$$

(iii) $(x_1, y_1) = (2, 4)$, $(x_2, y_2) = (-1, -4)$

Equation of line in two-point form is:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Substituting the values, we have:

$$\frac{y - 4}{-4 - 4} = \frac{x - 2}{-1 - 2} \Rightarrow \frac{y - 4}{-8} = \frac{x - 2}{-3}$$

$$-3(y - 4) = -8(x - 2)$$

$$\Rightarrow -3y + 12 = -8x + 16$$

$$\Rightarrow 8x - 3y - 4 = 0$$

(iv) x -intercept = $a = 6$, y -intercept = $b = -4$

$$\frac{x}{a} + \frac{y}{b} = 1$$

Substituting the values, we have:

$$\frac{x}{6} + \frac{y}{-4} = 1$$

$$\Rightarrow -3x + 6y = -24$$

$$-4x + 6y + 24 = 0$$

$$-2(2x - 3y - 12) = 0$$

$$\text{Or } 2x - 3y - 12 = 0$$

Example 5

Find the equation of straight line:

(i) through $(3, -2)$ and perpendicular to the line with slope 3. Answer

(ii) having y -intercept = -7 and parallel to the line with slope 4. Answer

(iii) through $(-5, 1)$ and perpendicular to the line passing through $(0, 9)$ and $(-2, 6)$. Answer

(iv) If length of perpendicular from the origin is 6 units inclined at 60° . Answer

Solution

(i) $(x_1, y_1) = (3, -2)$



Slope of given line = 3

As, the required line is perpendicular to the given line, therefore, slope of required line is:

$$m = -\frac{1}{3}$$

Equation of required line is: $y - y_1 = m(x - x_1)$

$$y - 2 = -\frac{1}{3}(x - 2) \quad (\text{point-slope form})$$

$$\Rightarrow 3(y - 2) = -(x - 2)$$

$$\Rightarrow 3y + 6 + x - 2 = 0$$

$$\Rightarrow x + 3y + 3 = 0$$

(ii) y -intercept = -7

Slope of given line = 4

As, the required line is parallel to the given line, therefore, slope of required line is:

$$m = 4$$

Equation of required line is: $y - y_1 = m(x - x_1)$

$$y = 4x - 7 \quad (\text{slope intercept form})$$

$$\Rightarrow 4x - y - 7 = 0$$

(iii) $(x_1, y_1) = (-5, 1)$ and perpendicular the line passing through $(0, 9)$ and $(-2, 6)$.

$$\text{Slope of given line} = \frac{6-9}{-2-0} = \frac{-3}{-2} = \frac{3}{2}$$

As, the required line is perpendicular to the given line, therefore, slope of required line is:

$$m = -\frac{2}{3}$$

$\therefore$ Equation of required line is:

$$y - 1 = -\frac{2}{3}(x - 1) \quad (\text{point-slope form})$$

$$\Rightarrow 3y - 3 = -2x - 10$$

$$\Rightarrow 2x + 3y + 7 = 0$$

(iv) $p = 0$ and $\theta = 60^\circ$

Equation of line in normal form is:

$$x \cos 60^\circ + y \sin 60^\circ = p$$

$$\Rightarrow x \cdot \frac{1}{2} + y \cdot \frac{\sqrt{3}}{2} = 6$$

$$\Rightarrow x + \sqrt{3}y = 12$$

$$\text{Or } x + \sqrt{3}y - 12 = 0$$

General Form of Equation of Straight Line

General form of a linear equation in two variable x and y is:

$$ax + by + c = 0$$

Where a , b and c are real numbers and, a and b cannot be both zero.

A linear equation $ax + by + c = 0$, in two variables x and y represents a straight line. Here a , b , c are constants and a and b are not simultaneously zero.

Proof

$$ax + by + c = 0 \quad \dots\dots\dots (1)$$

Case 1: When $a \neq 0$ and $b \neq 0$ then equation (1) takes the form,

$$by + c = 0 \text{ or } y = -\frac{c}{b}$$

which represents equation of line parallel to x -axis.

Case 2: When $a \neq 0$ and $b = 0$ then equation (1) takes the form,

$$ax + c = 0 \text{ or } x = -\frac{c}{a}$$

which represent equation of line parallel to y -axis.

Case 3: When $a \neq 0$ and $b \neq 0$ then equation (1) takes the form,

$$by = -ax - c \quad \text{or}$$

$$y = -\frac{a}{b}x - \frac{c}{b} = \left(-\frac{a}{b}x\right) + \left(-\frac{c}{b}\right) \text{ which}$$

represents slope-intercept form of equation of line with:

$$\text{Slope} = -\frac{a}{b} \text{ and } y\text{-intercept} = -\frac{c}{b}$$

Reduction of General Form of Equation of a Straight Line to Other Standard Form

We know that general form of equation of straight line in two variable is:

$$ax + by + c = 0 \quad \dots\dots\dots (1)$$

We can reduce equation (1) into various standard forms as follows:

1. **Slope-Intercept Form**

From equation (1), we can write:

$$by = -ax - c \text{ or}$$

$$y = -\frac{a}{b}x - \frac{c}{b} = \left(-\frac{a}{b}x\right) + \left(-\frac{c}{b}\right)$$

which represents slope-intercept form of equation of line with:

$$\text{slope} = -\frac{a}{b} \text{ and } y\text{-intercept} = -\frac{c}{b}$$

2. **Point-Slope Form**

$$\text{Slope of equation (1) is } m = -\frac{a}{b}$$

$$\text{A point on equation (1) is } (x_1, y_1) = \left(0, -\frac{c}{b}\right)$$

$\therefore$ Equation of line in point-slope form becomes:

$$y - \frac{-c}{b} = -\frac{a}{b}\left(x - \left(-\frac{c}{b}\right)\right) \text{ or } y - \frac{-c}{b} = -\frac{a}{b}\left(x + \frac{c}{b}\right)$$

3. **Two-Point Form**

Two points on equation (1) can be taken as

$$(x_1, y_1) = \left(0, -\frac{c}{b}\right) \text{ and } (x_2, y_2) = \left(-\frac{c}{a}, 0\right)$$

$\therefore$ Equation of line in two-point form is:

$$\frac{y - \frac{-c}{b}}{-\frac{c}{b} - 0} = \frac{x - 0}{-\frac{c}{a} - 0} \text{ or } y - \frac{-c}{b} = -\frac{b}{a}\left(x - 0\right)$$

4. **Two-Intercept Form**

From equation (1)

$$ax + by = -c \text{ or } \frac{ax}{-c} + \frac{by}{-c} = 1 \text{ or}$$

$$\frac{x}{-c/a} + \frac{y}{-c/b} = 1$$

Which is equation in two-intercept form with:

$$x\text{-intercept} = -\frac{c}{a} \text{ and } y\text{-intercept} = -\frac{c}{b}$$

5. **Symmetric Form**



Slope of equation (1) is $m = \tan \theta = -\frac{a}{b}$

$$\sin \theta = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \cos \theta = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\text{A point on equation (1) is } (x_1, y_1) = \left(0, -\frac{c}{b}\right)$$



$\therefore$ Equation of line in symmetric form is:

$$\frac{x - 0}{\cos \theta} = \frac{y - \frac{-c}{b}}{\sin \theta} \text{ or } \frac{x - 0}{\frac{b}{\sqrt{a^2 + b^2}}} = \frac{y + \frac{c}{b}}{\frac{a}{\sqrt{a^2 + b^2}}}$$

6. **Normal Form**

Equation in normal form is:

$$x \cos \theta + y \sin \theta = p \quad \dots\dots\dots (2)$$

From equation (1), we have:

$$ax + by = -c \quad \dots\dots\dots (3)$$

Equations (2) and (3) are identical, therefore:

$$\begin{aligned} \frac{a}{\cos \theta} &= \frac{b}{\sin \theta} = \frac{-c}{p} \\ \Rightarrow \frac{p}{-c} &= \frac{\cos \theta}{a} = \frac{\sin \theta}{b} \\ &= \frac{\sqrt{\cos^2 \theta + \sin^2 \theta}}{a\sqrt{a^2 + b^2}} = \frac{1}{a\sqrt{a^2 + b^2}} \\ \Rightarrow p &= \frac{-c}{a\sqrt{a^2 + b^2}} \end{aligned}$$

Now, dividing both sides by $a\sqrt{a^2 + b^2}$, we get





$$\frac{ax}{x\sqrt{a^2+b^2}} + \frac{by}{y\sqrt{a^2+b^2}} = \frac{-c}{x\sqrt{a^2+b^2}}$$

$$x\left(\frac{a}{x\sqrt{a^2+b^2}}\right) + y\left(\frac{b}{y\sqrt{a^2+b^2}}\right) = \frac{-c}{x\sqrt{a^2+b^2}}$$

Example 6

- Reduce the equation $6x - 5y + 15 = 0$ into
- (i) slope-intercept form (answer)
 - (ii) two-intercept form (answer)
 - (iii) point-slope form (answer)
 - (iv) two-point form (answer)
 - (v) normal form (answer)
 - (vi) symmetric form (answer)

Solution

(i) $6x - 5y + 15 = 0$
 $\Rightarrow 5y = 6x + 15$

$\Rightarrow y = \frac{6}{5}x + 3$

$y = mx + c$

Where $m = \frac{6}{5}$ and $c = 3$

(ii) $6x - 5y + 15 = 0$
 $\Rightarrow 6x - 5y = -15$
 $\Rightarrow \left(\frac{6x}{-15}\right) - \left(\frac{5y}{-15}\right) = 1$

$\Rightarrow \frac{6}{-15}x + \frac{5}{15}y = 1$

$\Rightarrow \frac{x}{-2.5} + \frac{y}{3} = 1$

where x -intercept $= -\frac{5}{2}$ and y -intercept $= 3$

(iii) A point on the line is $(x_1, y_1) = (0, 3)$.

Also slope of line is $m = -\left(\frac{6}{-5}\right) = \frac{6}{5}$

Equation of line can be written as

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{6}{5}(x - 0)$$

(iv) Two points on the line are $(x_1, y_1) = (3, 2)$

and $(x_2, y_2) = \left(\frac{-5}{2}, 0\right)$

Equation of line passing through both points is:

$$\frac{y-2}{2-0} = \frac{x-3}{-\frac{5}{2}-3}$$

(v) Given equation can be written as

$6x - 5y = -15$ or
 $-6x - 5y = 15$ (Make right side positive)

$$\sqrt{\frac{6}{-6}} \cdot \frac{6x}{-6} + \sqrt{\frac{5}{-5}} \cdot \frac{5y}{-5} = \sqrt{\frac{15}{15}}$$

Dividing both sides by $\sqrt{61}$

$$\frac{-6}{\sqrt{61}}x + \frac{5}{\sqrt{61}}y = \frac{15}{\sqrt{61}}$$

$$x\left(\frac{-6}{\sqrt{61}}\right) + y\left(\frac{5}{\sqrt{61}}\right) = \frac{15}{\sqrt{61}}$$



Which is normal from equation.

(vi) Given equation is $6x - 5y + 15 = 0$

$m = \tan\theta = \frac{6}{5}$ which implies $\sin\theta = \frac{6}{\sqrt{61}}$

$$\sin\theta = \frac{6}{\sqrt{61}}$$

A point on the line is $(x_1, y_1) = (0, 3)$.

Thus equation in symmetric form is:

$$\frac{x-0}{\frac{6}{\sqrt{61}}} = \frac{y-3}{\frac{5}{\sqrt{61}}}$$



Exercise 8.2

Q.1 Find the equation of horizontal line passing through:

(i) (2, 3) (answer) (ii) (8, 0) (answer)

(iii) (-5, -9) (answer) (iv) $\left(\frac{-3}{2}, \frac{6}{2}\right)$ (answer)

(i) **Solution**

Equation of horizontal line

$$y = y\text{-coordinate}$$

$$\Rightarrow y = 3$$

(ii) **Solution**

$$(8, 0)$$

Equation of horizontal line

$$y = y\text{-coordinate}$$

$$y = 0$$

(iii) **Solution**

$$(-5, -9)$$

Equation of horizontal line

$$y = y\text{-coordinate}$$

$$y = -9$$

(iv) **Solution**

$$\left(\frac{-3}{2}, \frac{6}{2}\right)$$

Equation of horizontal line

$$y = y\text{-coordinate}$$

$$y = \frac{3}{2}$$

Q.2 Find the equation of vertical line

passing through:

(i) (1, 5) (answer) (ii) (9, 4) (answer)

(iii) (-4, -7) (answer) (iv) $\left(\frac{1}{4}, \frac{3}{4}\right)$ (answer)

(i) **Solution**

$$(1, 5)$$

Equation of vertical line

$$x = x\text{-coordinate}$$

$$x = 1$$

(ii) **Solution**

$$(9, 4)$$

Equation of vertical line

$$x = x\text{-coordinate}$$

$$x = 9$$

(iii) **Solution**

$$(-4, -7)$$

Equation of vertical line

$$x = x\text{-coordinate}$$

$$x = -4$$

(iv) **Solution**

$$\left(\frac{1}{4}, \frac{3}{4}\right)$$

Equation of vertical line

$$x = x\text{-coordinate}$$

$$x = \frac{1}{4}$$

Q.3 Find the equation of the line with the following information:

(i) slope $= 2$, y -intercept $= -3$ (answer)

(ii) through $(-5, 7)$ with slope 4 (answer)

(iii) through $(4, -5)$ with slope 0 (answer)

(iv) through $(-2, 9)$ with slope undefined (answer)

(v) through $(-6, 1)$ and $(2, -4)$ (answer)

(vi) through $(2, -4)$ and $(8, 4)$ (answer)

(vii) x -intercept $= -4$, y -intercept $= 2$ (answer)

(viii) slope $= -1$, x -intercept $= 11$ (answer)

(i) **Solution**

Slope $= 2$, y -intercept $= -3$

Slope $= m = 2$, y -intercept $= c = -3$

Equation of line, $y = mx + c$

$$y = (2)x + (-3)$$

$$y = 2x - 3$$

$$\Rightarrow 2x - y - 3 = 0$$

(ii) **Solution**

Equation of line through $(-5, 7)$ with slope 4

$\therefore x_1 = -5, y_1 = 7, m = 4$

Equation of line in point slope form

$$y - y_1 = m(x - x_1)$$



$$\begin{aligned} y - 7 &= 4(x - (-5)) \\ y - 7 &= 4(x + 5) \\ y - 7 &= 4x + 20 \\ \Rightarrow 4x + 20 - y + 7 &= 0 \\ 4x - y + 27 &= 0 \end{aligned}$$

(iii) Solution

Equation of line through (4, -5) with slope 0
 $x_1 = 4, y_1 = -5, m = 0$

Equation of line in point slope form:
 $y - y_1 = m(x - x_1)$
 $y - (-5) = 0(x - 4)$
 $y + 5 = 0$

(iv) Solution

Equation of line through (-2, 9) with slope undefined

$$\begin{aligned} x_1 &= -2, y_1 = 9, m = \infty \\ \text{Equation of line in point slope form:} \\ y - y_1 &= m(x - x_1) \\ y - 9 &= \infty(x - (-2)) \\ y - 9 &= \frac{1}{0}(x + 2) \therefore \omega = \frac{1}{0} \\ 0(y - 9) &= 1(x + 2) \\ 0 &= x + 2 \\ \Rightarrow x + 2 &= 0 \end{aligned}$$

(v) Solution

Equation of line through (-6, 1) and (2, -4)
 $x_1 = -6, y_1 = 1, x_2 = 2, y_2 = -4$
 Equation of line in two point form

$$\begin{aligned} \frac{y - y_1}{y_2 - y_1} &= \frac{x - x_1}{x_2 - x_1} \\ \frac{y - 1}{-4 - 1} &= \frac{x - (-6)}{2 - (-6)} \\ \frac{y - 1}{-5} &= \frac{x + 6}{8} \\ \frac{y - 1}{-5} &= \frac{x + 6}{8} \\ 8(y - 1) &= -5(x + 6) \\ 8y - 8 &= -5x - 30 \\ 8y - 8 + 5x + 30 &= 0 \\ 5x + 8y + 22 &= 0 \end{aligned}$$

(vi) Solution

Equation of line through (2, -4) and (8, 4)
 $x_1 = 2, y_1 = -4, x_2 = 8, y_2 = 4$

Equation of line in two point form

$$\begin{aligned} \frac{y - y_1}{y_2 - y_1} &= \frac{x - x_1}{x_2 - x_1} \\ \frac{y - (-4)}{4 - (-4)} &= \frac{x - 2}{8 - 2} \\ \frac{y + 4}{4 + 4} &= \frac{x - 2}{8 - 2} \\ \frac{y + 4}{8} &= \frac{x - 2}{6} \end{aligned}$$

$$6(y + 4) = 8(x - 2)$$

$$6y + 24 = 8x - 16$$

$$\Rightarrow 8x - 16 - 6y - 24 = 0$$

$$8x - 6y - 40 = 0$$

$$2(4x - 3y - 20) = 0$$

$$\Rightarrow 4x - 3y - 20 = 0 \quad (\because 2 \neq 0)$$

(vii) Solution

x - intercept = -4

y - intercept = 3

Equation of line in two - intercept form:

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\therefore a = -4$$

$$b = 3$$

$$\frac{x}{-4} + \frac{y}{3} = 1$$

$$\Rightarrow \frac{-x}{4} + \frac{y}{3} = 1$$

$$\frac{-3x + 4y}{20} = 1$$

$$-3x + 4y = 20$$

$$-3x + 4y - 20 = 0$$

$$\Rightarrow -1(3x - 4y + 20) = 0$$

$$\Rightarrow 3x - 4y + 20 = 0 \quad (\because -1 \neq 0)$$



(viii) Solution

Slope = m = -1

x-intercept = 11, y = 0

$$\Rightarrow (x_1, y_1) = (11, 0)$$

Equation of line in point slope form is

$$y - y_1 = m(x - x_1)$$

Putting the values we get

$$y - 0 = -1(x - 11)$$

$$y = -x + 11$$

$$x + y - 11 = 0$$

Q.4 Find the equation of line in symmetric form when:

(i) $(x_1, y_1) = (-4, 2)$ and $\tan \theta = \frac{3}{4}$ answer

(ii) $(x_1, y_1) = (6, -6)$ and $\tan \theta = 30^\circ$ answer

(i) Solution

$$(x_1, y_1) = (-4, 2) \text{ and } \tan \theta = \frac{3}{4}$$

$$x_1 = -4, y_1 = 2 \text{ and } \tan \theta = \frac{3}{4}$$

$$\Rightarrow a = 3, b = 4$$

Equation of line in symmetric form:

$$\begin{aligned} \frac{x - x_1}{\cos \theta} &= \frac{y - y_1}{\sin \theta} \\ \Rightarrow \frac{x - x_1}{\frac{x}{a} + \frac{y}{b}} &= \frac{y - y_1}{\frac{y}{a} + \frac{x}{b}} \\ \Rightarrow \frac{x - (-4)}{4} &= \frac{y - 2}{1} \\ \frac{x + 4}{\sqrt{3^2 + 4^2}} &= \frac{y - 2}{\sqrt{3^2 + 4^2}} \\ \Rightarrow \frac{x + 4}{5} &= \frac{y - 2}{5} \\ \frac{x + 4}{5} &= \frac{y - 2}{5} \end{aligned}$$

(ii) Solution

$(x_1, y_1) = (6, -6)$ and $\tan \theta = 30^\circ$

$$x_1 = 6, y_1 = -6$$

Equation of line in symmetric form:

$$\begin{aligned} \frac{x - x_1}{\cos \theta} &= \frac{y - y_1}{\sin \theta} \\ \frac{x - 6}{\cos 30^\circ} &= \frac{y - (-6)}{\sin 30^\circ} \\ \frac{x - 6}{\cos 30^\circ} &= \frac{y + 6}{\sin 30^\circ} \end{aligned}$$

Q.5 Find the equation of line in normal form when:

(i) $p = 5$ and $\theta = 120^\circ$ answer

(ii) $p = 10$ and $\tan \theta = 1$ answer

(i) Solution

$$p = 5 \text{ and } \theta = 120^\circ$$

Equation in normal form:

$$x \cos \theta + y \sin \theta = p$$

Putting the values, we get

$$x \cos (120^\circ) + y \sin (120^\circ) = 5$$

$$x \left(\frac{-1}{2} \right) + y \left(\frac{\sqrt{3}}{2} \right) = 5$$

$$-x + y(\sqrt{3}) = 10$$

$$-x + y\sqrt{3} = 10$$

$$-x + \sqrt{3}y - 10 = 0$$

$$-1(x - \sqrt{3}y + 10) = 0$$

$$\Rightarrow x - \sqrt{3}y + 10 = 0 \quad (\because -1 \neq 0)$$

(ii) Solution

$$p = 10, \tan \theta = 1$$

$$\tan \theta = 1 \Rightarrow \theta = \tan^{-1}(1) = 45^\circ$$

Equation in normal form

$$x \cos \theta + y \sin \theta = p$$

Putting the values, we get

$$x \cos (45^\circ) + y \sin (45^\circ) = 10$$

$$x \left(\frac{1}{\sqrt{2}} \right) + y \left(\frac{1}{\sqrt{2}} \right) = 10$$



Multiplying both side by $\sqrt{2}$

$$\Rightarrow x + y = 10\sqrt{2}$$

$$x + y - 10\sqrt{2} = 0$$

Q.6 Find the equation of straight line:

(i) through $(-4, -4)$ and parallel to the line with slope -5 . answer

(ii) through $(5, -1)$ and perpendicular to the line with slope $\frac{1}{4}$. answer

(iii) having y-intercept $= 4$ and parallel to the line with slope $\frac{1}{2}$. answer

(iv) having x-intercept $= -2$ and perpendicular to the line with slope 4 . answer

(v) through $(-1, 4)$ and perpendicular to the line passing through $(3, 8)$ and $(1, -2)$. answer

(vi) through $(6, -4)$ and parallel to the line passing through $(-3, 2)$ and $(3, 6)$.

(i) Solution

Equation of line through $(-4, -4)$ and parallel to the line with slope -5 .

$$x_1 = -4, y_1 = -4, m = -5$$

Equation of straight line:

$$y - y_1 = m(x - x_1)$$

$$y - (-4) = -5[x - (-4)]$$

$$y + 4 = -5(x + 4)$$

$$y + 4 = -5x - 20$$

$$y + 4 + 5x + 20 = 0$$

$$5x + y + 24 = 0$$

(ii) Solution

Equation of line through $(5, -1)$ and

perpendicular to the line with slope $\frac{1}{4}$.

$$x_1 = 5, y_1 = -1$$

Slope of line perpendicular to line with slope

$$\frac{1}{4} \Rightarrow m = -4$$

$$\text{Equation of line: } y - y_1 = m(x - x_1)$$

Putting the values

$$y - (-1) = -4[x - (5)]$$

$$y + 1 = -4(x - 5)$$

$$y + 1 = -4x + 20$$

$$y + 4x + 1 - 20 = 0$$

$$4x + y - 19 = 0$$

(iii) Solution

y-intercept $= 4$

$$\text{slope} = m = \frac{1}{2}$$

Equation of line:

$$y = mx + c$$

$$y = \frac{1}{2}x + 4$$

Multiplying both side by 2

$$2y = x + 8$$

$$\Rightarrow x - 2y + 8 = 0$$

(iv) Solution

If x-intercept $= -2$, $y = 0$

$$\Rightarrow (x_1, y_1) = (-2, 0)$$

Perpendicular to the line, so slope $m = -\frac{1}{4}$

Equation of line in point slope form is

$$y - y_1 = m(x - x_1)$$

Putting the value we get

$$y - 0 = -\frac{1}{4}(x + 2)$$

$$4y = -x - 2$$

$$x + 4y + 2 = 0$$

(v) Solution

Equation of line through $(-1, 4)$ and perpendicular to the line passing through $(3, 8)$ and $(1, -2)$

$$x_1 = 3, y_1 = 8 \text{ and } x_2 = 1, y_2 = -2$$

$$\text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-2 - 8}{1 - 3} = \frac{-10}{-2} = 5$$

The line is perpendicular to given line so its slope is $m = -\frac{1}{5}$

$$(-1, 4) \Rightarrow x_1 = -1, y_1 = 4$$

Equation of line:

$$y - y_1 = m(x - x_1)$$

Putting the values we get

$$y - 4 = -\frac{1}{5}(x - (-1))$$

$$y - 4 = -\frac{1}{5}(x + 1)$$

$$y - 4 = -\frac{x}{5} - \frac{1}{5}$$

$$y + \frac{x}{5} - 4 + \frac{1}{5} = 0$$

$$x + y - 2 = 0$$

(vi) Solution

Equation of line through $(6, -4)$ and parallel to line passing through $(-3, 2)$ and $(3, 6)$

$$\text{Let } x_1 = -3, y_1 = 2 \text{ and } x_2 = 3, y_2 = 6$$

$$\text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 2}{3 - (-3)} = \frac{4}{6} = \frac{2}{3}$$

Line is parallel to the given line so its slope is

$$m = \frac{2}{3}$$

Equation of line:

$$y - y_1 = m(x - x_1)$$

$$y - (-4) = \frac{2}{3}(x - 6)$$

$$y + 4 = \frac{2}{3}(x - 6)$$

$$3(y + 4) = 2(x - 6)$$

$$3y + 12 = 2x - 12$$

$$\Rightarrow 2x - 3y - 24 = 0$$

$$2x - 3y - 24 = 0$$

Q.7 Find equation of the line through Point $(3, 7)$ parallel to the line $4x - 3y + 1 = 0$. answer

Solution

First we find slope of given line

$$4x - 3y + 1 = 0$$

$$4x + 1 = 3y$$

$$\frac{4x + 1}{3} = y$$

$$\Rightarrow y = \frac{4}{3}x + \frac{1}{3}$$

$$y = mx + c$$

$$\text{Slope of given line} = m_1 = \frac{4}{3}$$

$$\text{Slope of required } \perp \text{ line} = m_2 = -\frac{3}{4}$$

The point slope form of the equation of line is

$$y - y_1 = m_2(x - x_1)$$

Since line passes through point $(x_1, y_1) = (3, 7)$.

$$y - 7 = -\frac{3}{4}(x - 3)$$

$$3(y - 7) = -4(x - 3)$$

$$3y - 21 = -4x + 12$$

$$\Rightarrow 0 = 4x - 12 - 3y + 21$$

$$\Rightarrow 4x - 3y + 9 = 0$$

Q.8 Find equation of the line through $(-2, -1)$ and perpendicular to the line

$$x - 2y = 0$$

Solution

First we find slope of given line

$$x - 2y = 0$$

$$x = 2y$$

$$\Rightarrow y = \frac{1}{2}x + 0$$

$$y = mx + c$$

$$\text{Slope of given line} = m_1 = \frac{1}{2}$$

$$\text{Slope of required } \perp \text{ line} = m_2 = -2$$

The point slope form of the equation of line is

$$y - y_1 = m_2(x - x_1)$$

Since line passes through point $(x_1, y_1) = (-2, -1)$

$$y - (-1) = -2[x - (-2)]$$

$$y + 1 = -2(x + 2)$$



$$y+1 = -2x+4$$

$$y+1-2x+4=0$$

$$2x+y-5=0$$

Q.9 Find equation of perpendicular bisector of line segment joining $(0,6)$ and $(2,-2)$.

Solution

As perpendicular bisector passes through the mid-point of line segment.

Then we find the midpoint of given points

$$(0,6), (2,-2)$$

$$\therefore M(x, y) = M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\text{Midpoint} = \left(\frac{0+2}{2}, \frac{6+(-2)}{2}\right) = \left(\frac{2}{2}, \frac{4}{2}\right) = (1, 2)$$

Slope of line segment joining given points

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-6}{2-0} = \frac{-8}{2} = -4$$

Slope of the line perpendicular to given line segment is $m_2 = m_1 \times -1$

$$m_2 = \frac{-1}{m_1} = \frac{-1}{-4} = \frac{1}{4}$$

The point slope form of the required equation $\perp$ to line given line segment is

$$y - y_1 = m_2(x - x_1)$$

Since line passes through the mid-point $(x_1, y_1) = (1, 2)$

$$y - 2 = \frac{1}{4}(x - 1)$$

$$4(y - 2) = (x - 1)$$

$$4y - 8 = x - 1$$

$$\Rightarrow 0 = x - 4y + 7 = 0$$

$$\Rightarrow x - 4y + 7 = 0$$

Q.10 Find equation of medians and altitudes of triangle with vertices

$$A(0, 4), B(4, 6) \text{ and } C(-2, -2)$$

Solution:

(i) Equation of Medians

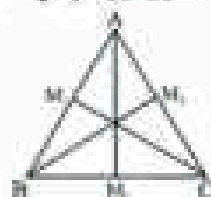
(a) Equation of median $\overline{AM_1}$

Given vertices $A(0, 4)$, $B(4, 6)$ and $C(-2, -2)$

First we find midpoint M_1 of side BC using

$$\therefore M(x, y) = M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$M_1\left(\frac{4+(-2)}{2}, \frac{6+(-2)}{2}\right) = M_1\left(\frac{2}{2}, \frac{4}{2}\right) = (1, 2)$$



Slope of median $\overline{AM_1}$:

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-4}{1-0} = \frac{-2}{1} = -2$$

$$m_1 = -2$$

The point slope form of the required median is

$$y - y_1 = m_1(x - x_1)$$

Since midpoint $(1, 2)$ lies on the median so

$$\text{put } x_1 = 1 \text{ and } y_1 = 2$$

$$y - 2 = -2(x - 1)$$

$$y - 2 = -2x + 2$$

$$2x + y - 2 - 2 = 0$$

$$2x + y - 4 = 0$$

(b) Equation of median $\overline{BM_2}$

First we find midpoint M_2 of side AC

$$M_2\left(\frac{0+(-2)}{2}, \frac{4+(-2)}{2}\right) = M_2\left(\frac{-2}{2}, \frac{2}{2}\right) = M_2(-1, 1)$$

Slope of median $\overline{BM_2}$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-6}{-1-4} = \frac{-5}{-5} = 1$$

$$m_2 = 1$$

The point slope form of the required median is

$$y - y_1 = m_2(x - x_1)$$

Since midpoint $(-1, 1)$ lies on the median so

$$\text{put } x_1 = -1 \text{ and } y_1 = 1$$

$$y - 1 = 1(x - (-1))$$

$$y - 1 = x + 1$$

$$\Rightarrow 0 = x - y + 1 + 1$$

$$\Rightarrow x - y + 2 = 0$$

(c) Equation of median $\overline{CM_1}$

First we find midpoint M_1 of side AB

$$M_1\left(\frac{0+4}{2}, \frac{4+6}{2}\right) = M_1\left(\frac{4}{2}, \frac{10}{2}\right) = M_1(2, 5)$$

Slope of median $\overline{CM_1}$

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-2)}{2-(-2)} = \frac{7}{4}$$

The point slope form of the required median is

$$y - y_1 = m_1(x - x_1)$$

Since midpoint $(2, 5)$ lies on the median so

$$\text{put } x_1 = 2 \text{ and } y_1 = 5$$

$$y - 5 = \frac{7}{4}(x - 2)$$

$$4(y - 5) = 7(x - 2)$$

$$4y - 20 = 7x - 14$$

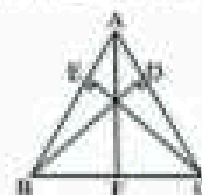
$$0 = 7x - 14 - 4y + 20$$

$$\Rightarrow 7x - 4y + 6 = 0$$

(ii) Equation of Altitudes

Given vertices $A(0, 4)$, $B(4, 6)$ and $C(-2, -2)$

$$\text{Slope of line } = \frac{y_2 - y_1}{x_2 - x_1}$$



First we find the slope of sides of $\triangle ABC$

$$\text{Slope of side } \overline{AB} = \frac{6-4}{4-0} = \frac{2}{4} = \frac{1}{2}$$

$$\text{Slope of side } \overline{BC} = m_1 = \frac{-2-6}{-2-4} = \frac{-8}{-6} = \frac{4}{3}$$

$$\text{Slope of side } \overline{AC} = m_1 = \frac{-2-4}{-2-0} = \frac{-6}{-2} = 3$$

(a) Equation of altitude $\overline{EC}$

$$\text{Slope of altitude } \overline{EC} = \frac{-1}{\text{Slope of } \overline{AB}}$$

$$m_2 = \frac{-1}{\left(\frac{1}{2}\right)} = -2$$

The point slope form of the required median is

$$y - y_1 = m_2(x - x_1)$$

Since vertex $C(-2, -2)$ lies on the line of altitude so put $x_1 = -2$ and $y_1 = -2$

$$y - (-2) = -2[x - (-2)]$$

$$y + 2 = -2(x + 2)$$

$$y + 2 = -2x - 4$$

$$2x + 4 + y + 2 = 0$$

$$2x + y + 6 = 0$$

(b) Equation of altitude $\overline{BD}$

$$\text{Slope of altitude } \overline{BD} = \frac{-1}{\text{Slope of } \overline{AC}} = \frac{-1}{3}$$

$$m_2 = \frac{-1}{3}$$

The point slope form of the required median is

$$y - y_1 = m_2(x - x_1)$$

Since vertex $B(4, 6)$ lies on the line of altitude so put $x_1 = 4$ and $y_1 = 6$

$$y - 6 = \frac{-1}{3}(x - 4)$$

$$3(y - 6) = -1(x - 4)$$

$$3y - 18 = -x + 4$$

$$x + 3y - 18 - 4 = 0$$

$$x + 3y - 22 = 0$$



(c) Equation of altitude $\overline{AF}$

Slope of altitude $\overline{AF} = \frac{-1}{\text{Slope of } BC} = \frac{-1}{\left(\frac{4}{3}\right)} = \frac{-3}{4}$

$$m_1 = \frac{-3}{4}$$

The point slope form of the required median is:

$$y - y_1 = m_1(x - x_1)$$

Since vertex $A(0,4)$ lies on the line of altitude so put $x_1 = 0$ and $y_1 = 4$,

$$y - 4 = \frac{-3}{4}(x - 0)$$

$$4(y - 4) = -3(x)$$

$$4y - 16 = -3x$$

$$3x + 4y - 16 = 0$$

Q.11 Reduce the equations

(a) $6x + 8y - 11 = 0$ (b) $4x - 3y + 9 = 0$ into:

(i) slope-intercept form answer

(ii) two-intercept form answer

(iii) point-slope form answer

(iv) two-point form answer

(v) normal form answer

(vi) symmetric form answer

(i) Solution

$$(a) 6x + 8y - 11 = 0$$

First we find the slope of given equation

Slope intercept form: $y = mx + c$

$$8y = -6x + 11$$

$$y = \frac{-6x + 11}{8}$$

$$y = \frac{-6}{8}x + \frac{11}{8}$$

$$y = \frac{-3}{4}x + \frac{11}{8}$$

$$y = mx + c$$

$$\Rightarrow m = \frac{-3}{4}, c = \frac{11}{8}$$

(ii) Solution

Two-intercept form: $\frac{x}{a} + \frac{y}{b} = 1$

$$6x + 8y - 11 = 0$$

$$6x + 8y = 11$$

$$\frac{6x}{11} + \frac{8y}{11} = 1 \quad (\text{Dividing by } 11)$$

$$\frac{x}{\frac{11}{6}} + \frac{y}{\frac{11}{8}} = 1$$

Where, x -intercept = $\frac{11}{6}$, y -intercept = $\frac{11}{8}$

(iii) Solution

Point-slope form: $y - y_1 = m(x - x_1)$

To get x point on given line put $x = 0$ in given equation, $6(0) + 8y - 11 = 0$

$$8y - 11 = 0 \Rightarrow y = \frac{11}{8}$$

Since point $\left(0, \frac{11}{8}\right)$ lies on the line so put $x_1 = 0$

, $y = \frac{11}{8}$ and slope of line, $m = \frac{-3}{4}$ in equation.

Equation of line:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{11}{8} = \frac{-3}{4}(x - 0)$$

(iv) Solution

Two-point on the line

First we get two point on the given line.

$$\text{If } x = 0 \text{ then } y = \frac{11}{8}$$

$$\text{Similarly, if } y = 0 \text{ then } x = \frac{11}{6}$$

Two points on the line are $\left(\frac{11}{6}, 0\right)$ and $\left(0, \frac{11}{8}\right)$

To get equation of line:

$$\text{put } x_1 = 0, y_1 = \frac{11}{8}, x_2 = \frac{11}{6}, y_2 = 0$$

$$\text{in the equation } \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$



$$\frac{y - \frac{11}{8}}{0 - \frac{11}{8}} = \frac{x - 0}{\frac{11}{6} - 0}$$

(v) Solution

Normal form: $x \cos \theta + y \sin \theta = p$

$$6x + 8y - 11 = 0$$

$$6x + 8y = 11$$

First we normalize the coefficient of x and y

$$\sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

$$\frac{6}{10}x + \frac{8}{10}y = \frac{11}{10} \quad (\text{Dividing by } 10)$$

$$x\left(\frac{3}{5}\right) + y\left(\frac{4}{5}\right) = \frac{11}{10} \quad (i)$$

Which is normal form of equation.

(vi) Solution

$$\text{Symmetric form: } \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta}$$

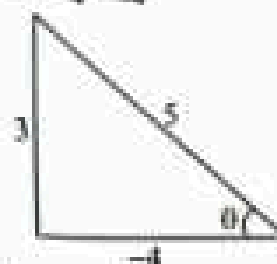
$$6x + 8y - 11 = 0$$

The slope of given equation is

$$y = \frac{-3}{4}x + \frac{11}{8}$$

$$m = \frac{-3}{4}$$

$$\tan \theta = m = \frac{-3}{4} = \frac{3}{-4}$$



$$\cos \theta = \frac{3}{5}, \sin \theta = \frac{-4}{5}$$

A point $\left(0, \frac{11}{8}\right)$ lies on the line because if $x = 0$

$$\text{then } y = \frac{11}{8}$$

Equation of symmetric form

$$\frac{x - 0}{\frac{11}{6}} = \frac{y - \frac{11}{8}}{\frac{11}{8}}$$

(b) $4x - 3y + 9 = 0$

Solution

(i) Slope intercept form: $y = mx + c$

$$4x - 3y + 9 = 0$$

$$\Rightarrow 3y = 4x + 9$$

$$y = \frac{4}{3}x + \frac{9}{3}$$

$$y = \frac{4}{3}x + 3$$

$$y = mx + c$$

$$\Rightarrow m = \frac{4}{3}, c = 3$$

(ii) Two intercept form, $\frac{x}{a} + \frac{y}{b} = 1$

$$4x - 3y + 9 = 0$$

$$4x - 3y = -9$$

$$\frac{4x}{-9} - \frac{3y}{-9} = 1 \quad (\text{Dividing by } -9)$$

$$\left(\frac{-4}{9}\right)x + \frac{1}{3}y = 1 \Rightarrow \frac{1}{\sqrt{\left(\frac{-4}{9}\right)^2} + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{16 + 9}}{\sqrt{25}} = \frac{5}{5}$$

$$\frac{x}{-\frac{9}{4}} + \frac{y}{3} = 1$$

$$x\text{-intercept} = \frac{-9}{4}, y\text{-intercept} = 3$$

(iii) Point slope form: $y - y_1 = m(x - x_1)$

$$4x - 3y + 9 = 0$$

A point (0, 3) lies on the line because if $x = 0$ then $y = 3$.

Slope of given equation $= m = \frac{4}{3}$

Put $x_1 = 0, y_1 = 3$

Equation of line

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{4}{3}(x - 0)$$

(iv) Two point form

First we find any two point on the given line. If

$x = 0$ then $y = 3$. Similarly if $y = 0$ then $x = \frac{-9}{4}$

Thus two point lying on the line are

$$(x_1, y_1) = (0, 3) \text{ and } (x_2, y_2) = \left(\frac{-9}{4}, 0\right)$$

Equation of line:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 3}{0 - 3} = \frac{x - 0}{\frac{-9}{4} - 0}$$

(v) Normal form, $x \cos \theta + y \sin \theta = p$

$$4x - 3y + 9 = 0$$

$$4x - 3y = -9$$

$$\Rightarrow -4x + 3y = 9$$

$$-\frac{4}{5}x + \frac{3}{5}y = \frac{9}{5}$$

$$x\left(-\frac{4}{5}\right) + y\left(\frac{3}{5}\right) = \frac{9}{5} \dots \dots \dots (i)$$

Which is normal form of equation.

(vi) Symmetric form, $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta}$

From slope of equation $4x - 3y + 9 = 0$

$$4x + 9 = 3y$$



$$y = \frac{4}{3}x + 3$$

$$y = mx + c$$

$$m = \frac{4}{3}$$

$$\tan \theta = m$$

$$\Rightarrow \tan \theta = \frac{4}{3}$$



$$\sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$\sin \theta = \frac{4}{5}, \cos \theta = \frac{3}{5}$$

A point (0, 3) lies on the line because if $x = 0$ then $y = 3$ for given equation.

So put $x_1 = 0$ and $y_1 = 3$.

$$\text{equation of symmetric } \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta}$$

$$\frac{x - 0}{\frac{3}{5}} = \frac{y - 3}{\frac{4}{5}}$$

Angle between two coplanar intersecting lines

Theorem

Let l_1 and l_2 be two non-vertical $\perp$ perpendicular coplanar lines making angles α and β with positive x-axis respectively $\perp$ shown in figure.

If m_1 and m_2 are the slope of l_1 and l_2 respectively, then the angle θ from l_1 to l_2 is

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$



Proof: In $\triangle ABC$, β and θ are two interior angles and α is non adjacent exterior angle.

$$\therefore \alpha = \beta + \theta$$

$$\Rightarrow \theta = \alpha - \beta$$

$$\Rightarrow \tan \theta = \tan (\alpha - \beta)$$

$$\Rightarrow \tan \theta = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

As $\tan \alpha = m_2$ and $\tan \beta = m_1$,

$$\therefore \tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Example 7

Find the measure of angle from l_1 to l_2 if: slope of $l_1 = -1$ and slope of $l_2 = 2$

Solution

Slope of $l_1 = m_1 = -1$ and slope of $l_2 = m_2 = 2$

If θ be the angle from l_1 to l_2 then:

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{2 - (-1)}{1 + 2(-1)} = -3$$

$$\theta = \tan^{-1}(-3) = 108.43^\circ$$

Example 8

Find the interior angles of triangle whose vertices are $A(-2, 0)$, $B(3, 0)$ and $C(5, 5)$.

Solution

Let the slopes of sides AB , BC and AC are denoted by m_1 , m_2 and m_3 respectively.

$$\therefore m_1 = \frac{0 - 0}{3 - 2} = 0, m_2 = \frac{5 - 0}{5 - 3} = \frac{5}{2}$$

$$m_3 = \frac{5 - 0}{5 - (-2)} = \frac{5}{7}$$



Let α , β and γ be interior angles of $\triangle ABC$ at vertex A , B and C respectively, then:

(i) Angle measured from AB to AC .

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3} = \frac{\frac{5}{7} - 0}{1 + \left(\frac{5}{7}\right)(0)} = \frac{5}{7}$$



$$\Rightarrow \alpha = \tan^{-1}\left(\frac{5}{7}\right) = 35.7^\circ$$

(ii) Angle measured from BC to BA

$$\tan \beta = \frac{m_1 - m_2}{1 + m_2 m_1} = \frac{0 - \frac{5}{2}}{1 + \left(\frac{5}{2}\right)\left(\frac{5}{7}\right)} = -\frac{2}{3}$$

$$\Rightarrow \beta = \tan^{-1}\left(\frac{2}{3}\right) = 33.69^\circ$$

As angle β lies in Q.II therefore

$$\beta = 180^\circ - 33.69^\circ = 146.31^\circ$$

(iii) Angle measured from CA to CB .

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_3 m_2} = \frac{\frac{5}{2} - \frac{5}{7}}{1 + \left(\frac{5}{7}\right)\left(\frac{5}{2}\right)} = \frac{\frac{25}{14}}{\frac{49}{14}} = \frac{25}{49}$$

$$\Rightarrow \gamma = \tan^{-1}\left(\frac{25}{49}\right) = 27^\circ$$

Point of Intersection of Two Straight Lines

Let l_1 and l_2 two non-parallel straight lines such that:

$$l_1: a_1x + b_1y + c_1 = 0 \text{ and}$$

$$l_2: a_2x + b_2y + c_2 = 0$$



(ii) l_1 joining $(-2, 1)$ and $(3, 4)$

l_2 joining $(-1, 3)$ and $(4, 8)$

Solution

$$\therefore \text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } l_1 = m_1 = \frac{4-1}{3-(-2)} = \frac{3}{5}$$

$$\text{Slope of } l_2 = m_2 = \frac{8-3}{4-(-1)} = \frac{5}{5} = 1$$

If θ be the angle from l_1 to l_2 then

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan \theta = \frac{1 - \frac{3}{5}}{1 + (1)\left(\frac{3}{5}\right)} = \frac{\frac{2}{5}}{\frac{8}{5}} = \frac{2}{8}$$

$$\tan \theta = \frac{2}{8} = \frac{1}{4}$$

$$\theta = \tan^{-1}\left(\frac{1}{4}\right) = \boxed{14.0^\circ}$$

(iii) l_1 joining $(-5, -4)$ and $(3, 1)$

l_2 joining $(-3, 2)$ and $(8, 8)$

Solution

$$\therefore \text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } l_1 = m_1 = \frac{1-(-4)}{3-(-5)} = \frac{5}{8} = \frac{1}{2}$$

$$\text{Slope of } l_2 = m_2 = \frac{8-2}{8-(-3)} = \frac{6}{11} = \frac{3}{5}$$

If θ be the angle from l_1 to l_2

then

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan \theta = \frac{\frac{3}{5} - \frac{1}{2}}{1 + (1)\left(\frac{3}{5}\right)} = \frac{\frac{1}{10}}{\frac{8}{5}} = \frac{1}{16}$$

$$\tan \theta = \frac{1}{16} = \frac{1}{16}$$

$$\theta = \tan^{-1}\left(\frac{1}{16}\right)$$

$$= \boxed{3.58^\circ}$$

(iv) l_1 joining $(2, -6)$ and $(5, -8)$

l_2 joining $(3, -5)$ and $(-10, -5)$

Solution

$$\therefore \text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } l_1 = m_1 = \frac{-8-(-6)}{5-2} = \frac{-2}{3} = -\frac{2}{3}$$

$$\text{Slope of } l_2 = m_2 = \frac{-5-(-5)}{-10-3} = \frac{0}{-13} = 0$$

If θ be the angle from l_1 to l_2

then

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan \theta = \frac{0 - (-\frac{2}{3})}{1 + (0)(-\frac{2}{3})} = \frac{\frac{2}{3}}{1} = \frac{2}{3}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\theta = 33.7^\circ$$

v) l_1 joining $(6, -5)$ and $(7, -9)$

l_2 joining $(2, -2)$ and $(-8, -12)$

Solution

$$\therefore \text{Slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } l_1 = m_1 = \frac{-9-(-5)}{7-6} = \frac{-4}{1} = -4$$

$$\text{Slope of } l_2 = m_2 = \frac{-12-(-2)}{-8-2} = \frac{-10}{-10} = 1$$

If θ be the angle from l_1 to l_2

then

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan \theta = \frac{1 - (-4)}{1 + (-4)(1)} = \frac{5}{-3} = -\frac{5}{3}$$

$$\tan \theta = -\frac{5}{3}$$

$$\theta = \tan^{-1}\left(-\frac{5}{3}\right)$$

$$= \boxed{-59.0^\circ}$$

Q.3 Find the interior angles of triangle ABC when:

(i) Slope AB = 0, Slope of BC = -1, Slope of AC = 1

(ii) Slope AB = 0.25, Slope of BC = 1.25, Slope of AC = 1

(iii) Slope AB = 0.4, Slope of BC = -1.5, Slope of AC = 1.667

(iv) Slope AB = -1, Slope of BC = 0.8, Slope of AC = 0

Solution

Let

Slope of AB = $m_1 = 0$,

Slope of BC = $m_2 = -1$,

Slope of AC = $m_3 = 1$

Let α, β, γ be interior angles of $\triangle ABC$ at vertex A, B and C respectively.

(i) Angle measured from $\overrightarrow{AB}$ to $\overrightarrow{AC}$:

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3} = \frac{1 - 0}{1 + (0)(1)} = 1$$

$$\tan \alpha = 1$$

$$\alpha = \tan^{-1}(1)$$

$$\boxed{\alpha = 45^\circ}$$

(ii) Angle measured from $\overrightarrow{BC}$ to $\overrightarrow{BA}$:

$$\tan \beta = \frac{m_1 - m_2}{1 + m_2 m_1}$$

$$\tan \beta = \frac{0 - (-1)}{1 + (0)(-1)} = \frac{1}{1} = 1$$

$$\beta = \tan^{-1}(1)$$

$$\boxed{\beta = 45^\circ}$$

(iii) Angle measured from $\overrightarrow{CA}$ to $\overrightarrow{CB}$:

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_3 m_2} = \frac{-1 - 1}{1 + (-1)(1)} = \frac{-2}{0} = \infty$$

$$\gamma = \tan^{-1}(\infty)$$

$$\boxed{\gamma = 90^\circ}$$

(iv) **Solution**

Let

Slope of AB = $m_1 = 0.25$

Slope of BC = $m_2 = 1.25$

Slope of AC = $m_3 = 1$

Let α, β, γ be interior angles of $\triangle ABC$ at vertex A, B and C respectively.

(i) Angle measured from $\overrightarrow{AB}$ to $\overrightarrow{AC}$:

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3} = \frac{1 - 0.25}{1 + (0.25)(1)} = \frac{0.75}{1.25}$$

$$\tan \alpha = \frac{3}{5}$$

$$\alpha = \tan^{-1}\left(\frac{3}{5}\right)$$

$$\boxed{\alpha = 31^\circ}$$

(ii) Angle measured from $\overrightarrow{BC}$ to $\overrightarrow{BA}$

Angle is measured from $\overrightarrow{AB}$ to $\overrightarrow{AC}$

$$\tan \beta = \frac{m_1 - m_2}{1 + m_2 m_1} = \frac{0.25 - 1.25}{1 + (1.25)(0.25)}$$

$$\tan \beta = \frac{-1}{1.3125}$$

$$\tan \beta = -\frac{16}{21}$$

$$\beta = \tan^{-1}\left(-\frac{16}{21}\right) = -37^\circ$$

As angle β lie in Q.II therefore

$$\beta = 180^\circ - 37^\circ = 143^\circ$$

$$\boxed{\beta = 143^\circ}$$

(iii) Angle measured from $\overrightarrow{CA}$ to $\overrightarrow{CB}$:

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_3 m_2}$$

$$\tan \gamma = \frac{1.25 - 1}{1 + (1.25)(1)} = \frac{0.25}{2.25} = \frac{1}{9}$$

$$\tan \gamma = \frac{1}{9}$$

$$\gamma = \tan^{-1}\left(\frac{1}{9}\right) = 6.3^\circ$$

$$\boxed{\gamma = 6.3^\circ}$$

$$\tan \gamma = \frac{1}{9}$$

$$\gamma = \tan^{-1}\left(\frac{1}{9}\right)$$

$$\boxed{\gamma = 6^\circ}$$

(iii) Solution

Let
Slope of $\overline{AB} = m_1 = 0.4$
Slope of $\overline{BC} = m_2 = -1.5$
Slope of $\overline{AC} = m_3 = 1.667$
Let α , β and γ be interior angles of $\triangle ABC$ at vertex A, B, C respectively.

(a) Angle measured from $\overline{AB}$ to $\overline{AC}$:

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3}$$

$$\tan \alpha = \frac{1.667 - 0.4}{1 + (0.4)(1.667)}$$

$$\tan \alpha = 0.7601$$

$$\alpha = \tan^{-1}(0.7601)$$

$$\Rightarrow \boxed{\alpha = 37^\circ}$$

(b) Angle measured from $\overline{BC}$ to $\overline{BA}$:

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{0.4 - (-1.5)}{1 + (0.4)(-1.5)}$$

$$\tan \beta = \frac{19}{4}$$

$$\beta = \tan^{-1}\left(\frac{19}{4}\right)$$

$$\boxed{\beta = 78^\circ}$$

(c) Angle measured from $\overline{CA}$ to $\overline{CB}$:

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$= \frac{-1.5 - 1.667}{1 + (-1.5)(1.667)}$$

$$\tan \gamma = 2.1106$$

$$\gamma = \tan^{-1}(2.1106)$$



$$\boxed{\gamma = 62^\circ}$$

(iv) Solution

Let
Slope of $\overline{AB} = m_1 = -1$
Slope of $\overline{BC} = m_2 = 0.8$
Slope of $\overline{AC} = m_3 = 0$
Let α , β and γ be interior angles of $\triangle ABC$ at vertex A, B, C respectively.

(a) Angle measured from $\overline{AB}$ to $\overline{AC}$:

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3} = \frac{0 - (-1)}{1 + (0)(-1)}$$

$$\tan \alpha = 1$$

$$\alpha = \tan^{-1}(1)$$

$$\boxed{\alpha = 45^\circ}$$

(b) Angle measured from $\overline{BC}$ to $\overline{BA}$:

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-1 - 0.8}{1 + (-1)(0.8)}$$

$$\tan \beta = -9$$

$$\beta = \tan^{-1}(-9)$$

$$\beta = -83.68^\circ = -94^\circ$$

As angle β lies in Q.II therefore
 $\beta = 180^\circ - 94^\circ = 86^\circ$

$$\boxed{\beta = 86^\circ}$$

(c) Angle measured from $\overline{CA}$ to $\overline{CB}$:

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\tan \gamma = \frac{0.8 - 0}{1 + (0.8)(0)}$$

$$\tan \gamma = 0.8$$

$$\gamma = \tan^{-1}(0.8)$$

$$\boxed{\gamma = 39^\circ}$$

Q.4 Find angle between lines:

- (i) $2x + 2y + 5 = 0$ and $2x - 3y + 8 = 0$
 (ii) $x + 2y - 4 = 0$ and $2x - 4y + 9 = 0$
 (iii) $6x - y + 1 = 0$ and $x - 7y + 12 = 0$

(i) Solution

$2x + 2y + 5 = 0$ and $2x - 3y + 8 = 0$
First we find the slope of these lines from given equations.

$$2y = -2x - 5 \quad \text{and} \quad 3y = 2x + 8$$

$$y = -\frac{2}{2}x - \frac{5}{2} \quad \text{and} \quad y = \frac{2}{3}x + \frac{8}{3}$$

$$m_1 = -\frac{2}{2} \quad \text{and} \quad m_2 = \frac{2}{3}$$

Let α be the angle between given two lines

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan \alpha = \frac{\frac{2}{3} - (-\frac{2}{2})}{1 + \left(-\frac{2}{2}\right)\left(\frac{2}{3}\right)} = \tan \alpha = \frac{\frac{2}{3} + \frac{2}{2}}{1 - \frac{2}{3}}$$

$$\Rightarrow \tan \alpha = \infty$$

$$\alpha = \tan^{-1}(\infty)$$

$$\Rightarrow \boxed{\alpha = 90^\circ}$$

(ii) Solution

$x + 2y - 4 = 0$ and $2x - 4y + 9 = 0$
First we find the slope of these lines from given equations.

$$2y = -x + 4 \quad \text{and} \quad 4y = 2x + 9$$

$$y = -\frac{1}{2}x + \frac{4}{2} \quad \text{and} \quad y = \frac{2}{4}x + \frac{9}{4}$$

$$y = -\frac{1}{2}x + 2 \quad \text{and} \quad y = \frac{1}{2}x + \frac{9}{4}$$



$$m_1 = -\frac{1}{2} \quad \Rightarrow \quad m_2 = \frac{1}{2}$$

Let α be the angle between two lines

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan \alpha = \frac{\frac{1}{2} - (-\frac{1}{2})}{1 + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

$$\tan \alpha = \frac{\frac{2}{2}}{1 - \frac{1}{4}}$$

$$\tan \alpha = \frac{1}{\frac{3}{4}}$$

$$\Rightarrow \tan \alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\boxed{\alpha = 53^\circ}$$

(iii) Solution

$6x - y + 1 = 0$ and $x - 7y + 12 = 0$
First we find the slope of these lines from given equations.

$$6x + 1 = y \quad \text{and} \quad x + 12 = 7y$$

$$\Rightarrow y = 6x + 1 \quad \text{and} \quad \Rightarrow 7y = x + 12$$

$$y = 6x + 1 \quad \text{and} \quad y = \frac{1}{7}x + 12$$

$$m_1 = 6 \quad \text{and} \quad m_2 = \frac{1}{7}$$

Let α be the angle between two lines

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan \alpha = \frac{\frac{1}{7} - 6}{1 + \left(\frac{1}{7}\right)(6)}$$



$$\tan \alpha = \frac{-41}{13}$$

$$\alpha = \tan^{-1}\left(\frac{-41}{13}\right) = -72^\circ$$

As angle α lies in Q.II therefore
 $\alpha = 180^\circ - 72^\circ = 108^\circ$

$$\boxed{\alpha = 108^\circ}$$

Q.5 Find the interior angles of triangle XYZ whose vertices are:

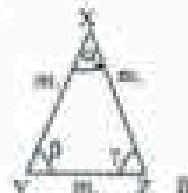
(i) $X(-2, 3), Y(-3, -4), Z(5, 7)$ answer

(ii) $X(-3, 2), Y(0, -4), Z(3, 3)$ answer

(iii) $X(-2, 0), Y(1, -4), Z(6, 6)$ answer

(iv) $X(-4, 1), Y(0, -3), Z(4, 3)$ answer

(i) Solution



$X(-2, 3), Y(-3, -4), Z(5, 7)$

Let the slopes of sides $\overline{XY}, \overline{YZ}$ and $\overline{XZ}$ be denoted by m_1, m_2 and m_3 respectively

$$m_1 = \frac{-4-3}{-3-2} = \frac{-7}{-5} = \frac{7}{5}$$

$$m_2 = \frac{7-(-4)}{5-(-3)} = \frac{11}{8}$$

$$m_3 = \frac{3-(-4)}{-2-5} = \frac{-7}{-7} = 1$$

Let α, β, γ be interior angle of $\triangle XYZ$ at vertex X, Y and Z respectively.

(a) Angle measured from $\overline{XY}$ to $\overline{XZ}$.

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3}$$

$$\tan \alpha = \frac{1 - \frac{7}{5}}{1 + \left(-\frac{1}{7}\right)\left(\frac{7}{5}\right)}$$

$$\tan \alpha = \frac{-1-49}{1-1}$$

$$\tan \alpha = \frac{-50}{0}$$

$$\tan \alpha = \infty$$

$$\alpha = \tan^{-1}(\infty)$$

$$\boxed{\alpha = 90^\circ}$$

(b) Angle measured from $\overline{YZ}$ to $\overline{YX}$.

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan \beta = \frac{\frac{7}{5} - \frac{11}{8}}{1 + \left(\frac{7}{5}\right)\left(\frac{11}{8}\right)} = \frac{\frac{28-55}{40}}{1 + \frac{77}{40}} = \frac{-27}{118}$$

$$\tan \beta = -1$$

$$\beta = \tan^{-1}(-1)$$

$$\boxed{\beta = 45^\circ}$$

(c) Angle measured from $\overline{ZX}$ to $\overline{ZY}$.

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\tan \gamma = \frac{\frac{11}{8} - 1}{1 + \left(\frac{11}{8}\right)\left(1\right)} = \frac{\frac{3}{8}}{1 + \frac{11}{8}} = \frac{3}{19}$$

$$\tan \gamma = \frac{3}{19}$$

$$\gamma = \tan^{-1}\left(\frac{3}{19}\right)$$

$$\boxed{\gamma = 43^\circ}$$

(ii) Solution

$X(-3, 2), Y(0, -4), Z(3, 3)$

Let the slopes of sides $\overline{XY}, \overline{YZ}$ and $\overline{XZ}$ be denoted by m_1, m_2 and m_3 respectively

$$m_1 = \frac{-4-2}{0-(-3)} = \frac{-6}{3} = -2$$



$$m_2 = \frac{3-(-4)}{3-0} = \frac{7}{3}$$

$$m_3 = \frac{2-(-4)}{-3-3} = \frac{6}{-6} = -1$$

Let α, β, γ be the interior angles of $\triangle XYZ$ at vertex X, Y, Z respectively.

(a) Angle measured from $\overline{XY}$ to $\overline{XZ}$.

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3} = \frac{-1 - (-1)}{1 + \left(-\frac{1}{6}\right)(-1)} = \frac{0}{\frac{5}{6}} = 0$$

$$\tan \alpha = \frac{0}{\frac{5}{6}} = 0$$

$$\tan \alpha = 1.4$$

$$\alpha = \tan^{-1}(1.4)$$

$$\boxed{\alpha = 34^\circ}$$

(b) Angle measured from $\overline{YZ}$ to $\overline{YX}$.

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-1 - \frac{7}{3}}{1 + \left(-1\right)\left(\frac{7}{3}\right)} = \frac{\frac{-3-7}{3}}{1 - \frac{7}{3}} = \frac{-10}{\frac{-4}{3}} = \frac{25}{2}$$

$$\tan \beta = \frac{-7}{-1} = 7$$

$$\beta = \tan^{-1}(7)$$

$$\boxed{\beta = 82^\circ}$$

(c) Angle measured from $\overline{ZX}$ to $\overline{ZY}$.

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{\frac{7}{3} - (-1)}{1 + \left(\frac{7}{3}\right)(-1)} = \frac{\frac{10}{3}}{1 - \frac{7}{3}} = \frac{10}{\frac{-4}{3}} = \frac{-15}{2}$$

$$\tan \gamma = \frac{21}{22}$$

$$\alpha = \tan^{-1}\left(\frac{21}{22}\right) \Rightarrow \boxed{\gamma = 44^\circ}$$

(iii) Solution

$X(-2, 0), Y(1, -4), Z(6, 6)$

Let the slopes of sides $\overline{XY}, \overline{YZ}, \overline{XZ}$ be denoted by m_1, m_2, m_3 respectively.

$$m_1 = \frac{-4-0}{1-(-2)} = \frac{-4}{3}$$

$$m_2 = \frac{6-(-4)}{6-1} = \frac{10}{5} = 2$$

$$m_3 = \frac{6-0}{6-(-2)} = \frac{6}{8} = \frac{3}{4}$$

Let α, β, γ be the interior angle of $\triangle XYZ$ at vertex X, Y, Z respectively.

(a) Angle measured from $\overline{XY}$ to $\overline{XZ}$.

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3}$$

$$\tan \alpha = \frac{\frac{3}{4} - \frac{-4}{3}}{1 + \left(\frac{-4}{3}\right)\left(\frac{3}{4}\right)} = \frac{\frac{9+16}{12}}{1-1} = \frac{25}{0}$$

$$\tan \alpha = \tan^{-1}(\infty)$$

$$\Rightarrow \boxed{\alpha = 90^\circ}$$

(b) Angle measured from $\overline{YZ}$ to $\overline{YX}$.

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{-4}{3} - 2}{1 + \left(\frac{-4}{3}\right)(2)} = \frac{\frac{-4-6}{3}}{1 - \frac{8}{3}} = \frac{-10}{\frac{-5}{3}} = 6$$

$$\tan \beta = \frac{-10}{-5} = \frac{10}{5} = 2$$

$$\tan \beta = 2$$

$$\beta = \tan^{-1}(2)$$

$$\Rightarrow \boxed{\beta = 63^\circ}$$

(c) Angle measured from $\overline{ZX}$ to $\overline{ZY}$.

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{2 - \frac{3}{4}}{1 + \left(2\right)\left(\frac{3}{4}\right)} = \frac{\frac{5}{4}}{1 + \frac{3}{2}} = \frac{5}{\frac{5}{2}} = 2$$



$$\tan \gamma = \frac{1}{3}$$

$$\gamma = \tan^{-1}\left(\frac{1}{3}\right) = 17^\circ$$

$$\boxed{\gamma = 17^\circ}$$

Or: Solution

$X(-4, 1), Y(0, -3), Z(4, 3)$

Let the slope of sides $\overline{XY}, \overline{YZ}, \overline{XZ}$ be denoted by m_1, m_2, m_3 respectively.

$$m_1 = \frac{-3-1}{0-(-4)} = \frac{-4}{4} = -1$$

$$m_2 = \frac{3-(-3)}{4-0} = \frac{6}{4} = \frac{3}{2}$$

$$m_3 = \frac{3-1}{4-(-4)} = \frac{2}{8} = \frac{1}{4}$$

Let α, β, γ be the interior angle of $\triangle XYZ$ at vertex X, Y, Z respectively.

(a) Angle measured from $\overline{XY}$ to $\overline{XZ}$:

$$\tan \alpha = \frac{m_3 - m_1}{1 + m_1 m_3} = \frac{\frac{1}{4} - (-1)}{1 + \left(-1\right)\left(\frac{1}{4}\right)} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}$$

$$\tan \alpha = \frac{5}{3}$$

$$\tan \alpha = \frac{5}{3}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{5}{3}\right)$$

$$\Rightarrow \boxed{\alpha = 59^\circ}$$

(b) Angle measured from $\overline{YZ}$ to $\overline{YX}$:

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-1 - \frac{3}{2}}{1 + (-1)\left(\frac{3}{2}\right)} = \frac{\frac{-2-3}{2}}{\frac{2-3}{2}} = \frac{-5}{-1} = 5$$

$$\tan \beta = 5 \Rightarrow$$

$$\beta = \tan^{-1}(5) \quad \boxed{\beta = 79^\circ}$$

(c) Angle measured from $\overline{ZX}$ to $\overline{ZY}$:

$$\tan \gamma = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{\frac{3}{2} - \frac{1}{4}}{1 + \left(\frac{3}{2}\right)\left(\frac{1}{4}\right)} = \frac{\frac{5}{4}}{\frac{13}{8}} = \frac{10}{13}$$

$$\tan \gamma = \frac{10}{13}$$

$$\gamma = \tan^{-1}\left(\frac{10}{13}\right)$$

$$\boxed{\gamma = 42^\circ}$$

Q.6 Find the point of intersection of lines:

(i) $2x + y + 1 = 0$ and $x - y - 4 = 0$

(ii) $x + y + 3 = 0$ and $2x - 5y + 8 = 0$

(iii) $2x + 5y + 3 = 0$ and $3x - 4y - 5 = 0$

(i) Solution

The point of intersection of two lines is the simultaneous solution of their equations.

$$2x + y + 1 = 0 \dots (i)$$

$$x - y - 4 = 0 \dots (ii)$$

Adding (i) & (ii)

$$2x + y + 1 = 0$$

$$x - y - 4 = 0$$

$$3x - 3 = 0$$

$$3x = 3$$

$$x = \frac{3}{3}$$

$$x = 1$$

$$\boxed{x = 1}$$

By putting $x = 1$ in equation (i)

$$2(1) + y + 1 = 0$$

$$2 + y + 1 = 0$$

$$2 + 3 = 0$$

$$\boxed{y = -3}$$

Then required point of intersection is $(1, -3)$

(ii) Solution

$$x + y + 3 = 0 \dots (i)$$



$$2x - 5y + 8 = 0 \dots (ii)$$

Multiplying equation (i) by 2 we get

$$2x + 2y + 6 = 0 \dots (iii)$$

Subtract equation (iii) from (ii)

$$2x - 5y + 8 = 0$$

$$2x + 2y + 6 = 0$$

$$-7y + 2 = 0$$

$$-7y = -2$$

$$\Rightarrow 7y = 2$$

$$\boxed{y = \frac{2}{7}}$$

By putting $y = \frac{2}{7}$ in equation (i)

$$x + \frac{2}{7} + 3 = 0$$

$$x = -3 - \frac{2}{7}$$

$$x = \frac{-21-2}{7}$$

$$\boxed{x = \frac{-23}{7}}$$

Required point of intersection is $\left(\frac{-23}{7}, \frac{2}{7}\right)$

(iii) Solution

$$2x + 5y + 3 = 0 \dots (i)$$

$$3x - 4y - 5 = 0 \dots (ii)$$

Multiplying equation (i) by 4 and equation (ii) by 5 then adding we get

$$8x + 20y + 12 = 0$$

$$15x - 20y - 25 = 0$$

$$23x - 13 = 0$$

$$23x = 13$$

$$\boxed{x = \frac{13}{23}}$$

$$\boxed{y = \frac{13}{23}}$$

$$\boxed{y = \frac{13}{23}}$$

By putting $x = \frac{13}{23}$ in equation (i) we get

$$2\left(\frac{13}{23}\right) + 5y + 3 = 0$$

$$\frac{26}{23} + 5y + 3 = 0$$

$$5y = -3 - \frac{26}{23}$$

$$5y = \frac{-69-26}{23}$$

$$5y = \frac{-95}{23}$$

$$y = \frac{-95}{23 \times 5}$$

$$\boxed{y = \frac{-19}{23}}$$

Required point of intersection is $\left(\frac{13}{23}, \frac{-19}{23}\right)$

Q.7 Find equation of line passing through point of intersection of lines.

$3x + 2y + 1 = 0, x - 2y + 3 = 0$ and

(a) passing through point $(-1, 0)$

(b) parallel to $3x - 4y + 3 = 0$

Solution

(a) Equation of line

$$3x + 2y + 1 = 0, x - 2y + 3 = 0$$

$$3x + 2y + 1 = 0 \dots (i)$$

$$x - 2y + 3 = 0 \dots (ii)$$

We know that

$$(3x + 2y + 1) + k(x - 2y + 3) = 0$$

$$3x + 2y + 1 + kx - 2ky + 3k = 0$$

$$(3 + k)x + 2y(1 - k) + (1 + 3k) = 0 \dots (iii)$$

As the equation (iii) passes through $(-1, 0)$

$$(3 + k)(-1) + 2(0)(1 - k) + (1 + 3k) = 0$$

$$-3 - k + 0 + 1 + 3k = 0$$

$$2k - 2 = 0$$

$$2k = 2$$

$$\Rightarrow \boxed{k = 1}$$

By putting $k = 1$ in equation (iii)

$$(3 + 1)x + 2y(1 - 1) + [1 + 3(1)] = 0$$

$$4x + 2y(0) + 4 = 0$$



$$4x + 4 = 0$$

$$4(x+1) = 0 \quad (\because 4 \neq 0)$$

$$\Rightarrow x + 1 = 0$$

b) Solution

$$\text{Parallel to } 3x - 4y + 3 = 0 \quad \dots\dots (iv)$$

Slope of line (iii)

$$m_1 = -\left(\frac{3+k}{3(1-k)}\right) = m_2 = -\left(\frac{3+k}{2-2k}\right)$$

$$\text{Slope of line (iv) } 3x - 4y + 3 = 0$$

$$3x + 3 = 4y$$

$$\Rightarrow y = \frac{3}{4}x + \frac{3}{4}$$

$$m_2 = \frac{3}{4}$$

Lines are parallel, so $m_1 = m_2$

$$-\left(\frac{3+k}{2-2k}\right) = \frac{3}{4}$$

$$\frac{3+k}{2-2k} = -\frac{3}{4}$$

$$4(3+k) = -3(2-2k)$$

$$12 + 4k = -6 + 6k$$

$$12 + 6 = 6k - 4k$$

$$18 = 2k$$

$$\Rightarrow k = \frac{18}{2} \Rightarrow \boxed{k=9}$$

By putting the value of $k = 9$ in equation (iii)

$$(3+9)x + 2y(1-9) + [1+3(9)] = 0$$

$$12x + 2y(-8) + (1+27) = 0$$

$$12x - 16y + 28 = 0$$

$$4(3x - 4y + 7) = 0$$

$$\Rightarrow 3x - 4y + 7 = 0 \quad (\because 4 \neq 0)$$

Q.8 Find the equation of family of lines passing through point of intersection of $6x + 5y + 3 = 0$ and $2x - 5y + 13 = 0$ with slope 1.

Solution

$$6x + 5y + 3 = 0 \rightarrow (i)$$

$$2x - 5y + 13 = 0 \rightarrow (ii)$$

We know that

$$(6x + 5y + 3) + k(2x - 5y + 13) = 0$$

$$(6 + 2k)x + (5 - 5k)y + (3 + 13k) = 0 \quad \dots\dots (iii)$$

Slope of line of equation (iii)

$$m_1 = -\left(\frac{6+2k}{5-5k}\right)$$

Slope of the given line is

$$m_2 = 1$$

$$\therefore m_1 = m_2$$

$$-\left(\frac{6+2k}{5-5k}\right) = 1$$

$$\frac{6+2k}{5-5k} = -1$$

$$6 + 2k = -3(5 - 5k)$$

$$6 + 2k = -15 + 15k$$

$$6 + 15 = 15k - 2k$$

$$21 = 13k$$

$$\Rightarrow \boxed{k = \frac{21}{13}}$$

By putting $k = \frac{21}{13}$ in equation (iii)

$$\left[6 + 2\left(\frac{21}{13}\right)\right]x + \left[5 - 2\left(\frac{21}{13}\right)\right]y + \left[3 + 13\left(\frac{21}{13}\right)\right] = 0$$

$$\left(\frac{78+42}{13}\right)x + \left(\frac{65-105}{13}\right)y + [3+21] = 0$$

$$\left(\frac{120}{13}\right)x + \left(\frac{-40}{13}\right)y + [24] = 0$$

$$120x - 40y + 312 = 0 \quad (\text{Multiplying by 13})$$

$$8(15x - 5y + 39) = 0$$

$$15x - 5y + 39 = 0 \quad (\because 8 \neq 0)$$

Q.9 Find equation of family of line passing through point of intersection of lines.

$$2x - 5y + 4 = 0, 6x - 4y + 3 = 0 \text{ and}$$

(a) parallel to x-axis

(b) parallel to y-axis

Solution

$$2x - 5y + 4 = 0 \rightarrow (i)$$

$$6x - 4y + 3 = 0 \rightarrow (ii)$$

Multiplying equation (i) by 3 and subtract from equation (ii)

$$6x - 15y + 12 = 0$$

$$\frac{6x - 4y + 3 = 0}{-11y + 7 = 0}$$

$$\Rightarrow 7 = 11y$$

$$\Rightarrow y = \frac{7}{11}$$

By putting $y = \frac{7}{11}$ in equation (i)

$$2x - 3\left(\frac{7}{11}\right) + 4 = 0$$

$$2x - \frac{35}{11} + 4 = 0$$

By multiplying both sides by 11

$$\Rightarrow 22x - 35 + 44 = 0$$

$$22x + 9 = 0$$

$$22x = -9$$

$$x = \frac{-9}{22}$$

$$x = \frac{-9}{22}$$

The point of intersection of the line is

$$\left(\frac{-9}{22}, \frac{7}{11}\right)$$

(b) Equation of line parallel to x-axis

$$\text{so } x = 0$$

$$y = \frac{7}{11}$$

$$11y = 7$$

$$\Rightarrow 11y - 7 = 0$$

(b) Equation of line parallel to y-axis

$$\text{so } y = 0$$

$$x = \frac{-9}{22}$$

$$22x = -9$$

$$22x + 9 = 0$$

Q.18 Find equation of line passing through point of intersection of lines.

$$2x - y + 2 = 0, x - 2y + 1 = 0 \text{ and}$$

(a) parallel to $x - 2y + 11 = 0$

(b) perpendicular to $2x + 5y + 3 = 0$.

Solution

$$2x - y + 2 = 0 \rightarrow (i)$$

$$x - 2y + 1 = 0 \rightarrow (ii)$$

Multiplying (ii) by 2 then subtract from (i)

$$\frac{2x - y + 2 = 0}{\underline{2x - 4y + 2 = 0}}$$

$$\frac{3y = 0}{3y = 0}$$

$$3y = 0$$

$$\Rightarrow \boxed{y = 0}$$

By putting $y = 0$ in equation (ii)

$$x - 2(0) + 1 = 0$$

$$x + 1 = 0$$

$$\boxed{x = -1}$$

Point of intersection is $(-1, 0)$

(a) Parallel to $x - 2y + 11 = 0$ in slope intercept form

$$x + 11 = 2y$$

$$\Rightarrow y = \frac{1}{2}x + \frac{11}{2}$$

$$\boxed{m = \frac{1}{2}}$$

$$m = \frac{1}{2} \text{ and } (x_1, y_1) = (-1, 0)$$

Point-slope form

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{1}{2}(x - (-1))$$

$$y = \frac{1}{2}(x + 1)$$

$$2y = x + 1$$



$$\Rightarrow x - 2y + 1 = 0$$

(b) Perpendicular to $2x - 5y + 1 = 0$

Slope intercept form

$$5y = -2x - 1$$

$$y = -\frac{2}{5}x - \frac{1}{5}$$

The slope of line is $m = -\frac{2}{5}$

The slope of the line perpendicular to will be

$$m = \frac{5}{2}$$

Taking point of intersection $(x_1, y_1) = (-1, 0)$

Point slope form

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{5}{2}(x - (-1))$$

$$y = \frac{5}{2}(x + 1)$$

$$2y = 5x + 5$$

$$5x - 2y + 5 = 0$$

Q.11 Find equation of line passing through point of intersection of lines.

$$x - 2y + 4 = 0, 3x - y - 3 = 0 \text{ and}$$

(a) parallel to line passing through $(2, -3)$ and $(8, 4)$. answer

(b) perpendicular to line passing through $(2, -3)$ and $(8, 4)$. answer

Solution

$$x - 2y + 4 = 0 \dots (i)$$

$$3x - y - 3 = 0 \dots (ii)$$

Point intercept form of line (i) and (ii)

$$(x - 2y + 4) + k(3x - y - 3) = 0 \dots (iii)$$

$$\Rightarrow x - 2y + 4 + 3kx - ky - 3k = 0$$

$$(1 + 3k)x - (2 + k)y + (4 - 3k) = 0 \dots (iv)$$

Slope of line (iv) is

$$m_1 = \frac{1 + 3k}{2 + k}$$

Slope of line between $(2, -3)$ and $(8, 4)$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-3)}{8 - 2} = \frac{7}{6}$$

$$m_2 = \frac{7}{6}$$

a) For parallel lines slope are same so,

$$m_1 = m_2$$

$$\frac{1 + 3k}{2 + k} = \frac{7}{6}$$

$$2(1 + 3k) = -7(2 + k)$$

$$2 + 6k = -14 - 7k$$

$$6k + 7k = -14 - 2$$

$$13k = -16$$

$$k = -\frac{16}{13}$$

By putting $k = -\frac{16}{13}$ in equation (iv)

$$(x - 2y + 4) + \left(-\frac{16}{13}\right)(3x - y - 3) = 0$$

$$\Rightarrow 13x - 26y + 52 - 48x + 16y + 48 = 0$$

$$-35x - 10y + 100 = 0$$

$$-5(7x + 2y - 20) = 0$$

$$\Rightarrow 7x + 2y - 20 = 0$$

b) Perpendicular to line

$$m_1 \cdot m_2 = -1$$

$$\left(\frac{1 + 3k}{2 + k}\right)\left(\frac{7}{6}\right) = -1$$

$$\frac{-7 - 23k}{4 + 2k} = -1$$

$$\Rightarrow -7 - 23k = -4 - 2k$$

$$-7 + 4 = -2k + 23k$$

$$-3 = 19k$$

$$k = -\frac{3}{19}$$

Putting $k = -\frac{3}{19}$ in equation (iv)

$$(x - 2y + 4) + \left(-\frac{3}{19}\right)(3x - y - 3) = 0$$

Multiplying both side by 19, we get

$$19(x - 2y + 4) - 3(3x - y - 3) = 0 \times 19$$

$$\Rightarrow 19x - 38y + 76 - 9x + 3y + 9 = 0$$

$$10x - 35y + 85 = 0$$

$$5(2x - 7y + 17) = 0$$

$$\Rightarrow 2x - 7y + 17 = 0 \quad (\div 5 \neq 0)$$

Example 11

In the linear equation $y = 1.12x + 8$ if " x " represents the number of kilometers and " y " represents the cost of the bus fare.

i) What will be the cost of bus fare after travelling 50 km? answer

ii) Ash paid a bus fare of Rs.490 in a journey. Find number of kilometers travelled by him? answer

Solution

$$\text{Given: } y = 1.12x + 8 \dots (i)$$

i) Substituting $x = 50$ in equation (i), we have:

$$y = 1.12 \times 50 + 8 = 64$$

$$\therefore \text{Cost of bus fare} = \text{Rs.64}$$

ii) Substituting $y = 490$ in equation (i), we have:

$$490 = 1.12x + 8 \Rightarrow 1.12x = 490 - 8 = 472$$

$$x = 470 \div 1.12 = 421.43$$

$$\therefore \text{Number of kilometers} = 421.43$$

Example 12

Sania buys mangoes @ Rs. 150 per kilogram and melon @ Rs.80 per kilogram. She has Rs.620 to spend on fruit. Write an equation in standard form that describes the situation. If she buys 2 kilograms of mangoes, how many kilograms of melon can she buy? answer

Solution

Suppose x denotes number of kilograms of mangoes and y denotes number of kilograms of melon.

The equation that describes this situation is

$$150x + 80y = 620 \dots (i)$$

As she buys 2 kilograms of mangoes, substituting $x = 2$ in the equation (i) we get:

$$150(2) + 80y = 620 \text{ or}$$

$$300 + 80y = 620$$

$$\Rightarrow 80y = 620 - 300 = 320 \Rightarrow y = 320 \div 80 = 4$$

$\therefore$ Sania can buy 4 kilograms of melon.

Example 13

The pollution index in a large city increases in an approximately linear fashion from 8 am until 3 pm in autumn season. On 24th November, the reading at 10:00 hours is around 80 parts per million (ppm) and at 14:00 hours, the reading is 110 ppm.

a) Write an equation for this situation. answer

b) What does the gradient (slope) mean in terms of pollution? answer

c) What does the y-intercept mean in terms of pollution? answer

d) What will be the pollution index at 12 pm? answer

Solution

a) If t denotes time and p denotes pollution index then:

$$(t_1, p_1) = (10, 80) \text{ and } (t_2, p_2) = (14, 110)$$

Using equation of line in two-point form, we have:

$$\frac{t - t_1}{t_2 - t_1} = \frac{p - p_1}{p_2 - p_1}$$

Substituting the values, we have:

$$\frac{t - 10}{14 - 10} = \frac{p - 80}{110 - 80} \Rightarrow \frac{t - 10}{4} = \frac{p - 80}{30}$$

$$\Rightarrow 30(t - 10) = 4(p - 80)$$

$$\Rightarrow 30t - 300 = 4p - 320$$

$$\Rightarrow 30t - 4p - 20 = 0$$

$$\Rightarrow 15t - 2p + 10 = 0 \dots (i)$$

Which is required equation.



b) From (i), We have:

$$2p = 12c + 18 \Rightarrow p = 7.5c + 9 \dots (ii)$$

$$\Rightarrow m = 7.5 \text{ and } c = 9$$

Here the gradient 7.5 means that pollution index is increasing @7.5 ppm per hour.

c) y -intercept $= c = 9$ means that pollution index at 00.00 hours (12am) is 9 ppm.

You can plot graph to understand in a better way.

d) Substituting $t = 12$ in equation (ii), we get:
 $p = 7.5 \times 12 + 9 = 99 \text{ ppm}$

Exercise 2.4

Q.1 Nasir sold 'fruits' @Rs.120 per dozen and 'Shakri Malta' @Rs.150 per dozen and earned Rs. 1200. Write an equation in standard form that describes the situation. If he sold 4 dozens of 'Shakri Malta', how many dozens of 'fruits' he sold?

Solution:

Suppose x denotes dozens fruit and y denotes dozens shakri malta.

Equation that describes the situation:

$$120x + 150y = 1200$$

$$12x + 15y = 120$$

$$3(4x + 5) = 120$$

Dividing both sides by 3 we get

$$4x + 5y = 40 \dots\dots\dots (i)$$

Shakri malta $= y = 4$ dozens

$$4x + 5(4) = 40$$

$$4x + 20 = 40$$

$$4x = 40 - 20$$

$$4 = \frac{20}{4}$$

$$\boxed{x = 5}$$

5 dozens of fruit sold.

Q.2 The linear equation $y = 1450x + 2000$ describes the total cost for staying in a hotel for one day, where Rs.2000 is the rent of room for maximum 8 people and Rs.1450 is the food cost per person.

- Find the cost paid to hotel if a group of people stays for one day.
- How many people can stay in the hotel for Rs.1450 for one day?

Solution

$$y = 1450x + 2000$$

$$i) \text{ No. of people } = x = 7$$

$$y = 1450(7) + 2000$$

$$= 10150 + 2000$$

$$y = 12150$$

12150 cost paid to hotel if a group of 7 people stays for one day.

$$ii) y = 1450x + 2000$$

$$\text{cost price} = y = 13600$$

$$13600 = 1450x + 2000$$

$$13600 - 2000 = 1450x$$

$$\Rightarrow 1450x = 11600$$

$$x = \frac{11600}{1450}$$

$$\boxed{x = 8}$$

8 people can stay in the hotel for Rs.13600 per one day.

Q.3 If one company provides Rs.2500 per week along with extra bonus of Rs.700 of the other offers Rs.800 per day. Convert data into linear equations and tell which is better deal for two weeks?

Solution

Let,

$$\text{No. of weeks} = x = 2$$

$$\text{Total amount earned} = y$$

$$\Rightarrow y = 2500x + 700 \text{ (first company)}$$

$$y = 2500(2) + 700$$

$$y = 1100 + 700$$

$$y = 1800$$

Second company

$$\text{Offer per day} = \text{Rs.}800x$$

$$\text{Offer per week } 800(7x)$$

$$x = 800(7x)$$

$$\text{For } x = 2$$

$$x = 800(7)(2)$$

$$x = 11200$$

The first deal from company 'A' is better for two weeks.

Q.4 A man earns Rs.10 per hour. He has Rs.500 in addition with him.

i) Write a linear equation for the situation and tell how much will he get after 12 hours?

ii) What does slope show in this situation?

iii) What does y -intercept represent here?

Solution:

i) Since the man earns Rs.10 per hour, the slope of the line that represents the situation is 10. Rs.500 addition with him, so the y -intercept $= 500$

By putting $m = 10$ and $C = 500$ into the slope

$$y = mx + c$$

$$y = 10x + 500 \text{ (answer)}$$

By putting $x = 12$

$$y = 120(12) + 500$$

$$= 1440 + 500$$

$$y = 1940$$

The man will get Rs.1940 after 12 hours

ii) Slope shows earning per hour in this situation.

iii) y - intercept represent charge per hour.



Q.5 Ali shifted in a rented house on first September. The electric meter of house was showing 44 units on that day. If the average electricity consumed is 18 units per day:

i) Represent the situation through linear equation.

ii) How many units are consumed till 30 September?

iii) What will be the bill after one month @ Rs.20 per unit?

iv) After how many days, the meter shows 404 units?

Solution:

No. of days since 1st September $= x$

Average consumed 18 units $= 18$ units

i) No. of days $= x = 30$ No. of units consumed $= 18x$

$$\text{Total no. of units} = y = 18x + 44$$

ii) No. of units $= y = 18x + 44$

$$y = 18(30) + 44$$

$$y = 540 + 44$$

$$y = 584$$

Consumed units till 30 Sep. $= 584$ units.

iii) Unit price $= \text{Rs.}20$

$$\text{Bill after one month} = 584 \times 20$$

$$\text{Rs. } 11680$$

iv) Total no. of units $= y = 404$

$$\therefore y = 18x + 44$$

$$404 = 18x + 44$$

$$404 - 44 = 18x$$

$$\Rightarrow 18x = 360$$

$$x = \frac{360}{18}$$

$$\boxed{x = 20}$$

After 20 days, the meter shows 404 units.

Q.6 Ali hired a taxi with a fixed charge of Rs.1500 plus Rs.450 per 30 minutes.

i) Represent the relation as a linear equation.

iii) What will be the cost of taxi fare after 5 hours? answer

iii) What is the slope of equation in the case? answer

Solution:

Q Let x be the number of hours y represent the total cost.

If amount per 30 minutes interval is Rs.450 then amount per hour will be
 $\text{Rs.}450 \times 2 = \text{Rs.}900$

The linear equation representing this situation is

$$y = 900x + 1500 \text{ --- (i)}$$

ii) To find the cost of taxi fare after 5 hours, put $x = 5$ in equation (i).

$$y = 900 \times 5 + 1500$$

$$y = 4500 + 1500$$

$$y = 6000$$

The cost of taxi fare after 5 hours will be Rs.6000.

iii) In this case slope of the equation is 900, which represents that charges per hour are Rs.900.

Q.7.1) Derive the relation between Fahrenheit and Celsius scales in slope-intercept form. answer

ii) What does y-intercept and slope show in the equation? answer

iii) What temperature in Fahrenheit when temperature in Celsius is 5°C ? answer

Solution:

(i) Relation between Fahrenheit and Celsius:

$$\text{We know that } F = \frac{9}{5}C + 32$$

$$\text{Taking } F = y, C = x$$

$$\Rightarrow y = \frac{9}{5}x + 32$$

$$y = \frac{9}{5}x + 32$$

$$\text{Compare with } y = mx + c$$



$$\text{Slope } m = \frac{9}{5}, y\text{-intercept } c = 32$$

(ii) The y-intercept and Slope:

The y-intercept $c = 32$ means when temperature in degree centigrade is zero the temperature in Fahrenheit is 32 degrees. Also that, slope shows the rate of increase of temperature in Fahrenheit. Here when temperature in Fahrenheit increases $\frac{9}{5}$ degree

for every increase of one degree in centigrade.

(iii) Temperature in Fahrenheit when $C = 5^{\circ}$

$$y = \frac{9}{5}x + 32$$

$$\text{By putting } x = 5$$

$$y = \frac{9}{5}(5) + 32$$

$$y = 9 + 32 = 41$$

Temperature in Fahrenheit is 41°F when the temperature in Celsius is 5°C .

Q.8 A cricket team scores 96 runs in 6 overs and 180 runs in 30 overs.

i) What is an equation of line for this situation. answer

ii) What does the gradient mean in terms of scores? answer

iii) What does the y-intercept mean in terms of scores? answer

iv) What will be the predicted score after 40 overs? answer

v) After how many overs, the predicted score will be 240? answer

Solution:

Let number of overs and scores are x and y respectively. By given information 8 possible ordered pairs are (6,96) and (30,180).

(ii) Equation of line:

First we find the slope of line passing through these points using the following

Formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Using ordered pairs (6,96) and (30,180)

$$m = \frac{180 - 96}{30 - 6} = \frac{84}{24} = 3.5$$

To find the y-intercept of line we can use slope

$m=3.5$ and any of ordered pair in slope intercept form $y = mx + c$

We take $m = 3.5$ and $(x,y) = (6,96)$

$$96 = 16 + 6 + c$$

$$96 = 96 + c$$

$$96 - 96 = c \Rightarrow \boxed{c = 0}$$

The required equation of line is obtained by putting the values in $y = mx + c$

$$y = 3.5x + 0$$

$$y = 3.5x \text{ --- (i)}$$

(ii) Gradient of line:

The gradient of line is $m = 3.5$ which means scores are increasing at the rate of 3.5 scores per over.

(iii) y-intercept:

The intercept of 0 means at the start of match total scores are zero, i.e. scores starts from zero.

(iv) Predicted score after 45 overs:

To predict the score after 45 overs put $x = 45$ in the equation (i)

$$y = 3.5x$$

$$y = 3.5(45)$$

$$y = 157.5$$

After 45 overs, predicted score are 157.5.

(v) Finding overs for predicted scores:

To find the overs for predicted scores 240, put $y = 240$ in equation (i).

$$240 = 3.5x$$



$$\frac{240}{3.5} = x$$

$$40 = x$$

$$\Rightarrow \boxed{x = 40}$$

The predicted scores, 240 will be after 40 overs.

Q.9 Abdullah rented a truck for one day. The truck company charged Rs.5000 per day and some additional money per kilometre. He drives 125 kilometres and paid Rs.30,000.

i) Write an equation in point-slope form that describes this situation. answer

ii) Find the amount per kilometre the truck rental company charges and relate it with slope. answer

iii) How much would it cost if Abdullah drives 180km? answer

Solution:

i) Let Number of km driven be x and total cost be y .

The fixed amount per day $= c = \text{Rs.}5000$

Additional amount per km $= p$

$$\text{(ii) Total cost} = (\text{Cost per km}) \times (\text{No. of km}) + \text{Fixed amount}$$

$$y = Px + 5000 \text{ --- (i)}$$

Put $y = 30,000$ and $x = 125$ in equation (i)

$$\Rightarrow 30,000 = P(125) + 5000$$

$$30,000 - 5000 = P(125)$$

$$25,000 = P(125)$$

$$\frac{25000}{125} = P$$

$$\Rightarrow \boxed{P = 200}$$

Truck company charge Rs.200 per km.

$$y = 200x + 5000$$

The slope of line is 200 which implies that total cost increases at the rate of Rs.200 per km.



(ii) Rewriting equation in point slope.

$$(x_1, y_1) = (0, 3000)$$

Point slope form,

$$y - y_1 = m(x - x_1)$$

$$y - 3000 = 200(x - 0)$$

(iii) Cost for 180 km

Putting $x = 180$ in equation (i)

$$y = 2000 + 200(180)$$

$$= 2000 + 36000$$

$$y = \text{Rs. } 41,000$$

It Rs. 41,000 cost if Abdulah drove 180km.

Q.18 A ship starts travelling from Karachi with latitude of 25° N and longitude 67° E. If the ship travel in a straight line and reaches a destination with latitude of 32° N and longitude 54° E, then derive the equation of line in point-slope form. If the ship moves to another location with a latitude of 39° N, what is longitude of that location? *answer*

Solution

(i) Let latitude be x and longitude be y .

$$(x_1, y_1) = (25^\circ, 67^\circ)$$

$$(x_2, y_2) = (32^\circ, 54^\circ)$$

$$\text{Slope of line} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{54^\circ - 67^\circ}{32^\circ - 25^\circ}$$

$$m = -\frac{13}{7}$$

Substituting $m = -\frac{13}{7}$ and $(x_1, y_1) = (25^\circ, 67^\circ)$

into point slope form we get

$$y - y_1 = m(x - x_1)$$

$$y - 67^\circ = -\frac{13}{7}m(x - 25^\circ)$$

(ii) By putting $x = 39^\circ$

$$y - 67^\circ = -\frac{13}{7}(39^\circ - 25^\circ)$$

$$y - 67^\circ = -\frac{13}{7}(14^\circ)$$

$$y - 67^\circ = -13(2^\circ)$$

$$y = -26^\circ + 67^\circ + 41^\circ$$

Thus longitude of the location is 41° .

Q.19 Length and width of a plot are in the ratio 2:1.

(i) Write equation of line and find the length of plot if the width of plot is 30 feet. *answer*

(ii) What is slope in this case and what does it mean? *answer*

Solution

Let Length and width of the rectangle be y and x respectively.

Ratio of length and width = 2:1

$$y : x = 2 : 1$$

$$\frac{y}{x} = \frac{2}{1}$$

$$\Rightarrow y = 2x$$

By putting $x = 30$ ft we get

$$y = 2(30)$$

$$y = 60 \text{ ft}$$

Length of plot is 60 ft.

(ii) $(x_1, y_1) = (0, 0)$

$$(x_2, y_2) = (30, 60)$$

$$\text{Slope of line} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{60 - 0}{30 - 0}$$

$$\frac{60}{30}$$

$$m = \frac{60}{30} \Rightarrow m = 2$$

The slope is 2 which means that length of the plot is 2 feet for every 1 foot of width.



Miscellaneous Exercise 5

Encircle the correct option in the following:

- Slope of two lines are 0 and m . What is angle between both lines? *answer*
(a) 0° (b) 60°
(c) 90° (d) 180°
- One angle of right triangle ABC is determined by the line joining A(1, 2) and B(2, 3). *answer*
(a) 30° (b) 45°
(c) 60° (d) 80°
- Slope of $(-1, 0)$ and $(1, y)$ is 3. What is y ?
(a) 10 (b) 11
(c) -12 (d) 12
- The line $3x - 4y - 3 = 0$ passes through (1, 2). What is k ? *answer*
(a) 1 (b) -1
(c) -2 (d) 2
- The line $3x - 6 = 0$ represents a line; *answer*
(a) parallel to x-axis
(b) parallel to y-axis
(c) passing through origin
(d) touching both axis
- Line $y = 0$ represent; *answer*
(a) x-axis (b) y-axis
(c) a plane (d) a point
- The line $y = k$, is above the x-axis, if
(a) $k = 0$ (b) $k < 0$
(c) $k > 0$ (d) $k = 0$
- The line $x = a$, is left to the y-axis, if *answer*
(a) $a = 0$ (b) $a < 0$
(c) $a > 0$ (d) $a < 0$
- Slope of a line l is -4. What is slope of a line perpendicular to l ? *answer*
(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$
(c) 4 (d) -4
- Which of the following line has slope $\frac{2}{3}$? *answer*

$$(a) 2x - 3y = 0$$

$$(b) 2x - 3y = 2$$

$$(c) 3x + 2y = 1$$

$$(d) 3x + 2y = 3$$

11. The line $y = 2x - 5$ is written in the form;

(a) point-slope *answer*

(b) two-intercept

(c) slope-intercept

(d) two-point

12. x-intercept of the line $x + y = 5$ is; *answer*

$$(a) 1$$

$$(b) -1$$

$$(c) 5$$

$$(d) -5$$

13. A line intercepts both axis at (2, 0) and

(0, 7) respectively. Its y-intercept is; *answer*

$$(a) -7$$

$$(b) 2$$

$$(c) -7$$

$$(d) 7$$

14. Point of intersection of lines $9x - 7y = 0$ and

$8x - 11y = 0$, is; *answer*

$$(a) (0, 0)$$

$$(b) (0, 7)$$

$$(c) (9, 0)$$

$$(d) (7, 9)$$

Answer Key

1	a	2	b	3	d	4	a
5	b	6	a	7	c	8	d
9	a	10	b	11	c	12	c
13	d	14	a				

Q.2 Prove that A(3, -10), B(1, 4) and C(2, -3) are collinear.

We can check if the slopes between each pair of points are equal. *answer*

Solution

$$\Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$$

First, calculate the slope between points A(3, -10) and B(1, 4).

$$\Rightarrow m_{AB} = \frac{4 - (-10)}{1 - 3} = \frac{14}{-2} = -7$$

Next, calculate the slope between points A(3, -10) and C(2, -3).



$$\Rightarrow m_{AB} = \frac{-3 - (-100)}{2 - 3} = \frac{7}{-1} = -7$$

Since $m_{AB} = m_{AC}$, the slopes between points A and B, and points A and C are equal, indicating that lines AB and AC are part of the same straight line. Therefore, points A, B and C are collinear.

Q.3 x-intercept of a line is double of y-intercept. Find equation of line if it passes through (2, 1). *****

Solution

Let the y-intercept be b , so the x-intercept is $2b$. The equation of a line with x-intercept a and y-intercept b is given by:

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

Substituting $a = 2b$, we get:

$$\frac{x}{2b} + \frac{y}{b} = 1$$

Multiplying both sides "2b".

$$\Rightarrow 2b \left(\frac{x}{2b} \right) + 2b \left(\frac{y}{b} \right) = 2b$$

$$x + 2y = 2b \quad \text{--- (1)}$$

Given the line passes through (2, 1), substitute

$$x = 2 \text{ and } y = 1$$

$$\Rightarrow x + 2y = 2b$$

$$\Rightarrow 2 + 2(1) = 2b$$

$$\Rightarrow 4 = 2b$$

$$\Rightarrow 2b = 4 \quad \Rightarrow b = 2$$

Substitute $b = 2$ back into $x + 2y = 2b$ (1)

$$\Rightarrow x + 2y = 2(2)$$

$$\Rightarrow x + 2y = 4$$

Q.4 Reduce $5x - 2y + 1 = 0$ into slope intercept form and two intercept form. *****

Solution

(a) Slope intercept form

To convert $5x - 2y + 1 = 0$ into slope-intercept

form $y = mx + c$, solve for y

$$\Rightarrow 5x + 1 = 2y$$

$$\Rightarrow y = \frac{5}{2}x + \frac{1}{2}$$

Therefore, the slope-intercept form is

$$y = \frac{5}{2}x + \frac{1}{2} \quad \text{--- (1)}$$

(b) Two intercept form:

To convert into two-intercept form

$$\left(\frac{x}{a} + \frac{y}{b} = 1 \right) \text{ where } a \text{ and } b \text{ are the x-intercept}$$

and y-intercept respectively, find the intercepts by setting $y=0$ to find a and $x=0$ to find b .

• For x-intercept (a)

$$\Rightarrow 5a - 2y + 1 = 0$$

By putting $x = a$ and $y = 0$

$$\Rightarrow 5a - 2(0) + 1 = 0$$

$$\Rightarrow 5a + 1 = 0$$

$$\Rightarrow a = -\frac{1}{5}$$

• For y-intercept (b)

$$\Rightarrow 5a - 2y + 1 = 0$$

(By setting $x = 0$ and $y = b$)

$$\Rightarrow 5(0) - 2b + 1 = 0$$

$$\Rightarrow -2b + 1 = 0$$

$$\Rightarrow b = \frac{1}{2}$$

Therefore, the two-intercept form is:

$$\frac{x}{-\frac{1}{5}} + \frac{y}{\frac{1}{2}} = 1$$

Q.5 (i) Reduce $x - 2y = 3$ into intercept form and find x and y-intercepts. *****

Solution

$$\text{The equation in intercept form, } \frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow x - 2y = 3$$

$$\text{Dividing both sides by 3, } \Rightarrow \frac{x}{3} + \frac{y}{-1} = 1$$

The x-intercept $a = 3$ and the y-intercept $b = -1$.

(ii) Find its slope and transform the equation into point slope form. *****

(a) Finding Slope:

$$\Rightarrow x - 2y = 3$$

Slope intercept form is $y = mx + c$.

$$\Rightarrow -2y = -x + 3$$

Multiply both sides by -1

$$\Rightarrow y = \frac{1}{2}x - 1 \quad \text{--- (1)}$$

Hence, slope $= m = \frac{1}{2}$ and y-intercept $c = -1$

(b) Equation into point slope form:

$$x - 2y = 3$$

To get a point on line, put $x = 0$ in equation

$$0 - 2y = 3$$

$$y = -1$$

$$\Rightarrow (x_1, y_1) = (0, -1)$$

Equation of line passing through (0, -1) and

$$\text{having slope } m = \frac{1}{2}$$

$$\Rightarrow y - y_1 = m(x - x_1)$$

$$\Rightarrow y + 1 = \frac{1}{2}(x - 0)$$

Q.6 Normal form of an equation of line is $x \cos 150^\circ + y \sin 150^\circ = 10$. Transform the equation into slope-intercept form and find its slope and y-intercept. *****

Solution

Given the normal form of the equation of the line is:

$$\Rightarrow x \cos 150^\circ + y \sin 150^\circ = 10 \quad \text{--- (1)}$$

First we find the values of $\cos 150^\circ$ and $\sin 150^\circ$:

$$\cos 150^\circ = \cos (180^\circ - 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$$

Substitute these values into the equation (1)

$$\Rightarrow x \left(-\frac{\sqrt{3}}{2} \right) + y \left(\frac{1}{2} \right) = 10$$

$$\Rightarrow -\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 10$$

To transform this into the slope-intercept form $y = mx + c$, isolate y :

$$\Rightarrow \frac{1}{2}y = \frac{\sqrt{3}}{2}x + 10$$

Multiplying by 2 we get

$$\Rightarrow 2 \left(\frac{1}{2} \right) y = 2 \left(\frac{\sqrt{3}}{2} \right) x + 2(10)$$

$$\Rightarrow y = \sqrt{3}x + 20$$

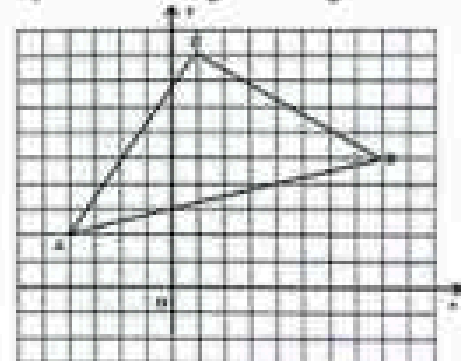
By comparing with $y = mx + c$ we observe that the slope is $m = \sqrt{3}$ and the y-intercept is $c = 20$.

Q.7

(i) Find vertices of triangle ABC. *****

(ii) Calculate slopes of its sides. *****

(iii) Find interior angles of triangle. *****



Solution

(i) Vertices of $\triangle ABC$:

To find the vertices of triangle ABC, we will determine the coordinates of points A, B and C by drawing perpendiculars on x-axis and y-axis respectively from these vertices.

$$\Rightarrow A(-4, 2), B(3, 5), C(1, 9)$$

**(ii) Slope of sides:**

The slope of a line segment between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope side of AB

Use the points $A(-4, 2)$ and $B(8, 5)$

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{8 - (-4)} = \frac{3}{8 + 4} = \frac{3}{12} = \frac{1}{4}$$

Slope of BC

Use the points $B(8, 5)$ and $C(1, 9)$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 5}{1 - 8} = \frac{4}{-7} = -\frac{4}{7}$$

Slope of AC

Use the points $A(-4, 2)$ and $C(1, 9)$

$$m_3 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 2}{1 - (-4)} = \frac{7}{1 + 4} = \frac{7}{5}$$

Therefore, $m_1 = \frac{1}{4}$, $m_2 = -\frac{4}{7}$, $m_3 = \frac{7}{5}$

(iii) Interior angles of $\triangle ABC$:

The slopes of sides AB, BC and AC are denoted by m_1 , m_2 and m_3 and are given below.

$$AB = m_1 = \frac{1}{4}; BC = m_2 = -\frac{4}{7}; AC = m_3 = \frac{7}{5}$$

Let α , β and γ be interior angles of $\triangle ABC$ at the vertices A, B and C respectively.

Finding $m \angle A$

$$\tan \alpha = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{\frac{7}{5} - \frac{1}{4}}{1 + \left(\frac{7}{5}\right)\left(-\frac{4}{7}\right)}$$

$$\tan \alpha = \frac{\frac{28 - 5}{20}}{1 - \frac{28}{20}} = \frac{\frac{23}{20}}{\frac{20 - 28}{20}} = \frac{23}{-8}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{23}{-8}\right) = 40^\circ$$

Finding $m \angle B$

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{1}{4} - \left(-\frac{4}{7}\right)}{1 + \left(\frac{1}{4}\right)\left(-\frac{4}{7}\right)}$$

$$\tan \beta = \frac{\frac{7 + 16}{28}}{\frac{28 - 4}{28}} = \frac{23}{24}$$

$$\Rightarrow \beta = \tan^{-1}\left(\frac{23}{24}\right) = 44^\circ$$

Finding $m \angle C$

$$\tan \gamma = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{\frac{7}{5} - \frac{1}{4}}{1 + \left(\frac{7}{5}\right)\left(\frac{1}{4}\right)}$$

$$\tan \gamma = \frac{\frac{-20 - 49}{20}}{1 + \frac{7}{20}} = \frac{-69}{\frac{20 + 7}{20}} = \frac{-69}{\frac{27}{20}}$$

$$\Rightarrow \gamma = \tan^{-1}\left(\frac{-69}{27}\right) = 44^\circ$$

As γ lies in II quadrant

$$\text{So, } \beta = 180^\circ - 44^\circ = 96^\circ$$

Therefore, the interior angles of $\triangle ABC$ are 40° , 44° , 96° .

Q.8 Two points $P(4, -1)$ and $Q(8, 3)$ lie on a line. Find:

- Coordinates of mid-point M of PQ
- Slope of PM
- The equation of line parallel to PQ through $(-2, 2)$.
- The equation of line perpendicular to PQ through $(-2, 3)$.

Solution

(i) Coordinates of the midpoint of PQ

To find the coordinates of the midpoint M of PQ, we use the midpoint formula.

$$= M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Putting the given coordinates $P(4, -1)$ and $Q(8, 3)$, we get

$$= M\left(\frac{4 + 8}{2}, \frac{-1 + 3}{2}\right)$$

$$= M\left(\frac{12}{2}, \frac{2}{2}\right)$$

$$= M(6, 1)$$

(ii) Slope of PM

To find the slope of PM, use the slope

$$\text{formula: } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Putting the coordinates of P and M we get

$$= m_{PM} = \frac{1 - (-1)}{6 - 4}$$

$$= m_{PM} = \frac{2}{2}$$

$$\Rightarrow m_{PM} = 1$$

$$\Rightarrow \text{The slope of } \overline{PM} = 1$$

(iii) Equation of line parallel to PQ

The slopes of parallel lines are same.

The line also passes through $(-2, 2)$, using the point-slope form, we get

$$\Rightarrow y - y_1 = m(x - x_1)$$

Substitute the slope (m) and the point $(-2, 2)$

$$\Rightarrow y - 2 = 1(x + 2)$$

$$\Rightarrow y - 2 = x + 2$$

$$\Rightarrow x - y + 2 + 2$$

$$\Rightarrow x - y + 4 = 0$$

(iv) Equation of line perpendicular to PQ

The slope of the line perpendicular to PQ is the negative reciprocal of m_{PQ} :



$$\Rightarrow m_{\text{perpendicular}} = \frac{1}{m_{PQ}} = \frac{1}{-1} = -1$$

Using the point-slope form for the line passing through $(-2, 2)$

$$\Rightarrow y - y_1 = m_{\text{perpendicular}}(x - x_1)$$

Put $x_1 = -2$, and $y_1 = 2$

$$\Rightarrow y - 2 = -1(x + 2)$$

$$\Rightarrow y - 2 = -x - 2$$

$$\Rightarrow y = -x$$

$$\Rightarrow y + x = 0$$

$$\Rightarrow x + y = 0$$

Q.9 Two points $A(2, -2)$ and $B(4, 6)$ lie on a line. Find:

- Length of AB.
- Slope of BA.
- Values of a and b when the line AB is $ax + by - 10 = 0$.
- The equation of line parallel to AB passing through $(0, 3)$.

Solution**(i) Length of AB**

To find the length of AB, we use the distance formula.

$$\Rightarrow AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Putting the coordinates of A $(2, -2)$, B $(4, 6)$ we get

$$\Rightarrow AB = \sqrt{(2 - 4)^2 + (-2 - 6)^2}$$

$$\Rightarrow AB = \sqrt{(2)^2 + (8)^2}$$

$$\Rightarrow AB = \sqrt{4 + 64}$$

$$\Rightarrow AB = \sqrt{68} = \sqrt{4 \times 17} = 2\sqrt{17} \text{ units}$$

(ii) Slope of BA

To find the slope of BA, we use the slope formula



$$\Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$$

Putting the coordinates of B and A we get

$$\Rightarrow m_{BA} = \frac{6 - (-2)}{4 - 2}$$

$$\Rightarrow m_{BA} = \frac{8}{2}$$

$$\Rightarrow m_{BA} = 4$$

Therefore, the slope of $\overline{BA} = 4$

(iii) Value of a and b :

Given that line AB is $ax + by - 10 = 0$ and it passes through $A(2, -2)$ and $B(4, 6)$, we can substitute these into the equation to find a and b .

For point $A(2, -2)$ put $x = 2$ and $y = -2$

$$\Rightarrow ax + by - 10 = 0$$

$$\Rightarrow a(2) + b(-2) - 10 = 0$$

$$\Rightarrow 2a - 2b = 10$$

$$\Rightarrow \frac{2a}{2} - \frac{2b}{2} = \frac{10}{2}$$

$$\Rightarrow a - b = 5 \dots\dots\dots (i)$$

For point $B(4, 6)$ put $x = 4$ and $y = 6$

$$\Rightarrow ax + by - 10 = 0$$

$$\Rightarrow a(4) + b(6) - 10 = 0$$

$$\Rightarrow 4a + 6b = 10$$

$$\Rightarrow \frac{4a}{2} + \frac{6b}{2} = \frac{10}{2}$$

$$\Rightarrow 2a + 3b = 5 \dots\dots\dots (ii)$$

Now, we solve equations (i) and (ii) simultaneously. Multiply equation (i) by 2.

$$\Rightarrow 2a - 2b = 10 \dots\dots\dots (iii)$$

Now subtracting equation (ii) from (iii) we get

$$\Rightarrow (2a - 2b) - (2a + 3b) = 10 - 5$$

$$\Rightarrow 2a - 2b - 2a - 3b = 10 - 5$$

$$\Rightarrow -5b = 5 \Rightarrow b = \frac{5}{-5} \Rightarrow b = -1$$

Substitute $b = -1$ in equation (i).

$$\Rightarrow a - b = 5$$

$$\Rightarrow a - (-1) = 5$$

$$\Rightarrow a + 1 = 5 \Rightarrow a = 4$$

Therefore, $a = 4$ and $b = -1$.

(iv) The equation of line parallel to $\overline{AB}$:

The slope of the line parallel to $\overline{AB}$ is the same

as the slope of $\overline{AB}$, which is $m_{AB} = 4$.

To find the equation of the line parallel to $\overline{AB}$ through $(0, 3)$, we use the point-slope form.

$$y - y_1 = m(x - x_1)$$

Substitute the slope $m = 4$ and the point $(0, 3)$.

$$\Rightarrow y - 3 = 4(x - 0)$$

$$\Rightarrow y - 3 = 4x$$

$$\Rightarrow 0 = 4x - y + 3$$

$$\Rightarrow 4x - y + 3 = 0$$

The equation of the line parallel to $\overline{AB}$ passing $(0, 3)$ is $4x - y + 3 = 0$

Q.10 Find the equation of line passing through mid-point of $(4, 4)$ and $(8, 0)$ parallel to the line having slope $\frac{2}{3}$. **Solve**

Solution

First, we find the midpoint M of the segment joining the points $(4, 4)$ and $(8, 0)$

$$\Rightarrow M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\Rightarrow M\left(\frac{4 + 8}{2}, \frac{4 + 0}{2}\right)$$

$$\Rightarrow M\left(\frac{12}{2}, \frac{4}{2}\right)$$

$$\Rightarrow M(6, 2)$$

Two parallel lines have same slope.

Using the point-slope form with the midpoint

$M(6, 2)$ and slope $\frac{2}{3}$, the equation of the line is



$$\Rightarrow y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 2 = \frac{2}{3}(x - 6)$$

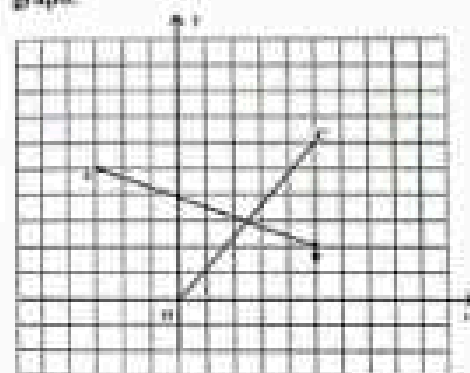
$$3(y - 2) = 2(x - 6)$$

$$3y - 6 = 2x - 12$$

$$6 = 2x - 12 - 3y + 18$$

$$2x - 3y - 2 = 0$$

Q.11 Lines $\overline{OC}$ and $\overline{AB}$ are shown in the graph.



Find

(i) Coordinates of end points of $\overline{OC}$ and $\overline{AB}$. **Solve**

(ii) Slopes of both lines. **Solve**

(iii) Equations of both lines. **Solve**

(iv) Coordinates of point of intersection of both lines. **Solve**

Solution

(i) Coordinates of endpoints

• For $\overline{OC}$: $O(0, 0)$, $C(5, 6)$

• For $\overline{AB}$: $A(-3, 5)$, $B(5, 2)$

(ii) Slopes of both lines:

• For of $\overline{OC}$:

$$\Rightarrow m_{OC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 0}{5 - 0} = \frac{6}{5}$$

• For of $\overline{AB}$:

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{5 - (-3)} = \frac{-3}{8}$$

Therefore, Slope $\overline{OC} = \frac{6}{5}$.

Slope of $\overline{AB} = -\frac{3}{8}$

(iii) Equation of both lines

Using point-slope form.

$y - y_1 = m(x - x_1)$ to find the equations of both lines:

(a) Equation of $\overline{OC}$: $O(0, 0)$, $C(5, 6)$

$$\Rightarrow y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 0 = \frac{6}{5}(x - 0) \Rightarrow y = \frac{6}{5}x$$

$$\Rightarrow 5(y) = 5\left(\frac{6}{5}x\right) \Rightarrow 5y = 6x$$

$$\Rightarrow 6x - 5y = 0$$

(b) Equation of $\overline{AB}$: $A(-3, 5)$, $B(5, 2)$

$$\Rightarrow y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 5 = -\frac{3}{8}(x - (-3))$$

$$\Rightarrow y - 5 = -\frac{3}{8}(x + 3)$$

$$\Rightarrow 8(y - 5) = -3(x + 3)$$

$$\Rightarrow 8y - 40 = -3x - 9$$

$$\Rightarrow 8y + 3x - 40 + 9 = 0$$

$$\Rightarrow 3x + 8y - 31 = 0$$

(iv) point of intersection:

To find the point of intersection we solve these equations simultaneously.

$$\text{Equation of } \overline{OC}: 6x - 5y = 0 \dots\dots (i)$$

$$\text{Equation of } \overline{AB}: 3x + 8y = 31 \dots\dots (ii)$$

Multiplying equation (ii) by 2, we get

$$2(3x + 8y) = 2(31)$$

$$6x + 16y = 62 \dots\dots\dots (iii)$$



Subtracting equation (ii) from (iii)

$$\begin{aligned} 6x + 16y &= 62 \\ \underline{+ 6x + 2y &= 0} \\ 21y &= 62 \\ y &= \frac{62}{21} \end{aligned}$$

Put $y = \frac{62}{21}$ in equation (i)

$$\begin{aligned} 6x - 5y &= 0 \\ 6x - 5\left(\frac{62}{21}\right) &= 0 \\ 6x &= \frac{310}{21} \\ x &= \frac{310}{21 \times 6} \\ x &= \frac{310}{126} = \frac{155}{63} \Rightarrow x = \frac{155}{63} \end{aligned}$$

Therefore, coordinates of point of intersection of given lines are $\left(\frac{155}{63}, \frac{62}{21}\right)$

Q.12 Locate two points on the coordinate plane that satisfy the equation $x - 2y = 2$. Find:

- The slope of segment l connecting two points. reason
- Slope of segment p perpendicular to l . reason
- Mid-point of the segment l . reason
- Equation of line passing through mid-point of segment l and slope of p . reason

Solution

Locating two points:

To locate two points that satisfy $x - 2y = 2$, let's choose two values for x and solve for y .

$$\begin{aligned} \bullet \text{ put } x &= 0, \\ \Rightarrow x - 2y &= 2 \\ \Rightarrow 0 - 2y &= 2 \\ \Rightarrow -2y &= 2 \quad \Rightarrow y &= -1 \end{aligned}$$

So, one point is $A(0, -1)$

$$\begin{aligned} \bullet \text{ Now } x &= 2, \\ \Rightarrow x - 2y &= 2 \\ \Rightarrow 2 - 2y &= 2 \\ \Rightarrow 2 - 2 &= 2y \\ \Rightarrow 0 &= 2y \Rightarrow y &= 0 \end{aligned}$$

So, another point is $B(2, 0)$

Two points on line are $A(0, -1)$, $B(2, 0)$

(i) **Slope of line segment connecting two points**

To find the slope of line segment l connecting $A(0, -1)$ and $B(2, 0)$ we midpoint formula:

$$\Rightarrow m_l = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-1)}{2 - 0}$$

Therefore, the slope of $l = \frac{1}{2}$.

(ii) **The slope of a line segment p perpendicular to l**

The relation for slope of a line perpendicular to another line is:

$$\begin{aligned} m_p \times m_l &= -1 \\ m_p \times \frac{1}{2} &= -1 \\ m_p &= -2 \end{aligned}$$

Thus the slope of perpendicular segment is -1

(iii) **Midpoint of segment l**

To find the midpoint of segment l between $A(0, -1)$ and $B(2, 0)$ we midpoint formula:

$$\begin{aligned} \Rightarrow M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ \Rightarrow M\left(\frac{0 + 2}{2}, \frac{-1 + 0}{2}\right) \\ \Rightarrow M\left(\frac{2}{2}, \frac{-1}{2}\right) = M\left(1, \frac{-1}{2}\right) \end{aligned}$$



Thus the mid-point of segment l is $M\left(1, \frac{-1}{2}\right)$

(iv) **The equation of line passing through midpoint:**

To find the equation of the line passing through the midpoint $M\left(1, \frac{-1}{2}\right)$ of segment

l with the slope of segment $p = m_p = -2$, we use point-slope form:

$$y - y_1 = m(x - x_1)$$

Using point $M\left(1, \frac{-1}{2}\right)$ and $m_p = -2$:

$$\Rightarrow y - y_1 = m(x - x_1)$$

$$\Rightarrow y - \left(\frac{-1}{2}\right) = -2(x - 1)$$

Multiplying both sides by 2, we get

$$\begin{aligned} \Rightarrow 2y + 1 &= -4(x - 1) \\ \Rightarrow 2y + 1 &= -4x + 4 \\ \Rightarrow 2y + 4x + 1 - 4 &= 0 \\ \Rightarrow 4x + 2y - 3 &= 0 \end{aligned}$$

So, the equation of the required line is

$$4x + 2y - 3 = 0$$



UNIT 09

Geometry and Polygons

Demonstrative Geometry

It is a branch of geometry in which geometrical statements are proved through logical reasoning.

Reasoning

Reasoning is a way of proving results. The statements without logical reasoning are not acceptable. Different forms of reasoning are accepted in different cases. There are two types of reasoning.

i) Inductive reasoning ii) Deductive reasoning

i) Inductive Reasoning

In inductive reasoning we work from some specific observations to a general result.

ii) Deductive Reasoning

In deductive reasoning we work from some general result to a specific conclusion.

Example 1

Prove that $3x + 9 = 3(x + 3)$. *****

Solution

Let $x = 1$

$$\text{LHS} = 3(1) + 9 = 12$$

$$\text{RHS} = 3(1+3) = 3(4) = 12$$

Similarly, the statement is true for all real numbers.

Hence proved.

Example 2

Find $(102)^2 - 7$ *****

$$(a+b)^2 = ?$$

Solution

$$\text{General: } (a+b)^2 = a^2 + b^2 + 2ab$$

$$= 100^2 + 2^2 + 2 \times 100 \times 2$$

$$= 10000 + 4 + 400 = 10404$$

Mathematical Statement

A mathematical statement is a meaningful composition of words that can be either true or false. It may contain words and symbols.

i) Sum of 2 and 5 is 7.

ii) Square root of 36 is 6.

Proposition

A statement which may or may not be true is called a proposition.

There are three parts of proposition.

i) The premise

It is assumption that something is true.

ii) The argument

The logical chain of reasoning that leads from the premises to the conclusion is known as an argument.

iii) The conclusion

The result obtained after giving argument to the premises is called conclusion. The conclusion must be true if the premises are true and the argument is valid.

Fundamental assumptions

Fundamental assumptions are statements which are regarded true without any proof.

1. Axiom

An axiom is a mathematical statement which is assumed to be true without any proof. Axioms are truths that have been derived on the basis of everyday experience and form the basis for all other derivations. Axioms deal with numbers and their relations.

For example:

A number is always equal to itself (reflexive property). i.e. if x is any real number then $x = x$.



1. Postulate

Axioms related to geometry are called postulates. Some postulates of geometry are given below.

i) Infinite number of lines can pass through a point.

ii) Through two different points one and only one line can pass.

2. Conjecture

A statement that is believed to be true but its truth has not been proved is called a conjecture.

Proof

A proof is a series of conjectures and axioms (postulates) and proved theorems that combine together to give a true result.

Theorem

A mathematical statement which can be proved or supposed provable through logical reason is called a theorem.

Important Steps of Proof of a Theorem

i) Statement

A description of a geometrical theorem in words should be written first. It is called statement of the theorem.

ii) Figure

After writing the statement a neat figure should

be drawn to explain or understand the given information and the result to be proved.

iii) Given

In this step only given information should be written symbolically so that it becomes easy to use in the proof.

iv) Required to Prove

After given information it is necessary to write the result symbolically which is to be proved so that our attention does not divert from the main objective at any stage.

v) Construction

The necessary addition in the figure can be done to prove the theorem easily. This addition in the figure is called construction.

vi) Proof

It is the most important step. It consists of statements and facts along with their reasons through which we obtain the required result.

Corollary

Some results which can be deduced directly from theorems are called corollaries.

Converse of a Theorem

If given and to prove of a theorem are interchanged, the new statement is called converse of a theorem. It is not necessary that converse of a theorem is also theorem.

Exercise 9.1

Q.1 What is the difference between axiom and conjecture. *****

Solution

An axiom is a mathematical statement which is assumed to be true without any proof.

For example, If $a = b$ and $b = c$, then $a = c$ (transitive property).

Conjecture

A statement that is believed to be true but its truth has not been proved is called a conjecture.

For example, observe the following pattern of numbers: 4, 8, 12, 16 *****

What is next number in the pattern? One of the conjectures that, the next number is $16+4 = 20$.

Q.2 Which of the following mathematical statements?

i) Difference of 19 and 12 is 7. *****

ii) $-2+7-3=2$ *****

iii) $34+16=50$ *****

iv) $a+b=9$ *****

v) $(a+b)^2 = a^2 + b^2 + 2ab + b^2$ *****



- (i) $2 + 3 \times 2 = 6$ (closed)
 (ii) The product of x and y is smaller than 5. (open)
 (iii) If x is real then either $x < 0$ or $x > 0$ or $x = 0$. (closed)
 (iv) If $a > b$ and $b > c$ then $a < c$ (closed)
 (v) $xy + z = 12$ (open)
 (vi) $x - 1 = 4$ if $x = 4$ and $x = 0$ (closed)

Solution

The following statements are mathematical statements.

- (i) Difference of 19 and 12 is 7 (True)
 (ii) $-2 + 5 - 3 = 2$ (True)
 (iii) $34 - 16 = 18$ (False)
 (iv) (i)
 (v) $(a+b)^2 = a^2 + 2ab + b^2$ (True for all values of a and b)

- (vi) $2 + 3 \times 2 = 6$ (True)
 (vii) (i)
 (viii) If x is real then either $x < 0$ or $x > 0$ or $x = 0$ (True)
 (ix) If $a > b$ and $b > c$ then $a < c$. (True)
 (x) (i)
 (xi) $x - 1 = 4$ if $x = 4$ and $x = 0$ (True for given condition)

The following statements are not mathematical statements.

- (ii) The product of x and y is smaller than 5. (open)
 (iv) $a - b = 9$ (open)
 (v) $xy + z = 12$ (open)

Q.3 The sum of a and b is equal to 0. Is this sentence a mathematical statement? If not, how can we make it a mathematical statement? (closed)

Solution

"The sum of a and b is equal to 0".

This sentence is not a mathematical statement because it is an open sentence.

We can make the given sentence a mathematical statement by adding conditions

for a and b . For example, the sum of a and b is equal to 0 if $a = 5$ and $b = 5$.

Q.4 Prove $(x+1)^2 + 5 = x^2 + 2x + 6$ by taking $x = 1, 5$ and 10 . (closed)

Solution

$$(x+1)^2 + 5 = x^2 + 2x + 6$$

(i) When $x = 1$

$$\begin{aligned} \text{L.H.S.} &= (x+1)^2 + 5 \\ &= (1+1)^2 + 5 \\ &= (2)^2 + 5 \\ &= 4 + 5 \\ &= 9 \end{aligned}$$

$$\text{R.H.S.} = (x)^2 + 2(x) + 6 = 1 + 2 + 6 = 9$$

From (i) and (ii)

$$\text{L.H.S.} = \text{R.H.S.}$$

(ii) When $x = 5$

$$\begin{aligned} \text{L.H.S.} &= (x+1)^2 + 5 \\ &= (5+1)^2 + 5 \\ &= (6)^2 + 5 \\ &= 36 + 5 \\ &= 41 \end{aligned}$$

$$= 41 \text{ — (i)}$$

Now,

$$\begin{aligned} \text{R.H.S.} &= x^2 + 2x + 6 \\ &= (5)^2 + 2(5) + 6 \\ &= 25 + 10 + 6 \\ &= 41 \text{ — (ii)} \end{aligned}$$

From (i) and (ii) L.H.S. = R.H.S.

(iii) When $x = 10$

$$\begin{aligned} \text{L.H.S.} &= (x+1)^2 + 5 \\ &= (10+1)^2 + 5 = (11)^2 + 5 = (121) + 5 = 126 \text{ — (i)} \\ \text{R.H.S.} &= x^2 + 2x + 6 \\ &= (10)^2 + 2(10) + 6 \\ &= 100 + 20 + 6 \\ &= 126 \text{ — (ii)} \end{aligned}$$

From (i) and (ii) L.H.S. = R.H.S.

Conclusion: $(x+1)^2 + 5 = x^2 + 2x + 6$ is an identity equation and it is true for all values of variable x .

Q.5 Find the next number in the pattern using conjecture. 1, 3, 7, 15, 31, ____ State the conjecture used. (closed)

Solution

Pattern: 1, 3, 7, 15, 31

The next number in the pattern is $31 \times 2 + 1 = 63$

Q.6 Which of the following are axioms? How many of the axioms are postulates?

- (i) If $a = b$ then $b = a$ (axiom)
 (ii) 2 plus 2 make 4. (axiom)
 (iii) One and only one line can pass through two points. (axiom)
 (iv) If two sides of a triangle are equal then opposite angles are also equal. (axiom)
 (v) Product of two negative real numbers is always greater than zero. (axiom)
 (vi) All right angles are equal to one another. (axiom)
 (vii) The whole is greater than its part. (axiom)

- (viii) If $a > b$ and $c > d$ then $a+c > b+d$
 (ix) It is possible to extend a line segment continuously in both directions. (axiom)
 (x) When we add three consecutive even numbers, their sum is even. (axiom)

Solution

Statement	
(i) If $a = b$ then $b = a$	axiom
(ii) 2 plus 2 make 4	axiom
(iii) "One and only one line can pass through two points"	postulate
(iv) "If two sides of a triangle are equal then opposite angles are also equal"	neither axiom nor postulate
(v) "Product of two negative real number is always greater than zero"	axiom
(vi) "All right angles are equal to one another"	postulate
(vii) The whole is greater than its part	axiom
(viii) If $a > b$ and $c > d$ then $a+c > b+d$	axiom



(ix) It is possible to extend a line segment continuously in both directions.	Postulate
(x) When we add three consecutive even numbers, their sum is even.	axiom

Q.7 Explain all the steps of geometrical proof.

Solution

Important steps of geometrical proof are given below.

- (i) Statement
 (ii) Figure
 (iii) Given
 (iv) To prove
 (v) Construction
 (vi) Proof

See explanation on previous page 303.

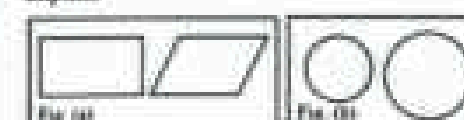
Similar Figures

Two or more figures that have the same shape but not the same size are called similar figures. Figures (i) and (ii) below represent the pairs of similar figures.



The symbol for similarity is '~'. Thus, if two figures A and B are similar, then we write $A \sim B$.

Example 3 See the following pairs of figures and explain.



Solution
 In Fig. (a), two quadrilaterals do not have the same shape. Hence, they are not similar. In Fig. (b), two circles have the same shape but not the same size, therefore they are similar.

**Example 4**

Solve

Check whether the following figures are similar.

All measurements are in centimetres.

i)



$$\frac{AC}{DF} = \frac{BC}{EF} = \frac{1}{2} \text{ and } \angle C = \angle F$$

So, $\triangle ABC \sim \triangle DEF$.

ii)



Solution

So $m\angle A = m\angle E$, $m\angle B = m\angle F$, $m\angle C = m\angle G$ and thus $m\angle D = m\angle H$.

Therefore, both quadrilaterals are similar.

Example 5

Solve

In the figure below $ALMN \sim \triangle XYZ$. Find values of x and y .



Solution

Since the two triangles are similar.

$$\begin{aligned} \frac{x}{4} &= \frac{y}{3} = \frac{5}{5} \\ \Rightarrow \frac{x}{4} &= \frac{y}{3} \text{ and } \frac{5}{5} = 1 \\ x &= 4 \text{ and } y = 3 \end{aligned}$$

Example 6

Solve

In the figure height of pole is 7.5 m. Find the height of wall.



Solution

Let height of wall = $CD = x$.

Join A to E and D.

Here $BE \parallel CD$ in the $\triangle ACD$.

$$\therefore \frac{AB}{AC} = \frac{BE}{CD}$$

$$\frac{15}{40} = \frac{7.5}{x}$$

$$15x = 40 \times 7.5$$

$$x = \frac{40 \times 7.5}{15} = 22.5$$

So, height of wall = 22.5 m.

Example 7

Solve

In $\triangle ABC$, $\overline{AB} = 10$ cm, $\overline{AC} = 3$ cm. Take D and E on $\overline{AB}$ and $\overline{AC}$ respectively such that $\overline{AD} = 4$ cm and $\overline{AE} = 2$ cm. Is $\overline{DE} \parallel \overline{BC}$?



Solution

$$\text{As } \frac{AD}{DB} = \frac{4}{6} = \frac{2}{3} \text{ [As } BD = AB - AD \text{]} \dots (i)$$

$$\text{And } \frac{AE}{EC} = \frac{2}{1} \text{ [As } EC = AC - AE \text{]} \dots (ii)$$

From (i) and (ii)

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

which shows that $\overline{DE}$ intersects $\overline{AB}$ and $\overline{AC}$ in the same ratio.

$$\therefore \overline{DE} \parallel \overline{BC}$$

Exercise 9.2

Q.1 Which of the following pairs of figures are similar?

Solve

Solution

i)



In right angled triangle acute angles are complementary. So,

$$m\angle A = 60^\circ \text{ and } m\angle R = 50^\circ$$

In $\triangle ABC \leftrightarrow \triangle PQR$

$$m\angle A = m\angle P = 60^\circ$$

$$m\angle B = m\angle Q = 90^\circ$$

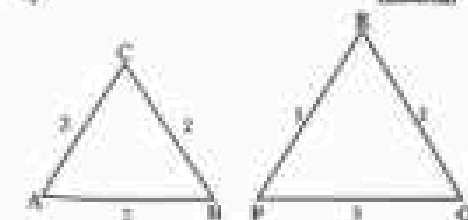
$$m\angle C = m\angle R = 30^\circ$$

All corresponding angles are equal. So

$$\triangle ABC \sim \triangle PQR$$

Given pair of triangles are similar.

ii)



In $\triangle ABC$ and $\triangle PQR$

$$\frac{AB}{PQ} = \frac{2}{3} \dots (i)$$

$$\frac{BC}{QR} = \frac{2}{3} \dots (ii)$$

$$\frac{CA}{RP} = \frac{2}{3} \dots (iii)$$

From (i), (ii) and (iii)

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = \frac{2}{3}$$

Since corresponding sides of given triangles are proportional to these triangles are similar.

iii)

Solve



Given quadrilateral do not have the same shape. Hence they are not similar.

iv)

Solve



In $\triangle ABC$ and $\triangle PQR$, $m\angle B = m\angle Q = 130^\circ$

$$\frac{AB}{PQ} = \frac{6}{4} = \frac{3}{2} = 1.5 \dots (i)$$

$$\frac{BC}{QR} = \frac{4.5}{3} = 1.5 \dots (ii)$$

From (i) and (ii)

$$\frac{AB}{PQ} = \frac{BC}{QR} \text{ and } m\angle B = m\angle Q$$

As two pairs of corresponding sides and one pair of corresponding angles are equal so, given triangles are similar.



In $\triangle ABC$ and $\triangle PQR$

$$\frac{mAB}{mPQ} = \frac{4}{6} = 0.6666... \quad (i)$$

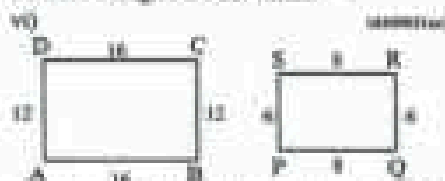
$$\frac{mBC}{mQR} = \frac{2}{3} = 0.6666... \quad (ii)$$

$$\frac{mCA}{mRP} = \frac{3}{4} = 0.75 \quad (iii)$$

From (i), (ii) and (iii)

$$\frac{mAB}{mPQ} \neq \frac{mBC}{mQR} \neq \frac{mCA}{mRP}$$

All corresponding sides are not proportional.
So these triangles are not similar.



In rectangle $ABCD$ and rectangle $PQRS$ opposite sides are equal.

$$\frac{mAB}{mPQ} = \frac{16}{8} = 2 \quad (i) \quad \frac{mBC}{mQR} = \frac{12}{6} = 2 \quad (ii)$$

$$\frac{mAD}{mSP} = \frac{12}{6} = 2 \quad (iii) \quad \frac{mCD}{mSR} = \frac{16}{8} = 2 \quad (iv)$$

In rectangle opposite sides are equal so all corresponding sides are proportional.

$$\frac{mAB}{mPQ} = \frac{mBC}{mQR} = \frac{mCD}{mRS} = \frac{mDA}{mSP}$$

Given figures are similarly



v)

In $\triangle ABC$ and $\triangle PQR$,

$$m\angle A = m\angle P = 30^\circ \quad (i)$$

$$\frac{mAB}{mPQ} = \frac{6}{9} = 0.666... \quad (ii)$$

$$\frac{mAC}{mPR} = \frac{5}{7.5} = 0.666... \quad (iii)$$

From (i) and (ii)

$$\frac{mAB}{mPQ} = \frac{mAC}{mPR} \text{ and } m\angle A = m\angle P$$

As two pairs of corresponding sides and one pair of corresponding angles are equal so, triangles are similar.

Q.2 Find the unknown quantities in the following similar figures. All measurements are in centimeters.

Solutions

i)



Given that triangles are similar so,

$$\frac{mAB}{mPQ} = \frac{mBC}{mQR} = \frac{mCA}{mRP}$$

$$\frac{8}{4} = \frac{3}{1.5} = \frac{5}{x}$$

$$\Rightarrow \frac{8}{4} = \frac{3}{1.5} \text{ and } \frac{8}{4} = \frac{5}{x}$$

$$2 = \frac{3}{1.5} \text{ and } 2 = \frac{5}{x}$$

$$\Rightarrow 10 = x \text{ and } 6 = y$$

$$\Rightarrow \boxed{x = 10 \text{ cm}} \text{ and } \boxed{y = 6 \text{ cm}}$$



ii)



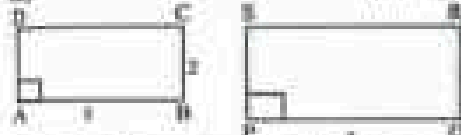
Given that triangles are similar so,

$$m\angle A = m\angle P \text{ (corresponding angles)}$$

$$30^\circ = x$$

$$\Rightarrow x = 30^\circ$$

iii)



Given that figures are similar so, ratio of corresponding sides are equal.

$$\frac{mAB}{mPQ} = \frac{mBC}{mQR}$$

$$\frac{1}{x} = \frac{2}{3}$$

$$\Rightarrow \frac{1}{x} = \frac{2}{3}$$

$$\Rightarrow x = \frac{3}{2} = 1.5$$

$$\Rightarrow x = \frac{1.5}{2} = 7.5 \text{ cm}$$

$$\Rightarrow \boxed{x = 7.5 \text{ cm}}$$

iv)



Given that figures are similar so, ratio of corresponding sides are equal.

$$\frac{mAB}{mPQ} = \frac{mBC}{mQR} = \frac{mCD}{mRS} = \frac{mDA}{mSP}$$

$$\frac{1}{x} = \frac{1.5}{3} = \frac{1}{6} = \frac{1}{6}$$

$$\Rightarrow \frac{1}{x} = \frac{1.5}{3} = \frac{1}{6}$$

$$\Rightarrow \frac{1}{x} = \frac{1}{6} \rightarrow (i)$$

$$\Rightarrow \frac{1.5}{y} = \frac{1}{6} \rightarrow (ii)$$

$$\Rightarrow \frac{1}{4} = \frac{1}{6} \rightarrow (iii)$$

From equation (i)

$$\frac{1}{x} = \frac{1}{6}$$

$$6 = 4 = 2x$$

$$\frac{24}{3} = x$$

$$8 = x$$

$$\boxed{x = 8 \text{ cm}}$$

From equation (ii)

$$\frac{y}{1.5} = \frac{1}{3}$$

$$3y = 1.5$$

$$y = \frac{1.5}{3}$$

$$y = 0.5 \text{ cm}$$

From equation (iii)

$$\frac{x}{4} = \frac{1}{6}$$

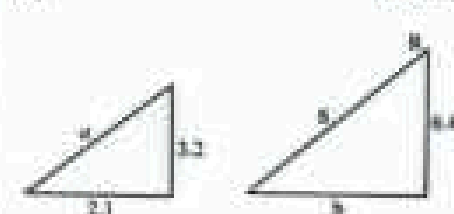
$$6x = 4$$

$$6x = 12$$

$$x = \frac{12}{6}$$

$$\boxed{x = 2 \text{ cm}}$$

v)



Given that figures are similar, so ratio of corresponding sides are equal.

$$\frac{m\overline{AB}}{m\overline{PQ}} = \frac{m\overline{BC}}{m\overline{QR}} = \frac{m\overline{CA}}{m\overline{RP}}$$

$$\frac{2.1}{b} = \frac{1.2}{6.4} = \frac{a}{8}$$

$$\Rightarrow \frac{2.1}{b} = \frac{1.2}{6.4} \text{ and } \frac{1.2}{6.4} = \frac{a}{8}$$

$$\Rightarrow \frac{b}{2.1} = \frac{6.4}{1.2} \text{ and } \frac{1}{2} = \frac{a}{8}$$

$$b = 2 \times 2.1 \text{ and } \frac{1}{2} \times 8 = a$$

$$\Rightarrow \boxed{b = 4.2 \text{ cm}} \text{ and } \boxed{a = 4 \text{ cm}}$$

v)



Given that triangles are similar, so ratio of corresponding sides are equal.

$$\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{BC}}{m\overline{EF}} = \frac{m\overline{CA}}{m\overline{FD}}$$

$$\frac{4}{8} = \frac{3}{b} = \frac{4}{10}$$

$$\Rightarrow \frac{4}{8} = \frac{3}{b} \text{ and } \frac{3}{b} = \frac{4}{10}$$

$$a = \frac{3}{4} \times 8 \text{ and } \frac{b}{3} = \frac{4}{10}$$

$$a = \frac{40}{10} \text{ and } \frac{b}{3} = 2$$

$$\boxed{a = 4 \text{ cm}} \text{ and } \boxed{b = 2 \times 3}$$

$$\boxed{b = 6 \text{ cm}}$$



Given that figures are similar, so ratio of corresponding sides are equal.

$$\frac{m\overline{MN}}{m\overline{NO}} = \frac{m\overline{NL}}{m\overline{NP}}$$

$$\frac{a}{8} = \frac{14}{16}$$

$$\frac{a}{8} = \frac{7}{8}$$

$$a = \frac{7}{8} \times 8$$

$$a = \frac{24}{2}$$

$$a = 12 \text{ cm}$$

Q.3 Two triangles ABC and DEF are similar and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = 2$

If $AB = 8 \text{ cm}$, $BC = 3 \text{ cm}$, $CA = 4 \text{ cm}$, find the lengths of sides of ADEF.

Solution

$$m\overline{AB} = 8 \text{ cm}, m\overline{BC} = 3 \text{ cm}, m\overline{CA} = 4 \text{ cm}$$

Given that

$$\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{BC}}{m\overline{EF}} = \frac{m\overline{CA}}{m\overline{FD}} = 2$$

$$\Rightarrow \frac{m\overline{AB}}{m\overline{DE}} = 2, \frac{m\overline{BC}}{m\overline{EF}} = 2, \frac{m\overline{CA}}{m\overline{FD}} = 2$$

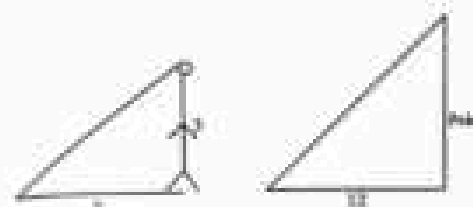
Putting the values, we get

$$\frac{8 \text{ cm}}{m\overline{DE}} = 2, \frac{3 \text{ cm}}{m\overline{EF}} = 2, \frac{4 \text{ cm}}{m\overline{FD}} = 2$$

$$\frac{8 \text{ cm}}{2} = m\overline{DE}, \frac{3 \text{ cm}}{2} = m\overline{EF}, \frac{4 \text{ cm}}{2} = m\overline{FD}$$

$$\Rightarrow \boxed{DE = 4 \text{ cm}}, \boxed{EF = 1.5 \text{ cm}}, \boxed{FD = 2 \text{ cm}}$$

Q.4 While going back to home from school, Wajid noted that his shadow was $\frac{2}{3}$ of his height. Find the height of people nearby having shadow of 12m.



Solution

Let height of pole is h.
Since given triangles are similar.
(Both are right angled Δ s).
Ratio of corresponding sides are equal.

$$\frac{h}{3} = \frac{12}{4}$$

$$h = 3 \times 3 \Rightarrow h = 9 \text{ m}$$

Thus height of pole is 9m.

Q.5 Find the length of the larger rope of the hanging bridge, where $x = y$.



Solution

Let length of rope = $l = ?$
As both triangles are right angled triangles in which given that $x = y$, being complementary angles of right triangles other angles are also equal.

Both figures are similar and ratio of their corresponding sides are equal.

$$\frac{l}{10 \text{ m}} = \frac{12 \text{ m}}{8 \text{ m}}$$

$$l = \frac{12}{8} \times 10 \text{ m}$$

$$l = \frac{120}{8} \text{ m}$$

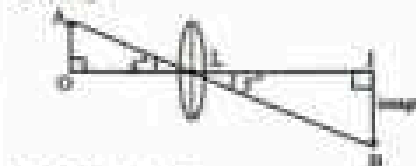
$$l = 15 \text{ m}$$

Thus length of larger rope is 15m.

Q.6 In the figure, OA is object lying in front of a convex lens at a distance of 10cm.

Find the distance of image from the lens if its size is twice that of the object.

Solution



In ΔLOA and ΔLOB
 $m\angle O = m\angle O \Rightarrow$ (both are right angle)
 $m\angle x' = m\angle y' \Rightarrow$ (vertically opposite angles)
 $m\angle A = m\angle B \Rightarrow$ (correspondence of acute angle is right Δ)
So, $\Delta LOA \sim \Delta LOB$

$$\text{Distance of image from lens} = m\overline{OL} = ?$$

$$\text{Distance of object from lens} = m\overline{OL} = 10 \text{ cm}$$

$$m\overline{OB} = 2 m\overline{OA} \text{ (given)}$$

$$\Rightarrow \frac{m\overline{OB}}{m\overline{OA}} = 2 \dots (1)$$

In similar triangle ratio of corresponding sides are equal.

$$\frac{m\overline{OB}}{m\overline{OA}} = \frac{m\overline{OL}}{m\overline{OA}}$$

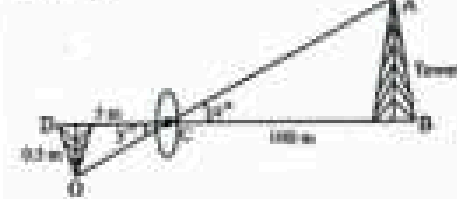
$$\frac{m\overline{OB}}{10 \text{ cm}} = 2 \text{ (from 1)}$$

$$m\overline{OB} = 2 \times 10 \text{ cm}$$

$$\Rightarrow \boxed{m\overline{OB} = 20 \text{ cm}}$$

Q.7 In the figure, an electricity tower is seen through the telescope. Find height of tower if height of its image is 8.5 m.

Solution





Height of tower = $m\overline{AB} = h$

Distance of tower from lens = $m\overline{BC} = 100m$

Distance of image from Lens = $m\overline{CD} = 5m$

Height of image = $m\overline{DE} = 0.5m$

In $\triangle ABC$ and $\triangle EDC$

$m\angle B = m\angle D$ (both right angles)

$m\angle C = m\angle C$ (vertically opposite $\angle$ s)

$m\angle A = m\angle E$ (complements of equal acute angles)

$\therefore \triangle ABC \sim \triangle EDC$

In similar triangle ratio of corresponding sides are equal.

$$\frac{m\overline{AB}}{m\overline{CD}} = \frac{m\overline{BC}}{m\overline{DC}}$$

$$\frac{h}{0.5m} = \frac{100m}{5m}$$

$$\frac{h}{0.5m} = 20$$

$$\frac{h}{0.5m} = 20$$

$$h = 20 \times 0.5m$$

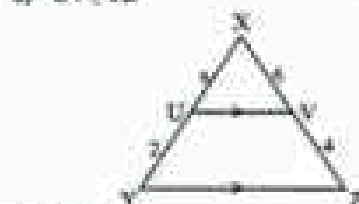
$$h = 10m$$

Thus height of tower is 10m.

Q.8 Find the values of unknown quantities in the following figures. All the measurements are in centimeters.

a) $\overline{UV} \parallel \overline{YZ}$

Answer



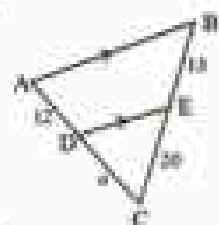
b) $\overline{BC} \parallel \overline{DE}$

Answer



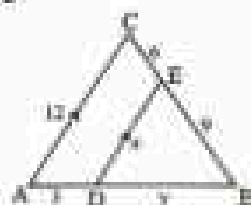
a) $\overline{AB} \parallel \overline{DE}$

Answer



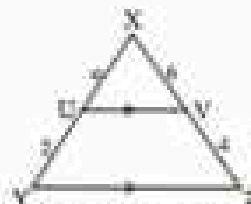
b) $\overline{AC} \parallel \overline{DE}$

Answer



Solution

a)



Since $\overline{UV} \parallel \overline{YZ}$ and divides other two sides of $\triangle XYZ$ proportionally i.e.

$$\frac{m\overline{XU}}{m\overline{UY}} = \frac{m\overline{XV}}{m\overline{VZ}}$$

$$\frac{3}{2} = \frac{4}{4}$$

$$4 = \frac{8}{4} \times 2$$

$$4 = \frac{8}{4} \times 2$$

$$4 = \frac{12}{4}$$

$$4 = 3cm$$

b) $\overline{BC} \parallel \overline{DE}$



Given that $\overline{BC} \parallel \overline{DE}$, $\overline{DE}$ divides other two sides of $\triangle ABC$ proportionally i.e.

$$\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}$$

$$\frac{12}{5} = \frac{8}{6}$$

$$\frac{12}{5} \times 6 = 8$$

$$\frac{12}{5} \times 6 = 8$$

$$14.4 = 8$$

$$8 = 14.4$$

$$\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}$$

$$\frac{12}{5} = \frac{8}{6}$$

$$12 \times 6 = 8 \times 5$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$



Given that $\overline{AB} \parallel \overline{DE}$ and $\overline{DE}$ divides other two sides of $\triangle ABC$ proportionally i.e.

$$\frac{m\overline{AD}}{m\overline{DC}} = \frac{m\overline{BE}}{m\overline{EC}}$$

$$\frac{12}{5} = \frac{8}{6}$$

$$12 \times 6 = 8 \times 5$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

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$$72 = 40$$

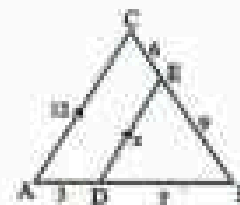
$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$

$$72 = 40$$



Given that $\overline{AC} \parallel \overline{DE}$

A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.

$$\frac{m\overline{BD}}{m\overline{DA}} = \frac{m\overline{BE}}{m\overline{EC}}$$

$$\frac{y}{3} = \frac{8}{6}$$

$$\frac{y}{3} = \frac{8}{6}$$

$$y = \frac{8}{6} \times 3$$

$$y = \frac{8}{6} \times 3$$

$$y = \frac{8}{6} \times 3$$

$$y = \frac{8}{6} \times 3$$

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$$y = \frac{8}{6} \times 3$$

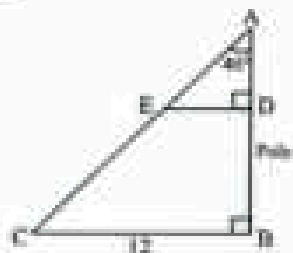
$$y = \frac{8}{6} \times 3$$



Q.9 In the following figure, find the measure of $\angle C$ and $\angle AED$, if $\overline{ED} \parallel \overline{CB}$.
(1000)



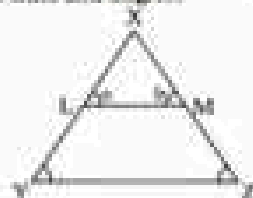
Solution:



In right $\triangle ADE$,
 $m\angle D = 90^\circ$, $m\angle A = 40^\circ$
 We know that
 $m\angle A + m\angle D + m\angle E = 180^\circ$
 $40^\circ + 90^\circ + m\angle E = 180^\circ$
 $130^\circ + m\angle E = 180^\circ$
 $m\angle E = 180^\circ - 130^\circ$
 $m\angle E = 50^\circ$ (i)
 In right $\triangle ABC$,
 $m\angle B = 90^\circ$, $m\angle A = 40^\circ$
 We know that
 $m\angle A + m\angle B + m\angle C = 180^\circ$
 $40^\circ + 90^\circ + m\angle C = 180^\circ$
 $130^\circ + m\angle C = 180^\circ$
 $m\angle C = 180^\circ - 130^\circ$
 $m\angle C = 50^\circ$ (ii)
 From (i) and (ii)

$m\angle E = m\angle C$ and $m\angle D = m\angle B$
 By property of parallel lines
 Since corresponding angles are equal in,
 $\overline{BC} \parallel \overline{ED}$

Q.10 In the figure, $\triangle XYZ$ is an equilateral triangle and $\overline{LM} \parallel \overline{YZ}$. Find measure of a and b . What type of triangle is XLM with respect to sides and angles? (1000)



Solution:

- (i) Given that $\triangle XYZ$ is an equilateral triangle.
 Equilateral $\triangle$ is also an equiangular $\triangle$ with each angle equal to 60° i.e.
 $m\angle X = 60^\circ$, $m\angle Y = 60^\circ$, $m\angle Z = 60^\circ$
 Given that $\overline{LM} \parallel \overline{YZ}$,
 $m\angle X = m\angle Y$ (corresponding angles)
 $a = 60^\circ$
 Now, $m\angle M = m\angle Z$ (corresponding angles)
 $b = 60^\circ$
 (ii) In $\triangle XLM$,
 $m\angle L = 60^\circ$, $m\angle M = 60^\circ$, $m\angle X = 60^\circ$
 (iii) $\triangle XLM$ is an acute angled triangle.
 $\triangle XLM$ is also an equilateral triangle.

Q.11 Find the height of shorter tree if longer one is 12m high from the ground level, where observer is at a distance of 25 m from the shorter tree and distance between both trees is 5m (1000)



Solution:



Height of shorter tree = $m\angle D = h = ?$
 Height of longer tree = $m\angle B = 12m$
 Distance of observer from shorter tree
 $= \overline{CO} = 25m$
 Distance between both trees = $m\angle BC = 5m$
 Distance of observer from longer tree
 $= \overline{OB} = (25+5) = 30m$

Given, $\overline{AB} \parallel \overline{CD}$

$$\frac{m\angle OB}{m\angle C} = \frac{m\angle AB}{m\angle D}$$

$$\frac{30m}{25m} = \frac{12m}{h}$$

$$\frac{6}{5} = \frac{12}{h}$$

$$6h = 5 \times 12m$$

$$h = \frac{60m}{6}$$

$$h = 10m$$

Thus height of shorter tree is 10m.

Area and volume of similar figures

Area of similar figures

1) In the figure, $\triangle ABC$ and $\triangle DEF$ are similar because:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{1}{2}$$



If A and A_1 denote the area of $\triangle ABC$ and $\triangle DEF$ respectively, then:

$$\therefore \text{Area of } \triangle = \frac{1}{2} (b \times h)$$

$$A_1 = \frac{1}{2} \times 4 \times 3 = 6\text{cm}^2 \text{ and}$$

$$A_2 = \frac{1}{2} \times 8 \times 6 = 24\text{cm}^2$$

$$\text{Now } \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{1}{2} \text{ and}$$

$$\frac{A_1}{A_2} = \frac{6}{24} = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

$$\text{Therefore: } \frac{A_1}{A_2} = \left(\frac{AB}{DE}\right)^2 = \frac{A_1}{A_2} = \left(\frac{BC}{EF}\right)^2 \text{ or}$$

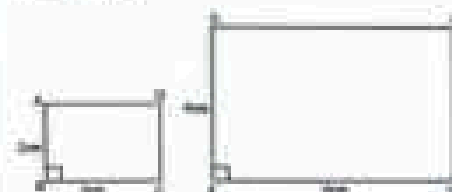
$$\frac{A_1}{A_2} = \left(\frac{CA}{FD}\right)^2$$

Conclusion

Ratio of area of two triangles is equal to the square of the ratio of any two corresponding sides of triangles.

2) In the figure, $ABCD$ and $EFGH$ are similar rectangles because:

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{1}{2}$$



If A_1 and A_2 denote the area of rectangles $ABCD$ and $EFGH$ respectively, then:

$$A_1 = 2 \times 3 = 6\text{cm}^2 \text{ and } A_2 = 4 \times 6 = 24\text{cm}^2$$

$$\text{Now } \frac{AB}{EF} = \frac{BC}{FG} = \frac{1}{2} \text{ and } \frac{A_1}{A_2} = \frac{6}{24} = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$



$$\text{Therefore: } \frac{A_1}{A_2} = \left(\frac{AB}{EF}\right)^2 = \frac{A_1}{A_2} = \left(\frac{BD}{FG}\right)^2$$

Conclusion

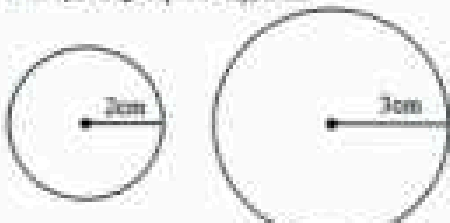
Ratio of areas of two quadrilaterals is equal to the square of the ratio of any two corresponding sides of quadrilaterals.

3) Figure shows two similar circles with radii 2 cm and 3 cm respectively.

If $r_1 = 2$ cm and $r_2 = 3$ cm, then:

$$\frac{r_1}{r_2} = \frac{2}{3}$$

If A_1 and A_2 denote the areas of circles with radii r_1 and r_2 respectively, then:



$$A_1 = \pi \times 2^2 = 4\pi \text{ cm}^2 \text{ and } A_2 = \pi \times 3^2 = 9\pi \text{ cm}^2$$

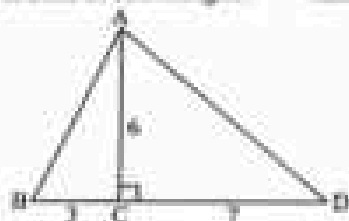
$$\text{Now } \frac{A_1}{A_2} = \frac{4\pi}{9\pi} = \frac{4}{9} = \left(\frac{2}{3}\right)^2$$

$$\text{Therefore: } \frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2$$

Example 8

Triangles ABC and ADC with bases 3 cm and 7 cm respectively have common height 5 cm.

Prove that the ratio of their areas is equal to ratio of bases of both triangles.



Solution

$$\text{Area of } \triangle = \frac{1}{2} (b \times h)$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times 3 \times 5 = 7.5 \text{ cm}^2$$

$$\text{Area of } \triangle ADC = \frac{1}{2} \times 7 \times 5 = 17.5 \text{ cm}^2$$

$$\frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle ADC} = \frac{7.5}{17.5} = \frac{3}{7} = \frac{\text{Base of } \triangle ABC}{\text{Base of } \triangle ADC}$$

Example 9

The ratio of the areas of the bases of two similar cylinders is 9:4.

The area of the base of the smaller cylinder is 240 cm².



i) What is the area of base of larger cylinder?

ii) Write the ratio of heights of both cylinders.

Solution

i) If A_1 and A_2 are areas of bases larger and smaller cylinders respectively, then:

$$\frac{A_1}{A_2} = \frac{9}{4} \Rightarrow \frac{A_1}{240} = \frac{9}{4}$$

$$\Rightarrow A_1 = \frac{9}{4} \times 240 = 540 \text{ cm}^2$$

ii) Again, if h_1 and h_2 heights of larger and smaller cylinders respectively, then:

$$\left(\frac{h_1}{h_2}\right)^2 = \frac{A_1}{A_2} \Rightarrow \left(\frac{h_1}{h_2}\right)^2 = \frac{540}{240} = \frac{9}{4}$$

$$\Rightarrow \frac{h_1}{h_2} = \frac{3}{2}$$

Example 10

In the figure, $\overline{DE} \parallel \overline{BC}$. Find:

$$\text{i) } \frac{DE}{BC} \quad \text{ii) } \frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle ABC}$$

iii) Find the area of $\triangle ADE$ if area $\triangle ABC = 256 \text{ cm}^2$.



i) Area of trapezium DBCE.



$$\text{i) } \frac{DE}{AB} = \frac{AD}{AB} = \frac{3}{5}$$

$$\text{ii) } \frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle ABC} = \left(\frac{AD}{AB}\right)^2 = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$$

Exercise 9.5

1. Following pairs of shapes are similar. Find unknown area in each case. All measurements are in cm and cm².

Solution

i)



Given that triangles are similar we know that

$$\frac{A_1}{A_2} = \left(\frac{a}{b}\right)^2$$

$$\frac{A_1}{32} = \frac{16}{64}$$

$$A_1 = \frac{16}{64} \times 32$$

$$A_1 = \frac{16}{4} \times 32 \text{ cm}^2 \quad (\because A_2 = 32 \text{ cm}^2)$$

$$A_1 = 4 \times 32 \text{ cm}^2$$

$$A_1 = 128 \text{ cm}^2$$

ii)



Given that shapes are similar, we know that

$$\frac{A_1}{A_2} = \left(\frac{a}{b}\right)^2$$

$$\frac{A_1}{72} = \frac{81}{36}$$

$$\Rightarrow \frac{A_1}{72} = \frac{16}{81} \quad (\text{Reciprocal})$$

$$A_1 = \frac{16}{81} \times 72$$

$$A_1 = \frac{16}{9} \times 162 \text{ cm}^2 (\because A_2 = 162 \text{ cm}^2)$$

$$A_1 = 16 \times 18 \text{ cm}^2$$

$$A_1 = 288 \text{ cm}^2$$

iii)



Given that shapes are similar. We know that in similar shapes



$$\frac{A_1}{A_2} = \left(\frac{3}{6}\right)^2$$

$$\frac{A_1}{A_2} = \frac{9}{36}$$

$$A_1 = \frac{9}{36} \times A_2$$

$$A_1 = \frac{1}{4} \times 900 \text{ cm}^2$$

$$A_1 = 225 \text{ cm}^2$$

iv)



Given that shapes are similar.

We know that in similar shapes

$$\frac{A_1}{A_2} = \left(\frac{3.6}{1.8}\right)^2$$

$$\frac{A_1}{A_2} = \left(\frac{3.6}{1.8}\right)^2$$

$$\frac{A_1}{A_2} = (2)^2$$

$$\frac{A_1}{A_2} = 4$$

$$A_1 = 4A_2$$

$$\Rightarrow A_1 = \frac{A_2}{4}$$

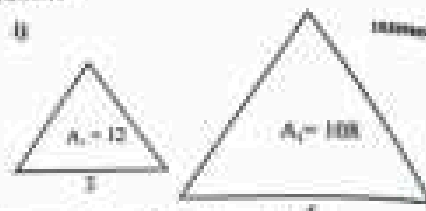
$$A_2 = \frac{180 \text{ cm}^2}{4}$$

$$A_2 = 45 \text{ cm}^2$$

Q.2 Following pairs of shapes are similar. Find unknown area in each case. All measurements are in cm and cm^2 .

Solutions

i)



Given that triangles are similar.
In similar triangles

$$\frac{A_1}{A_2} = \left(\frac{3}{x}\right)^2$$

$$\frac{A_1}{A_2} = \frac{3^2}{x^2}$$

$$\frac{12}{108} = \frac{9}{x^2}$$

$$\frac{x^2}{9} = \frac{108}{12}$$

$$\frac{x^2}{9} = 9$$

$$x^2 = 9 \times 9$$

$$x^2 = 81$$

$$\Rightarrow x^2 = 9^2$$

$$\Rightarrow x = 9 \text{ cm}$$

ii)



Given that shapes are similar.
In similar parallelograms

$$\frac{A_1}{A_2} = \left(\frac{4}{x}\right)^2$$

$$\frac{16}{256} = \frac{4^2}{x^2}$$

$$\frac{1}{16} = \frac{4^2}{x^2}$$

$$\frac{1}{4} \times 100 = x^2$$

$$25 = x^2$$



$$\Rightarrow x^2 = 25$$

$$x^2 = 5^2$$

$$\Rightarrow x = 5 \text{ cm}$$

iii)



Given that shapes are similar.
In similar shapes

$$\frac{A_1}{A_2} = \left(\frac{5}{x}\right)^2$$

$$\frac{100}{100} = \frac{5^2}{x^2}$$

$$1 = \frac{25}{x^2}$$

$$1 \times 100 = x^2$$

$$100 = x^2$$

$$\Rightarrow \sqrt{x^2} = \sqrt{100}$$

$$x = 10 \text{ cm}$$

iv)



Given that shapes are similar.
In similar shapes

$$\frac{A_1}{A_2} = \left(\frac{3}{x}\right)^2$$

$$\frac{36}{100} = \frac{(3)^2}{x^2}$$

$$\frac{36^2}{100} = \frac{256}{x^2}$$

$$2x^2 = 3 \times 256$$

$$x^2 = \frac{3 \times 256}{2}$$

$$x^2 = 3 \times 128$$

$$x^2 = 384$$

$$\sqrt{x^2} = \sqrt{384}$$

$$x = 19.6 \text{ cm}$$

Q.3 Radii of two spheres are 6 cm and 8 cm respectively. Find:

i) The ratio of areas of both spheres. (answer)

ii) Area of larger sphere if area of smaller sphere is 3600 cm^2 . (answer)

iii) Area of smaller sphere if area of larger sphere is 1600 cm^2 . (answer)

Solution

Radius of smaller sphere = $r_1 = 6 \text{ cm}$

Radius of larger sphere = $r_2 = 8 \text{ cm}$

Let A_1 and A_2 be the area of smaller and larger sphere respectively. Shapes of two sphere are similar.

i) Ratio of areas

$$\frac{A_1}{A_2} = \left(\frac{6}{8}\right)^2$$

$$\frac{A_1}{A_2} = \left(\frac{6}{8}\right)^2 = \frac{36}{64} = \frac{9}{16}$$

$$A_1 : A_2 = 9 : 16$$

ii) Area of larger sphere A_2 , if $A_1 = 360 \text{ cm}^2$

$$\frac{A_1}{A_2} = \frac{9}{16}$$

$$\frac{360}{A_2} = \frac{9}{16}$$

$$\Rightarrow \frac{A_2}{360} = \frac{16}{9} \quad (\text{Reciprocal})$$

$$A_2 = \frac{16}{9} \times 360$$

$$A_2 = 16 \times 40$$

$$A_2 = 640 \text{ cm}^2$$

iii) Area of smaller sphere A_1 , if $A_2 = 1600 \text{ cm}^2$

$$\frac{A_1}{A_2} = \frac{9}{16}$$

$$A_1 = \frac{9}{16} = A_2$$

$$A_2 = \frac{9}{16} \times 1600$$

$$A_2 = 9 \times 100$$

$$A_2 = 900 \text{ cm}^2$$

Q.4 Ratio of areas of two regular pentagons is 16:25. Find the ratio of sides of pentagons. Also find length of side of second pentagon if length of side of first pentagon is 8 cm.

Solution (answer)

$$\text{Ratio of areas of pentagons} = 16:25 = \frac{16}{25}$$

Let l_1 and l_2 be the side lengths of 1st and 2nd regular pentagons.

Regular pentagons are similar.

$$\therefore \frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$\frac{16}{25} = \left(\frac{l_1}{l_2}\right)^2$$

$$\Rightarrow \sqrt{\frac{l_1}{l_2}} = \sqrt{\frac{16}{25}}$$

$$\frac{l_1}{l_2} = \frac{4}{5}$$

$$l_1 : l_2 = 4 : 5$$

$$\text{ii) Find } l_2 \text{ when } l_1 = 8 \text{ cm}$$

$$\Rightarrow \frac{8}{l_2} = \frac{4}{5}$$

$$\Rightarrow l_2 = \frac{5}{4} \times 8$$

$$l_2 = \frac{40}{4}$$

$$\Rightarrow l_2 = 10 \text{ cm}$$

Q.5 In the figure, $\overline{AB} \parallel \overline{YZ}$. If areas of triangles XAB and XYZ are in the ratio



25:36, find $\frac{XB}{XZ}$ and $\frac{AB}{YZ}$.



(answer)

Are the ratios equal?

Solution



$$\text{Given that } \frac{\text{Area of } \triangle XAB}{\text{Area of } \triangle XYZ} = \frac{25}{36} \quad \text{--- (i)}$$

$$\text{ii) Since } \overline{AB} \parallel \overline{YZ} \Rightarrow \triangle XAB \sim \triangle XYZ$$

$$\frac{\text{Area of } \triangle XAB}{\text{Area of } \triangle XYZ} = \left(\frac{XB}{XZ}\right)^2$$

$$\frac{25}{36} = \left(\frac{XB}{XZ}\right)^2$$

$$\Rightarrow \sqrt{\frac{XB}{XZ}} = \sqrt{\frac{25}{36}}$$

$$\frac{XB}{XZ} = \frac{5}{6}$$

$$\frac{XB}{XZ} = \frac{5}{6} \quad \text{--- (ii)}$$

$$\text{iii) } \frac{\text{Area of } \triangle XAB}{\text{Area of } \triangle XYZ} = \left(\frac{AB}{YZ}\right)^2$$

$$\frac{25}{36} = \left(\frac{AB}{YZ}\right)^2$$

$$\Rightarrow \sqrt{\frac{AB}{YZ}} = \sqrt{\frac{25}{36}}$$

$$\frac{AB}{YZ} = \frac{5}{6}$$

$$\frac{AB}{YZ} = \frac{5}{6} \quad \text{--- (iii)}$$

$$\frac{XB}{XZ} = \frac{AB}{YZ} = \frac{5}{6}$$

Q.6 In a map, length of a 10m wall is shown by 5cm. If area of wall shown on the map is 1400 cm², find the area of actual wall. (answer)

Solution

Actual length of wall = $l_1 = 10 \text{ m}$

Length of wall on map = $l_2 = 5 \text{ cm}$

Area of wall on map = $A_2 = 1400 \text{ cm}^2$

Actual area of wall = $A_1 = ?$

Actual wall and shape of wall on map are similar.

So,

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$\frac{A_1}{1400} = \left(\frac{10}{5}\right)^2$$

$$\frac{A_1}{1400} = \left(\frac{100}{25}\right)$$

$$A_1 = 4 \times 1400 \text{ cm}^2$$

$$A_1 = 5600 \text{ cm}^2$$

Q.7 Two cuboids are similar. Height of smaller cuboid is one-third of bigger one.

i) Find the ratio of surface area of larger cuboid to that of smaller one. (answer)

ii) Find the surface area of bigger cuboid if surface area of smaller cuboid is 350 cm². (answer)

Solution

$$\text{Let height of smaller cube} = h_2 = \frac{1}{3}x$$

$$\text{height of larger cube} = h_1 = x$$

If A_1 and A_2 are surface areas of larger and smaller cubes respectively, then

i) Ratio of surface areas

$$\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2$$

$$\frac{A_1}{A_2} = \left(\frac{x}{\frac{1}{3}x}\right)^2$$



$$\frac{A_1}{A_2} = \frac{x^2}{\frac{x^2}{9}}$$

$$\frac{A_1}{A_2} = \frac{9x^2}{x^2}$$

$$A_1 : A_2 = 9 : 1$$

$$\text{ii) } A_1 = 9 \text{ if } A_2 = 350 \text{ cm}^2$$

We know that

$$\frac{A_1}{A_2} = 9$$

$$A_1 = 9 \times A_2$$

$$A_1 = 9 \times 350 \text{ cm}^2$$

$$A_1 = 3150 \text{ cm}^2$$

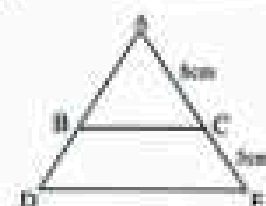
Q.8 In the figure, $\overline{BC} \parallel \overline{DE}$. Find:

$$\text{i) } \frac{BC}{DE} \text{ and } \frac{AB}{AD} \quad \text{(answer)}$$

$$\text{ii) } \frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle ADE} \quad \text{(answer)}$$

$$\text{iii) The area of } \triangle ABC \text{ if area of } \triangle ADE \text{ is } 507 \text{ cm}^2. \quad \text{(answer)}$$

iv) area of quadrilateral BDEC. What type of the quadrilateral is? (answer)



Solution

i) (a)

$$\text{Given that } \overline{BC} \parallel \overline{DE}$$

So, $\triangle ABC \sim \triangle ADE$

So similar triangles,

$$\frac{BC}{DE} = \frac{AC}{AE}$$

$$\frac{BC}{DE} = \frac{1}{3}$$

$$\frac{BC}{DE} = \frac{1 \text{ cm}}{3 \text{ cm}}$$



$$\therefore \overline{AE} = 8 \times 5 = 40$$

$$\frac{\overline{mBC}}{\overline{mDE}} = \frac{8}{13} = k$$

(b)

$$\text{Now, } \frac{\overline{mAB}}{\overline{mAD}} = \frac{\overline{mAC}}{\overline{mAE}}$$

$$\frac{\overline{mAB}}{\overline{mAD}} = \frac{8\text{cm}}{13\text{cm}}$$

$$\frac{\overline{mAB}}{\overline{mAD}} = \frac{8}{13}$$

$$\text{ii) } \frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle ADE} = \left(\frac{\overline{mAB}}{\overline{mAD}} \right)^2$$

$$= \left(\frac{8\text{cm}}{13\text{cm}} \right)^2$$

$$= \frac{64\text{ cm}^2}{169\text{ cm}^2}$$

$$\frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle ADE} = \frac{64}{169} \Rightarrow 64 : 169$$

iii) Area of $\triangle ABC$

We know that

$$\frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle ADE} = \frac{64}{169}$$

$$\frac{\text{Area of } \triangle ABC}{507\text{ cm}^2} = \frac{64}{169}$$

$$\therefore \text{Area of } \triangle ABC = 507\text{ cm}^2$$

$$\text{Area of } \triangle ABC = \frac{64}{169} \times 507\text{ cm}^2$$

$$\text{Area of } \triangle ABC = 64 \times 3\text{ cm}^2$$

$$\text{Area of } \triangle ABC = 192\text{ cm}^2$$

iv) Area of quadrilateral BDEC

Area of quadrilateral BDEC

$$= \text{Area of } \triangle ADE - \text{Area of } \triangle ABC$$

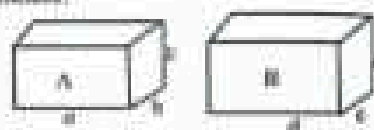
$$= 507\text{ cm}^2 - 192\text{ cm}^2$$

$$= 315\text{ cm}^2 \text{ (Trapezium)}$$

Volume of Similar Solids

Two solids are said to be similar if they have same shape. The ratio of corresponding lengths (sides etc.) of two similar solids is constant called the scale factor of the solids.

1) In the figure two cuboids are similar. Therefore:



$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = k$$

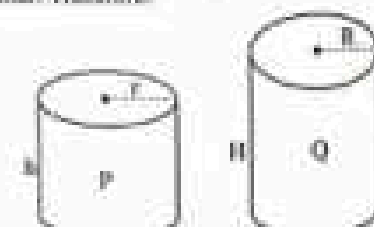
$$\Rightarrow a = dk, b = ek, c = fk$$

Now if V_1 is volume of cuboid A and V_2 is volume of cuboid B, then

$$\frac{V_1}{V_2} = \frac{abc}{def} = \frac{dk \cdot ek \cdot fk}{d \cdot e \cdot f} = k^3 \Rightarrow V_1 = k^3 V_2$$

If two cuboids are similar then volume of one cuboid is k^3 times the volume of the other cuboid.

2) In the figure two cylinders P and Q are similar. Therefore:



$$\frac{r}{R} = \frac{h}{H} = k$$

$$\Rightarrow \frac{r}{R} = k \text{ and } \frac{h}{H} = k$$

Now if V_1 is volume of cylinder P and V_2 is the volume of cylinder Q, then

$$\frac{V_1}{V_2} = \frac{\pi r^2 h}{\pi R^2 H} = \left(\frac{r}{R} \right)^2 \left(\frac{h}{H} \right)$$

$$= (k)^2 (k) = k^3$$

$$\Rightarrow V_1 = k^3 V_2$$



Example 11

Check whether the prisms are similar or not?



Solution

Let $a = 3, b = 4, c = 5$ and $d = 4.5, e = 6, f = 7.5$.

$$\frac{a}{d} = \frac{3}{4.5} = \frac{2}{3}$$

Then:

$$\frac{b}{e} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{c}{f} = \frac{5}{7.5} = \frac{2}{3}$$

Therefore, the two prisms are similar.

Example 12

Find x , if the following solids are similar. Also find the ratio of volumes of both. How many times is the volume of larger solid than smaller one?



Solution

Let $h_1 = 2\text{cm}, h_2 = 4\text{cm}, a_1 = 3\text{cm}, a_2 = ?$

As the figures are similar therefore:

$$\frac{h_1}{h_2} = \frac{a_1}{a_2} \Rightarrow a_2 = \frac{a_1 h_2}{h_1} = \frac{3 \times 4}{2} = 6\text{cm}$$

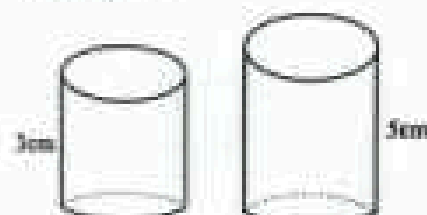
$$\text{Now, } \frac{V_1}{V_2} = \left(\frac{h_1}{h_2} \right)^3 = \left(\frac{2}{4} \right)^3 = \frac{1}{8}$$

Hence, $V_2 = 8V_1$ volume of larger solid is 8 times the volume of smaller one.

Example 13

In the figure two geometrically similar cylinders are shown. Find:

i) The ratio of volume of smaller cylinder to larger cylinder.



ii) Curved surface area of smaller cylinder if that of larger one is 250 cm^2 .

iii) volume of larger cylinder if volume of smaller one is 162 cm^3 .

Solution

i) v_1 and v_2 are volumes of smaller and larger cylinders respectively, then:

$$\frac{v_1}{v_2} = \left(\frac{3}{5} \right)^3 = \frac{v_1}{v_2} = \frac{27}{125}$$

ii) If A_1 and A_2 are areas of smaller and larger cylinders respectively, then:

$$\frac{A_1}{A_2} = \left(\frac{3}{5} \right)^2$$

$$\frac{A_1}{250} = \frac{9}{125}$$

$$\Rightarrow A_1 = \frac{9}{125} \times 250 = 90\text{ cm}^2$$

$$\text{iii) } \frac{162}{v_1} = \left(\frac{3}{5} \right)^3$$

$$\Rightarrow \frac{162}{v_1} = \frac{27}{125}$$

$$\Rightarrow 27 \times v_1 = 162 \times 125$$

$$\Rightarrow v_2 = \frac{162 \times 125}{27} = 750\text{ cm}^3$$



Exercise 9.4

Q.1 Determine whether the solids are similar or not?

Solution

(i)



Let $a = 3\text{cm}$, $b = 5\text{cm}$, $c = 6\text{cm}$, $d = 10\text{cm}$

$$\frac{a}{c} = \frac{3\text{cm}}{6\text{cm}} = \frac{1}{2} \quad \text{--- (i)}$$

$$\frac{b}{d} = \frac{5\text{cm}}{10\text{cm}} = \frac{1}{2} \quad \text{--- (ii)}$$

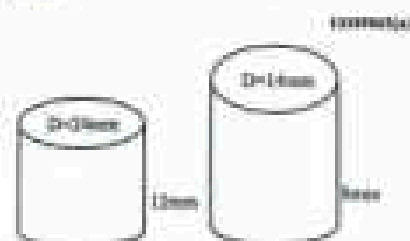
From (i) and (ii)

$$\frac{a}{c} = \frac{b}{d}$$

Since ratios of corresponding sides are equal, so given cuboid are similar.

Solution

(ii)



Let $L = 12\text{mm}$, $D = 24\text{mm}$

$r = 8\text{mm}$, $d = 14\text{mm}$

$$\frac{L}{r} = \frac{12\text{mm}}{8\text{mm}} = \frac{3}{2} \quad \text{--- (i)}$$

$$\text{Now, } \frac{D}{d} = \frac{24\text{mm}}{14\text{mm}} = \frac{12}{7} \quad \text{--- (ii)}$$

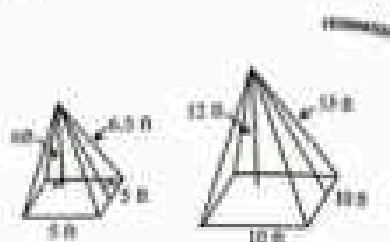
From (i) and (ii)

$$\frac{L}{r} \neq \frac{D}{d}$$

So given cylinders are not similar.

Solution

(iii)



Let base $= b = 5\text{ ft}$, Base $= B = 10\text{ ft}$

Height $= h = 6\text{ ft}$, height $H = 12\text{ ft}$

Slant height $= S = 6.5\text{ ft}$, Slant height $= 13\text{ ft}$

$$\frac{b}{B} = \frac{5\text{ ft}}{10\text{ ft}} = \frac{1}{2} \quad \text{--- (i)}$$

$$\frac{h}{H} = \frac{6\text{ ft}}{12\text{ ft}} = \frac{1}{2} \quad \text{--- (ii)}$$

$$\frac{S}{S'} = \frac{6.5\text{ ft}}{13\text{ ft}} = \frac{1}{2} \quad \text{--- (iii)}$$

From (i), (ii) and (iii)

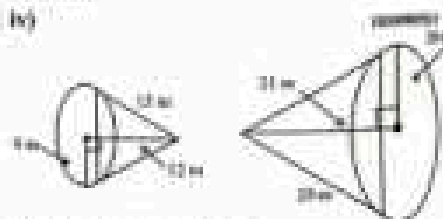
$$\frac{b}{B} = \frac{h}{H} = \frac{S}{S'}$$

Corresponding measurements are proportional.

So, given pyramids are similar.

Solution

(iv)



Radius $= r = 9\text{m}$, $R = 20\text{m}$

Vertical height $= h = 12\text{m}$, $H = 21\text{m}$

Slant length $= l = 15\text{m}$, $L = 29\text{m}$

$$\frac{r}{R} = \frac{9\text{m}}{20\text{m}} = \frac{9}{20} \quad \text{--- (i)}$$

$$\frac{h}{H} = \frac{12\text{m}}{21\text{m}} = \frac{4}{7} \quad \text{--- (ii)}$$



$$\frac{r}{L} = \frac{15\text{m}}{29\text{m}} = \frac{15}{29} \quad \text{--- (iii)}$$

From (i), (ii) and (iii)

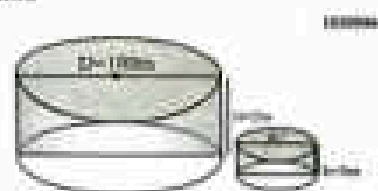
$$\frac{r}{R} \neq \frac{h}{H} \neq \frac{l}{L}$$

So given cones are not similar.

Q.2 Solids are similar. Find the values of unknowns. Also find the ratios of volume of solids.

Solution

(i)



(ii) Value of unknown d .

As given cylinder are similar, So, Ratios of corresponding lengths are equal.

$$(a) \frac{D}{d} = \frac{H}{h}$$

$$\frac{100\text{mm}}{d} = \frac{100\text{mm}}{25\text{mm}}$$

$$d = \frac{25}{100} \times 100\text{mm}$$

$$d = 25 = 4\text{m}$$

$$d = 20\text{ m}$$

Solution

(b) Ratio of volumes of solids

V_1 and V_2 volume of larger and smaller cylinders respectively then

$$\frac{V_1}{V_2} = \left(\frac{H}{h}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{25\text{m}}{5\text{m}}\right)^3$$

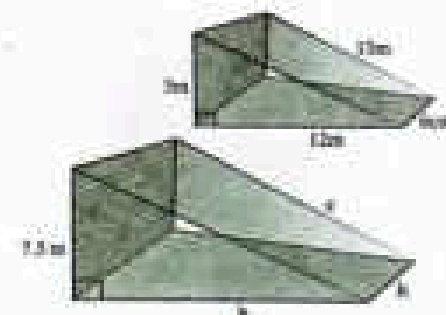
$$\frac{V_1}{V_2} = \left(\frac{5}{1}\right)^3$$

$$\frac{V_1}{V_2} = \frac{125}{1}$$

$$V_1 : V_2 = 125 : 1$$

(ii) Given that solids are similar, so

(a) Value of unknown



Solution

$$\frac{b}{12\text{m}} = \frac{c}{12\text{m}} = \frac{h}{12\text{m}} = \frac{7.5}{12\text{m}}$$

So,

$\frac{a}{12\text{m}} = \frac{7.5}{12\text{m}}$	$\frac{b}{12\text{m}} = \frac{7.5}{12\text{m}}$	$\frac{h}{12\text{m}} = \frac{7.5}{12\text{m}}$
$\frac{a}{12\text{m}} = \frac{5}{8}$	$\frac{b}{12\text{m}} = \frac{5}{8}$	$\frac{h}{12\text{m}} = \frac{5}{8}$
$a = \frac{7.5 \times 12\text{m}}{12\text{m}}$	$b = \frac{7.5 \times 12\text{m}}{12\text{m}}$	$h = \frac{7.5 \times 12\text{m}}{12\text{m}}$
$a = 7.5 \times 12\text{m}$	$b = 7.5 \times 12\text{m}$	$h = 7.5 \times 12\text{m}$
$a = 19.5\text{m}$	$b = 9\text{m}$	$h = 18\text{m}$

(b) Ratio of volumes of solids

$$\frac{V_1}{V_2} = \left(\frac{7.5\text{m}}{12\text{m}}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{5}{8}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{125}{512}\right)$$

$$\frac{V_1}{V_2} = \left(\frac{125}{512}\right)$$

$$\frac{V_1}{V_2} = \left(\frac{125}{512}\right)$$

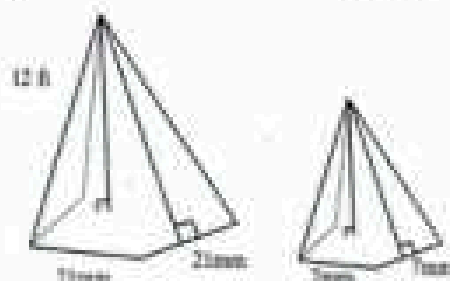


$$\frac{V_1}{V_2} = \frac{27}{8} = 27:8$$

Q.3 Solids are similar. Find the unknown volume.

Solution

(i)



Given that solids are similar.

If V_1 and V_2 be volume of larger and smaller pyramids then

$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{21\text{cm}}{7\text{cm}}\right)^3$$

$$\frac{V_1}{V_2} = (3)^3$$

$$\frac{V_1}{V_2} = 27$$

$$\Rightarrow V_1 = \frac{V_2}{27}$$

Put $V_2 = 9000 \text{ mm}^3$

$$V_1 = \frac{9000}{27}$$

$$V_1 = 333.33 \text{ mm}^3$$

Solution

(ii)



Given that solids are similar.
If V_1 and V_2 are volume of smaller and large cylinder respectively, then

$$\frac{V_1}{V_2} = \left(\frac{r_1}{r_2}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{10}{5}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{2}{1}\right)^3$$

$$\frac{V_1}{V_2} = \frac{8}{1}$$

$$V_1 = \frac{128}{216} \times V_2$$

$$V_1 = \frac{128}{216} \times 360 \text{ m}^3$$

$$V_1 = 208.33 \text{ m}^3$$

Solution

(iii)



Given that solids are similar, $V_1 = 468 \text{ cm}^3$



If V_1 and V_2 are volumes of smaller and larger cones respectively, then

$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{6\text{cm}}{15\text{cm}}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{2}{5}\right)^3$$

$$\frac{V_1}{V_2} = \frac{8}{125}$$

$$V_1 = \frac{8}{125} \times V_2$$

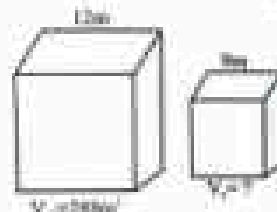
$$V_1 = \frac{8}{125} \times 468 \text{ cm}^3$$

$$V_1 = \frac{3600}{125}$$

$$V_1 = 28.8 \text{ cm}^3$$

Solution

(iv)



If V_1 and V_2 are volumes of larger and smaller cuboid respectively, then

Given solids are similar.

$$\frac{V_1}{V_2} = \left(\frac{L_1}{L_2}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{12\text{m}}{6\text{m}}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{4}{1}\right)^3$$

$$\frac{V_1}{V_2} = \frac{64}{27}$$

Put $V_1 = 288 \text{ cm}^3$

$$\frac{288}{V_2} = \frac{64}{27}$$

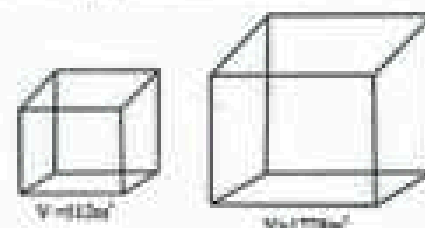
$$\Rightarrow \frac{V_2}{288} = \frac{27}{64}$$

$$V_2 = \frac{27}{64} \times 288$$

$$V_2 = \frac{7776}{64}$$

$$V_2 = 121.5 \text{ cm}^3$$

Q.4 Find the ratio of scale factors of the following pairs of similar solids.



Solution

Let $V_1 = 128 \text{ m}^3$, $V_2 = 1728 \text{ m}^3$

Scale factor = Ratio of corresponding measurements of similar object.

$$k = \frac{V_1}{V_2} = \frac{128\text{m}^3}{1728\text{m}^3}$$

$$k = \frac{V_1}{V_2} = 4:27$$

Q.5 Two swimming pools are similar with a scale factor of 4:5. The amount of chlorine mixture to be added is proportional to the volume of water in the pool.

Solution

Scale factor of swimming pools = 4:5

Chlorine mixture for small pool = $m_1 = 3$ cups

Chlorine mixture for larger pool = $m_2 = 7$

By scale factor, $K = 4:5$

$$\frac{m_1}{m_2} = k$$



$$\begin{aligned}\frac{2}{m_1} &= \left(\frac{4}{3}\right)^3 \\ \frac{2}{m_1} &= \frac{64}{125} \\ \Rightarrow \frac{m_1}{2} &= \frac{125}{64} \quad (\text{Reciprocal}) \\ m_1 &= \frac{125}{64} \times 2 \\ m_1 &= \frac{250}{64} = 3.9 \\ m_1 &= 3.9 \text{ (approx.)}\end{aligned}$$

Q.6 A model bus is built with a scale of 1:10. The model bus has a volume of 30m^3 . What is the volume of the actual bus?

Solution

$$\begin{aligned}\text{Scale } 1:10 &= \frac{1}{10} \\ \text{Volume of model bus} &= V_1 = 30\text{m}^3 \\ \text{Volume of actual bus} &= V_2 = ? \\ \text{By condition} \\ \frac{V_1}{V_2} &= k^3 \\ \frac{V_2}{V_1} &= \left(\frac{1}{10}\right)^3 \\ \Rightarrow \frac{30\text{m}^3}{V_2} &= \frac{1}{1000} \\ \frac{V_2}{30\text{m}^3} &= \frac{1000}{1} \quad (\text{Reciprocal}) \\ V_2 &= 1000 \times 30\text{m}^3 \\ V_2 &= 30,000\text{m}^3\end{aligned}$$

Q.7 Solid A shown below is similar to solid B (not shown) with a scale factor 2:3. Find the surface area and volume of solid B if surface area and volume of solid A are 130cm^2 and 280cm^3 respectively.



Solution

$$\begin{aligned}\text{Scale factor} &= k = 2:3 = \frac{2}{3} \\ \text{Surface area of A} &= A_1 = 130\text{cm}^2 \\ \text{Volume of solid A} &= V_1 = 280\text{cm}^3 \\ \text{Surface area of B} &= A_2 = ? \\ \text{Volume of solid B} &= V_2 = ? \\ \text{i) By scale factor, ratio of areas} \\ \frac{A_1}{A_2} &= k^2 \\ \frac{130\pi}{A_2} &= \left(\frac{2}{3}\right)^2 \\ \frac{130\pi}{A_2} &= \frac{4}{9} \\ \Rightarrow \frac{A_2}{130\pi} &= \frac{9}{4} \\ A_2 &= \frac{9}{4} \times 130\pi = \frac{1170}{4}\pi = 292.5\pi\text{cm}^2 \\ A_2 &= 292.5\pi\text{cm}^2\end{aligned}$$

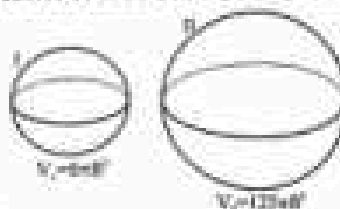
Ratio of volumes:

$$\begin{aligned}\text{ii) By scale factor: } k &= 2:3 = \frac{2}{3} \\ \frac{V_1}{V_2} &= \left(\frac{2}{3}\right)^3 \\ \frac{280\pi}{V_2} &= \frac{8}{27} \\ \Rightarrow \frac{V_2}{280\pi} &= \frac{27}{8}\end{aligned}$$



$$\begin{aligned}\Rightarrow V_2 &= \frac{27}{8} \times 280\pi\text{cm}^3 \\ V_2 &= \frac{7560\pi\text{cm}^3}{8} \\ V_2 &= 945\pi\text{cm}^3\end{aligned}$$

Q.8 Solid I is similar to solid II. Find the scale factor of solid I to solid II.



Solution

Solid I is similar to solid II.

$$\begin{aligned}\frac{V_1}{V_2} &= k^3 \\ k^3 &= \frac{4\pi\text{cm}^3}{125\pi\text{cm}^3} \\ k^3 &= \frac{4}{125} \\ k^3 &= \left(\frac{2}{5}\right)^3 \\ k &= \frac{2}{5} \\ k &= 2:5\end{aligned}$$

Q.9 Solid A and solid B are mathematically similar. The volume of solid A is 32cm^3 . The volume of solid B is 108cm^3 . The height of solid A is 16cm. Find the height of solid B.

Solution

$$\begin{aligned}\text{Volume of solid A} &= V_1 = 32\text{cm}^3 \\ \text{Volume of solid B} &= V_2 = 108\text{cm}^3 \\ \text{Height of solid A} &= h_1 = 16\text{cm} \\ \text{Height of solid B} &= h_2 = ?\end{aligned}$$

Given that solid A and solid B are mathematically similar. The ratio of volumes of two similar solids is equal to the cube of the ratio of any two corresponding lengths of two solids.

$$\begin{aligned}\frac{V_1}{V_2} &= \left(\frac{h_1}{h_2}\right)^3 \\ \frac{32\text{cm}^3}{108\text{cm}^3} &= \left(\frac{16\text{cm}}{h_2}\right)^3 \\ \frac{8}{27} &= \frac{1000\text{cm}^3}{(h_2)^3} \\ (h_2)^3 &= \frac{27}{8} \times 1000\text{cm}^3 \\ (h_2)^3 &= \frac{27000}{8}\text{cm}^3 \\ (h_2)^3 &= 3375\text{cm}^3 \rightarrow (\text{Taking cube root of both sides}) \\ \sqrt[3]{(h_2)^3} &= \sqrt[3]{3375\text{cm}^3} \\ h_2 &= 15\text{cm} \\ \text{Thus height of solid B is } 15\text{cm}.\end{aligned}$$

Q.10 P and Q are two similar solids. Solid P has surface area and volume 100cm^2 and 135cm^3 respectively. Find volume of solid Q if Q has surface area 300cm^2 .

Solution

$$\begin{aligned}\text{Surface area of solid P} &= A_1 = 100\text{cm}^2 \\ \text{Volume of solid P} &= V_1 = 135\text{cm}^3 \\ \text{Surface area of solid Q} &= A_2 = 300\text{cm}^2 \\ \text{Volume of solid Q} &= V_2 = ?\end{aligned}$$

If l_1 and l_2 are the corresponding lengths of solid P and solid Q respectively, then

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

Putting the values,



$$\begin{aligned} \frac{108\text{cm}^3}{300\text{cm}^3} &= \left(\frac{h_1}{h_2}\right)^3 \\ \Rightarrow \left(\frac{h_1}{h_2}\right)^3 &= \frac{9}{25} \\ \left(\frac{h_1}{h_2}\right)^3 &= \left(\frac{3}{5}\right)^3 \\ \left(\frac{h_1}{h_2}\right)^3 &= \left(\frac{3}{5}\right)^3 \\ \Rightarrow \frac{h_1}{h_2} &= \frac{3}{5} \end{aligned}$$

We know that

$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$$

Putting values

$$\begin{aligned} \frac{135}{V_2} &= \left(\frac{3}{5}\right)^3 \\ \frac{135}{V_2} &= \frac{27}{125} \\ \Rightarrow \frac{V_2}{135} &= \frac{125}{27} \\ V_2 &= \frac{125}{27} \times 135 \\ V_2 &= 125 \times 5 \\ V_2 &= 625\text{cm}^3 \end{aligned}$$

Thus volume of solid Q is 625cm^3 .

Q.11 X and Y are two similar cylinders such that: base area of X : base area of Y = 16:25. Find:

- Ratio of heights of both cylinders. answer
- Ratio of areas of curved surface of cylinder. answer
- Ratio of volume of cylinders. answer

Solution

Let A_1 and A_2 be base areas of cylinder X and Y respectively.

Given that

Base area of X : base area of Y = 16:25

$$A_1:A_2 = 16:25$$

(i) If h_1 and h_2 are corresponding heights of cylinders and respectively, then

$$\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2 \quad \dots (i)$$

Given that

$$\frac{A_1}{A_2} = \frac{16}{25} \quad \dots (ii)$$

Comparing (i) and (ii)

$$\left(\frac{h_1}{h_2}\right)^2 = \frac{16}{25}$$

$$\left(\frac{h_1}{h_2}\right)^2 = \frac{16}{25}$$

$$\left(\frac{h_1}{h_2}\right)^2 = \left(\frac{4}{5}\right)^2$$

$$\sqrt{\left(\frac{h_1}{h_2}\right)^2} = \sqrt{\left(\frac{4}{5}\right)^2}$$

$$\Rightarrow \frac{h_1}{h_2} = \frac{4}{5} \quad \dots (iii)$$

$$h_1:h_2 = 4:5$$

(ii) **Ratio of areas of curved surfaces**

If a_1 and a_2 are the areas of curved surface of

cylinder x and y respectively then $\frac{a_1}{a_2} = \left(\frac{h_1}{h_2}\right)^2$

Putting the value. From (iii)

$$\frac{a_1}{a_2} = \left(\frac{4}{5}\right)^2$$

$$\frac{a_1}{a_2} = \frac{16}{25}$$

$$a_1:a_2 = 16:25$$

(iii) **Ratio of volumes**

If V_1 and V_2 are the volumes of cylinder X and cylinder Y respectively, then



$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$$

$$\frac{V_1}{V_2} = \left(\frac{4}{5}\right)^3$$

$$\frac{V_1}{V_2} = \frac{64}{125}$$

$$V_1/V_2 = 64/125$$

Q.12 The volume of one right circular cone is 8 times the other one. If the radius of larger cone is 12cm, find the radius of the smaller one. answer

Solution

Radius of larger cone = R = 12cm

Radius of smaller cone = r = ?

If V_1 and V_2 are volume of right circular cones, then

$$V_1 = 8V_2$$

$$\frac{V_1}{V_2} = 8 \quad \dots (i)$$

Both right circular cones are similar.

$$\frac{V_1}{V_2} = \left(\frac{R}{r}\right)^3 = \frac{R^3}{r^3}$$

Putting the value

$$\begin{aligned} 8 &= \frac{(12)^3}{r^3} \\ \Rightarrow 8r^3 &= 1728 \end{aligned}$$

$$r^3 = \frac{1728}{8}$$

$$r^3 = 216$$

Taking cube root of both sides

$$\sqrt[3]{r^3} = \sqrt[3]{216}$$

$$r = 6\text{ cm}$$

Thus radius of smaller right circular cone is 6 cm.

Q.13 Mass of two similar objects are 8kg and 17kg respectively. If the height of first object is 2m, what is the height of second object? answer

Solution

Mass of 1st object = $m_1 = 8\text{kg}$

Mass of 2nd object = $m_2 = 17\text{kg}$

Height of 1st object = $h_1 = 2\text{m}$

Height of 2nd object = $h_2 = ?$

Note: "If both solids are made from the same materials, then ratio of their masses is the same as the ratio of their volumes".

$$\frac{V_1}{V_2} = \frac{m_1}{m_2} \quad \dots (i)$$

For similar objects

$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$$

$$\frac{m_1}{m_2} = \left(\frac{h_1}{h_2}\right)^3 \quad \dots (ii)$$

From (i) and (ii)

$$\frac{m_1}{m_2} = \left(\frac{h_1}{h_2}\right)^3$$

$$\frac{8\text{kg}}{17\text{kg}} = \left(\frac{2}{h_2}\right)^3$$

$$\frac{8}{27} = \left(\frac{2}{h_2}\right)^3$$

$$\Rightarrow (h_2)^3 = \frac{8 \times 27}{8}$$

$$(h_2)^3 = 27$$

Taking cube root of B.S.

$$\sqrt[3]{(h_2)^3} = \sqrt[3]{27}$$

$$h_2 = 3\text{m}$$

Thus height of 2nd object is 3m.

Polygons

'Polygon' is the Greek word where 'poly' means 'many' and 'gon' means 'angles'.

A polygon is a two-dimensional convex figure that has a finite number of sides (at least three sides). The point where two line segments meet is called vertex of polygon.



Regular Polygons

If all the sides and interior angles of the polygons are equal, they are known as regular polygons. Regular polygons are also equiangular.

The examples of regular polygons are equilateral triangle, square, pentagon, hexagon etc.

Irregular Polygon

If in a polygon at least one angle or one side is not equal in size then the polygon is called irregular.

Key Fact

- A polygon with 3 sides is called pentagon.
- A polygon with 6 sides is called hexagon.
- A polygon with 7 sides is called heptagon.
- A polygon with 8 sides is called octagon.
- A polygon with 9 sides is called nonagon.
- A polygon with 10 sides is called decagon.

Interior angle of a regular polygon

Sum of measures of all the interior angles of a polygon having n sides is $:(n-2) \times 180^\circ$

The measure of each interior angle of n -sided regular polygon is given by the formula:

$$= \frac{(n-2) \times 180^\circ}{n}$$

Example 14

Find a interior angle of a regular nonagon

Solution

A nonagon has 9 sides, therefore $n = 9$

If θ is an interior angle of the nonagon then:

$$\theta = \frac{(9-2) \times 180^\circ}{9} = \frac{1260^\circ}{9} = 140^\circ$$

Example 15

Find the value of x in the following polygons.

(i)



(ii)



Solution

(i) The given polygon is pentagon and sum of measures of its interior angles 720° .

$$\therefore 90^\circ + 125^\circ + 112^\circ + x + 140^\circ = 720^\circ$$

$$\Rightarrow x + 595^\circ = 720^\circ$$

$$\Rightarrow x = 720^\circ - 595^\circ = 125^\circ$$

(ii) The given polygon is heptagon and sum of measures of its interior angles is 900° .

$$\therefore (x + 18^\circ) + x + 138^\circ + (x + 18^\circ) + 130^\circ + 130^\circ + 132^\circ = 900^\circ$$

$$\Rightarrow 3x + 558^\circ = 900^\circ$$

$$\Rightarrow 3x = 900^\circ - 558^\circ = 342^\circ$$

$$\Rightarrow x = \frac{342^\circ}{3} = 114^\circ$$

Sum of measures of exterior angles of polygon

Figure shows a regular hexagon. An interior angle of a regular hexagon is 120° and sum of measures of an interior and exterior angle is 180° , therefore:

$$\text{Interior angle} + \text{Exterior angle} = 180^\circ$$

$$120^\circ + \text{Exterior angle} = 180^\circ$$

$$\text{Exterior angle} = 180^\circ - 120^\circ = 60^\circ$$

Now, hexagon has 6 sides, therefore:

Sum of measures of exterior angles of regular

$$\text{hexagon} = 6 \times 60^\circ$$

$$= 360^\circ$$



For a regular polygon having n sides:

$$\text{Exterior angle} + \text{interior adjacent angle} = 180^\circ$$

Sum of all exterior angles = Sum of all interior angles



So, sum of all exterior angles

$$= n \times 180^\circ - (n-2) \times 180^\circ$$

$$= n \times 180^\circ - 180^\circ + 2 \times 180^\circ$$

$$= 180^\circ n - 180^\circ n + 360^\circ = 360^\circ$$

Exterior Angles of a Regular Polygon

The measure of each exterior angle of an n -sided regular polygon is:

$$= \frac{360^\circ}{n}$$

Example 16

Exterior angle of a regular polygon is 120° .

Identify the name of polygon.

Solution

Measure of exterior angles = 120°

Measure of one exterior angle = 120°

$$\text{Number of exterior angles} = \frac{360^\circ}{120^\circ} = 3$$

Since the polygon has 3 exterior angles therefore, it has 3 sides.

Hence, it is an equilateral triangle.

Example 17

In a certain polygon, the sum of the measures of all the interior angles is equal to twice that of the exterior angles. What is the name of that polygon?

Solution

Given that

Sum of interior angles = 2 × the sum of exterior angles

If n is number of polygon, then:

$$(n-2) \times 180^\circ = 2 \times 360^\circ$$

$$\Rightarrow n-2 = \frac{2 \times 360^\circ}{180^\circ} \Rightarrow n-2 = 4 \Rightarrow n = 6$$

Hence the polygon is hexagon.

Example 18

The measures of the exterior angles of a polygon are $(x-4)^\circ$, $(3x-4)^\circ$, $(7x-3)^\circ$, $(2x-3)^\circ$, $(8x-1)^\circ$ and $(9x+1)^\circ$.

(i) Identify polygon:

(ii) Find x .

(iii) Find the measure of each angle.

Solution

(i) Since there are exterior angles in the polygon, therefore polygon is hexagon.

(ii) As the sum of measures of exterior angles in a polygon is 360° , therefore:

$$(x-4)^\circ + (3x-4)^\circ + (7x-3)^\circ + (2x-3)^\circ + (8x-1)^\circ + (9x+1)^\circ = 360^\circ$$

$$\Rightarrow x-4^\circ + 3x-4^\circ + 7x-3^\circ + 2x-3^\circ + 8x-1^\circ + 9x+1^\circ = 360^\circ$$

$$\Rightarrow 30x-160^\circ = 360^\circ$$

$$\Rightarrow 30x = 360^\circ \Rightarrow x = 12^\circ$$



iii) Substituting the value of x in above expressions of angles, we get

$$(12+4x)(3)=12-4x, (7+12-3)x, (3+12+3)x, \\ (8+12-1)x, (9+12+1)x \\ \text{Or } 16^\circ, 32^\circ, 81^\circ, 27^\circ, 95^\circ, 109^\circ$$

Exercise 9.5

Q.1 Find the number of sides of a regular polygon if each of its exterior angle is:

- a) 45° answer b) 60° answer
c) 120° answer d) 40° answer

Solution

a) Each exterior angle = 45°

We know that

Each exterior angle of n -sided regular polygon

$$= \frac{360^\circ}{n}$$

$$45^\circ = \frac{360^\circ}{n}$$

$$\Rightarrow n = \frac{360^\circ}{45^\circ} = 8$$

Since polygon has 8 sides, therefore the polygon is regular octagon.

b) 60°

Each exterior angle = 60°

We know that

Each exterior angle of n -sided

$$\text{Regular polygon} = \frac{360^\circ}{n}$$

$$60^\circ = \frac{360^\circ}{n}$$

$$\Rightarrow n = \frac{360^\circ}{60^\circ} = 6$$

Hence, polygon is regular hexagon

c) 120°

Each exterior angle = 120°

Each exterior angle of

$$n\text{-sided regular polygon} = \frac{360^\circ}{n}$$

$$120^\circ = \frac{360^\circ}{n}$$

$$n = \frac{360^\circ}{120^\circ}$$

$$n = 3$$

Since, number of sides are 3, therefore the polygon is an equilateral triangle.

d) 40°

Each exterior angle = 40°

Each exterior angle of

$$n\text{-sided regular polygon} = \frac{360^\circ}{n}$$

$$40^\circ = \frac{360^\circ}{n}$$

$$\Rightarrow n = \frac{360^\circ}{40^\circ}$$

$$n = 9$$

Since, number of sides is 9, therefore the polygon is a regular nonagon.

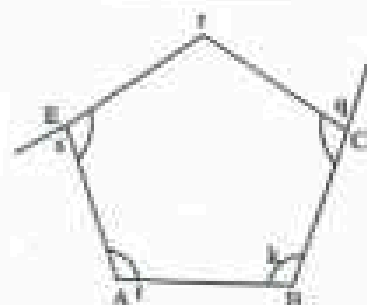
2. Draw a regular pentagon whose exterior angles are p, q, r, s, t and each interior angle is k . What is the measure of

(a) each interior angle answer

(b) Each exterior angle answer

Solution

Drawing Regular pentagon



number of sides of regular pentagon = $n=5$

a) Each interior angle = K

We know that

Each interior angle of regular

$$\text{pentagon} \quad K = \frac{(n-2) \times 180^\circ}{n}$$

$$K = \frac{(5-2) \times 180^\circ}{5}$$

$$K = \frac{3 \times 180^\circ}{5}$$

$$K = \frac{540^\circ}{5}$$

$$K = 108^\circ$$

b) We know that

in the regular pentagon

$$\text{Each exterior angle} = \frac{360^\circ}{n}$$

$$\text{Each exterior angle} = \frac{360^\circ}{5} = 72^\circ$$

Thus

$$p = q = r = s = t = 72^\circ$$

Q.3 Each interior angle of a polygon is five times the exterior angle of the polygon. Find the number of sides. answer

Solution

Let each exterior angle be x

Each interior angle = $5x$

Sum of interior angle and adjacent exterior =

180°

$$x + 5x = 180^\circ$$

$$6x = 180^\circ$$

$$x = \frac{180^\circ}{6}$$

$$x = 30^\circ$$

We know that if n is number of sides, then

$$\text{each exterior angle} = \frac{360^\circ}{n}$$

$$= 30^\circ = \frac{360^\circ}{n}$$

$$\Rightarrow n = \frac{360^\circ}{30^\circ}$$

$$n = 12$$

Thus number of sides is 12.

Q.4 Find the minimum interior angle and maximum exterior angles possible in a regular polygon. Give reason to support your answer. answer

Solution

Minimum interior angle = 60°

Maximum interior angle = $(79.999 \dots) (< 180^\circ)$

If maximum interior angle is 180° , then adjacent sides of polygon become collinear.

Since the interior and exterior angles must sum to 180° . In a regular polygon as the number of sides increases, the interior angle approaches 180° and exterior angle approach 0° .

Q.5 Find the exterior angle of a regular polygon of: answer

a) 5 sides b) 9 sides

c) 15 sides d) 28 sides

Solution

a) 5 sides, $n = 5$

$$\text{Each exterior angle} = \frac{360^\circ}{n} = \frac{360^\circ}{5} = 72^\circ$$

b) 9 sides, $n = 9$

$$\text{Each exterior angle} = \frac{360^\circ}{n} = \frac{360^\circ}{9} = 40^\circ$$

c) 15 sides, $n = 15$

$$\text{Each exterior angle} = \frac{360^\circ}{n} = \frac{360^\circ}{15} = 24^\circ$$

d) 28 sides, $n = 28$

$$\text{Each exterior angle} = \frac{360^\circ}{n} = \frac{360^\circ}{28} = 12^\circ$$

Q.6 The ratio between an exterior angle and the interior angle of a regular polygon is 1:2. Find:

a) the measure of each exterior angle. answer

b) the measure of each interior angle. answer



c) The number of sides in the polygon.

Solution

Ratio between exterior and interior angles = 1:2

Sum of elements of ratio = $1+2 = 3$

Sum of exterior and interior angles = 180°

$$a) \text{ Exterior angle} = \frac{1}{3} \times 180^\circ = 60^\circ$$

$$b) \text{ Interior angle} = \frac{2}{3} \times 180^\circ = 120^\circ = 2 \times 60^\circ$$

c) If n is number of sides then

$$\text{each exterior angle} = \frac{360^\circ}{n}$$

$$60^\circ = \frac{360^\circ}{n} \Rightarrow n = 6$$

$$\Rightarrow n = \frac{360^\circ}{60^\circ}$$

$$n = 6$$

Thus number of sides in the polygon is 6.

Q.7 Is it possible to have a regular polygon each of whose exterior angle is 50° ? Give reason to support your answer.

Solution

Each angle = 50°

If n is number of sides of a regular polygon, then

$$\text{Each exterior angle} = \frac{360^\circ}{n}$$

$$50^\circ = \frac{360^\circ}{n}$$

$$\Rightarrow n = \frac{360^\circ}{50^\circ} = 7.2$$

Since number of sides can never be in fraction so, 50° can not be exterior angle of any regular polygon.

Q.8 Name the polygon whose sum of interior angles is equal to the sum of its exterior angles.

Solution

If n is number of sides of a regular polygon, then

Sum of exterior angles = 360° ... (i)

Sum of interior angles = $(n-2) \times 180^\circ$... (ii)

By condition, both sum are equal

$$(n-2) \times 180^\circ = 360^\circ$$

$$n-2 = \frac{360^\circ}{180^\circ}$$

$$n-2 = 2$$

$$n = 4$$

Hence, regular polygon having 4 sides is a square.

Q.9 The sum of all the interior angles of a regular polygon is four times the sum of its exterior angles. Identify polygon.

Solution

If n is number of sides of a regular polygon, then

Sum of exterior angles = 360°

Sum of interior angles = $(n-2) \times 180^\circ$

By condition

Sum of interior angles

$$= 4 \times \text{sum of exterior angles}$$

$$(n-2) \times 180^\circ = 4 \times 360^\circ$$

$$n-2 = \frac{4 \times 360^\circ}{180^\circ}$$

$$n-2 = 4 \times 2$$

$$n = 8 + 2$$

$$\boxed{n = 10}$$

The regular polygon having 10 equal sides is a decagon.

Q.10 An exterior angle of a regular polygon is 12° . What is the sum of all its interior angles?

Solution

An exterior angle = 12°

If n is number of sides of a regular polygon, then

$$\text{Each exterior angle} = \frac{360^\circ}{n}$$

$$12^\circ = \frac{360^\circ}{n}$$

$$n = \frac{360^\circ}{12^\circ} = 30$$

$$\boxed{n = 30}$$

Sum of interior angles = $(n-2) \times 180^\circ$

$$= (30-2) \times 180^\circ$$

$$= 28 \times 180^\circ$$

$$= 5040^\circ$$

Q.11 Prove that each interior angle and its corresponding exterior angles in any polygon are supplementary.

Solution

If n is a number of sides of regular polygon, then

Sum of interior angles = $(n-2) \times 180^\circ$

$$\text{One interior angle} = \frac{(n-2) \times 180^\circ}{n} \dots (i)$$

Sum of exterior angles = 360°

$$\text{One exterior angle} = \frac{360^\circ}{n} \dots (ii)$$

By adding eq. (i) and (ii), we get

one interior angle + one exterior angle

$$= \frac{(n-2) \times 180^\circ}{n} + \frac{360^\circ}{n}$$

$$= \frac{180^\circ \times 360^\circ + 360^\circ}{n}$$

$$= \frac{180^\circ \times n}{n}$$

$$= 180^\circ$$

Since, the sum of interior angle and its corresponding exterior angle is 180° hence, they are supplementary angles.

Q.12 Find the number of sides in a regular polygon when the measure of each exterior angle is 72° .

Solution

Each exterior angle = 72°

If n is number of sides of regular polygon, then

$$\text{Each exterior angle} = \frac{360^\circ}{n}$$

$$72^\circ = \frac{360^\circ}{n}$$

$$\Rightarrow n = \frac{360^\circ}{72^\circ} = 5$$

Thus number of sides are 5.

Q.13 The exterior angles of a pentagon are $(y+5)^\circ$, $(2y+3)^\circ$, $(3y+2)^\circ$, $(4y+1)^\circ$ and $(5y+4)^\circ$ respectively. Find the measure of each angle.

Solution

In a regular polygon,

Sum of all exterior angles = 360°

$$(y+5)^\circ + (2y+3)^\circ + (3y+2)^\circ + (4y+1)^\circ + (5y+4)^\circ = 360^\circ$$

$$y+5+2y+3+3y+2+4y+1+5y+4 = 360$$

$$15y+15 = 360$$

$$15y = 360-15$$

$$15y = 345$$

$$y = \frac{345}{15}$$

$$y = 23$$

$$1^{\text{st}} \text{ angle} = (y+5)^\circ = (23+5)^\circ = 28^\circ$$

$$2^{\text{nd}} \text{ angle} = (2y+3)^\circ = (2 \times 23+3)^\circ = 49^\circ$$

$$3^{\text{rd}} \text{ angle} = (3y+2)^\circ = (3 \times 23+2)^\circ = (69+2)^\circ = 71^\circ$$

$$4^{\text{th}} \text{ angle} = (4y+1)^\circ = (4 \times 23+1)^\circ = (92+1)^\circ = 93^\circ$$

$$5^{\text{th}} \text{ angle} = (5y+4)^\circ = (5 \times 23+4)^\circ = (115+4)^\circ = 119^\circ$$

Q.14 A convex polygon has 14 diagonals. Find the number of sides of the polygon.

Solution

Number of diagonals = 14

If n is number of sides of a regular polygon, then

$$\text{Number of diagonals} = \frac{n(n-3)}{2}$$

$$14 = \frac{n(n-3)}{2}$$

$$28 = n^2-3n$$



$$\Rightarrow x^2 - 7x + 4x - 28 = 0$$

By factorisation, we get

$$x^2 - 7x + 4x - 28 = 0$$

$$x(x-7) + 4(x-7) = 0$$

$$(x-7)(x+4) = 0$$

$$x-7 = 0 \text{ or } x+4 = 0$$

$$x = 7 \text{ or } x = -4$$

Since, n is a number sides, so $n = -4$ (negative value) is not possible. So

$$n = 7$$

Thus number of sides of polygon are 7.

Q.15 Find the sum of all the interior angles of a polygon having 13 sides. *****

Solution

$$\text{number of sides} = n = 13$$

$$\begin{aligned} \text{Sum of interior angles} &= (n-2) \times 180^\circ \\ &= (13-2) \times 180^\circ \\ &= 11 \times 180^\circ \\ &= 1980^\circ \end{aligned}$$

Q.16 The sum of all the interior angles of a polygon is 2880° . How many sides does the polygon have? *****

Solution

$$\text{Sum of interior angles} = 2880^\circ$$

We know that

$$\text{Sum of all interior angles} = (n-2) \times 180^\circ$$

$$2880^\circ = (n-2) \times 180^\circ$$

$$\frac{2880^\circ}{180^\circ} = n-2$$

$$16 = n-2$$

$$16+2 = n$$

$$18 = n$$

$$\Rightarrow n = 18$$

Thus polygon has 18 sides.

Real life problems involving regular polygons

Perimeter of Regular Polygon

Perimeter of regular polygon

$$= (\text{number of sides}) \times (\text{length of one side})$$

$$P = n \times s$$

Area of Regular Polygon

Area of a regular polygon is defined as:

$$A = \frac{1}{2}(P)(a)$$

Where 'P' is perimeter of regular polygon and 'a' is called apothem. Substituting the value of 'P', we have:

$$A = \frac{1}{2}(n \times s)(a) = \frac{n}{2}(sa)$$

Where 's' denotes the length of side.

Example 19

Basement of a water tank is in the shape of regular pentagon having the side length 8 ft. Its apothem is 7 feet. Find the perimeter and area of basement of tank. *****

Solution

Number of sides of basement = 5

Side length of basement = 8 ft

Apothem of basement = 7 ft

Perimeter of basement = 5 × 8 = 40 ft

$$\text{Area of basement} = \frac{1}{2}(40)(7) = 140 \text{ ft}^2$$

Exercise 9.6

Q.1 A lawn is in the shape of equilateral triangle. Find the perimeter of lawn if length of one side is 5m. Also find the cost of boundary wall of lawn @Rs.220 per meter. *****

Solution

Shape of lawn = equilateral triangle

Length of one side = 5 = 5m

Perimeter of equilateral Δ = 3s

$$\begin{aligned} \text{Perimeter of lawn} &= 3 \times 5\text{m} \\ &= 15 \text{ m} \end{aligned}$$

Cost of 1 m boundary wall = Rs. 220

$$\begin{aligned} \text{Cost of 15m boundary wall} &= \text{Rs. } 220 \times 15 \\ &= \text{Rs. } 3,300 \end{aligned}$$



Q.2 Cricket ground in a village is in the shape of parallelogram. One side of ground is 65m long and distance between parallel side having length 65m is 42m. Find the cost of planting grass @ Rs.10 per square metre. *****

Solution

Shape of ground = parallelogram

Base length = b = 65m

Height = h = 42 m

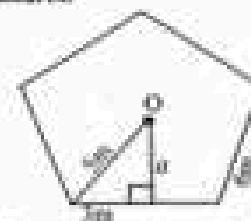


$$\begin{aligned} \text{Area of figm} &= b \times h \\ &= 65\text{m} \times 42\text{m} \\ &= 2,730 \text{ m}^2 \end{aligned}$$

Cost of planting grass at $1\text{m}^2 = \text{Rs. } 10$

$$\begin{aligned} \text{Cost of planting grass at } 2,730\text{m}^2 &= \text{Rs. } 10 \times 2730 \\ &= \text{Rs. } 27,300 \end{aligned}$$

Q.3 Base of a minaret of a mosque is built in the shape of regular pentagon as shown in the figure. Find the perimeter and area of base of minaret. *****



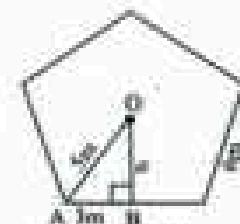
Shape of base of minaret is regular pentagon

Side length = 5 = 5m

Perimeter of pentagon = 5s

Perimeter of base = 5 × 5 m

$$P = 25 \text{ m}$$



$$\text{Area of regular pentagon} = \frac{1}{2}(p)(a) \dots (i)$$

First we find value of a

In Δ AOB, by Pythagoras theorem

$$(\overline{AO})^2 = (\overline{AB})^2 - (\overline{BO})^2$$

$$5\text{m}^2 = 3\text{m}^2 + a^2$$

$$25\text{m}^2 - 9\text{m}^2 = a^2$$

$$16\text{m}^2 = a^2$$

$$\Rightarrow \sqrt{a^2} = \sqrt{16\text{m}^2}$$

$$a = 4\text{m}$$

Putting the value of p and a in eq. (i)

$$\begin{aligned} \text{Area} &= \frac{1}{2}(p)(a) \\ &= \frac{1}{2}(25\text{m})(4\text{m}) \\ &= \frac{1}{2}(100\text{m}^2) \\ &= 50 \text{ m}^2 \end{aligned}$$

Q.4 A plot in the shopping area is in the shape of regular hexagon. One side of the plot is 5m long. Find:

i) The cost of fencing the plot @ Rs.100 per metre. *****

ii) Area of the plot. *****

iii) the cost of filling the plot @ Rs.500 per m^2 . *****

Solution

Shape of plot is regular hexagon.

One side of plot = 5 = 5m

i) Cost of fencing

First we find perimeter of hexagonal plot



Length of boundary of plot = $p = 65$

$$P = 6 \times 10m$$

$$p = 60m$$

Cost of fencing 1m of plot = Rs. 160

Cost of fencing 60m of plot = Rs. 160 $\times$ 60

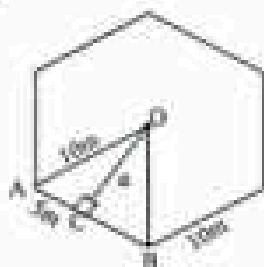
$$\text{Rs. 9,600}$$

ii) Area of Plot

$$\text{Area of hexagonal plot} = \frac{1}{2}(p)(a) \dots (i)$$

First we find value of a

In hexagon ΔOAB is an equilateral Δ with each side 10m.



Here $OC \perp AB$ as it bisects AB

In ΔOCA , by pythagoras theorem

$$(OA)^2 = (AC)^2 + (OC)^2$$

$$(10m)^2 = (5m)^2 + a^2$$

$$100m^2 - 25m^2 = a^2$$

$$75m^2 = a^2$$

$$75m^2 = a^2$$

$$\sqrt{a^2} = \sqrt{75m^2}$$

$$a = 5\sqrt{3}m$$

Putting the value of " p " and a in eq. (i)

$$\text{Area of plot} = \frac{1}{2}(60m)(5\sqrt{3}m)$$

$$= (10m)(5\sqrt{3}m)$$

$$= 150\sqrt{3}m^2$$

$$= 259.81m^2$$

iii) Cost of Filling the Plot

$$\text{Filling rate} = \text{Rs. 200 per } m^2$$

$$\text{Cost of filling } 1m^2 = \text{Rs. 200}$$

$$\text{Cost of filling } 259.81m^2 = \text{Rs. 200} \times 259.81$$

$$= \text{Rs. 129,900}$$

Q.5 A room is in the shape of square having perimeter of 48 feet. Find:

i) The cost of carpeting the floor @Rs.350 per ft^2 . *****

ii) The inner area of each wall if room is 10 feet high. *****

iii) The cost of painting inner sides of the room @ Rs.100 per ft^2 . *****

Solution

Shape of room is square.

$$\text{Perimeter of room} = P = 48 \text{ ft}$$

i) Cost of carpeting the floor

If L is length of floor then

$$P = 4L$$

$$48 = 4 \times L$$

$$\frac{48}{4} = L$$

$$12 = L$$

$$\Rightarrow L = 12 \text{ ft.}$$

$\Rightarrow$ Length of each side of room is 12ft

$$\text{Area of room} = L \times L$$

$$= 12ft \times 12ft$$

$$= 144 \text{ ft}^2$$

Given that carpeting rate = Rs.350 per ft^2

Cost of $1ft^2$ carpeting = Rs.350

Cost of $144ft^2$ carpeting = Rs. 350 $\times$ 144

$$= \text{Rs. 50,400}$$

ii) Area of each wall

Given that height of wall = $h = 10ft$

$$\text{Length of wall} = L = 12ft$$

$$\text{Area of each wall} = L \times h$$

$$= 12ft \times 10ft$$

$$= 120ft^2$$

iii) Cost of painting

Painting rate = Rs.100 per ft^2

There are 4 walls in a room.

$$\text{Area of 1 wall} = 120ft^2$$



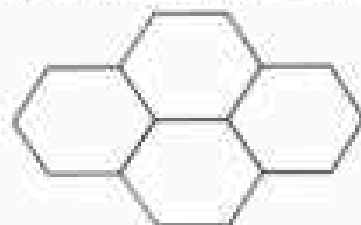
$$\begin{aligned} \text{Total area of 4 walls} &= 4 \times 120ft^2 \\ &= 480 \text{ ft}^2 \end{aligned}$$

$$\text{Cost of painting } 1ft^2 = \text{Rs.100}$$

$$\text{Cost of painting } 480 \text{ ft}^2 = \text{Rs. 100} \times 480$$

$$= \text{Rs. 48,000}$$

Q.6 A tile is in the shape of regular hexagon. Each side of the tile is one foot long. Find the perimeter of four tiles joined together as shown in the figure. *****



Solution

$$\text{Each side of tile} = s = 1 \text{ ft}$$

$$\text{Number of tiles} = 4$$

As shape of each tile is hexagon

$$\text{Total number of sides} = 6 \times 4 = 24$$

$$\text{Number of adjoining sides} = 5 \times 2 = 10$$

$$\text{Number of sides on the boundary} = 24 - 10 = 14$$

If length of each side is 1ft, then perimeter of shape is 14ft.

$$p = 14ft$$

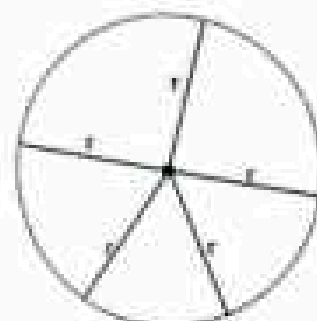
Locus

A locus is the set of points that satisfy a given condition. It is a path traced by a moving point which moves according to some given geometrical condition.

Locus of a point from a fixed point

A circle with centre O and radius r cm is the locus of a point P moving in a plane in such a way that its distance from a fixed point O is always equal to r cm.

This theorem helps to determine the region formed by all the points which are located at the same distance from a single point.

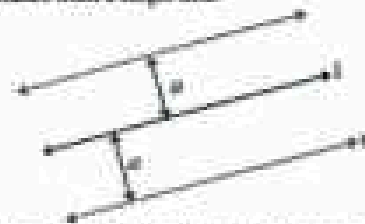


Locus of point

Locus of a point from a given straight line

Locus of a point that moves in such a way that its distance from a fixed line l is always equal to a cm. As clear from the figure, the locus of moving point is a pair of straight lines m and n each parallel to l and are located on either side of l at a distance a cm from the line l .

This theorem helps to find the region formed by all the points which are located at the same distance from a single line.



Locus of a point equidistant from two given points

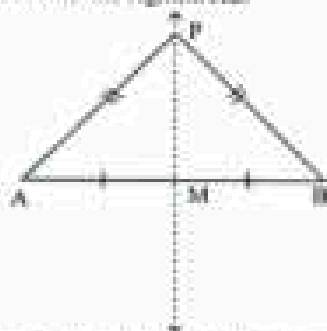
The locus which is equidistant from the two given points say A and B , is the perpendicular bisector of the line segment that joins the two points.

Here, in the figure PS is the perpendicular bisector of AB .

This theorem helps to determine the region formed by all the points which are located at the same distance from points A and B .



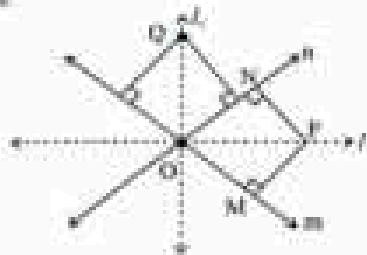
The region formed should be the perpendicular bisector of the line segment AB.



Locus of the point equidistant from two given intersecting lines

The locus of point which is equidistant from the two intersecting lines say m and n , is considered to be a pair of lines that bisect the angle formed by the two lines m and n .

In the figure, locus of point P forms a line l which bisects the angle formed by two intersecting lines m and n , and is equidistant from both lines. We say P lies on the bisector of angle formed by m and n . Similarly, locus of point Q forms a line l_1 which bisects the angle formed by lines m and n , and is equidistant from both lines. Thus, l and l_1 are the pairs of lines that bisect the angle formed at O . We say Q lies on the bisector of angle formed by m and n . This theorem helps to find the region formed by all the points which are located at the same distance from the two intersecting lines.



Example 20

Locus of three points A, B and C are equidistant from a fixed point O. Prove that they form concentric circles. Also sketch the figure.



Solution

Since the loci of the three points A, B and C are equidistant from a fixed point O, therefore, point O is the centre of circles formed by the movement of these points.

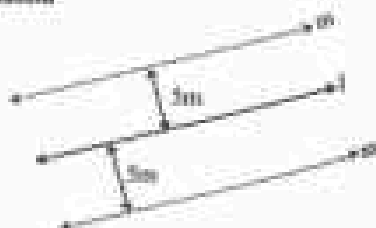
Hence the three circles formed are concentric circles.

Example 21

A point P is moving parallel to a straight line at a distance of 5m.

- Draw the locus of the path.
- Prove that the point is moving in straight line.
- How many paths are possible for the moving points P ?
- Another point Q is moving 8m away from the line l . Explain why the point Q is not the part of locus?

Solution



- Locus of the path of point P is possible along the lines m and n that are 5m apart from l .



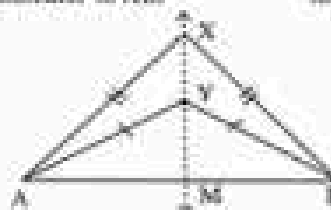
- As the point P is moving parallel to straight line l , therefore it follows straight path too.

- Two paths which are along m and n .

- As the point Q is moving along straight line 8m away from l , therefore it is not part of locus of point P which is 5m away from l .

Example 22

Figure shows two isosceles triangles XAB and YAB on the same base AB . Show the produced line XY bisects AB and is perpendicular to AB .



Solution

Draw line segments XA and YA intersecting AB at M . As $\triangle XAB$ is isosceles, therefore:

$$XA = XB \quad \text{--- (i)}$$

Which shows that X is equidistant from A and B and hence lies on the perpendicular bisector of AB . Similar, $\triangle YAB$ is isosceles, therefore:

$$YA = YB$$

Which shows that Y is equidistant from A and B and hence lies on the perpendicular bisector of AB .

This shows that XY is perpendicular bisector of AB .

Also, (i) and (ii) show that every point on XY is equidistant from A and B .

Since M lies on XY , therefore M is mid-point AB and consequently XYM is perpendicular bisector of AB .

Example 23

Take three non-collinear points and find a point O which is equidistant from these non-collinear points. Draw the locus of point passing through these three non-collinear points. What could be the name of this locus? Give the point O a specific name.

Solution

Let A , B and C be three non-collinear points in the plane.

Join AB and BC .

Draw perpendicular bisector m of AB and n of BC .

Both bisectors meet at O .

As O lies on perpendicular bisector of AB , therefore:

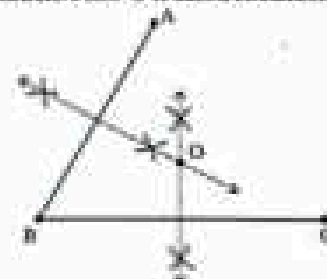
$$OA = OB \quad \text{(i)}$$

Again, as O lies on perpendicular bisector of BC , therefore:

$$OB = OC \quad \text{(ii)}$$

$\therefore OA = OB = OC$ (from (i) and (ii))

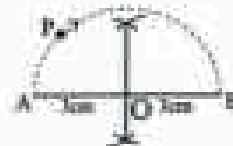
Hence O is equidistant from points A , B and C , and if we draw a circle with centre O , we get a locus of circle passing through these non-collinear points A , B and C called circumcircle. Point O is called circumcentre.



Exercise 9.5

- Q.1 Draw $AB = 6\text{cm}$. Bisect AB at O and draw locus of point P equidistant from O and above AB . What could be the name of locus?

Solution



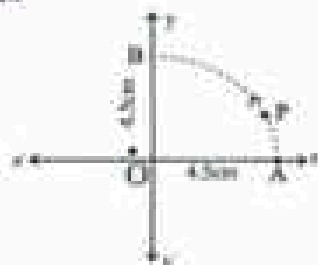


- Draw $m\overline{AB} = 4\text{cm}$.
- Draw right bisector of $m\overline{AB}$ which passes through midpoint O of $\overline{AB}$.
- With centre O , draw a semicircle of radius $m\overline{OA}$ above $\overline{AB}$.
- Locus of point P equidistant from point O is the semicircle above $\overline{AB}$.

Q.2 Draw coordinate axes. Take two points A and B on x -axis and y -axis respectively at a distance of 4.5cm each. Draw a locus of points from A to B equidistant from origin. What is specific name of this locus? How many such loci can be drawn around the origin? answer

Solution

- Draw



- Draw coordinate axes.
- Take two points A and B on x -axis and y -axis respectively which are 4.5cm from origin O .
- With at origin O , draw an arc of radius 4.5cm from A to B which is the required locus of point P .
- The name of locus is quarter circle.
- Four such loci can be drawn around the origin.

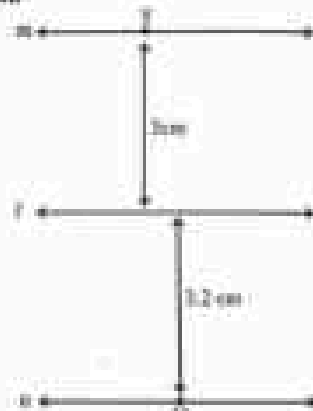
Q.3 Draw a horizontal line l .

- Take a point T above l at a distance of 3cm and draw a locus of points through T parallel to l . answer

- Take a point T above l at a distance 3.2cm and draw a locus of points through T parallel to l . answer

- What is distance between both loci? answer

Solution



Draw a horizontal line l .

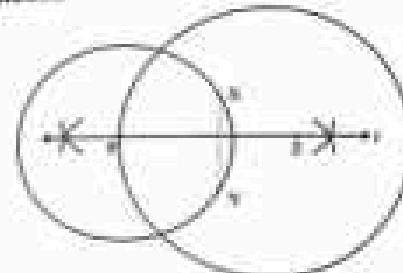
- Take a point T above l at a distance of 3cm . Draw a line m parallel to l passing through point T . Which is the locus of T parallel to line l .
- Take another point Q below l at a distance of 3.2cm . Draw a line n parallel to l passing through Q . Which is the locus of Q parallel to line l .
- The distance between both loci:
 $3\text{cm} + 3.2\text{cm} = 6.2\text{cm}$.

Q.4 Diagram shows a circle with centre P . X and Y are two points on the circumference of the circle.

- Draw locus of points which are equidistant from X and Y through P . answer
- Take another point Z on the locus outside the circle and draw another circle of radius PZ . answer

- What is the relation of this circle with given circle. answer

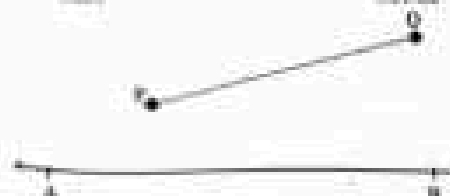
Solution



- The locus of points which are equidistant from X and Y is the right bisector l of chord $\overline{XY}$.
- Take another point Z on the locus l outside the circle and draw another circle of radius $m\overline{PZ}$ which passes through centre P .
- The circle with centre Z intersects the circle with centre P these circles are intersecting circles.

Q.5 Draw a line AB . Let P and Q be two points not on AB but coplanar with AB . Draw the locus of points from P to Q .

- What is the name of the locus? answer
- Is there any point on the locus PQ if extended which lies on line AB ? answer
- How can we take points P and Q such that no point of the above locus lies on line AB ? answer
- How can we take points P and Q such that every point of line AB may lie on locus? answer



- Draw $\overline{AB}$. Take two points P and Q not on AB . Join P to Q .

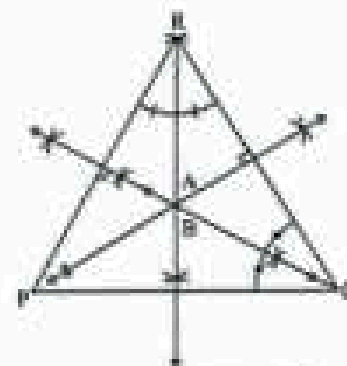
The locus of point from P to Q is line segment $\overline{PQ}$.

- Yes, if locus $\overline{PQ}$ is extended, then its intersecting point will be on $\overline{AB}$.
- If P and Q are at the same distance from the line then locus will be parallel to line AB and no point of locus will be on $\overline{AB}$.
- By taking P and Q on the $\overline{AB}$ we can make it possible that every point of locus is on the $\overline{AB}$.

Q.6 Draw an equilateral triangle PQR of suitable measurement.

- Draw right bisectors of any two sides and locate a point A where both bisectors meet. answer
- Draw angle bisectors of any two vertices and locate a point B where both bisectors meet. answer
- What is the relation between locus of A and B ? answer

Solution



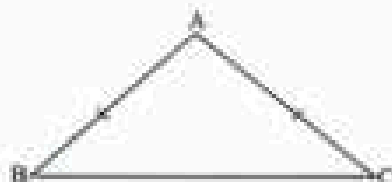
- The right bisectors of $\overline{PQ}$ and $\overline{PR}$ meet at point A .



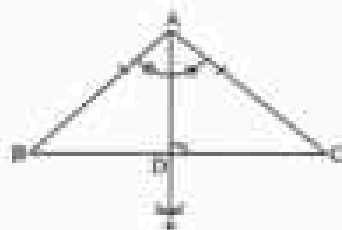
- ii) The angle bisectors of $\angle Q$ and $\angle R$ meet at a point S .

iii) Points A and B coincide.

Q.7 Figure shows an isosceles triangle ABC . Prove that the locus of bisector of angle A is right bisector of side BC . *Answer*



Solution



- Draw an isosceles $\triangle ABC$ in which $\overline{mAB} = \overline{mAC}$.
- Draw $\overline{AD}$, the bisector of $\angle A$. Which rule the side $\overline{BC}$ at point D .
- By measuring $\overline{BD}$ and $\overline{CD}$ we see that $\overline{mBD} = \overline{mCD}$ i.e. D is midpoint of $\overline{BC}$.

iv) By measuring $\angle ADC$ and $\angle ADB$ we see that $m\angle ADC = m\angle ADB = 90^\circ$.

Hence, locus of bisector of $\angle A$ is right bisector of $\overline{BC}$.

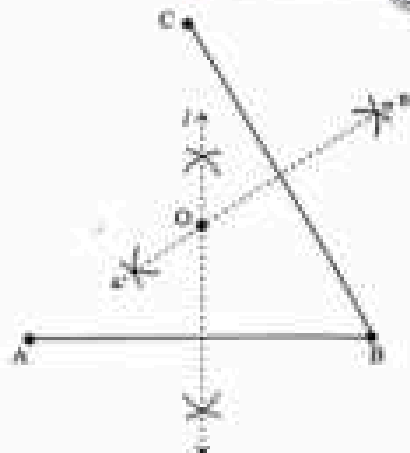
Q.8 Draw three non-collinear points in the plane. Find the locus of the point which are equidistant from these three points. How many such points exist? *Answer*

Solution

- Take let A, B and C be three non-collinear points in a plane.
- Join A to B and B to C .
- Draw right bisector l of $\overline{AB}$ and m of $\overline{BC}$.
- The right bisectors of $\overline{AB}$ and $\overline{BC}$ meet each other at point O .
- Since, point O is on right bisector of $\overline{AB}$ so, $\overline{mOA} = \overline{mOB}$(i)
- Similarly point O is also on the right bisector of $\overline{BC}$.
So $\overline{mOB} = \overline{mOC}$(ii)
From (i) and (ii)
 $\overline{mOA} = \overline{mOB} = \overline{mOC}$
Hence point O is equidistant from three non-collinear points.
- There is only one point which is equidistant from three non-collinear points.

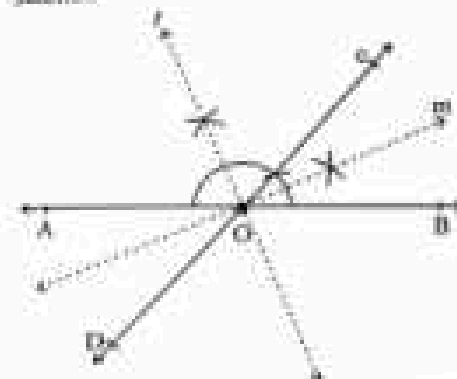
Q.9 Take two lines AB and CD inclined at 60° intersecting at O .

- Draw a locus of points which are equidistant from both lines. *Answer*
- Draw bisector of 60° . *Answer*
- What is relation between locus of point equidistant from lines and angle bisector? *Answer*



- iv) Draw bisector of adjacent angle at O find the relation between both angle bisectors. *Answer*

Solution



Miscellaneous Exercise 9

Q.1 Encircle correct option in the following:

- Which of the following is polygon? *Answer*
(a) circle (b) pyramid
(c) quadrilateral (d) sphere
- Which of the following regular polygon? *Answer*
(a) kite (b) rhombus
(c) rectangle (d) square
- If two triangles are similar, their corresponding sides are: *Answer*
(a) proportional (b) equal
(c) congruent (d) parallel
- What is the sum of interior angles for an irregular hexagon? *Answer*
(a) 120° (b) 720°
(c) 135° (d) 360°
- What is the sum of interior angles for a irregular 12 sides polygon? *Answer*
(a) 1800° (b) 2160°
(c) 1980° (d) 360°
- How many sides a regular polygon has if its exterior angle is 15° *Answer*

Draw two lines AB and CD inclined at 60° intersecting at O .

- The angle bisectors l and m of $\angle BOC$ and $\angle AOC$ are locus of points which are equidistant from both lines.
- Locus n is bisector of 60° .
- The locus l and angle bisector are same.
- Both angle bisectors are perpendicular.

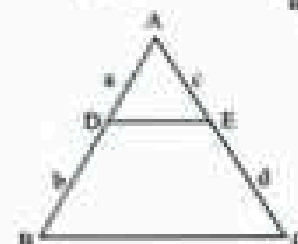
- 20°
- 24°
- 21°
- 27°

7. In the figure, $AB = 21$ cm and P divides AB in the ratio $3:4$. What is length of $\overline{AP}$? *Answer*



- 5 cm
- 10 cm
- 12 cm
- 9 cm

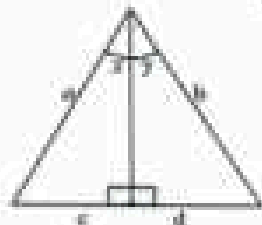
8. In the figure, $\frac{a}{b} = \frac{c}{d}$. Which one is true? *Answer*



- $DE = BC$
- $DE > BC$
- $DE < BC$
- $DE \neq BC$

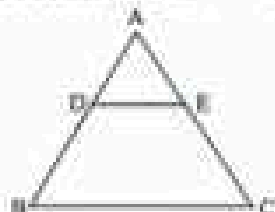


9. In the figure if $x = y$. Then, the value of b is: answer



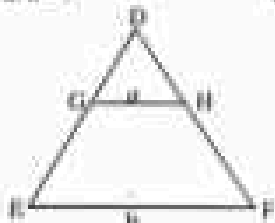
- (a) $\frac{ac}{d}$ (b) $\frac{ad}{c}$
(c) $\frac{cd}{a}$ (d) $\frac{c}{ad}$

10. In the figure, $\triangle ABC$ is equilateral and $\overline{DE} \parallel \overline{BC}$, then $\triangle ADE$ is: answer



- (a) right angled (b) scalene
(c) isosceles (d) equilateral

11. In the figure, $\overline{GH} \parallel \overline{EF}$.
Then $a : b = ?$ answer



Additional MCQ

18. What is key characteristic of similar figures? answer
- (a) They have the same size
(b) They have different shapes

- (a) $DG : DE$ (b) $DG : DH$
(c) $DG : GE$ (d) $DE : EF$
12. What is the sum of all the exterior angles of a 13-sided polygon whose one interior angle is equal to x° ? answer
- (a) 90° (b) 360°
(c) 360° (d) 180°
13. Which polygon has both its interior and exterior angles the same? answer
- (a) pentagon (b) triangle
(c) square (d) hexagon
14. The formation or expression of an axiom or theory without sufficient evidence for proof is known as: answer
- (a) axiom (b) conjecture
(c) corollary (d) theorem
15. A mathematical statement that is proved true based on already accepted axioms is called: answer
- (a) axiom (b) conjecture
(c) postulate (d) theorem
16. A mathematical statement that is assumed to be true without proof is called: answer
- (a) axiom (b) conjecture
(c) postulate (d) theorem
17. Two solids with equal ratio of corresponding linear measures: answer
- (a) are similar
(b) are congruent
(c) have different area
(d) have same volume



- (a) They have the same shape
(b) They have different angles
19. Which criteria are used to establish the similarity of two polygons? answer
- (a) only angle ratios
(b) only side length ratios
(c) both angle ratios and side length ratio
(d) number of sides
20. If two triangles are similar and the ratio of their corresponding sides length is 3:2, what can you conclude about the ratio of their areas? answer
- (a) the ratio of their areas is 9:4
(b) the ratio of their areas is 3:2
(c) the ratio of their areas is 6:4
(d) the ratio of their areas cannot be determined
21. If two rectangular prisms are similar, and the ratio of their side lengths is 2:1, what is the ratio of their volumes? answer
- (a) The ratio of their volumes is 4:1
(b) The ratio of their volumes is 2:1
(c) The ratio of their volumes is 8:1
(d) The ratio of their volumes is 1:2
22. Which of the following statements about the volume of similar solids is true? answer
- (a) Similar solids always have the same volume
(b) The ratio of their volumes is the cube of the ratio of their linear dimensions
(c) Similar solids have different shapes
(d) The volume of similar solids depends on their masses

Answer Key

1	c	2	d	3	a	4	b
5	a	6	c	7	d	8	d
9	b	10	d	11	a	12	b
13	c	14	b	15	d	16	a
17	a	18	c	19	c	20	a
21	a	22	b				

- Q.2 Find the exterior angle of a polygon with 6 sides. answer

Solution

Number of sides = $n = 6$

$$\begin{aligned} \text{Each exterior angle} &= \frac{360^\circ}{n} \\ &= \frac{360^\circ}{6} \\ &= 60^\circ \end{aligned}$$

- Q.3 Is it possible to have a polygon, in which sum of interior angles is 9 right angles? answer

Solution

Let we have n -sided polygon whose sum of interior angles is right angles = $9 \times 90^\circ = 810^\circ$.

Sum of interior angles = $(n-2) \times 180^\circ$

$$810^\circ = (n-2) \times 180^\circ$$

$$\frac{810^\circ}{180^\circ} = n-2$$

$$4.5 = n-2$$

$$4.5 + 2 = n$$

$$\Rightarrow n = 6.5$$

As n is number of sides which is not possible in fraction. So, there is no polygon whose sum of interior angles is 9 right angles.

- Q.4 Is it possible to have a polygon whose sum of interior angles is 7200° ? answer

Solution

Let we have where n -sided polygon whose sum of interior angles is 7200° .

Sum of interior angles = $(n-2) \times 180^\circ$

$$7200^\circ = (n-2) \times 180^\circ$$

$$\frac{7200^\circ}{180^\circ} = (n-2)$$

$$40 = n-2$$

$$40 + 2 = n$$

$$42 = n$$

$$n = 42$$

Yes, a polygon having 42 equal sides has sum of interior angles equal to 7200° .



Q.5 Find the measure of each angle of a regular nonagon.

Nonagon has 9 equal sides, $n = 9$

Each interior angle = $\frac{(n-2) \times 180^\circ}{n}$

$$= \frac{(9-2) \times 180^\circ}{9} = \frac{7 \times 180^\circ}{9}$$

$$= \frac{1260^\circ}{9}$$

$$= 140^\circ$$

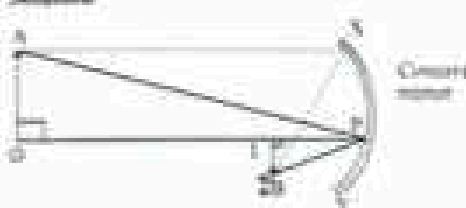
Q.6 In the figure XY is a concave mirror OA is object IB is its image.

i) Show that $\triangle OAP \sim \triangle IBP$

ii) Find height of object if IB = 2 cm, OP = 10 cm, IP = 4 cm.



Solution



i) In $\triangle OAP$ and $\triangle IBP$

$\angle OAP = \angle IBP$ (both are 90°)

$\angle OPA = \angle IPB$ (angles of incident and reflection are same)

$\angle POA = \angle PBI$ (complement of acute angles of right $\triangle$)

$\triangle OAP \sim \triangle IBP$ corresponding angles are equal

ii) Height of object $\overline{OA} = h$

$\overline{IB} = 2 \text{ cm}$, $\overline{OP} = 10 \text{ cm}$

$$\overline{IP} = 4 \text{ cm}$$

In similar triangles

$$\frac{\overline{OA}}{\overline{IB}} = \frac{\overline{OP}}{\overline{IP}}$$

Putting the values

$$\frac{h}{2 \text{ cm}} = \frac{10 \text{ cm}}{4 \text{ cm}}$$

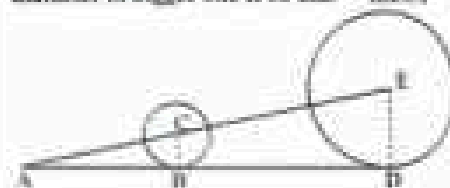
$$h = \frac{10}{4} \times 2 \text{ cm}$$

$$h = \frac{20}{4} \text{ cm}$$

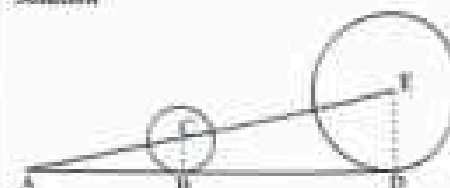
$$h = 5 \text{ cm}$$

Thus height of object is 5 cm.

Q.7 In the figure, $\overline{AB} = 6 \text{ cm}$, $\overline{BD} = 9 \text{ cm}$. Find the diameter of smaller circle, if diameter of bigger one is 30 cm.



Solution



Given:

$$\overline{AB} = 6 \text{ cm}$$

$$\overline{BD} = 9 \text{ cm}$$

Diameter of bigger circle = 30 cm

Radius of bigger circle = $\overline{BD} = 15 \text{ cm}$

Radius of smaller circle = $r = \overline{AB}$

As $\triangle ABC$ and $\triangle ADE$ are similar, and ratio of corresponding sides are equal.



$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{AD}}{\overline{AE}}$$

$$\frac{6}{15} = \frac{9}{15} \quad (\overline{AD} = 6 + 9 = 15 \text{ cm})$$

$$r = \frac{6}{15} \times 15$$

$$r = \frac{240}{15} \text{ cm}$$

$$r = 16 \text{ cm}$$

$$\begin{aligned} \text{Diameter of smaller circle, } d &= 2r \\ &= 2 \times 16 \text{ cm} \\ &= 32 \text{ cm} \end{aligned}$$

Q.8 Prove that if vertex angles of two isosceles triangles are equal then the two triangles are similar.

Solution

Let $\triangle ABC$ and $\triangle PQR$ be two isosceles triangles having vertex angles of equal measurement i.e. $\angle A = \angle P$ (i)



In isosceles triangle angles opposite to equal sides are equal.

$$\angle B = \angle C = x^\circ \text{ and } \angle Q = \angle R = y^\circ$$

In $\triangle ABC$,

$$\angle A + \angle B + \angle C = 180^\circ$$

$$\angle A + x^\circ + x^\circ = 180^\circ$$

$$2x^\circ = 180^\circ - \angle A$$

$$x^\circ = \frac{180^\circ - \angle A}{2} \quad \text{--- (ii)}$$

In $\triangle PQR$,

$$\angle P + \angle Q + \angle R = 180^\circ$$

$$\angle P + y^\circ + y^\circ = 180^\circ$$

$$2y^\circ = 180^\circ - \angle P$$

$$y^\circ = \frac{180^\circ - \angle P}{2}$$

$$y^\circ = \frac{180^\circ - \angle A}{2}$$

$$\text{ii) } (\angle B = \angle Q)$$

From (i) and (ii)

$$x^\circ = y^\circ$$

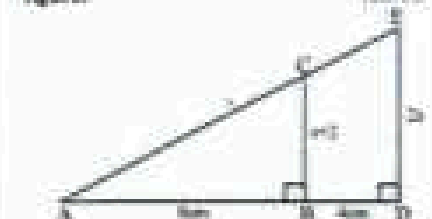
$$\Rightarrow \angle B = \angle Q \text{ and } \angle C = \angle R$$

Also $\angle A = \angle P$ (Given)

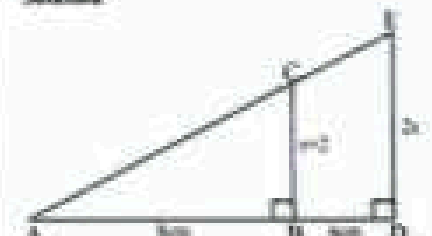
The all corresponding angles of two isosceles triangle are equal which proves that two isosceles triangles are similar.

$$\triangle ABC \sim \triangle PQR$$

Q.9 Find the length of $\overline{BC}$ and $\overline{DE}$ in the figure.



Solution



$$\overline{BC} = x + 2$$

$$\overline{DE} = 2x$$

If $\overline{AB} = 6 \text{ cm}$, $\overline{BD} = 4 \text{ cm}$

Then $\overline{AD} = 6 \text{ cm} - 4 \text{ cm} = 2 \text{ cm}$

Since, $\triangle ABC \sim \triangle ADE$,

So,



$$\frac{m\overline{AB}}{m\overline{AD}} = \frac{m\overline{BC}}{m\overline{DE}}$$

$$\frac{8\text{cm}}{12\text{cm}} = \frac{x+2}{2x}$$

$$\frac{2}{3} = \frac{x+2}{2x}$$

$$\Rightarrow 2(2x) = 3(x+2)$$

$$4x = 3x + 6$$

$$2x - 3x = 6$$

$$x = 6$$

$$\text{Length of } \overline{BC} = x + 2$$

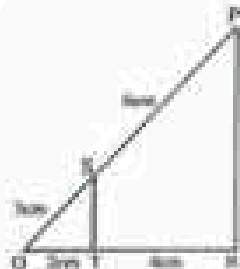
$$= 6 + 2 = 8\text{cm}$$

$$\text{Length of } \overline{DE} = 2x = 2(6) = 12\text{cm}$$

Q.10 Triangle SQT and PQR are similar. Find the ratio of area of triangle SQT to that of triangle PQR.



Solution



$$m\overline{QS} = 3\text{cm}, m\overline{SP} = 6\text{cm}$$

$$m\overline{PQ} = 8\text{cm}$$

Let A_1 and A_2 be the areas of $\triangle SQT$ and $\triangle PQR$ respectively.

Given that $\triangle SQT \sim \triangle PQR$

So,

$$\frac{A_1}{A_2} = \left(\frac{m\overline{QS}}{m\overline{PQ}}\right)^2$$

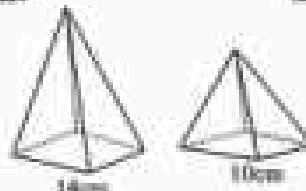
$$\frac{A_1}{A_2} = \left(\frac{3\text{cm}}{8\text{cm}}\right)^2$$

$$\frac{A_1}{A_2} = \left(\frac{3}{8}\right)^2$$

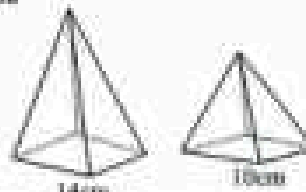
$$\frac{A_1}{A_2} = \frac{1}{9} \Rightarrow A_1 : A_2 = 1 : 9$$

Q.11 The following pyramids are similar and larger pyramid has a surface area of 392 cm^2 .

What is the surface area of smaller pyramid?



Solution



Let A_1 and A_2 be surface areas of smaller and larger pyramids respectively. $A_2 = 392\text{ cm}^2$

Also given that pyramids are similar,

$$\left(\frac{A_1}{A_2}\right) = \left(\frac{s_1}{s_2}\right)^2$$

$$\frac{A_1}{392} = \left(\frac{10\text{cm}}{14\text{cm}}\right)^2$$

$$\frac{A_1}{392} = \left(\frac{5}{7}\right)^2$$

$$A_1 = \frac{25}{49} \times 392$$

$$A_1 = 25 \times 8$$

$$A_1 = 200\text{ cm}^2$$

Q.12 The two cylinders are similar. What is the volume of the larger cylinder if the volume of smaller cylinder is 400π cm³.



Solution



Let V_1 and V_2 be the volume of larger and smaller cylinders respectively.

So,

$$V_1 = 40\pi\text{ cm}^3, r = 2\text{ cm}$$

$$V_2 = 3, R = 5\text{ cm}$$

Given that cylinders are similar,

$$\frac{V_1}{V_2} = \left(\frac{R}{r}\right)^3$$

$$\frac{V_1}{40} = \left(\frac{5}{2}\right)^3$$

$$V_1 = \frac{27}{4} \times 40$$

$$V_1 = 27 \times 5$$

$$V_1 = 135\pi\text{ cm}^3$$

Q.13 The following pairs of solids are similar. Find the surface area of red solid.

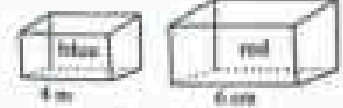
Q



Surface of area = 136 m^2

Solution

Q



Surface of area = 136 m^2

Let A_1 and A_2 be the surface areas of larger (red) and smaller (blue) cuboid respectively.

$$A_1 = 7, L = 8\text{ cm}$$

$$A_2 = 336\text{ cm}^2, l = 4\text{ cm}$$

Given that solids are similar,

So,

$$\frac{A_1}{A_2} = \left(\frac{L}{l}\right)^2$$

$$\frac{A_1}{336\text{ cm}^2} = \left(\frac{8\text{ cm}}{4\text{ cm}}\right)^2$$

$$\frac{A_1}{336\text{ cm}^2} = \left(\frac{2}{1}\right)^2$$

$$A_1 = \frac{9}{4} \times 336\text{ cm}^2$$

$$A_1 = 9 \times 84\text{ cm}^2$$

$$A_1 = 756\text{ cm}^2$$

Thus surface area of larger (red) solid is 756 cm^2 .

Q



Let A_1 and A_2 be the surface areas of larger (blue) and smaller (red) solid respectively.

$$A_1 = 1800\text{ inch}^2, L = 20\text{ inch}$$

$$A_2 = 7, l = 15\text{ inch}$$

Given that solids are similar,

So,

$$\frac{A_1}{A_2} = \left(\frac{L}{l}\right)^2$$

$$\frac{1800}{A_2} = \left(\frac{20}{15}\right)^2$$

$$\frac{1800}{A_2} = \left(\frac{4}{3}\right)^2$$

$$\frac{1800}{A_2} = \frac{16}{9}$$

$$\Rightarrow \frac{A_2}{1800} = \frac{9}{16}$$

$$A_2 = \frac{9}{16} \times 1800$$

$$A_2 = 9 \times 101.25$$

$$A_2 = 1012.5\text{ inch}^2$$

Thus surface area of smaller solid is 1012.5 inch^2



UNIT 10

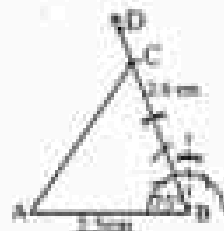
Practical Geometry

Construction of triangle when Measures of two sides and included angle are given

Example 1

Construct a triangle ABC when $\overline{AB} = 2.5$ cm, $\overline{BC} = 2.8$ cm and $m\angle B = 75^\circ$.

Solution:



Steps of construction

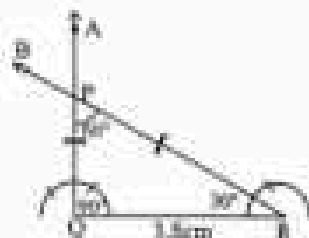
- Draw $\overline{AB} = 2.5$ cm.
- Construct an angle of 75° at point B with the help of ruler and compasses and draw $\overline{BD}$.
- With center B, draw an arc of radius 2.8 cm intersecting $\overline{BD}$ at C.
- Join C to A.
- ABC is required triangle.

Construction of triangle when Measures of one side and two angles are given

Example 2

Construct a triangle PQR when $m\angle Q = 90^\circ$, $m\angle P = 60^\circ$ and $m\angle R = 30^\circ$.

Solution:



Angle R is required for construction. We know that:

$$m\angle P + m\angle Q + m\angle R = 180^\circ$$

$$60^\circ + 90^\circ + m\angle R = 180^\circ$$

$$m\angle R = 30^\circ$$

Steps of construction

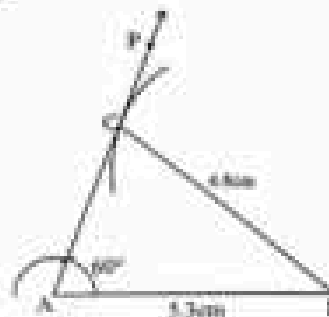
- Draw $m\angle Q = 90^\circ$.
- Construct an angle of 60° at Q with the help of ruler and compasses and draw $\overline{QA}$.
- Construct an angle of 30° at R and draw $\overline{RB}$.
- Both $\overline{QA}$ and $\overline{RB}$ intersect at point P.

Thus, PQR is required triangle.

Example 3

Construct a triangle ABC such that $\overline{AB} = 4.5$ cm and $m\angle A = 60^\circ$.

Solution:



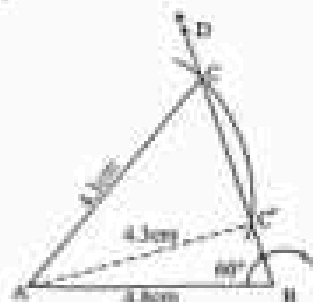
Steps of construction

- Draw $m\angle A = 60^\circ$.
- Construct $m\angle BAP = 60^\circ$.
- With center B, draw an arc of radius 3.5 cm which intersects $\overline{AP}$ at point C.

Example 4

Construct a triangle having length of two sides 4.5 cm and 4.1 cm and angle of 60° opposite to the side 4.5 cm long.

Solution:



Q.1 Construct the following triangles.

- $\triangle ABC$ when $m\overline{AB} = 5.8$ cm, $m\overline{AC} = 4.2$ cm, $m\angle A = 90^\circ$.

Steps of construction

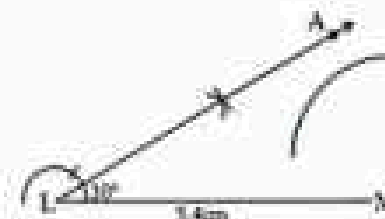
- Draw a line segment $\overline{AB}$ of length 4.8 cm.
- Construct $m\angle ABD = 60^\circ$.
- With center A, draw an arc of radius 4.3 cm intersecting $\overline{BD}$ at points C and C'.
- Join C and C' to A.

Thus two triangles ABC and ABC' are constructed.

Example 5

Construct a triangle having length of two sides as 2.5 cm and 5.4 cm and angle of 30° is opposite to the side 2.5 cm.

Solution:

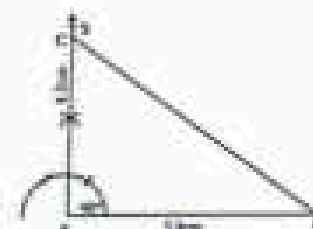


Steps of construction

- Draw a line segment LM of length 5.4 cm.
- Construct $m\angle MLA = 30^\circ$.
- With center M, draw an arc of radius 2.5 cm which does not intersect $\overline{LA}$ at any point.

$\therefore$ Construction of triangle is not possible.

Exercise 10.1





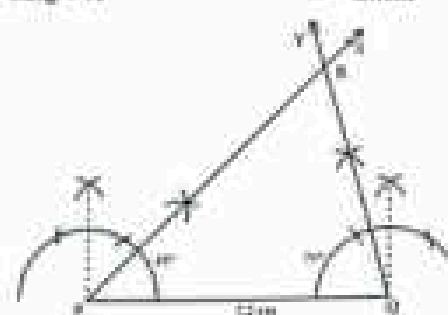
Steps of construction

- Draw $m\overline{AB} = 5.5\text{ cm}$
 - Construct an angle of 90° at point A with the help of compass and ruler and draw $m\overline{AX}$.
 - With center A, draw an arc of radius 4.2 cm intersecting $m\overline{AX}$ at C.
 - Join C to B.
- $\triangle ABC$ is required triangle.
- $\triangle LMN$ when $m\overline{LM} = 4\text{ cm}$, $m\overline{MN} = 4.7\text{ cm}$, $m\angle M = 120^\circ$



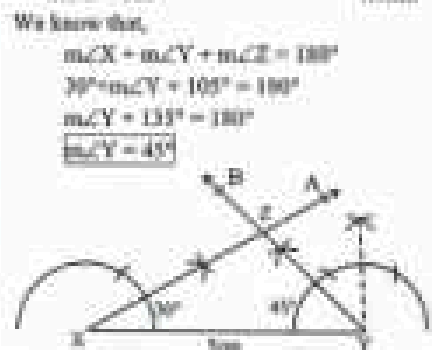
Steps of construction

- Draw $m\overline{LM} = 4\text{ cm}$
 - Construct an angle of 120° at point M with the help of compass and ruler and draw $m\overline{MX}$.
 - With center M, draw an arc of radius 4.7 cm intersecting $m\overline{MX}$ at N.
 - Join L to N.
- $\triangle LMN$ is required triangle.
- $\triangle PQR$ when $m\overline{PQ} = 7.2\text{ cm}$, $m\angle P = 45^\circ$, $m\angle Q = 75^\circ$



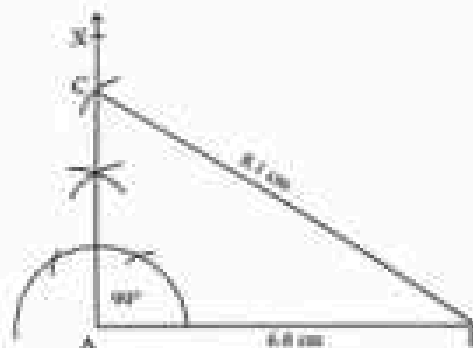
Steps of construction

- Draw $m\overline{PQ} = 7.2\text{ cm}$
 - Construct an angle of 45° at point P with the help of compass and ruler and draw $m\overline{PX}$.
 - Construct an angle of 75° at point Q with the help of compass and ruler and draw $m\overline{QY}$.
 - Both $m\overline{PX}$ and $m\overline{QY}$ intersect at point R.
- $\triangle PQR$ is required triangle.
- $\triangle XYZ$ when $m\overline{XY} = 5\text{ cm}$, $m\angle X = 30^\circ$, $m\angle Z = 105^\circ$



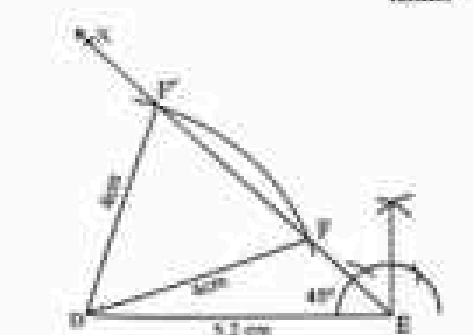
Steps of construction

- Draw $m\overline{XY} = 5\text{ cm}$
 - Calculate $m\angle Y$ which is 45° .
 - Construct an angle of 30° at point X with the help of compass and ruler and draw $m\overline{XA}$.
 - Construct an angle of 45° at point Y with the help of compass and ruler and draw $m\overline{YB}$.
 - Both $m\overline{XA}$ and $m\overline{YB}$ intersect at point Z.
- $\triangle XYZ$ is required triangle.
- Q.2 Construct the following triangles where possible.
- $m\overline{AB} = 6.8\text{ cm}$, $m\overline{BC} = 8.1\text{ cm}$, $m\angle A = 90^\circ$



Steps of construction

- Draw $m\overline{AB} = 6.8\text{ cm}$
 - Construct an angle of 90° at point A with the help of compass and ruler and draw $m\overline{AX}$.
 - With center B, draw an arc of radius 8.1 cm intersecting $m\overline{AX}$ at C.
 - Join C to B.
- $\triangle ABC$ is required triangle.
- $m\overline{DE} = 5.2\text{ cm}$, $m\overline{DF} = 4\text{ cm}$, $m\angle E = 45^\circ$



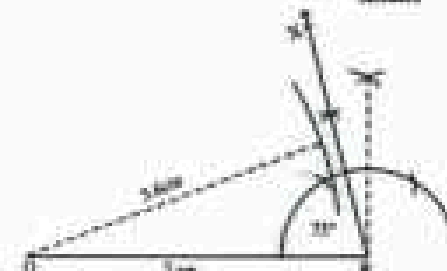
Steps of construction

- Draw $m\overline{DE} = 5.2\text{ cm}$
- Construct $m\angle E = 45^\circ$ with the help of compass and ruler and draw $m\overline{EX}$.
- With center D, draw an arc of radius 4 cm intersecting $m\overline{EX}$ at F and F'.

- Join D to F and F'

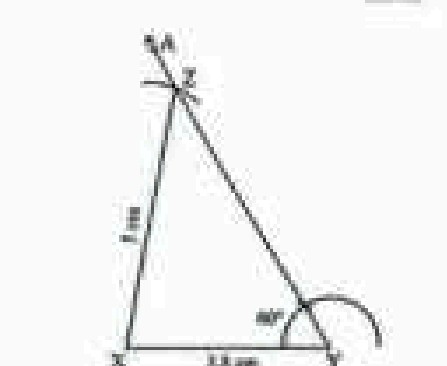
Thus two triangles DEF and DEF' are constructed.

- $m\overline{QR} = 7.8\text{ cm}$, $m\overline{PQ} = 5.6\text{ cm}$, $m\angle R = 75^\circ$



Steps of construction

- Draw $m\overline{QR} = 7.8\text{ cm}$
 - Construct $m\angle R = 75^\circ$ with the help of compass and ruler and draw $m\overline{RX}$.
 - With center Q, draw an arc of radius 5.6 cm which does not intersect $m\overline{RX}$ at any point.
- Thus construction of required triangle is not possible.
- $m\overline{XY} = 3.8\text{ cm}$, $m\overline{XZ} = 5\text{ cm}$, $m\angle Y = 60^\circ$





Steps of construction

Draw $m\overline{XY} = 3.5$ cm

- Construct $m\angle Y = 60^\circ$ with the help of compass and ruler and draw $m\overline{YZ}$.
- With centre X, draw an arc of radius 3 cm which intersects $m\overline{YZ}$ at point Z.
- Join X to the point Z.

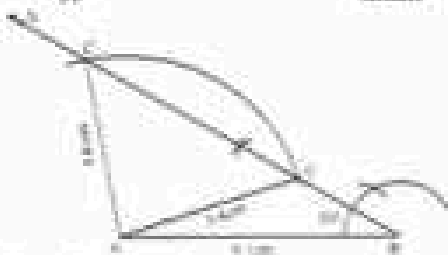
$\triangle XYZ$ is required triangle.
Let $m\overline{AB} = 3.5$ cm, $m\overline{AC} = 3.5$ cm,
 $m\angle B = 107^\circ$



Steps of construction

- Draw $m\overline{AB} = 3.5$ cm.
- Construct $m\angle B = 107^\circ$ with the help of compass and ruler and draw $m\overline{BC}$.
- With centre A, draw an arc of radius 3.5 cm which intersects $m\overline{BC}$ at C.
- Join A to the point C.

$\triangle ABC$ is required triangle.
Let $m\overline{AB} = 3.5$ cm, $m\overline{AC} = 3.5$ cm, $m\angle B = 107^\circ$



Steps of construction

- Draw $m\overline{AB} = 4.1$ cm
- Construct $m\angle B = 30^\circ$ with the help of compass and ruler and draw $m\overline{BC}$.
- With centre A, draw an arc of radius 3.8 cm intersecting $m\overline{BC}$ at C and C'.
- Join A to C and C'.

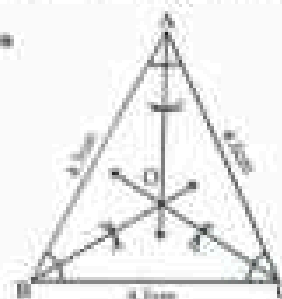
Directions of angles of a triangle

A ray which divides a given angle into two equal parts is called bisector of that angle.

Example 6

Construct a triangle ABC such that $m\overline{AB} = 4.7$ cm, $m\overline{BC} = 4.5$ cm and $m\overline{AC} = 4.6$ cm. Draw bisectors of these angles. Are they concurrent?

Solution



Steps of construction

- Draw $m\overline{BC} = 4.5$ cm.
- With centre B, draw an arc of radius 4.7 cm.
- With centre C, draw another arc of radius 4.6 cm intersecting first arc at A. Join A to B and C. $\triangle ABC$ is the required triangle.
- Draw bisectors of $\angle A$, $\angle B$ and $\angle C$ which meet at point O.

Conclusion

Angle bisectors of a triangle are concurrent.

Altitude of a triangle

A perpendicular line segment from a vertex to opposite sides of triangle is called altitude of triangle.

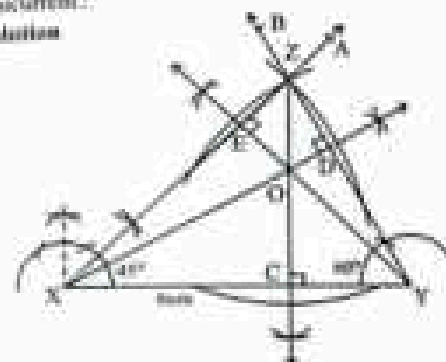
Example 7

Construct $\triangle XYZ$ when:

$m\overline{XY} = 6$ cm, $m\angle X = 45^\circ$, $m\angle Y = 60^\circ$

Draw altitudes of triangle. Are they concurrent?

Solution



Steps of construction

- Draw $m\overline{XY} = 6$ cm.
- Construct $\angle YXA = 45^\circ$ and $\angle XYB = 60^\circ$.
- $\overline{XA}$ and $\overline{XB}$ meet at Z.
- Draw altitudes $\overline{XD} \perp \overline{YZ}$, $\overline{YE} \perp \overline{XZ}$ and $\overline{ZF} \perp \overline{XY}$.
- The three altitudes meet at O.

Conclusion: $\therefore$ Altitudes of a triangle are concurrent.

Note:

- Altitudes of an acute angled triangle meet inside the triangle.
- Altitude of a right-angled triangle meet at the vertex of right angle.
- Altitudes of an obtuse angled triangle meet outside the triangle.

Perpendicular bisectors of sides of triangle

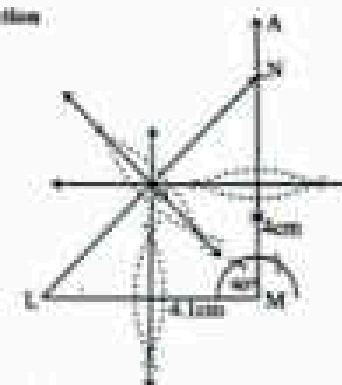
Example 8

Construct a $\triangle LMN$ when $LM = 4.1$ cm, $MN = 4$ cm and $\angle M = 90^\circ$. Draw perpendicular



bisectors of sides of triangle. Are they concurrent?

Solution



Steps of construction

- Draw $LM = 4.1$ cm.
- Construct $\angle LMA = 90^\circ$.
- With centre M, draw an arc of radius 4 cm intersecting $\overline{LA}$ at N.
- Join N to L.
- Draw right bisectors $\overline{LM}$, $\overline{MN}$ and $\overline{LN}$ which meet at point O.

Conclusion: $\therefore$ Right bisectors of sides of a triangle are concurrent.

Note:

- Right bisectors of an acute angled triangle meet inside the triangle.
- Right bisectors of a right angled triangle meet at the midpoint of hypotenuse the triangle.
- Right bisectors of an obtuse angled triangle meet outside the triangle.

Median of a triangle

Median of a triangle is a line segment joining the midpoint of a side to the opposite vertex.

**Key Fact**

- The point where angle bisectors of triangle meet is called **incribed centre** or **in-centre**.
- The point where altitudes of a triangle meet is called **orthocentre**.
- The point where perpendicular bisectors of sides of a triangle meet is called **circumcentre**.
- The point where medians of a triangle meet is called **centroid** of triangle. In an equilateral triangle incribed centre, circumcentre, orthocentre and centroid all lie at a single point (coincide).
- In any triangle circumcentre, orthocentre and centroid are collinear (lie on a same line).
- There are infinite of bisectors of any line segment. Only one of them is perpendicular to the line segment.
- Centre of gravity of a triangular shaped body lies on points of intersection of medians.

Example 9

Construct a triangle and prove that the medians

of a triangle are concurrent?

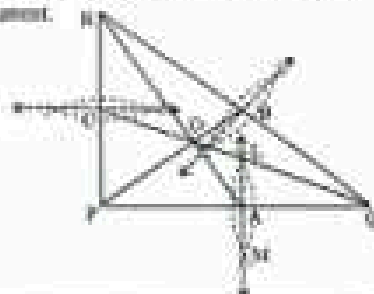
1001002

Solution

Step of construction

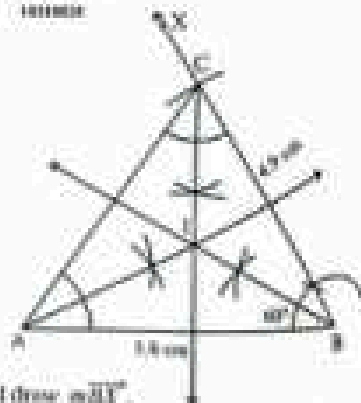
- Construct ΔPQR
- With centres P and Q, draw arcs of equal radius (more than half of PQ), which meet at two points L and M. $\overline{LM}$ bisect $\overline{PQ}$ at A.
- Similarly find midpoints B and C of the sides QR and PR respectively.
- $\overline{AP}$, $\overline{BQ}$ and $\overline{CR}$ are medians of triangle which meet at point O.

Conclusion: Thus medians of a triangle are concurrent.

**Exercise 10.2**

Q.1 Show that angle bisectors of the following triangles are concurrent.

- (a) ΔABC when $m\overline{AB} = 5.8\text{ cm}$, $m\overline{BC} = 4.9\text{ cm}$, $m\angle B = 60^\circ$ 1001002

**Steps of construction**

- Draw $m\overline{AB} = 5.8\text{ cm}$.
- Construct $m\angle B = 60^\circ$ with the help of compass and ruler and draw $m\overline{BX}$.
- With centre B, draw an arc of radius 4.9 cm intersecting $m\overline{BX}$ at point C.



- (c) Join A to C and get required ΔABC .

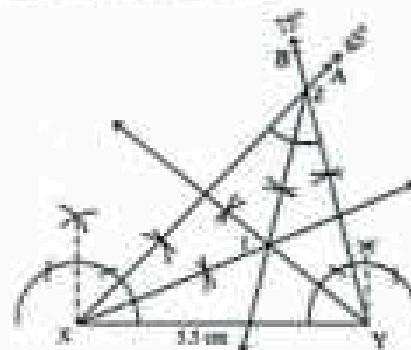
- (d) Draw bisectors of $\angle A$, $\angle B$ and $\angle C$ which meet each other at point I.

Conclusion

Angle bisectors of a triangle are concurrent.

- (b) ΔXYZ when $m\overline{XY} = 5.5\text{ cm}$, $m\angle X = 45^\circ$, $m\angle Y = 75^\circ$

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**Steps of construction**

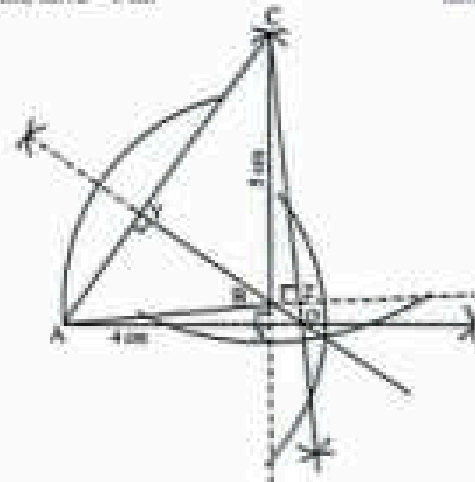
- Draw $m\overline{XY} = 5.5\text{ cm}$.
- Construct $m\angle X = 45^\circ$ with the help of compass and ruler and draw $m\overline{XZ}$.
- Construct $m\angle Y = 75^\circ$ with the help of compass and ruler and draw $m\overline{YZ}$.
- Both $m\overline{XZ}$ and $m\overline{YZ}$ intersect at point Z and required ΔXYZ is constructed.
- Draw bisectors of $\angle X$, $\angle Y$ and $\angle Z$ which meet each other at point I.

Conclusion

Angle bisectors of a triangle are concurrent.

Q.2 Show that altitudes of the following angles are concurrent.

- (a) ΔABC when $m\overline{AB} = 4\text{ cm}$, $m\overline{BC} = 5\text{ cm}$, $m\overline{AC} = 6\text{ cm}$ 1001002



Steps of construction

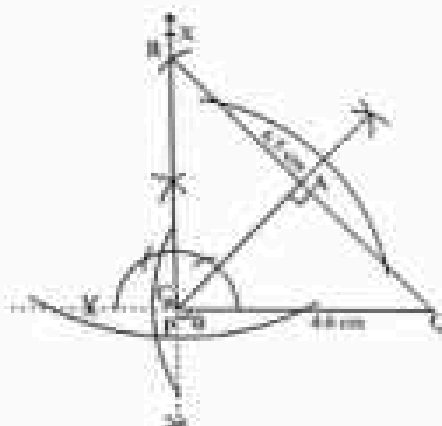
- Draw $m\overline{AB} = 4\text{ cm}$.
- With center A, draw an arc of radius 6 cm.
- With center B, draw an other arc of radius 5 cm which cuts the previous arc at C.
- Join point C to A, C to B and get required $\triangle ABC$.
- Draw altitudes $\overline{AX} \perp \overline{BC}$, $\overline{BY} \perp \overline{AC}$, and $\overline{CZ} \perp \overline{AB}$ which meet each other at point O.

Conclusion

Altitudes of a triangle are concurrent.

- (b) $\triangle PQR$ when $m\overline{PQ} = 4.6\text{ cm}$, $m\overline{QR} = 6.5\text{ cm}$, $m\angle P = 90^\circ$

EXAMPLE

Steps of construction

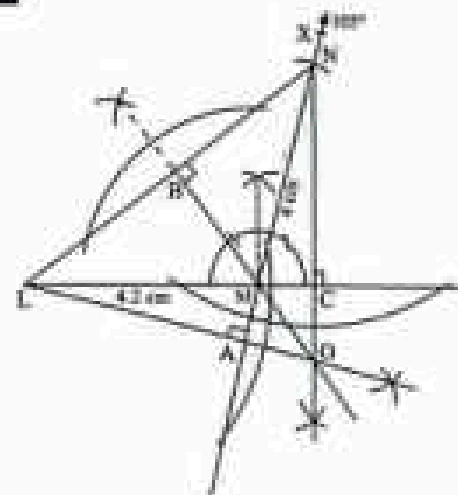
- Draw $m\overline{PQ} = 4.6\text{ cm}$.
- Construct $m\angle P = 90^\circ$ with the help of compass and ruler and draw $m\overline{PR}$.
- With center Q, draw an arc of radius 6.5 cm which cuts $m\overline{PR}$ at R.
- Join point Q to R and get required $\triangle PQR$.
- Draw altitudes $\overline{PA} \perp \overline{QR}$, $\overline{QB} \perp \overline{PR}$, and $\overline{RC} \perp \overline{PQ}$ which meet each other at point O, the vertex of right angle.

Conclusion

Altitudes of a triangle are concurrent.

- (c) $\triangle LMN$ when $m\overline{LM} = 4.2\text{ cm}$, $m\overline{MN} = 4\text{ cm}$, $m\angle M = 100^\circ$

EXAMPLE

Steps of construction

Draw $m\overline{LM} = 4.2\text{ cm}$

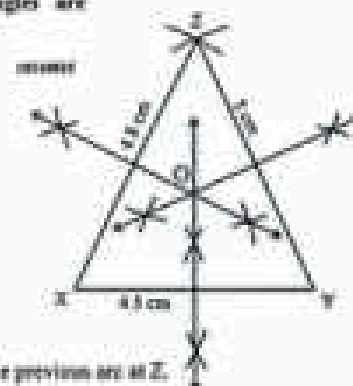
- Construct $m\angle M = 100^\circ$ with the help of compass and ruler and draw $m\overline{MN}$.
- With center L, draw an arc of radius 4 cm which cuts $m\overline{MN}$ at N.
- Join point L to N and get required $\triangle LMN$.
- Draw altitudes $\overline{LA} \perp \overline{MN}$, $\overline{MB} \perp \overline{LN}$, and $\overline{NC} \perp \overline{LM}$ which meet each other at point O.

Conclusion

Altitudes of a triangle are concurrent.

Q.3 Show that right bisectors of the following triangles are concurrent.

- (a) $\triangle XYZ$ when $m\overline{XY} = 4.5\text{ cm}$, $m\overline{YZ} = 5\text{ cm}$, $m\overline{ZX} = 4.8\text{ cm}$

Steps of construction

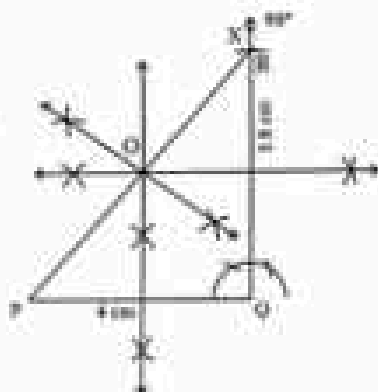
- Draw $m\overline{XY} = 4.5\text{ cm}$.
- With center X, draw an arc of radius 4.8 cm.
- With center Y, draw an other arc of radius 5 cm which cuts the previous arc at Z.
- Join point Z to X, Z to Y and get required $\triangle XYZ$.
- Draw right bisectors of sides $\overline{XY}$, $\overline{YZ}$ and $\overline{ZX}$ which meet each other at point O.

Conclusion

Right bisectors of sides of a triangle are concurrent.

- (b) $\triangle PQR$ when $m\overline{PQ} = 4\text{ cm}$, $m\overline{QR} = 5.8\text{ cm}$, $m\angle Q = 90^\circ$

EXAMPLE



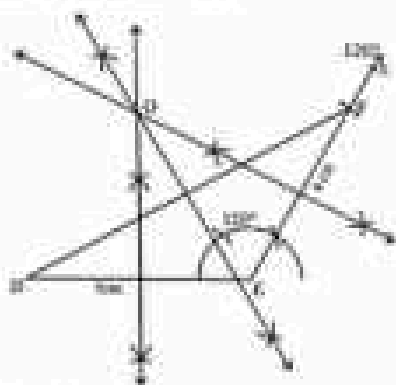
Steps of construction

- Draw $\overline{PQ} = 4 \text{ cm}$.
- Construct $\angle Q = 90^\circ$ with the help of compass and ruler and draw $\overline{QR}$.
- With center Q, draw an arc of radius 3 cm which cuts $\overline{QR}$ at R.
- Join point P to R, and get required $\triangle PQR$.
- Draw right bisectors of sides $\overline{PQ}$, $\overline{QR}$ and $\overline{PR}$ which meet each other at point O, the midpoint of hypotenuse.

Conclusion

Right bisectors of sides of a triangle are concurrent.

- (c) $\triangle DEF$ when $m\overline{DE} = 5 \text{ cm}$, $m\overline{EF} = 4 \text{ cm}$, $m\angle E = 120^\circ$



Steps of construction

- Draw $\overline{DE} = 5 \text{ cm}$.
- Construct $\angle E = 120^\circ$ with the help of compass and ruler and draw $\overline{EF}$.
- With center E, draw an arc of radius 4 cm which cuts $\overline{EF}$ at F.
- Join point F to D and get required $\triangle DEF$.
- Draw right bisectors of sides $\overline{DE}$, $\overline{EF}$ and $\overline{FD}$ which meet each other at point O.

Conclusion

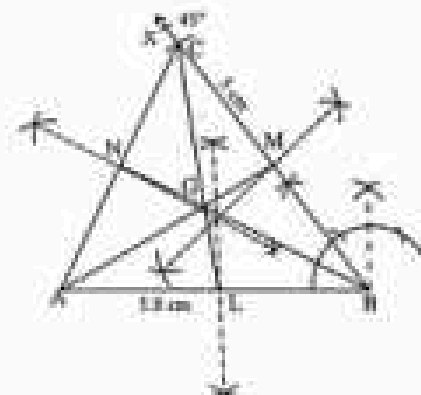
Right bisectors of sides of a triangle are concurrent.



Q.4 Show that medians of the following triangles are concurrent.

- (a) $\triangle ABC$ when $m\overline{AB} = 5.8 \text{ cm}$, $m\overline{BC} = 5 \text{ cm}$, $m\angle B = 45^\circ$

Answer



Steps of construction

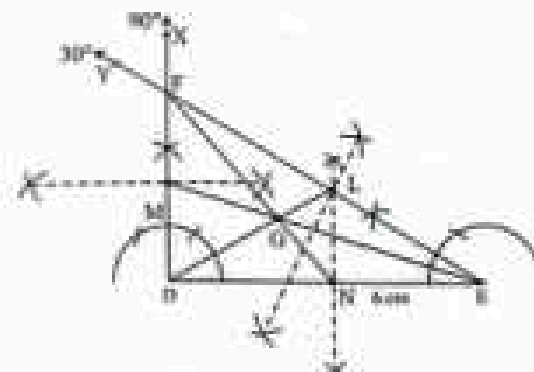
- Draw $\overline{AB} = 5.8 \text{ cm}$.
- Construct $\angle B = 45^\circ$ with the help of compass and ruler and draw $\overline{BC}$.
- With center B, draw an arc of radius 5 cm which cuts $\overline{BC}$ at C.
- Join point C to A and get required $\triangle ABC$.
- Draw right bisectors of sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ to find their midpoints L, M and N respectively.
- Join A to M, B to N and C to L.
- Thus $\overline{AM}$, $\overline{BN}$ and $\overline{CL}$ are required medians of $\triangle ABC$ which meet at O.

Conclusion

Medians of a triangle are concurrent.

- (b) $\triangle DEF$ when $m\overline{DE} = 6 \text{ cm}$, $m\angle D = 90^\circ$, $m\angle E = 30^\circ$

Answer





Steps of construction

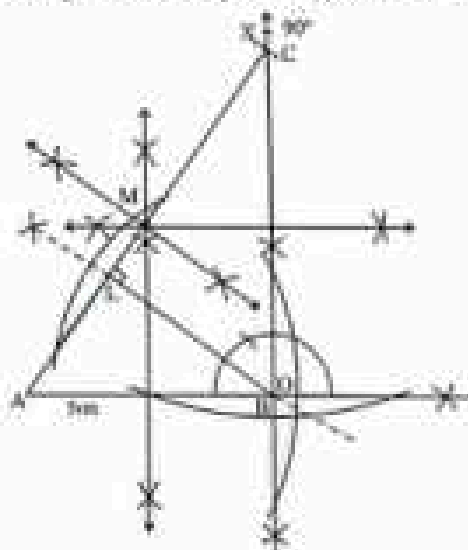
- Draw $m\overline{DE} = 6\text{ cm}$
- Construct $m\angle D = 90^\circ$ with the help of compass and ruler and draw $m\overline{DX}$.
- Construct $m\angle E = 30^\circ$ with the help of compass and ruler and draw $m\overline{EX}$.
- Both $m\overline{DX}$ and $m\overline{EX}$ cut each other at point F and $\triangle DEF$ is required triangle.
- Draw right bisectors of sides $\overline{DE}$, $\overline{EF}$ and $\overline{FD}$ to find their midpoints N, L and M respectively.
- Join D to L, E to M and F to N.
- Thus $\overline{DL}$, $\overline{EM}$ and $\overline{FN}$ are required medians of $\triangle DEF$ which meet at G.

Conclusion

Medians of a triangle are concurrent.

Q.5 Construct a right angled $\triangle ABC$ such that $m\angle B = 90^\circ$.

- Find its orthocentre, where does it lie?
- Find circumcentre M of the triangle. Is M the midpoint of hypotenuse $\overline{AC}$? Is $m\overline{BM} = m\overline{CM}$?



Steps of construction

- Draw $m\overline{AB} = 5\text{ cm}$
- Construct $m\angle B = 90^\circ$ with the help of compass and ruler and draw $m\overline{BX}$.
- With center B, draw an arc of radius 5 cm which cuts $m\overline{BX}$ at C.
- Join point A to C and get required $\triangle ABC$.
- Here, $\overline{BC} \perp \overline{AB}$ and $\overline{AB} \perp \overline{BC}$. Draw $\overline{BL} \perp \overline{AC}$ which meet at B the vertex of right angle.
- Draw right bisectors of side $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ which meet at M.

We observe that $m\overline{BM} = m\overline{CM}$ i.e. M is the midpoint of hypotenuse.

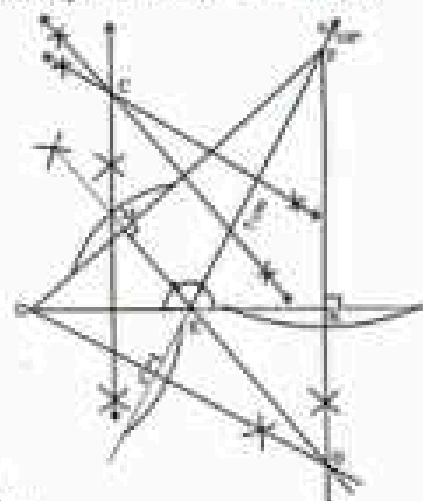
Conclusion

- Altitudes of a right triangle meet at the vertex of right angle.
- Right bisectors of the sides of a right triangle meet at the midpoint of hypotenuse.



Q.6 Construct an obtuse angled triangle DEF. Find its.

- Circumcentre
- Orthocentre check whether they lie inside or outside the triangle?



Steps of construction

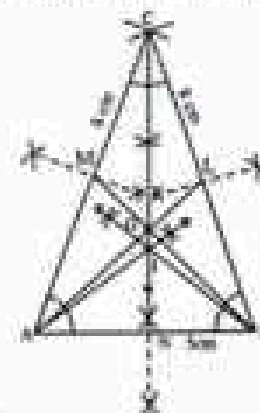
- Draw $m\overline{DE} = 4\text{ cm}$
- Construct $m\angle E = 120^\circ$ with the help of compass and ruler, and draw $m\overline{EX}$.
- With center E, draw an arc of radius 7 cm which cuts $m\overline{EX}$ at F.
- Join D to F and required $\triangle DEF$.
- Draw altitudes $\overline{DL} \perp \overline{DF}$, $\overline{EM} \perp \overline{FD}$, and $\overline{FN} \perp \overline{DE}$ which meet each other at point O (orthocenter), outside the triangle.
- Draw right bisectors of sides $\overline{DE}$, $\overline{EF}$ and $\overline{FD}$ which meet at point C (circumcenter), outside the triangle.

Conclusion

The circum center and ortho center of an obtuse angled triangle lie outside the triangle.

Q.7 Construct an isosceles triangle and Find its.

- Centroid
- Inscribed centre (In-centre)





Steps of construction

- Draw $\overline{AB} = 5\text{ cm}$
- With centre A, draw an arc of radius 8 cm.
- With centre B, draw another arc of radius 8 cm which cuts the previous arc at C.
- Join point C to A, C to B and get required triangle $\triangle ABC$.
- Draw right bisectors of sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ to find their midpoints N, L and M respectively.
- Join A to L, B to M and C to N and get medians $\overline{AL}$, $\overline{BM}$ and $\overline{CN}$ which meet each other at point G.
- Draw the bisectors of $\angle A$, $\angle B$ and $\angle C$ which meet each other at point I.

Conclusion

We observe that Centroid and bi-centre of a triangle are concurrent.

Q.8 Recognize altitudes, perpendicular bisectors and medians in the following triangles.

Solution

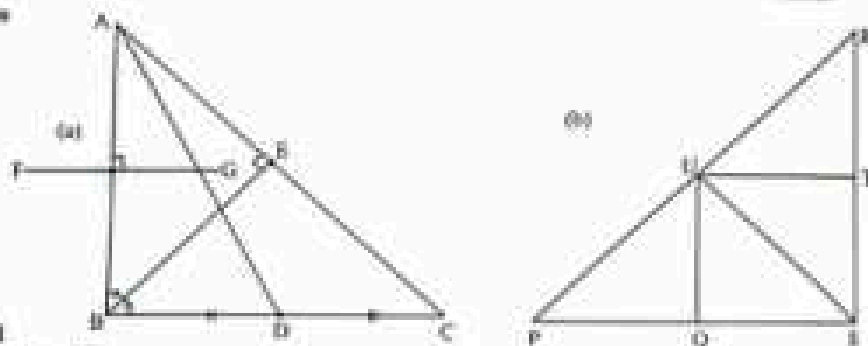


Figure-1

- Altitude is $\overline{BE}$
- Perpendicular bisector is $\overline{FG}$
- Median is $\overline{AD}$

Figure-2

Perpendicular bisectors, $\overline{UT}$, $\overline{US}$ and $\overline{UQ}$

Median is $\overline{UQ}$

Miscellaneous Exercise 10

Q.1 Encircle the correct option in the following:

- A line which bisects an angle into two equal parts is:
 - right bisector
 - angle bisector
 - altitude
 - median
- Which of the following is divided by a point called midpoint?
 - an angle
 - a line
 - a line segment
 - a ray

- A line passing through mid point and perpendicular to a side of a triangle is called:
 - right bisector
 - angle bisector
 - mid point
 - altitude
- Two sides making arms of a right angle in right triangle are:
 - altitudes
 - medians
 - vertices
 - bisectors



- Right bisectors of a right triangle meet at mid point of:
 - medians
 - altitude
 - side
 - hypotenuse
- Medians of a triangle are:
 - collinear
 - concurrent
 - perpendicular
 - parallel
- In an equilateral triangle angle bisectors, medians, altitudes and right bisectors:
 - are parallel
 - are perpendicular
 - coincide
 - are collinear
- Right bisectors of equilateral triangle are:
 - angle bisectors
 - altitudes
 - medians
 - All a, b, c
- If three lines meet at a point, they are called:
 - intersecting
 - perpendicular
 - collinear
 - concurrent
- Medians of a triangle intersect each other in the ratio:
 - 1:2
 - 1:3
 - 2:1
 - 2:3

Additional MCQs

- Which of the following is used to measure the angle?
 - compass
 - protractor
 - scale
 - set square
- Which of the following can be constructed by compass?
 - 103°
 - 123°
 - 130°
 - 55°
- Which of the following cannot be constructed with compass?
 - 15°
 - 30°
 - 45°
 - 95°
- A triangle having all sides equal is called:
 - isosceles
 - equilateral
 - right angled
 - acute angled

- isosceles
 - equilateral
 - right
 - acute
- A triangle having all sides different is called:
 - isosceles
 - scalene
 - equilateral
 - right
 - A triangle having two sides congruent is called:
 - scalene
 - right angled
 - equilateral
 - isosceles
 - The right bisectors of the three sides of a triangle are:
 - congruent
 - collinear
 - concurrent
 - parallel
 - The angle bisectors of the angles of a triangle are:
 - congruent
 - collinear
 - concurrent
 - parallel
 - The altitudes of a triangle are:
 - congruent
 - collinear
 - concurrent
 - parallel
 - The ___ altitudes of an isosceles triangle are congruent:
 - Two
 - Three
 - Four
 - All
 - How many congruent triangles can be made by joining the mid points of the sides of a triangle?
 - Three
 - Four
 - Five
 - Two
 - If the three altitudes of a triangle are congruent then the triangle is:
 - Equilateral
 - Right angled
 - isosceles
 - Acute angled
 - If two medians of a triangle are congruent then the triangle will be:
 - isosceles
 - Equilateral
 - Right angled
 - Acute angled



24. A line segment joining the vertex of a triangle to the midpoint of its opposite side is called its: **median**
- (a) Altitude
(b) Median
(c) Angle bisector
(d) Right bisector
25. A perpendicular line segment from a vertex of triangle to the opposite side, is called as _____ of the triangle: **altitude**
- (a) Altitude
(b) Median
(c) Angle bisector
(d) Right bisector
26. The point of concurrency of the three altitudes of a Δ is called its: **orthocentre**
- (a) Ortho centre
(b) In centre
(c) Circumcentre
(d) centroid
27. The bisectors of the angles of a triangle meet at a point called the _____ of the triangle: **centroid**
- (a) In centre
(b) Ortho centre
(c) Circumcentre
(d) centroid
28. The point of concurrency of the three right bisectors of the sides of a triangle is called its: **circumcentre**
- (a) Circumcentre
(b) In centre
(c) Ortho centre
(d) Centroid
29. Point of concurrency of three medians of a triangle is called its: **centroid**
- (a) In centre
(b) Ortho centre
(c) Centroid
(d) Circumcentre
30. Sum of interior angles of a triangle is: **180°**
- (a) 60°
(b) 120°
(c) 160°
(d) 240°
31. In right-angled triangle if one angle is 35°, then other angle will be: **55°**
- (a) 15°
(b) 65°
(c) 45°
(d) 55°

32. In right-angled triangle one angle is right, other two angles are: **acute**
- (a) right
(b) obtuse
(c) acute
(d) one acute, one obtuse
33. The altitudes of a right angled triangle are concurrent at the: **vertex**
- (a) Midpoint of hypotenuse
(b) Vertex of right angle
(c) Midpoint of base
(d) Vertical angle
34. In-centre is the point of concurrency of three _____ of triangle: **angle bisectors**
- (a) Right bisectors
(b) Angle bisectors
(c) Altitudes
(d) Medians
35. Circumcentre is point of concurrency of three _____ of triangle: **perpendicular bisectors**
- (a) right bisectors
(b) angle bisectors
(c) altitudes
(d) medians
36. Ortho centre is the point of concurrency of three _____ of triangle: **altitudes**
- (a) right bisectors
(b) angle bisectors
(c) altitudes
(d) medians
37. Centroid is the point of concurrency of three _____ of triangle: **medians**
- (a) right bisectors
(b) angle bisectors
(c) altitudes
(d) medians
38. There are more than three lines passing through the same point are called: **concurrent**
- (a) congruent
(b) concurrent
(c) parallel
(d) perpendicular



39. Which of the following represent the center of gravity of a triangle? **centroid**
- (a) In centre
(b) Ortho centre
(c) Centroid
(d) Circumcentre
40. If in-centre, circumcentre, orthocentre and centroid of a triangle coincide then triangle is: **equilateral**
- (a) Isosceles
(b) Equilateral
(c) Right angled
(d) Acute angled
41. The circum centre of an acute triangle lies _____ the triangle: **inside**
- (a) Inside
(b) Outside
(c) Midpoint
(d) vertex of
42. The circum centre of right triangle lies on the _____ of triangle: **hypotenuse**
- (a) vertex
(b) altitude
(c) hypotenuse
(d) base
43. The circumcentre of an obtuse triangle lies _____ the triangle: **outside**
- (a) Outside
(b) Inside
(c) Midpoint
(d) base of
44. The ortho centre of an acute triangle lies _____ of triangle: **inside**
- (a) Inside
(b) Outside
(c) Midpoint
(d) vertex of

45. The orthocentre of right triangle lies on the _____ of triangle: **vertex**
- (a) vertex
(b) altitude
(c) hypotenuse
(d) base
46. The orthocentre of an obtuse triangle lies _____ the triangle: **outside**
- (a) Outside
(b) Inside
(c) Midpoint
(d) base of
47. The in-centre of any triangle always lies _____ the triangle: **inside**
- (a) Outside
(b) Inside
(c) Midpoint
(d) on base of
48. The centroid of any triangle always lies _____ the triangle: **inside**
- (a) Outside
(b) Inside
(c) Midpoint
(d) on base of
49. If orthocentre of a triangle is outside the triangle then triangle is: **obtuse**
- (a) Acute angled
(b) Right angled
(c) Equilateral
(d) Obtuse angled
50. If orthocentre of a triangle is inside the triangle then triangle is: **acute**
- (a) Acute angled
(b) Right angled
(c) Isosceles
(d) Obtuse angled

Answer Key

1	b	2	c	3	a	4	a	5	d	6	b	7	c	8	d
9	d	10	a	11	b	12	a	13	d	14	c	15	b	16	d
17	c	18	c	19	a	20	a	21	b	22	a	23	a	24	b
25	a	26	a	27	a	28	a	29	c	30	c	31	d	32	c
33	b	34	b	35	a	36	c	37	d	38	b	39	c	40	b
41	a	42	c	43	a	44	a	45	a	46	a	47	b	48	b
49	d	50	a												

Q3 Construct the following triangles:

3. ΔPQR when $PQ = 6\text{cm}$, $\angle P = 30^\circ$, $\angle R = 90^\circ$. **10 marks**

Solution:

We know that

$$m\angle P + m\angle Q + m\angle R = 180^\circ$$

$$30^\circ + m\angle Q + 90^\circ = 180^\circ$$

$$m\angle Q = 180^\circ - 120^\circ$$

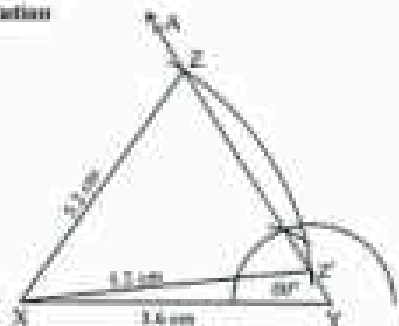
$$m\angle Q = 60^\circ$$



Steps of construction

- Calculate $m\angle Q = 60^\circ$
- Draw $m\overline{PQ} = 6\text{ cm}$
- Construct $m\angle P = 60^\circ$ with the help of compass and ruler and draw $m\overline{PX}$.
- Construct $m\angle Q = 60^\circ$ with the help of compass and ruler and draw $m\overline{QY}$.
- Both $m\overline{PX}$ and $m\overline{QY}$ meet at point R. $\triangle PQR$ is required triangle.
- $\triangle XYZ$ where $\overline{XY} = 5.6\text{ cm}$, $\overline{XZ} = 5.6\text{ cm}$.

Solution



Steps of construction

- Draw $m\overline{XY} = 5.6\text{ cm}$
- Construct $m\angle X = 60^\circ$ with the help of compass and ruler and draw $m\overline{XZ}$.
- With center Y, draw an arc of radius



5.6 cm intersecting $m\overline{XZ}$ at Z and Z' .

iv) Join X to Z and Z' .
Thus two triangles $\triangle XZ$ and $\triangle XZ'$ are constructed.

- $\triangle ABC$ where $\overline{BC} = 7\text{ cm}$, $\overline{AC} = 4.3\text{ cm}$, $m\angle B = 45^\circ$.



Steps of construction

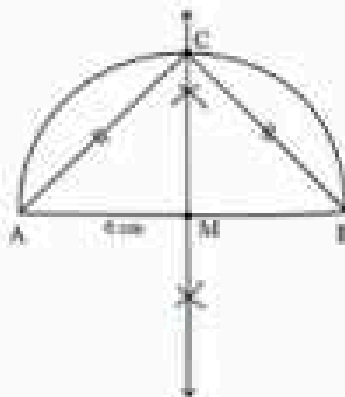
- Draw $m\overline{BC} = 7\text{ cm}$
- Construct $m\angle B = 45^\circ$ with the help of compass and ruler and draw $m\overline{BX}$.
- With center C, draw an arc of radius 4.3 cm which does not intersect $m\overline{BX}$ at any point. Thus construction of required triangle ABC is not possible.

Q.3 Construct a right isosceles triangle whose hypotenuse is 6 cm.

Solution

Hint - "The angle in a semicircle is right angle"

Hypotenuse = 6 cm



Steps of construction

- Draw $m\overline{AB} = 6\text{ cm}$
- Draw right bisector of $\overline{AB}$ to find its midpoint M.
- With center M, draw a semicircle of radius $m\overline{MA}$ or $m\overline{MB}$ which intersects the right bisector at point C.
- Join point C to A and B. $\triangle ABC$ is required right angled isosceles triangle.

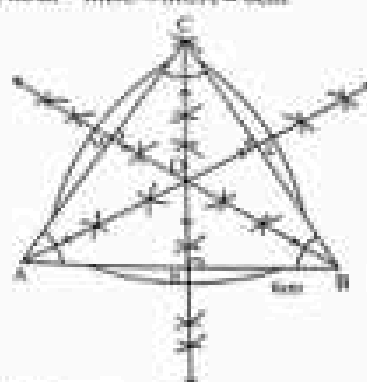
Q.4 Construct an equilateral triangle ABC. Find its:

- Incentre
- Circumcentre
- Orthocentre
- Centroid

Do they coincide?

Solution

Let $m\overline{AB} = m\overline{BC} = m\overline{CA} = 6\text{ cm}$



Steps of construction

- Draw $m\overline{AB} = 6\text{ cm}$
- With center A, draw an arc of radius 6 cm.
- With center B, draw another arc of radius 6 cm which cuts the previous arc at C.
- Join point C to A, C to B and get equilateral $\triangle ABC$.
- Draw angle bisectors of $\angle A$, $\angle B$ and $\angle C$ which meet at point O. So, O is the incentre of $\triangle ABC$.



- Draw right bisectors of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ which meet at point O. So, O is the circumcenter of $\triangle ABC$.
- Draw altitudes of $\triangle ABC$ which meet at point O. So, O is the Orthocenter of $\triangle ABC$.
- By joining the vertices of a triangle to the midpoint of sides of $\triangle ABC$ get medians of $\triangle ABC$ which meet at point O. So, O is the centroid of $\triangle ABC$.

Conclusion

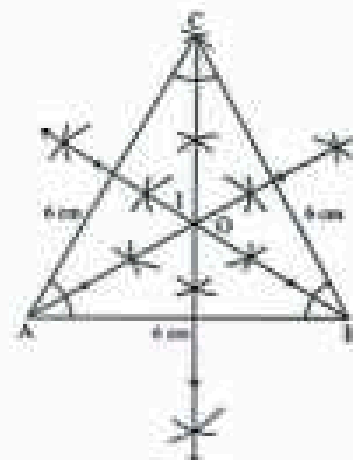
We observe that in-centre, circum center, orthocentre and centroid of an equilateral $\triangle ABC$ coincide.

Q.5 A student wants to find incentre of an equilateral triangle but he does not know how to bisect the angles. Can he find incentre by using any other method? Explain your answer by taking an example.

Solution

In case of equilateral triangle, incentre and circumcentre coincide. So he can find the incentre by finding the circumcentre. If he does not know how to bisect the angles, he can draw right bisectors of sides of a triangle.

Example: Let $\triangle ABC$ is an equilateral triangle in which $m\overline{AB} = m\overline{BC} = m\overline{CA} = 6\text{ cm}$



Steps of construction

- Draw $\overline{AB} = 6\text{ cm}$.
- With center A, draw an arc of radius 5 cm.
- With center B, draw another arc of radius 5 cm which cuts the previous arc at C.
- Join point C to A, C to B and get equilateral $\triangle ABC$.
- Draw angle bisectors of $\angle A, \angle B$ and $\angle C$ which meet at point L. So, L is the in-center of $\triangle ABC$.
- Draw right bisectors of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ which meet at point O. So, O is the circum center of $\triangle ABC$.

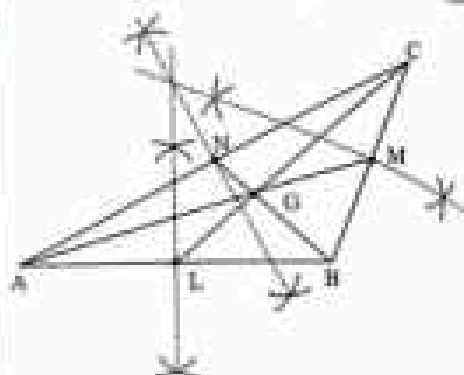
Conclusion

We observe that in-center and circum center of an equilateral $\triangle ABC$ coincide.

Q4 Construct a triangle and find its centre of gravity.

Solution

The centre of gravity of a triangular sheet of uniform thickness is its centroid. Centroid is the point of intersection of medians of a triangle.

Steps of construction

- Construct a $\triangle ABC$ of arbitrary measurements.
- Draw right bisectors of sides $\overline{AB}, \overline{BC}$ and $\overline{CA}$ to find their midpoints L, M and N respectively.
- Join A to M, B to N and C to L. Thus $\overline{AM}, \overline{BN}$ and $\overline{CL}$ are required medians of $\triangle ABC$ which meet at G.

Conclusion

Medians of a triangle are concurrent at point G which is a centre of gravity of $\triangle ABC$.



UNIT 11

Basic Statistics

Raw Data / Ungrouped Data

Data which has not been arranged in a systematic order is called raw data or ungrouped data.

Group Data

A tabular arrangement of data in which various items / values are arranged or distributed into some classes and the number of items falling in each class is stated is called grouped data or a frequency distribution.

Class Limits

A group of class is defined by two numbers. The smaller number is called lower class limit and larger one is called upper class limit.

Class Boundaries

The class boundaries can be formed from class limits as follows. First of all find difference between the lower class limit of second class and upper class limit of first class. Then subtract half of this difference from lower limits of all classes and add it to the upper limits of all classes.

Adjustment factor = $\frac{1}{2}$ Lower limit of a class
- upper limit of the preceding class = 0.5

Size of Class Interval

Size of a class interval is the difference of lower limits or upper limits of two consecutive classes. It is usually denoted by 'k'.
e.g. in 1 - 10, 11 - 20, 21 - 30, the size of class intervals is 10.

Class Marks / Mid Value

It is the average of lower and upper limits or boundaries of a class. Class marks are denoted by x . Class mark of class 0 - 10 is $\frac{0+10}{2} = 5$

and that 11 - 20 is $\frac{11+20}{2} = 15.5$

Frequency

Number of repetitions of an individual value in an ungrouped data is called its frequency. In a grouped data, number of values present in a group is called frequency of group.

Cumulative Frequency

The sum of frequency of a class and the frequencies of all the preceding classes is called cumulative frequency.

Construction of frequency distribution table

We can construct a frequency distribution table in two ways.

- Tally bar method
- Direct observation method

Tally bar method

Step-I: Decide how many groups should be made in the table.

Step-II: Find approximate size of class interval by the formula $h = \frac{A_2 - A_1}{n}$

Where h = size of class interval

A_2 = largest value in ungrouped data

A_1 = smallest value in ungrouped data

n = approximate number of classes

Step-III: Draw a table and write the class limits in first column according to the size of class interval.

Step-IV: Choose the value from ungrouped data one by one and draw small tally marks in second column against the respective group. If 5 or more values fall in a same group, then draw 5th tally mark diagonally. It will make the counting of tally marks easier.

Step-V: Count tally marks of each group and write their number against the respective class in third column (column of frequencies).



Step-VII: Find sum of all frequencies and check that it must be equal to the number of values present in ungrouped data i.e.

Number of total items in ungrouped data (n) = sum of frequencies in grouped data (Σf)

Example-1

The following are masses in pounds of 40 boys at a gym.

147, 138, 164, 150, 132, 144, 125, 149, 157, 146, 158, 140, 156, 148, 152, 144, 168, 126, 138, 178, 163, 126, 154, 165, 146, 173, 142, 147, 133, 153, 140, 135, 161, 145, 135, 142, 150, 156, 145, 128.

Solution

Step-I: Let us distribute the data in 5 classes.

Direct Observation Method

Mass in pounds	Observation	Frequencies
125-135	132, 125, 126, 128, 133, 135, 135, 128	8
136-146	138, 144, 146, 140, 136, 144, 138, 146, 142, 140, 145, 142, 145	13
147-157	147, 150, 149, 157, 148, 152, 154, 147, 133, 150, 154	11
158-168	164, 158, 168, 163, 163, 161	6
169-179	176, 173	2
Total	—	40

Discrete Frequency Distribution

If data is not in fractions and some values occur most often to constitute the whole data, then data is called a discrete data there is no need of class boundaries and size of class interval is 1.

Example-2

A cubical die is rolled 30 times and numbers of dots on front face recorded each time are as follows.

2, 6, 2, 3, 2, 4, 5, 4, 3, 1, 2, 6, 5, 3, 4, 6, 2, 2, 4, 6, 5, 1, 3, 4, 6, 3, 1, 1, 6, 3

Construction of Histogram

In histogram, data are represented by adjacent rectangles having bases along horizontal axis (marked off by class boundaries) and areas proportional to the class frequencies. If the class interval size is the

Step-II: $x_1 = 176, x_2 = 125, n = 5$

$$h = \frac{x_2 - x_1}{5} = \frac{176 - 125}{5} = 10.2 \approx 11$$

Step-III:

Mass in pounds	Tally marks	Frequencies
125-135		8
136-146		13
147-157		11
158-168		6
169-179		2
Total		40

Solution

Mass in pounds	Tally marks	Frequencies	Cumulative frequency
1		4	4
2		6	4+6=10
3		5	4+6+5=15
4		5	4+6+5+5=20
5		4	4+6+5+5+4=24
6		6	4+6+5+5+4+6=30
Total		30	



same, the heights of rectangles are proportional to the class frequencies. If the class interval size is unequal, then the heights of rectangles are to be adjusted.

Frequency density

The height of rectangle may be found by dividing the frequency by width of class interval. The ratio $\frac{\text{frequency}}{\text{width}}$ is called frequency density.

Example-3

The heights in cm 44 students of a school is given below.

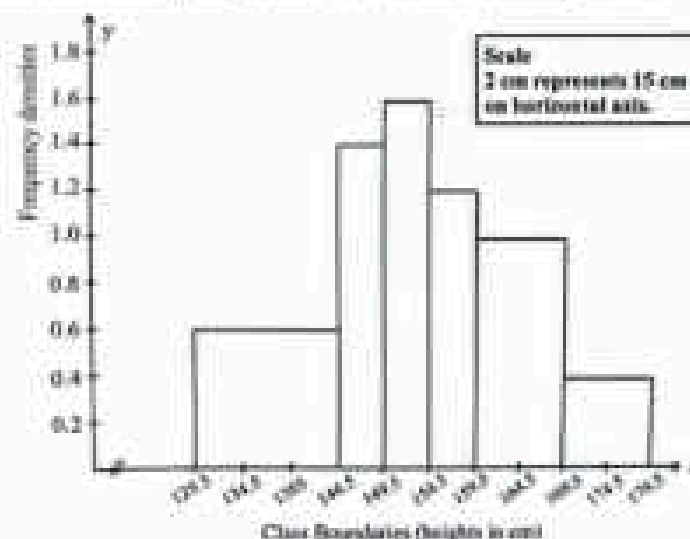
Height	130-144	145-149	150-154	155-159	160-169	170-179
No. of students	9	7	8	6	10	4

Represent the data by a histogram.

Solution

Height (cm)	f	Size of class interval	F.D	Class boundaries
130-144	9	15	$9 \div 15 = 0.6$	129.5-144.5
145-149	7	5	$7 \div 5 = 1.4$	144.5-149.5
150-154	8	5	$8 \div 5 = 1.6$	149.5-154.5
155-159	6	5	$6 \div 5 = 1.2$	154.5-159.5
160-169	10	10	$10 \div 10 = 1$	159.5-169.5
170-179	4	10	$4 \div 10 = 0.4$	169.5-179.5
Total	44	—	—	—

Title: A histogram showing heights (cm) of 44 students in a secondary school.





Construction of a Frequency Polygon

Polygon is a many-sided closed figure. To draw a frequency polygon mid points (class marks) are marked on the horizontal axis and frequencies are on the vertical axis. Points are plotted with class frequencies and their corresponding class marks. These points are joined by straight line segments. To complete a closed polygon we add extra classes at both ends with zero frequencies. In this way two extra points on x-axis are obtained on both ends. The points are joined by their mean plotted points. Finally a closed frequency polygon is obtained. It is easy to draw a frequency polygon on a histogram. To do so we find mid points of the top edges of rectangles and join them by segments. Then find two extra points on both ends on x-axis to complete the closed polygon.

Example-4

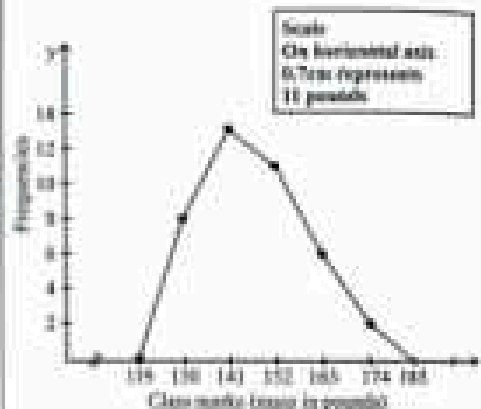
At a farm house there are 40 lambs, their masses are recorded as given below. Represent the following frequency distribution by a frequency polygon.

Masses in pounds	No. of lambs
125 – 135	8
136 – 146	13
147 – 157	11
158 – 168	6
169 – 179	2
Total	40

Solution

Masses	Frequency	Class Marks
114 – 124	0	119
125 – 135	8	130
136 – 146	13	141
147 – 157	11	152
158 – 168	6	163
169 – 179	2	174
180 – 192	0	185
Total	40	

Title: A frequency polygon showing masses of 40 lambs at a farm house.



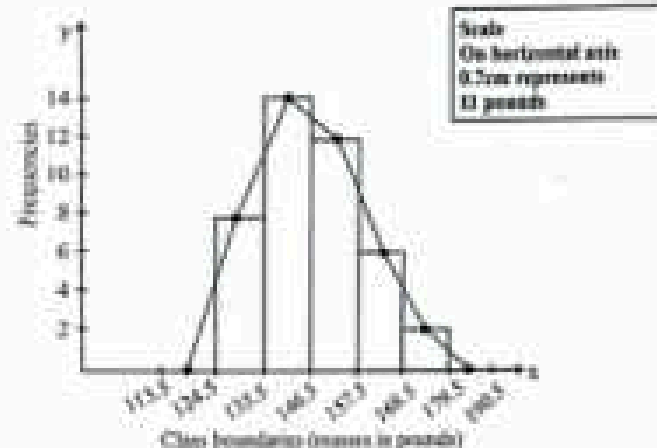
Example-5

Use the frequency distribution in example 4 and draw a frequency polygon on a histogram.

Solution

Masses	Frequency	Class boundaries	Class Marks
125 – 135	8	124.5 – 135.5	130
136 – 146	13	135.5 – 146.5	141
147 – 157	11	146.5 – 157.5	152
158 – 168	6	157.5 – 168.5	163
169 – 179	2	168.5 – 179.5	174

Title: A frequency polygon on a histogram showing masses of 40 lambs in pounds.



Exercise 11.3

Q.1 The given table shows number of regular readers of a library who have completely read the books.

Solution

Class Interval (books)	Frequency (No. of readers)	Class marks (midpoints)
1–10	5	5.5
11–20	4	15.5
21–30	8	25.5
31–40	9	35.5
41–50	2	45.5
51–60	2	55.5

$$\Sigma f = 30$$

- i) What is the total number of readers in the data? answer
 Sol. Total number of readers in the data are 30.
 ii) Which group contains highest number of readers? answer
 Sol. The group (31–40) contains highest number of readers i.e. 9.
 iii) Which group contains least number of readers? answer

Sol. No single group contains least number of readers.

iv) What is the lower limit of the last class? answer

Sol. The lower limit of last group is 51.

v) What is the lower boundary of the last class? answer

Sol. The lower boundary of last group is 50.5

vi) What is the size of the class interval? answer

Sol. The size of the class intervals is 10

vii) Find class marks (mid points) of all groups. answer

Sol. The class marks of all groups are given in 3rd column of above table.

Q.2 Number of gratitude cards made by 80 student of class 9 is given below.

79, 60, 74, 59, 55, 98, 61, 67, 89, 71, 71, 46, 63, 66, 69, 42, 75, 42, 71, 77, 78, 65, 87, 57, 78, 91, 82, 73, 65, 94, 48, 87, 62, 81, 63, 66, 65, 49, 45, 51, 49, 56, 54, 93, 63, 60, 68, 51, 73, 54, 58, 88, 76, 93, 48, 78, 48, 76, 85, 57,



63, 94, 82, 54, 88, 64, 77, 94, 72, 69, 51, 56, 87, 88, 81, 76, 81, 54, 66, 87.

ii) Prepare a frequency table to represent the above data using the classes 40-49 etc.

iii) Add cumulative frequency column in the table.

iii) How many students made less than 70 cards?

iv) What percent of students made less than 50 cards?

v) Which group has the greatest frequency?

vi) What is the size of the class interval?

Solution

Smallest value = 42

Largest value = 94

No. of observations = 40

Class Intervals	Tally marks	Frequency (f)	Cumulative frequency (C.F.)
40-49	/	7	7
50-59	/ / /	13	7+13 = 20
60-69	/ / / / /	22	20+22 = 42
70-79	/ / / /	17	42+17 = 59
80-89	/ / / /	13	59+13 = 72
90-99	/ /	8	72+8 = 80

Sol. 42 students made less than 70 cards.

iv) What percent of students made less than 50 cards?

Q.4 The ages of workers in a factory were recorded as below.

Ages in years	20-24	25-29	30-34	35-39	40-44	45-49
No. of workers	5	16	12	18	8	4

Draw a frequency polygon to represent the data.

Sol. 7 students made less than 30 cards.

In percentage = $\frac{7}{40} \times 100\% = \frac{70}{4}\% = 17.5\%$

v) Which group has the greatest frequency?

Sol. The group (60-69) has greatest frequency (22).

vi) What is the size of the class interval?

Sol. The size of each class interval is 10.

Q.3 The number of medals won by 40 players in a certain sports festival is given as:

0, 2, 1, 0, 1, 2, 3, 2, 6, 3, 2, 1, 3, 4, 2, 6, 1, 3, 1, 4, 3, 0, 1, 2, 3, 0, 0, 2, 3, 4, 1, 3, 6, 2, 4, 3, 1, 1, 4, 6, 2, 3, 1, 2, 3

Prepare a discrete frequency distribution. Also make a column of cumulative frequencies.

Smallest value = 0

Largest value = 6

No. of observations = n = 45

Solution

Class Intervals	Tally marks	Frequency (f)	Cumulative frequency (C.F.)
0	/	3	3
1	/ /	8	3+8 = 11
2	/ / /	10	11+10 = 21
3	/ / /	8	21+8 = 29
4	/ /	5	29+5 = 34
5	/ /	5	34+5 = 39
6	/	4	39+4 = 43

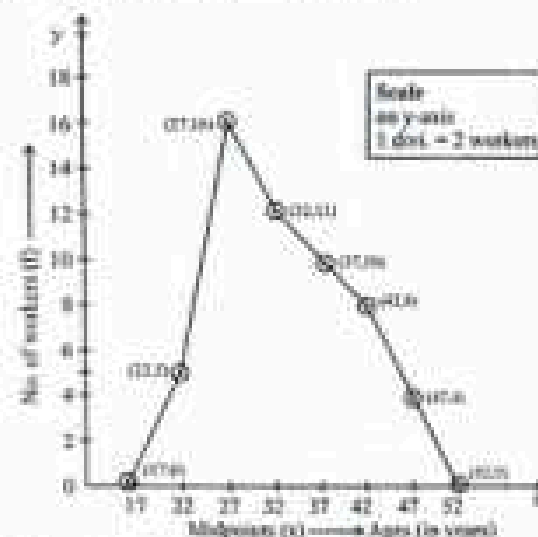


Solution

"A frequency polygon is many sided closed figure". In order to draw frequency polygon we take two extra groups of zero frequency. One before the first group and other after the last groups.

Ages in years	No. of workers	Midpoint (x)
15-19	0	17
20-24	5	22
25-29	16	27
30-34	12	32
35-39	10	37
40-44	8	42
45-49	4	47
50-54	0	52

Tab: Frequency polygon showing ages of workers in a factory.



Q.5 Represent the following data by a histogram.

Class Interval	0-5	5-15	15-20	20-40	40-45
Frequency	5	12	16	18	2

Solution

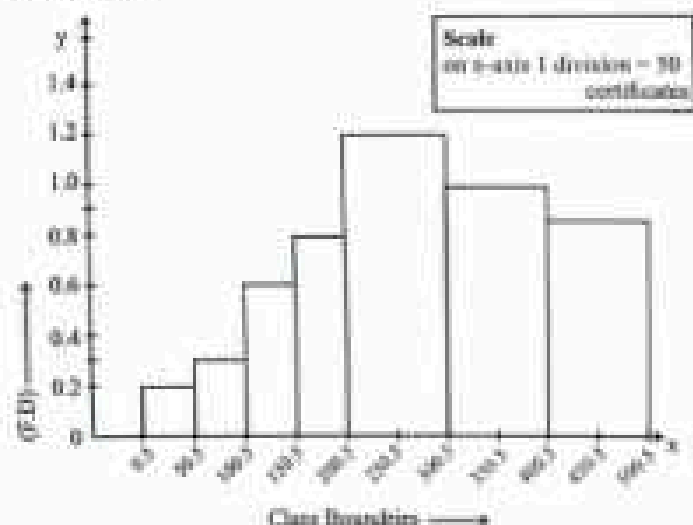
To make a histogram with unequal sizes of class intervals, we plot class boundaries on x-axis and frequency density on y-axis.

Frequency density (F.D) = $\frac{\text{Frequency of the class}}{\text{width of the class}}$

Class Intervals	Frequency (f)	Size of class intervals	Frequency density (F.D)	Class boundaries
0-5	5	5	$5 \div 5 = 1$	0-5
5-15	12	10	$12 \div 10 = 1.2$	5-15
15-20	16	5	$16 \div 5 = 3.2$	15-20
20-40	10	20	$10 \div 20 = 0.5$	20-40
40-45	2	5	$2 \div 5 = 0.4$	40-45

Note: In this case boundaries are same as the class limits.

The histogram of given data is:



Q.6 In a saving group, there are 400 members and the number of savings certificates held by them is shown in the following table. Construct a histogram of the distribution of saving certificates.

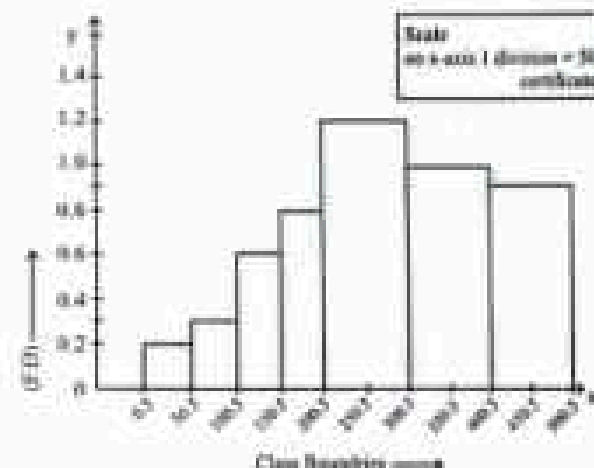
Class Interval	1-50	51-100	101-150	151-200	201-300	301-400	401-500
No. of members	10	15	20	40	120	100	85

Solution

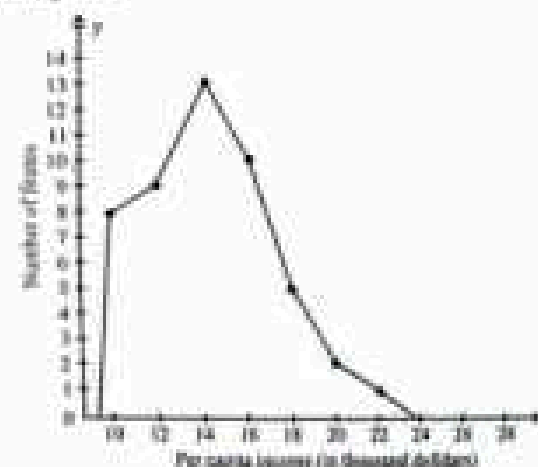
Class Intervals	Frequency (f)	Size of classes h	Frequency density (F.D.)	Class boundaries
1-50	10	50	$10 \div 50 = 0.2$	0.5-50.5
51-100	15	50	$15 \div 50 = 0.3$	50.5-100.5
101-150	20	50	$20 \div 50 = 0.4$	100.5-150.5

151-200	40	50	$40 \div 50 = 0.8$	150.5-200.5
201-300	120	100	$120 \div 100 = 1.2$	200.5-300.5
301-400	100	100	$100 \div 100 = 1$	300.5-400.5
401-500	85	100	$85 \div 100 = 0.85$	400.5-500.5

Histogram showing distribution of saving certificates



Q.7 The following frequency polygon represents the per-capita income of some states, recorded in a particular year.



i) What is the total number of states?

Hint:

Height of each dot along y-axis shows number of states.

Sol. Total number of states are 48

$$8+9+13+10+5+2+1 = 48$$



ii) What is the number of states which have the least per-capita income?

Sol. 3 states have least per-capita income.

iii) What is the number of states having highest per-capita income?

Sol. 1 state has highest per-capita income.

iv) What percent of states have per-capita income \$ 14000 and \$ 16000?

Sol. States having per-capita income and \$ 16000 are 13 and 10 respectively.
Their percentage = $\frac{23}{48} \times 100\% = 47.9\%$
 $\approx 48\%$

v) How many states have per-capita income \$ 18000 and above?

Sol.

vi) Represent the frequency polygon by

a frequency table.

Sol. Frequency table:

Per-capita income (in thousand dollars)	Number of states
10	8
12	9
14	13
16	10
18	5
20	2
22	1

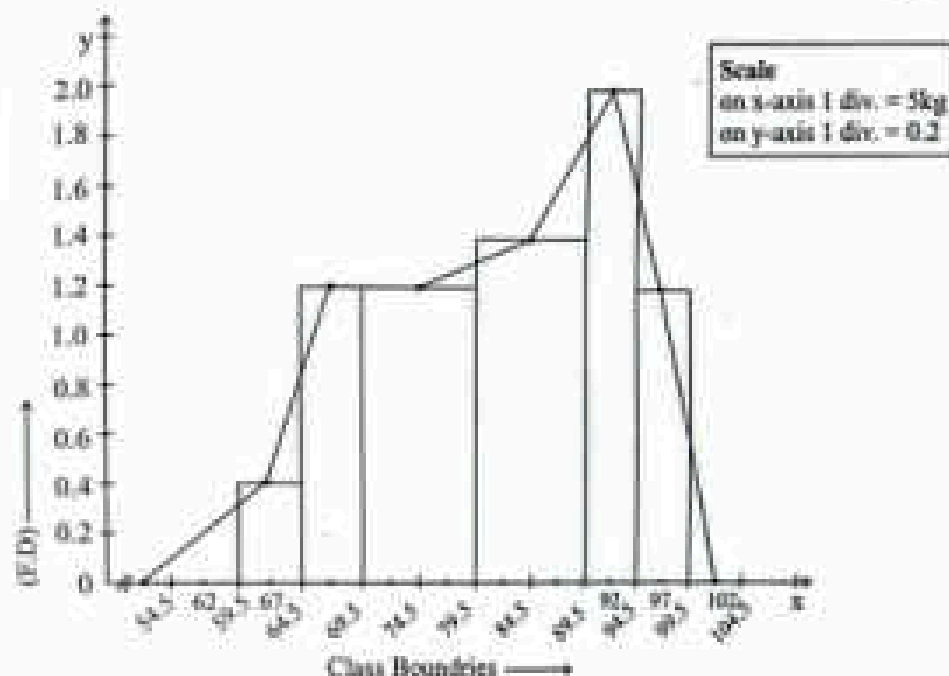
Q.8 The masses measured to the nearest kg of 50 boys are given in the frequency distribution

Mass(kg)	60-64	65-69	70-74	80-84	90-94	95-99
f	2	6	12	14	10	6

Solution

Mass (kg)	(f)	Size of classes (h)	Frequency Density $F.D = f/h$	Class Boundaries	Class marks (x)
60-64	2	5	$2 \div 5 = 0.4$	59.5 - 64.5	62
65-69	6	5	$6 \div 5 = 1.2$	64.5 - 69.5	67
70-74	12	5	$12 \div 5 = 2.4$	69.5 - 74.5	72
80-84	14	5	$14 \div 5 = 2.8$	79.5 - 84.5	82
90-94	10	5	$10 \div 5 = 2$	89.5 - 94.5	92
95-99	6	5	$6 \div 5 = 1.2$	94.5 - 99.5	97

To draw a frequency polygon on a histogram find mid points of the top edges of rectangles and join them by line segments. Then find two extra points on both ends on x-axis to complete the closed polygon.

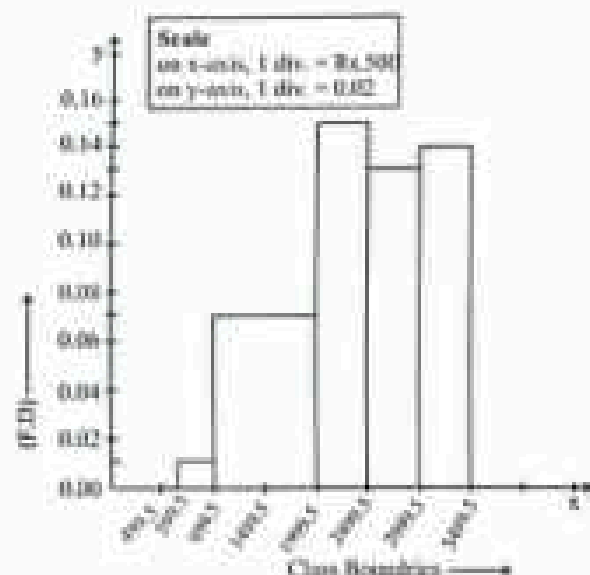


Q.9 At a sale, number of items placed according to the price range is given below:

Price (Rs)	600-999	1000-1999	2000-2499	2500-2999	3000-3499
f	50	70	75	65	70

Solution

Price Rs.	(f)	Size of classes (h)	$F.D = f/h$	Class boundaries
600 - 999	50	4000	$0.0125 = 0.01$	599.5 - 999.5
1000 - 1999	70	1000	0.07	999.5 - 1999.5
2000 - 2499	75	500	0.15	1999.5 - 2499.5
2500 - 2999	65	500	0.13	2499.5 - 2999.5
3000 - 3499	70	500	0.14	2999.5 - 3499.5



Measures of Central Tendency

A single value that represents a set of data is called an average. Since the averages tend to lie in the center of a data, they are called **measures of central tendency**. They are also called **measures of location** because they tend to be located at the center of the data. Here we will discuss 3 very common measures of central tendency.

i) Arithmetic Mean

ii) Median

iii) Mode

Arithmetic Mean

The ratio of sum of values of a data and number of values is called an arithmetic mean.

Arithmetic mean for ungrouped data

The values of data are usually denoted by x . In case of more than one data, their values are denoted by x, y or z etc. The sum of the values of the data ' x ' is denoted by ' Σx ' and number of values is denoted by n . The arithmetic mean of data x is symbolized as $\bar{x}$ and by above definition:

$$\bar{x} = \frac{\Sigma x}{n} = \frac{\text{sum of values}}{\text{number of values}}$$

Arithmetic Mean for grouped data

If values of data occur more than once, the sum of values can be found by using frequencies i.e.

If $x = 2, 3, 1, 2, 2, 3, 2, 3, 4, 2, 5, 2, 3, 4, 5, 2, 1, 3, 5, 1$

Then, $\Sigma x = 2 \times 3 + 1 \times 2 + 2 \times 3 + 2 \times 3 + 4 \times 2 + 5 \times 2 + 1 \times 3 + 3 \times 5 + 1 \times 5 = 55$



In this data frequency of 1 is 3 i.e. 1 occurs three times. Similarly frequencies of 2, 3, 4, 5 are 7, 5, 2, 1, respectively.

$$\Sigma x = 1 \times 3 + 2 \times 7 + 3 \times 5 + 4 \times 2 + 5 \times 1 = 3 + 14 + 15 + 8 + 5 = 55$$

$$\text{Also } n = \text{number of values} = \text{sum of frequencies} = 3 + 7 + 5 + 2 + 1 = 20$$

$$\therefore \bar{x} = \frac{\Sigma x}{n} = \frac{55}{20} = 2.75$$

The above data can be summarized in frequency distribution as follows:

x	1	2	3	4	5	Total
f	3	7	5	2	1	$\Sigma f = 20$
fx	$1 \times 3 = 3$	$2 \times 7 = 14$	$3 \times 5 = 15$	$4 \times 2 = 8$	$5 \times 1 = 5$	$\Sigma fx = 55$

$$\text{The arithmetic mean } \bar{x} = \frac{\text{sum of values}}{\text{number of values}} = \frac{\Sigma fx}{\Sigma f} = \frac{55}{20} = 2.75$$

Example 6

Find the arithmetic mean of the following data.

Marks	0-10	10-20	20-30	30-40	40-50
No. of students	2	5	7	3	8

Solution

Marks	f	Class marks	fx
0-10	2	5	10
10-20	5	15	75
20-30	7	25	175
30-40	3	35	105
40-50	8	45	360
Total	$\Sigma f = 25$	-	$\Sigma fx = 725$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{725}{25} = 29$$

Median

When data is arranged in some order, it can be represented by a single value that lies exactly at the centre of data. "The central or middle most value of an arranged data is called its median represented by M and as x value."

i. If n (number of values) is odd, then median = $\frac{n+1}{2}$ th value of the arranged data.

ii. If n is even, then Median = Arithmetic mean of the two middle values

$$= \frac{1}{2} \left(n^{\text{th}} \text{ value} + \frac{n+2}{2} \text{th value} \right)$$

Example 7

The selling prices of the seven articles sold by a shopkeeper in a week were 1075, 1070, 2050, 1030, 2045, 1090 and 1130. Find the median selling price of these articles.

Solution

Arranging these values in the ascending order as 1030, 1070, 1075, 1090, 1130, 2045, 2050.

Here $n = 7$ (odd), so

$$\text{Median} = \frac{n+1}{2} \text{th value} = \frac{7+1}{2} \text{th value} = \frac{8}{2} \text{th value} = 4^{\text{th}} \text{ value} = 1090$$

So, median selling price of seven articles is Rs. 1090

So, median selling price of seven articles is Rs. 1090

Example 8

Find the median of the following set of values.

40, 35, 45, 60, 50, 35, 40, 45, 35, 50

Solution

Arranged the numbers in ascending order:



35, 35, 35, 40, 40, 45, 45, 50, 50, 60.

Here $n = (\text{even}), \therefore$

Median = 5 = half of sum of two middle most values

$$= \frac{1}{2} \left(\frac{n}{2} \text{th value} + \frac{n+2}{2} \text{th value} \right)$$

$$= \frac{1}{2} \left(\frac{10}{2} \text{th value} + \frac{10+2}{2} \text{th value} \right)$$

$$= \frac{1}{2} (5\text{th value} + 6\text{th value})$$

$$= \frac{1}{2} (40 + 45) = \frac{85}{2} = 42.5$$

Median for Grouped Data

In a grouped data or frequency distribution the median is taken as $\frac{n}{2}$ th value. To find median

for a grouped data we first find the group which contains the median. To locate the median group we form cumulative frequency table. The formula used to find median of a group data is

$$\text{Median} = l + \frac{h}{f} \left(\frac{n}{2} - c \right)$$

Where l = frequency of median group

l = lower class boundary of median group

h = size of class interval

$n = \Sigma f$ = Number of items

c = Cumulative frequency of class preceding the median class.

Example 9

Marks obtained by 100 students in an examination are given below.

Marks	10-19	20-29	30-39	40-49	50-59
No. of students	5	25	40	20	10

Find the median marks.

Solution

$$\text{Median} = \frac{n}{2} \text{th value} = \frac{100}{2} \text{th value} = 50^{\text{th}} \text{ value.}$$

Marks	f	Class Boundaries	$c.f$
10-19	5	9.5-19.5	5
20-29	25	19.5-29.5	$C = 30$
30-39	$f = 40$	29.5-39.5	70
40-49	20	39.5-49.5	90
50-59	10	49.5-59.5	100

→ median group

From the above table, it is clear that 50^{th} value lies in the group 29.5-39.5.

So 29.5 - 39.5 is the median group.

$$\text{Median} = l + \frac{h}{f} \left(\frac{n}{2} - c \right)$$

Where $l = 29.5$, $h = 10$, $n = 100$, $f = 40$, $c = 30$

$$\text{Median} = 29.5 + \frac{10 \left(\frac{100}{2} - 30 \right)}{40}$$

$$= 29.5 + \frac{1}{4} (10 - 30)$$

$$= 29.5 + \frac{20}{4}$$

$$= 29.5 + 5 = 34.5 \text{ marks}$$

Mode

The most frequent (repeated) value of a data is called mode of the data.

Example 10

Find mode of following data sets.

i) 30, 25, 40, 30, 45, 25, 30, 40, 30

ii) 1, 2, 3, 2, 1, 3, 1, 2

iii) 125, 130, 115, 125, 135, 120, 130, 130

Solution

i) As 30 is the most frequent value so it is mode of the data. This data is called unimodal.

ii) As the value 1&2 occur equal number of times, so there are 2 modes. This data is called bimodal data.

iii) as the value 120, 125 and 130 occur two times each and frequencies of remaining values are lesser. So 120, 125 and 130 are three



modes of the data. This data is called trimodal data.

Mode of group data

To find mode for a frequency distribution or a group data, we first find the modal class. A class or group having largest number of items (frequency) is called modal group and it contains mode of the data. The mode of a grouped data can be found by the formula.

$$\text{Mode} = l + \frac{(f_m - f_1)}{2f_m - f_1 - f_3} \times h$$

Where l = lower class boundary of the modal class.

f_m = frequency of the modal class (the highest frequency)

f_1 = frequency of the class preceding the modal class

f_3 = frequency of the class following the modal class

h = size of class interval

Example 11

Fifty workers of a factory are distributed in 7 groups according to their wages in rupees per hour.

(10/100)

Wages in rupees	25-29	30-34	35-39	40-44	45-49	50-54	55-59
No. of workers	2	4	8	20	6	6	4

Find mode of the distribution.

Solution

Wages	f	Class boundaries
25 - 29	2	24.5 - 29.5
30 - 34	4	29.5 - 34.5
35 - 39	$f_1 = 8$	34.5 - 39.5
40 - 44	$f_m = 20$	39.5 - 44.5 → modal class
45 - 49	$f_3 = 6$	44.5 - 49.5
50 - 54	6	49.5 - 54.5
55 - 59	4	54.5 - 59.5

As 20 is the largest frequency, so (39.5 - 44.5) is the modal group.

$$l = 39.5, f_m = 20, f_1 = 8, f_3 = 6, h = 5$$

$$\text{Mode} = l + \frac{(f_m - f_1)}{2f_m - f_1 - f_3} \times h$$

$$\text{Mode} = 39.5 + \frac{(20 - 8)}{2 \times 20 - 8 - 6} \times 5$$

$$= 39.5 + \frac{12}{40 - 14} \times 5$$

Class having maximum frequency is modal class



$$= 39.3 + \frac{12}{26} \times 5 = 39.3 + \frac{60}{26}$$

$$= 39.3 + 2.31 = \text{Rs. } 41.61$$

Example 12

Find median and mode for the following discrete frequency distribution.

x	1	2	3	4	5
f	3	4	5	2	2

Solution

Here $n = \sum f = 16$

Median = class containing $\frac{n}{2}$ th observation

$= \frac{16}{2}$ th observation = 8th observation.

x	f	c.f.
1	3	3
2	4	7
3	5	12
4	2	14
5	2	16

$\rightarrow$ 8th observation

So Median = 3 (8th observation lies in the group where c.f. is 12.)

Hence Mode = most frequent observation = 3

Empirical Relation between Mean, Median and Mode

If any two of mean, median and mode of a data are given, then we find an approximately value of the third measure of central tendency by using an empirical relation.

$$\text{mode} = 3 \text{ median} - 2 \text{ mean}$$

$$\text{Or } \text{median} = \frac{1}{3}(\text{mode} + 2 \text{ mean})$$

$$\text{Or } \text{mean} = \frac{1}{2}(3 \text{ median} - \text{mode})$$

Weighted Mean

If it is required to find the average of some values which are not equally important, then we assign them certain numerical values to

express their relative importance. These numerical values are called weights.

If x denotes the items of a data and w denotes their respective weights, then arithmetic mean

$$\text{is } \bar{x} = \frac{\sum wx}{\sum w}$$

Example 14

The following are the monthly expenditure of a house according to their weights.

Items	Expenditure (Rs)	Weight Relative Importance
Food	8,000	20
House rent	5,000	10
Clothing	1,000	8
Utility bills	900	4
Education	600	2
Miscellaneous	100	1

Find the weighted mean.

Solution

Items	$X =$ Expenditure (Rs)	$W =$ weight w	wx
Food	8,000	20	1,60,000
House rent	5,000	10	50,000
Clothing	1,000	8	8,000
Utility bills	900	4	3,600
Education	600	2	1,200
Miscellaneous	100	1	100
Total	—	45	2,24,900

Weighted mean,

$$\bar{x}_w = \frac{\sum wx}{\sum w} = \frac{2,24,900}{45} = \text{Rs. } 50,17.22$$



Exercise 11.3

Q.1 Find arithmetic mean for the following data by definition.

a) $x = 2, 4, 6, 8, 10, 12$

Solution

Sol. $x = 2, 4, 6, 8, 10, 12$

No. of values = $n = 6$

We know that

$$\bar{x} = \frac{\sum x}{n} = \frac{2+4+6+8+10+12}{6} = \frac{42}{6} = 7$$

$$\bar{x} = 7$$

b) $y = 3, 4, -1, 7, -8, -5, 0$

Solution

Sol. $y = 3, 4, -1, 7, 8, -5, 0$

No. of values = $n = 7$

We know that

$$\bar{y} = \frac{\sum y}{n} = \frac{3+4+(-1)+7+(-8)+(-5)+0}{7}$$

$$\bar{y} = \frac{0}{7} = 0$$

$$\bar{y} = 0$$

c) $z = 0, 4, 8, 12, 16, 20, 24, 28$

Solution

Sol. $z = 0, 4, 8, 12, 16, 20, 24, 28$

No. of values = $n = 8$

We know that

$$\bar{z} = \frac{\sum z}{n} = \frac{0+4+8+12+16+20+24+28}{8}$$

$$\bar{z} = \frac{112}{8}$$

$$\bar{z} = 14$$

d) $u = 3.1, 4.2, 5.3, 6.4, 7.5, 8.6, 9.7, 10.8$

Solution

Sol. $u = 3.1, 4.2, 5.3, 6.4, 7.5, 8.6, 9.7, 10$

No. of values = $n = 8$

We know that

$$\bar{u} = \frac{\sum u}{n}$$

$$= \frac{3.1+4.2+5.3+6.4+7.5+8.6+9.7+10.8}{8}$$

$$\bar{u} = \frac{55.6}{8} = 6.95$$

$$\bar{u} = 6.95$$

e) $v = 1, 1, 1, 1, 1, 1, 1, 1$

Solution

Sol. $v = 1, 1, 1, 1, 1, 1, 1, 1$

No. of values = $n = 8$

$$\bar{v} = \frac{\sum v}{n} = \frac{1+1+1+1+1+1+1+1}{8}$$

$$\bar{v} = \frac{8}{8}$$

$$\bar{v} = 1$$

Q.2 Following are the scores made by two batsmen A and B in a series of innings.

A	13	18	6	73	7	19	109	36	64	29
B	47	12	76	48	4	51	37	48	13	8

Find arithmetic means of scores of the both players and state who is better as run getter?

Solution

Scores of batsman A

A	12	18	6	73	7	19	109	36	64	29
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No. of matches = $n = 10$

Sum of scores =

$$\sum x_1 = 12+18+6+73+6+7+19+109+36+64+29$$

$$\sum x_1 = 480$$

$$\text{Mean score} = \frac{\text{Sum of scores}}{\text{No. of matches}} = \frac{\sum x_1}{n}$$

$$\bar{x}_1 = \frac{480}{10}$$

$$\bar{x}_1 = 48 \dots (i)$$

The average score of batsman A is 48 score per match.

Scores of batsman B

B	47	12	76	48	4	51	37	48	13	8
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No. of matches $n = 10$

Sum of scores

$$\sum x_2 = 47+12+76+48+4+51+37+48+13+8$$

$$\sum x_2 = 336$$



$$\begin{aligned}\text{Mean score} &= \frac{\text{Sum of scores}}{\text{No. of matches}} = \frac{\sum x_i}{n} \\ &= \frac{336}{10} = 33.6, \text{ (ii)}\end{aligned}$$

From (i) and (ii), we conclude that batsman A is better than B as run getter.

Q.3 Marks of four students of class 9 in four subjects out of 100 are given below. The weights of subjects according to their importance are also given.

Subjects	Weights W	Marks			
		Hanazla	Musa	Zaki	Saleh
Mathematics	4	70	80	90	85
Physics	3	85	60	55	70
Chemistry	2	80	65	50	60
Biology	1	55	45	65	60

Find weighted mean of marks of each student.

Solution

Subjects	Weights W	Hanazla		Musa		Zaki		Saleh	
		marks x	wx	marks y	wy	Marks z	wz	marks v	wv
Mathematics	4	70	280	80	320	90	360	85	340
Physics	3	85	255	60	180	55	165	70	210
Chemistry	2	80	160	65	130	50	100	60	120
Biology	1	55	55	45	45	65	65	60	60
	$\sum w = 10$	$\sum wx = 780$		$\sum wy = 675$		$\sum wz = 690$		$\sum wv = 770$	

(i) Weighted mean of marks of Hanazla

$$\bar{x}_w = \frac{\sum wx}{\sum w} = \frac{780}{10} = 78$$

(ii) Weighted mean of marks of Musa

$$\bar{y}_w = \frac{\sum wy}{\sum w} = \frac{675}{10} = 67.5$$

(iii) Weighted mean of marks of Zaki

$$\bar{z}_w = \frac{\sum wz}{\sum w} = \frac{690}{10} = 69$$

(iv) Weighted mean of marks of Saleh

$$\bar{v}_w = \frac{\sum wv}{\sum w} = \frac{770}{10} = 77$$

Q.4 Following frequency distribution represents number of liters of fuel consumed against the distance in km. Find arithmetic mean of this data set. answer

Class intervals	0-5	5-10	10-15	15-20	20-25
Frequency	3	7	10	6	2

Solution

Class Intervals	Frequency (f)	Class marks (x)	fx
0-5	3	2.5	12.5
5-10	7	7.5	52.5
10-15	10	12.5	125
15-20	6	17.5	105
20-25	2	22.5	45
	$\sum f = 30$		$\sum fx = 340$

We know that



$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{340}{30} = 11.33$$

Q.5 Find arithmetic mean from the following data of number of a detergent packs sold against its mass in kg. answer

Class intervals	1-9	10-18	19-27	28-36	37-45
No. of packs	10	15	18	15	12

Solution

Classes (kg)	No. of packs (f)	Class marks (x)	fx
1-9	10	5	50
10-18	15	14	210
19-27	18	23	414
28-36	15	32	480
37-45	12	41	492
	$\sum f = 70$		$\sum fx = 1646$

We know that

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{1646}{70} = 23.51$$

Q.6 Find median for the following data. answer

Solutions

(i) 1, 4, 2, 5, 3, 7, 6

Arranged data in ascending order

$x = 1, 2, 3, 4, 5, 6, 7$

No. of values = $n = 7$ (odd)

As n is odd so

$$\text{Median} = \frac{7+1}{2} \text{th value}$$

$$i = \frac{7+1}{2} \text{th value}$$

$$i = \frac{8}{2} \text{th value}$$

$$i = 4 \text{th value}$$

As 4th value in arranged data is 4. So

$$\text{Median} = 4$$

(ii) 12, 14, 2, 6, 2, 8

answer

Arranged data in ascending order

$x = -8, -6, -4, -2, 2, 4, 6, 8$

No. of values = $n = 8$ (even)

As n is even, so

$$\text{Median} = \frac{1}{2} \left[\frac{n}{2} \text{th value} + \frac{n+2}{2} \text{th value} \right]$$

$$i = \frac{1}{2} \left[\frac{8}{2} \text{th value} + \frac{8+2}{2} \text{th value} \right]$$

$$i = \frac{1}{2} [4 \text{th value} + 5 \text{th value}]$$

$$i = \frac{1}{2} [-2 + 2] = \frac{1}{2} (0) = 0$$

$$\text{ii) } 0, 1, -2, -3, 4, 5, 6, 3$$

answer

Arranged data in ascending order

$x = -3, -2, 0, 1, 3, 4, 5, 6$

No. of values = $n = 8$ (Even)

As n is even so

$$\text{Median} = \frac{1}{2} \left[\frac{n}{2} \text{th value} + \frac{n+2}{2} \text{th value} \right]$$

$$i = \frac{1}{2} \left[\frac{8}{2} \text{th value} + \frac{8+2}{2} \text{th value} \right]$$

$$i = \frac{1}{2} [4 \text{th value} + 5 \text{th value}]$$

$$i = \frac{1}{2} [1 + 3] \quad \left[\begin{array}{l} 4 \text{th value} = 1 \\ 5 \text{th value} = 3 \end{array} \right]$$

$$i = \frac{1}{2} (4) = 2$$

(iv) 4, 3, 1, -3, 2, -3, 3, 4, 1

answer

Arranged data in ascending order

$x = -3, -3, 1, 1, 2, 3, 3, 4, 4$

No. of values = $n = 9$ (odd)

As n is odd, so

$$\text{Median} = \frac{n+1}{2} \text{th value}$$

$$i = \frac{9+1}{2} \text{th value}$$



$$k = \frac{10}{2} \text{th value}$$

$$k = 5^{\text{th}} \text{ value}$$

As 5^{th} value is arranged data is 2 so

$$\text{Median} = 2$$

$$v) \quad 4, 4, 4, 4, 4, 4$$

Arranged data

$$x = 4, 4, 4, 4, 4, 4$$

$$\text{No. of obs.} = n = 6 \text{ (Even)}$$

Q.7 In a practical of geometry and shapes students are required to construct a circle of any radius between 0 cm and 2 cm. The data given below is recorded.

Radius (cm)	1.8	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.2
No. of circles (f)	6	9	15	24	33	20	13	7	5

Find median radius of the circle.

Solution

Radius (x cm)	No. of circles (f)	Cumulative Frequency (C.F.)
1.8	6	6
1.6	9	6+9 = 15
1.4	15	15+15 = 30
1.2	24	30+24 = 54
1.0	33	54+33 = 87
0.8	20	87+20 = 107
0.6	13	107+13 = 120
0.4	7	120+7 = 127
0.2	5	127+5 = 132
	$\Sigma f = 132$	

$$\text{No. of values} = n = \Sigma f = 132 \text{ (Even)}$$

$$\text{Median} = \frac{1}{2} \left(\frac{n}{2} \text{th value} + \frac{n+1}{2} \text{th value} \right)$$

$$k = \frac{1}{2} \left(\frac{132}{2} \text{th value} + \frac{132+1}{2} \text{th value} \right)$$

$$k = \frac{1}{2} (66^{\text{th}} \text{ value} + 67^{\text{th}} \text{ value})$$

$$k = \frac{1}{2} (1.0 + 1.0) \quad \left(\begin{array}{l} \because 66^{\text{th}} \text{ value} = 1.0 \\ 67^{\text{th}} \text{ value} = 1.0 \end{array} \right)$$

As n is even, so

$$\text{Median} = \frac{1}{2} \left[\frac{n}{2} \text{th value} + \frac{n+1}{2} \text{th value} \right]$$

$$k = \frac{1}{2} (33^{\text{rd}} \text{ value} + 34^{\text{th}} \text{ value})$$

$$k = \frac{1}{2} (4+4)$$

$$k = \frac{1}{2} (8) = 4$$

$$k = \frac{1}{2} (2.0)$$

$$k = 1.0$$

Thus median radius of circle 1.0 cm.

Q.8 At an old age house, there are people of different age groups as given below.

Classes intervals	25-30	30-40	40-50	50-60	60-70
f	3	8	11	5	4



Find an average median age of people residing in that old age house.

Solution

Class Intervals	Frequency	Cumulative frequency (C.F.)
20-30	3	3
30-40	8	3+8 = 11 = c
40-50	f = 11	11+11 = 22 = Median group
50-60	5	22+5 = 27
60-70	4	27+4 = 31
	$n = \Sigma f = 31$	

The group containing $\frac{n}{2}$ th value is median group. From table, it is clear that

$$\frac{31}{2} \text{th} = 15.5^{\text{th}} \text{ value lies in the group } (40-50).$$

We know that

$$\text{Median} = l + \frac{h}{f} \left(\frac{n}{2} - c \right)$$

$$\text{Here, } l = 40, h = 10, f = 11, c = 11, n = 31$$

$$\text{Median} = 40 + \frac{10}{11} \left(\frac{31}{2} - 11 \right)$$

$$= 40 + \frac{10}{11} (15.5 - 11)$$

$$= 40 + \frac{10}{11} (4.5)$$

$$= 40 + \frac{45}{11}$$

$$= 40 + 4.09 = 44.09$$

Q.9 Find mode for the following data.

$$i) \quad 1, 2, 3, 4, 5, 6$$

Solution

As there is no repeated value in data have no mode.

$$ii) \quad 1, 4, 2, 3, 2, 5, 3, 2, 5, 4, 2$$

Solution

We observe that

2 occurs 5 times

3 occurs 2 times

4 occurs 2 times

5 occurs 2 times

The most frequent value is 2, so mode = 2

$$ii) \quad 120, 130, 140, 125, 125, 225, 130$$

Solution

120 occurs 2 times

125 occurs 1 time

130 occurs 1 time

140 occurs 1 time

225 occurs 2 times

We observe that

120 and 225 are most repeated value so mode = 120 and 225. (Data is bimodal)

$$ii) \quad 1, 4, 3, 5, 5, 3, 4, 2$$

Solution

We observe that each value is repeated in same frequency so data have no mode.

Q.10 Find mean, median and mode for the following discrete frequency distribution.

x	0	1	2	3	4	5
f	3	5	9	4	2	1

Solution

Finding mean

x	0	1	2	3	4	5
f	3	5	9	4	2	1
xf	0	5	18	12	8	5

We know that

$$\bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{48}{24} = 2$$

Finding median

x	0	1	2	3	4	5
f	3	5	9	4	2	1
cf	3	8	17	21	23	24

As $n = \Sigma f = 24$ (Even)

$$\text{Median} = \frac{1}{2} \left(\frac{n}{2} \text{th value} + \frac{n+1}{2} \text{th value} \right)$$



$$L = \frac{1}{2} \left(\frac{24}{2} \text{th value} + \frac{24+2}{2} \text{th value} \right)$$

$$L = \frac{1}{2} (12^{\text{th}} \text{ value} + 13^{\text{th}} \text{ value})$$

$$L = \frac{1}{2} (2+2)$$

$$\left(\begin{array}{l} 12^{\text{th}} \text{ value} = 2 \\ 13^{\text{th}} \text{ value} = 2 \end{array} \right)$$

$$L = \frac{1}{2} (4) = 2$$

$$\text{Median} = 2$$

Finding Mode

The most frequent value is 2

$$\therefore \text{Mode} = 2$$

Q.11 Find modal value for the following frequency distributions.

(a)

Classes	1-4	5-8	9-12	13-16	17-20
f	9	8	10	6	7

(b)

Classes	10-15	15-20	20-25	25-30	30-35
f	7	6	3	2	1

Solution

(a)

Class	f	Class Boundaries
1-4	9	0.5 - 4.5
5-8	$f_1 = 8$	4.5 - 8.5
9-12	$f_m = 10$	$f \rightarrow 9, 10, 11, 12.5$ modal group
13-16	$f_1 = 6$	12.5 - 16.5
17-20	7	16.5 - 20.5

As 10 is maximum frequency, so (9.5-12.5) is the modal group.

We know that

$$\text{Modal } l = l + \frac{(f_m - f_1)}{2f_m - f_1 - f_2} \times h$$

$$\text{Here, } l = 8.5, f_m = 10, f_1 = 8, f_2 = 6, h = 4$$

Putting these values we get.

$$\text{Mode} = 8.5 + \frac{(10-8) \times 4}{2(10)-8-6}$$

$$\text{Mode} = 8.5 + \frac{2 \times 4}{20-14}$$

$$\text{Mode} = 8.5 + \frac{8}{6}$$

$$\text{Mode} = 8.5 + 1.33$$

$$\text{Mode} = 9.83$$

(b) In this case class limits can be taken as class boundaries.

Classes	Frequency (f)	
	$f_1 = 0$ (no group)	
$l = 10-15$	$f_m = 7$	Modal Group
15-20	$f_1 = 6$	
20-25	3	
25-30	2	
30-35	1	

As 7 is the largest frequency, (10-15) is modal group.

We know that

$$\text{Mode} = l + \frac{(f_m - f_1)}{2f_m - f_1 - f_2} \times h$$

$$\text{Here, } l = 10, f_m = 7, f_1 = 0, f_2 = 6, h = 5$$

Putting the values,

$$\text{Mode} = 10 + \frac{(7-0)}{2(7)-0-6} \times 5$$

$$\text{Mode} = 10 + \frac{7}{14-6} \times 5$$

$$\text{Mode} = 10 + \frac{15}{8}$$

$$\text{Mode} = 10 + 1.875 = 11.875$$

$$\text{Mode} = 11.88$$

Probability

The measure of chance of an event is called probability and its values are 0 to 1.



- Probability of a new prophet after Hazrat Muhammad (SAW) is 0.
- Probability that Hakeem will win the race among Hakeem, Hajar & Hani is 1/3.
- Probability of sun setting is west is 1.

Single Event Probability

An event which has single sample point is called a single event or simple event or singular event. Eg. is tossing a coin the sample space is $\{H, T\}$, where $E_1 = \{T\}$, as only one out of 2 outcomes is possible. Here both E_1 and E_2 are single events.

Example 14

Write sample space for single roll of a die and enlist all the events. Discuss whether any compound event is present in this case. **Answer**

Solution

Here we notice if a single die is rolled once then any one of the outcomes may occur from 1, 2, 3, 4, 5 and 6. Therefore $S = \{1, 2, 3, 4, 5, 6\}$

Now all possible events are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$ from events listed above, is evident that all events are single. So no compound event in this case.

The probability of such events is obtained by the formula:

$$P(E) = \frac{\text{No. of favourable outcomes}}{\text{total no. of outcomes}}$$

Example 15

Write sample space for single spin of a spinner of 3 primary colours and enlist all the events. Discuss probability of all events. **Answer**

Solution

Here $S = \{\text{blue, red, green}\}$, the pointer of the spinner can stop at only one of the colour so $E_1 = \{\text{blue}\}, E_2 = \{\text{red}\}, E_3 = \{\text{green}\}$

$$P(E_1) = P(E_2) = \frac{1}{3}$$

Example 16

Ali picked a book at random from a library shelf which had 3 history books of grade 6, 4 geography books grade 7, 9 science books of grade 8, 3 science books of grade 7, 6 geography books of grade 8 and 3 science books of grade 6. Find the probability of getting:

- (a) a science book (b) a geography book
(c) a book of grade 7 (d) a book of grade 9
(e) a book of mathematics

Solution

Here total number of books = 30

a) Total science books are 17.

$$\text{So } P(E_1) = \frac{17}{30}$$

b) Total geography books are 10.

$$\text{So } P(E_2) = \frac{10}{30} = \frac{1}{3}$$

c) Total books of grade 7 are 7.

$$\text{So } P(E_3) = \frac{7}{30}$$

d) There is no book of grade 9.

$$\text{So } P(E_4) = \frac{0}{30} = 0$$

e) There is no book of mathematics

$$\text{So } P(E_5) = \frac{0}{30} = 0$$

Probability of an Event not occurring

An event which describes the non occurrence of another event is called non occurring event. If 4 boys Affan, Jaffar, Hassan and Maaz are playing ONG, then probability of winning of each boy is $\frac{1}{4}$. If the event of winning of Jaffar

is represented by E , then $P(E) = \frac{1}{4}$. Now the



event that Jaffer won't win is represented by $\bar{F}$

$$\text{and } P(\bar{F}) = 1 - \frac{1}{4} = \frac{3}{4}$$

The two events in which one occurs non-occurrence of other are called complementary events.

Example 17

There are 12 teachers in maths department among them 2 are higher secondary teachers, 4 are secondary teachers and 6 are senior elementary teachers. If good performance award is to be given to only one teacher then find probability.

- That a secondary teacher will get the award
- That a secondary teacher will not get the award.

Solution

Here total number of teachers are 12. If this is assumed that all teachers are equally hardworking, dedicated and showing excellent performance throughout their career then the events will be equally likely.

- Probability that a secondary teacher will

$$\text{get the award is } P(E) = \frac{4}{12} = \frac{1}{3}$$

- Probability that a secondary teacher will not get the award is $P(\bar{E}) = 1 - P(E)$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

Relative Frequency as an Estimate of Probability

The ratio between no. of times an event is occurring in an experimental trial and total number of trials in the experiment, i.e.

Relative Frequency =

$$\frac{\text{frequency of an event occurring}}{\text{total no. of trials in an experiment}}$$

- The relative frequency is often calculated after performing experiment so it is often

called experimental probability.

Example 18

A spinner of 5 equally likely coloured red, blue, green, yellow and white is spun 50 times and is landed 10 times on red, 12 times on blue, 9 times on green, 9 times on yellow and 10 times on white colour. Find the relative frequency for the event that it lands on green colour.

Solution

Total No. of trials = $n = 50$

No. of favourable outcomes for landing on green colour = $f = 9$

Relative Frequency =

$$\frac{\text{frequency of an event occurring}}{\text{total no. of trials in an experiment}} = \frac{9}{50}$$

Example 19

A fair coin is tossed 150 times and total comes up 70 times. What is the relative frequency of getting head up.

Solution

Total No. of trials = $n = 150$ No. of favourable outcomes for getting head up = $f = 70$

Relative Frequency =

$$\frac{\text{frequency of an event occurring}}{\text{total no. of trials in an experiment}} = \frac{70}{150} = 0.46\bar{6}$$

Example 20

5 students out of a section of 30 students are expected to be absent everyday in summer camp. What No. of students is expected to be absent every day from 12 such sections of same strength.

Solution

Total No. of students in 12 sections = n
 $= 12 \times 30 = 360$

$$\text{Probability of absent students} = p = \frac{5}{30} = \frac{1}{6}$$

Expected frequency =

$$\text{No. of trials} \times \text{probability of the event}$$

$$= n \times p = 360 \times \frac{1}{6} = 60$$



example 21

If probability of a person to be Haife & Qura is a village is $\frac{3}{10}$. What do you expect the no.

of Haife & Qura in 2500 such village? answer

Solution

Total No. of villages = $n = 2500$ and

$$\text{Probability} = p = \frac{3}{10}$$

Expected No. of Haife & Qura = No. of villages $\times$ probability of the event

$$= n \times p = 2500 \times \frac{3}{10} = 750$$

Exercise 11.3

Q.1 A letter is chosen randomly from the word 'ALLAH'. Now find the probability of getting

(i) a vowel answer (ii) an H answer

(iii) an L answer (iv) a consonant answer

Solution

Total number of letters of word ALLAH = 5

(i) Vowels in word ALLAH = 2

$$\text{Probability of getting a vowel} = P(V) = \frac{2}{5}$$

(ii) Number of H in word ALLAH = 1

$$\text{Probability of getting H} = P(H) = \frac{1}{5}$$

(iii) Number of 'L' in the word ALLAH = 2

$$\text{Probability of getting a L} = P(L) = \frac{2}{5}$$

(iv) Number of consonant in the word ALLAH = 3

$$\text{Probability of getting a consonant} = \frac{3}{5}$$

Q.2 A garment factory shipped an order which contained 7000 jackets, 2000 sweaters and 3000 trousers in which 7 trousers, 5 jackets and 9 sweaters are faulty. If the quality inspector unpacked one item at random, find the probability that it is:

(i) a trouser answer (ii) not a jacket answer

(iii) a faulty item answer

Solution

No. of jackets = 7,000

No. of sweaters = 2,000

No. of trousers = 3,000 =

Total number of items 12,000

(i) Probability of getting trouser = $P(T)$

$$= \frac{3,000}{12,000}$$

$$= \frac{3}{12}$$

$$= \frac{1}{4}$$

(ii) Total items = 12,000

Jackets = 7,000

No. of items that are not jackets

$$= 12,000 - 7,000$$

$$= 5,000$$

Probability of getting not a jacket =

$$P(\bar{T}) = \frac{5000}{12000} = \frac{5}{12}$$

(iii) No. of faulty jackets = 5

No. of faulty sweaters = 9

No. of faulty trousers = 7

Total no. of faulty items = $5 + 9 + 7 = 21$

Probability of getting a faulty item

$$= P(F) = \frac{21}{12000} = 0.00175$$

Q.3 A number wheel is divided into 8 equal sectors labelled as pansy, lily, orchid, tulip, jasmim, rose, marigold and



sunflower. Manu spins the wheel once. Find the probability that pointer:

- Stops at rose
- Stops at a 4 letter flower name
- Does not stop at marigold
- Stops at a 3 letter flower name
- Stops at a flower name

Solution:

Flower name:

Pansy, Lily, orchid, tulip, jasmine, rose, marigold, sunflower.

Number of flowers = 8

i) No. of rose = 1

Probability of pointer stopping at rose = $P(R)$
 $= \frac{1}{8}$

ii) No. of flowers having 4 letter name = 2
 Probability of pointer stopping at 4 letters flower name = $\frac{2}{8} = \frac{1}{4}$

iii) No. of flowers other than marigold = 7
 Probability of pointer does not stopping at marigold = $P(M') = \frac{7}{8}$

iv) No. of flowers having 3 letters of flower name = 0
 Probability of pointer stopping at 3 letters flower name = $\frac{0}{8} = 0$.

v) No. of total flowers = 8
 Probability of pointer stopping at flower name = $\frac{8}{8} = 1$.

Q.4 A decagonal die labelled 4, 4, 4, 4, 5, 5, 6, 7, 8, 8 is rolled once. Find the probability of getting:

- a 4
- an even number
- a multiple of 4
- not a 7
- an odd number
- a prime number

ii) LCM of 4 and 8

iii) HCF of 4 and 8

iv) factor of 12

Solution: 4, 4, 4, 4, 5, 5, 6, 7, 8, 8

i) Counting of 4 = 4

Probability of getting a 4 = $P(4) = \frac{4}{10} = \frac{2}{5}$

ii) Counting of Even numbers = 7

Probability of getting an even number =

$$P(E) = \frac{7}{10}$$

iii) Counting of multiples of 4 = 4

Probability of getting a multiple of 4 =

$$\frac{4}{10} = \frac{2}{5}$$

iv) Counting of digits other than 7 = 9

Probability of getting not a 7 = $\frac{9}{10}$

v) Counting of odd numbers $P(5) = 2$

Probability of getting an odd number = $P(O)$

$$= \frac{2}{10}$$

vi) Counting of digits that are prime = 3

Probability of getting a prime number = $\frac{3}{10}$

vii) No. of digits that are L.C.M. of 4 and 8 = 2

Probability of getting L.C.M. of 4 and 8 = $P(L)$

$$= \frac{2}{10} = \frac{1}{5}$$

viii) No. of digits that are H.C.F. of 4 and 8 = 4

Probability of getting H.C.F. of 4 and 8 = $P(H)$

$$= \frac{4}{10} = \frac{2}{5}$$

ix) No. of digits that are factors of 12 = 3

Probability of getting a factor of 12 $P(F) = \frac{3}{10}$

$$= \frac{1}{2}$$

Q.5 A one digit whole number is chosen at random. Find the probability that it is:

i) less than 5

ii) greater than 10

iii) not the largest 1-digit number

iv) additive identity of whole numbers

v) HCF of 3 and 5

vi) multiplicative identity of real numbers

vii) not a prime number

viii) factor of 9

Solution:

One digit whole numbers:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

No. of one digit whole numbers = 10

i) No. of one digit whole numbers less than

$$5 = 5$$

Probability of one digit whole numbers less than 5.

$$= P(L_5) = \frac{5}{10} = \frac{1}{2}$$

ii) No. of one-digit whole numbers greater than

$$10 = 0$$

Probability of one digit whole number greater

than 10 = $P(L_7) = \frac{0}{10} = 0$

iii) No. of one-digit numbers other than largest

1-digit number = 9

Probability of one digit whole number that is

$$\text{not largest 1-digit number} = \frac{9}{10}$$

iv) No. of additive identities of whole numbers =

$$1$$

Probability of additive identity of whole

$$\text{numbers } P(L_1) = \frac{1}{10}$$

v) No. of H.C.F. of 3 and 5 = 1

Probability of H.C.F. of 3 and 5 = $P(L_1) = \frac{1}{10}$

vi) No. of multiplicative identities of real

numbers = 1.



Probability of multiplicative identity $P(L_1)$ of real number = $\frac{1}{10}$

vii) No. of 1-digit numbers that are not prime = 6

Probability of 1-digit non-prime numbers $P(L_6)$

$$= \frac{6}{10} = \frac{3}{5}$$

viii) No. of 1-digit factors of 9 = 3

Probability of 1-digit factor of 9 = $P(L_3) = \frac{3}{10}$

Q.6 A coin is tossed 10 times. Complete the given table and answer the questions below the table.

Event	Frequency	Relative frequency
head	6	
tail	4	

i) What is the expected frequency of getting head, if it was tossed 520 times?

ii) What is the expected frequency of getting tail, if it was tossed 385 times?

Solution:

Event	Frequency	Relative frequency
head	6	$\frac{6}{10} = \frac{3}{5}$
tail	4	$\frac{4}{10} = \frac{2}{5}$

i) No. of trials = 520

Probability of event head = $\frac{3}{5}$

Expected frequency =

$$\text{No. of trials} \times \text{probability of Event}$$

$$= 520 \times \frac{3}{5}$$

$$= 104 \times 3 = 312$$



(i) No. of trials = 303

Probability of event trial = $\frac{2}{3}$

Expected frequency = No. of trials $\times$ probability of event.

$$= 303 \times \frac{2}{3}$$

$$= 61 \times 2$$

$$= 122$$

Q.7 Complete the table and give answers, if a die is rolled 120 times.

Number	Frequency	Relative frequency
1	15	...
2	25	...
3	15	...
4	20	...
5	20	...
6	15	...

(i) If it was rolled 6000 times, how many times occurrence of a 4 is expected? answer

(ii) If it was rolled 3000 times, how many times occurrence of a 1 is expected? answer

(iii) If it was rolled 400 times, how many times occurrence of a 2 is expected? answer

Solutions

Number	Frequency	Relative frequency
1	15	$\frac{15}{120} = \frac{1}{8}$
2	25	$\frac{25}{120} = \frac{5}{24}$
3	15	$\frac{15}{120} = \frac{1}{8}$
4	20	$\frac{20}{120} = \frac{1}{6}$
5	20	$\frac{20}{120} = \frac{1}{6}$

4	15	$\frac{15}{120} = \frac{1}{8}$
---	----	--------------------------------

(i) No. of trials = 6000

Expected frequency

= No. of trials $\times$ probability of 4

$$= 6000 \times \frac{1}{8}$$

$$= 750$$

(ii) No. of trials = 3000

Expected frequency

= No. of trials $\times$ probability of 1

$$= 3000 \times \frac{1}{6}$$

$$= 500$$

(iii) No. of trials = 400

Expected frequency

= No. of trials $\times$ probability of 2

$$= 400 \times \frac{5}{24}$$

$$= \frac{2000}{24}$$

$$= 83.33 \approx 83$$

Q.8 There are 6 girls sections of class 9 in a public school, namely teal, orchid, mauve, hazel, zaffre and denim. Only one girl is to be chosen randomly from all sections for national science Olympiad.

(a) Find the probability of selecting a girl from hazel section. answer

(b) Find the probability of not selecting a girl from hazel section. answer

(c) Verify that answers of both parts a and b add up to unity. answer

(d) If 6 girls are to be selected, find expected frequency of selecting from orchid section. answer

(e) If 60 girls are to be selected, what number of girls is expected to be chosen from Teal. answer

Solutions

6 girls section name:

Teal, orchid, mauve, hazel, zaffre, denim

(a) Total sections = 6

Probability of selecting a girl from hazel

$$\text{section } P(H) = \frac{1}{6}$$

(b) Probability of not selecting a girl from

$$\text{hazel section } P(H') = 1 - \frac{1}{6}$$

$$= \frac{6-1}{6} = \frac{5}{6}$$

(c) $P(H) + P(H') =$

$$= \frac{1}{6} + \frac{5}{6}$$

$$= \frac{1+5}{6}$$

$$= \frac{6}{6}$$

$$= 1$$

Result

Miscellaneous Exercise 11

Q.1 Encircle the correct option in the following:

1. Which of the following is a class mark of the interval (10-15)? answer

(a) 10

(b) 12.5

(c) 15

(d) 16

2. What is size of class intervals (4-7)? answer

(a) 4

(b) 5

(c) 6

(d) 7

3. Which of the following is chart of adjacent rectangles? answer

(a) bar graph

(b) frequency polygon

(c) histogram

(d) ogive

4. Which of the following is measure of central tendency? answer

(a) variance

(b) standard deviation

(c) range

(d) arithmetic mean

Sum of probabilities of 2 complementary events is unity.

(i) Girls to be selected = 6

Probability of selecting girls from orchid

$$\text{section} = P(O) = \frac{1}{6}$$

Expected frequency

= No. of trials $\times$ probability of event.

$$= 6 \times \frac{1}{6}$$

$$= 1$$

(ii) Girls to be selected = 60

Probability of selecting a girl from Teal section

$$= \frac{1}{6}$$

Expected frequency

= No. of trials $\times$ probability of event.

$$= 60 \times \frac{1}{6}$$

$$= 10$$

5. Which of the following is a formula of arithmetic mean for grouped data? answer

$$(a) \frac{\sum x}{n} \quad (b) \frac{\sum f \left(\frac{x}{2} - d \right)}{n}$$

$$(c) \frac{\sum fx}{\sum f} \quad (d) \frac{\sum (f_m - f) \cdot h}{2f_m - f_1 - f_l}$$

6. If arithmetic mean of 25 values is 10, then what is sum of values? answer

(a) 250

(b) 125

(c) 25

(d) 2.5

7. What is median of the data 4, 5, 6, 2, 1? answer

(a) 0

(b) 2

(c) 3

(d) 4

8. Which measure of central tendency can have more than one value? answer

(a) weighted mean



- (b) median
(c) mode
(d) arithmetic mean
9. The probability of getting N in "MUHAMMAD" is: none
(a) $\frac{1}{8}$ (b) $\frac{2}{8}$
(c) $\frac{3}{8}$ (d) none of these
10. Probability of picking an ace from a well shuffled pack of 52 playing cards is:
(a) $\frac{1}{32}$ (b) $\frac{1}{13}$ none
(c) $\frac{4}{13}$ (d) none of these
11. A normal fair die is rolled 6000 times. The expected number of 5 is: none
(a) 100 (b) 250
(c) 1000 (d) none of these
12. A fair coin is tossed 500 times. Expected number of tails is: none
(a) 100 (b) 250
(c) 1000 (d) none of these
13. In a group of 5 people, 4 like peach juice. The expected no. of people in a population of 1200 who like peach juice is: none
(a) 240 (b) 400
(c) 960 (d) none of these
14. How many times Hani should toss a fair coin if he expects to get 100 tail? none
(a) 100 (b) 200
(c) 1000 (d) none of these
15. An event which can never happen is called: improb
(a) sure event
(b) certain event
(c) possible event
(d) impossible event

16. Two such events whose probabilities are equal are called: none
(a) likely (b) equally likely
(c) unlikely (d) none of these
17. The probability of a certain event is:
(a) 0 (b) $\frac{1}{2}$ none
(c) $\frac{3}{4}$ (d) 1
18. The probability of an event can take the value: none
(a) 0 (b) 1
(c) $\frac{1}{2}$ (d) all of these
19. The probability of an event cannot take the value: none
(a) 0 (b) 1
(c) 2 (d) all of these
20. The probability of an impossible event is:
(a) 0 (b) 1 none
(c) $\frac{1}{2}$ (d) all of these

Answer Key

1	b	2	a	3	c	4	d	5	c
6	a	7	b	8	c	9	b	10	b
11	c	12	b	13	c	14	b	15	d
16	b	17	d	18	d	19	c	20	a

Q.2 For the following data, find mean, mode and median. none

Marks	5	10	15	20	25	30	35	40
f	2	4	6	8	10	7	5	3

Solution

i) Mean

x	f	fx
5	2	10
10	4	40
15	6	90
20	8	160
25	10	250
30	7	210



35	5	175
40	3	120
$\Sigma f = 45$		$\Sigma fx = 1055$

We know that

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{1055}{45} = 23.44$$

$$\bar{x} = 23.44$$

ii) Median

x	f	cf
5	2	2
10	4	2+4=6
15	6	6+6=12
20	8	12+8=20
25	10	20+10=30
30	7	30+7=37
35	5	37+5=42
40	3	42+3=45
$n = \Sigma$	$f = 45$	

No. of values = $n = 45$ (odd)

$$\text{Median} = \frac{n+1}{2} \text{th value}$$

$$= \frac{45+1}{2} \text{ value}$$

$$= \frac{46}{2} \text{th value}$$

$$= 23^{\text{rd}} \text{ value}$$

From cumulative

frequency (=f)

Column we observe

that 23rd value is 25.

So, Median = 25

iii) Mode

From frequency column we observe that most frequent value is 25 so

Mode = 25

Q.3 Find the median for the following frequency distribution. none

Marks	10-15	15-20	20-25	25-30	30-35	35-40
Frequency	3	8	11	5	4	

Solution

Median

Marks	(f)	Class Boundaries	cf	
30-35	3	29.5 - 35.5	3	
40-45	8	39.5 - 49.5	3+8=11	
50-55	11	49.5 - 59.5	11+11=22	Median group
60-65	5	59.5 - 69.5	22+5=27	
70-75	4	69.5 - 79.5	27+4=31	

From (C.F) column we observe that

(49.5-59.5) contains $\frac{31}{2}$ the value so,

(49.5-59.5) is a median group.

We know that

$$\text{Median} = l + \frac{\frac{n}{2} - c}{f}$$

$$\text{Here, } l = 49.5, b = 10, f = 11, n = 31, c = 11$$

Putting the values

$$l = 49.5 + \frac{10(\frac{31}{2} - 11)}{11}$$

$$l = 49.5 + \frac{10}{11}(15.5 - 11)$$

$$l = 49.5 + \frac{10}{11}(4.5)$$

$$l = 49.5 + \frac{45}{11}$$

$$l = 49.5 + 4.09$$

$$l = 53.59 \text{ marks}$$

Q.4 The arithmetic mean of 10 values is 35.5. If nine values are 20, 23, 37, 40, 29, 33, 45, 40, 45 find the tenth value. none

Solution

$$\bar{x} = 35.5$$

No. of observations = $n = 10$

Size of values = 20, 23, 37, 40, 29, 33, 45, 40, 45

Let tenth value = x



We know that

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$

$$15.5 = \frac{20 \times 25 + 25 \times 37 + 40 \times 29 + 35 \times 45 + 40 \times 49 + 5}{10}$$

$$15.5 \times 10 = 320 + y$$

$$155 = 320 + y$$

$$27 = y$$

$$\Rightarrow y = 27$$

Thus, the value is 27

Q.2 Find mean, median and mode for the following frequency distribution.

Classes	0-10	10-20	20-30	30-40	40-50
f	1	3	7	5	1

Solution

i) Mean

Classes	(f)	Midpoints(x)	(fx)
0-10	1	5	5
10-20	3	15	45
20-30	7	25	175
30-40	5	35	175
40-50	1	45	45
$\Sigma f = 10$			$\Sigma fx = 475$

We know that

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{475}{10} = 47.5$$

ii) Median

Classes	(f)	(cf)
0-10	1	1
10-20	3	1+3 = 4
20-30	7	4+7 = 11
30-40	5	11+5 = 16
40-50	1	16+1 = 17

Median group

From C.F column we observe that group

(20-30) contains $\frac{17}{2}$ th = 8.5th value, so median

group is (20-30)

We know that

$$\text{Median} = l + \frac{\frac{n}{2} - c}{f} \times h$$

$$\text{Here, } l = 20, h = 10, f = 7, c = 4, n = 10$$

Putting the values, we get

$$L = 20 + \frac{10/2 - 4}{7} \times 10$$

$$L = 20 + \frac{10}{7} \times (5 - 4)$$

$$L = 20 + \frac{10}{7} \times 1$$

$$L = 20 + \frac{25}{7}$$

$$L = 20 + 3.5$$

$$L = 23.5$$

iii) Mode

Classes	(f)
0-10	1
10-20	3+4 = 7
20-30	7+5 = 12
30-40	5+1 = 6
40-50	1

In this case class limits are taken as class boundaries.

From frequency column we observe that 7 is the largest frequency so (20-30) is a modal group.

$$\text{Mode} = l + \frac{(f_1 - f_2)}{2f_1 - f_2 - f_3} \times h$$

Putting values

$$\text{Mode} = 20 + \frac{(7 - 3) \times 10}{2(7) - 3 - 5}$$

$$\text{Mode} = 20 + \frac{2 \times 10}{14 - 8}$$

$$\text{Mode} = 20 + \frac{20}{4}$$



$$\text{Mode} = 20 + 5$$

$$\text{Mode} = 25$$

Q.4 Affan is learning to play dart. He plays it 20 times each day and probability of hitting bull's eye is $1/10$. Find:

a) Expected number of hits in 25 days.

b) Expected number of misses in 25 days.

Solution

$$\text{Probability of hitting bull's eye} = P(H) = \frac{1}{10}$$

$$\text{Probability of missing bull's eye} = P(M)$$

$$= 1 - \frac{1}{10}$$

$$= \frac{10-1}{10} = \frac{9}{10}$$

No. of trials in 1 day = 20

No. of trials in 25 days = $20 \times 25 = 500$

(a) Expected No. of hits in 25 days =

= No. of trials $\times$ probability of hitting

$$= 500 \times \frac{1}{10}$$

$$= 50$$

(b) Expected No. of misses in 25 days

No. of trials $\times$ probability of missing

$$= 500 \times \frac{9}{10}$$

$$= 50 \times 9$$

$$= 450$$

Q.7 Shifaa usually gets late for her school 2 days out of 6 working days of a week. What is the expected number of days of her late arrival in 4 weeks?

Solution

$$\text{Probability of getting late} = P(L) = \frac{2}{6} = \frac{1}{3}$$

Working days in 1 week = 6

Working days in 4 weeks $6 \times 4 = 24$

Expected No. of days getting late =

$$\text{Total working days} \times P(L)$$

$$= 24 \times \frac{1}{3}$$

$$= 8$$

Q.8 Aisha has tried her best to score 100% in each maths assessment, but the probability of getting this is 0.8. Find the expected number of assessments in which she will score 100% out of total 20 assessments.

Solution

Probability of getting 100% score = $P(H) = 0.8$

Total assessments = 20

Expected no. of assessments

$$= \text{Total no. of assessment} \times \text{probability of event}$$

$$= 20 \times 0.8$$

$$= 16 \text{ assessments}$$

Q.9 In a lab shelf there are 20 reports of covid 19. Among these 12 are females and 8 of the females are covid positive. If a report is picked at random, find:

(a) The probability of getting a female report

(b) The probability of getting a male report

(c) The probability of getting a female covid negative report

Solution

Total reports = 20

Reports of female = 12

Reports of male = $20 - 12 = 8$

No. of female with covid (+ve) = 2

No. of male with covid (-ve) = $8 - 2 = 6$

(a) Probability of getting female report

$$= P(F) = \frac{12}{20}$$

(b) Probability of getting male report

$$= P(M) = \frac{8}{20}$$

(c) Probability of getting female covid-ve report

$$= \frac{11}{20}$$



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Subject: Mathematics
Time Allowed: 20 minutes

OBJECTIVE

Class: 9th

SECTION - A (Marks:15)

Q.1 Fill the relevant bubble against each questions according to curriculum. Candidate Sign: _____

Questions	A	B	C	D	A	B	C	D
1. Which number is self-multiplicative among?	3	-3	-1	0	☐	☐	☒	☐
2. According to reflexive property, $y \sim y$ is	$y^2 = 17$	$y^2 = -17$	$y = 17$	$-17 = y^2$	☐	☒	☐	☐
3. $\log_{10} 100$ is	2	-2	0	Impossible	☐	☐	☐	☒
4. If $\log_{10} 25 = 4$ then x is	5	-5	± 5	Impossible	☒	☐	☐	☐
5. If $x(A) = P$ then $n(A \cap A) =$	P	2P	$2P^2$	P^2	☐	☐	☐	☒
6. Which is the highest common factor of $-12x^2y^3, 6xy^2, 24x^2y^2$	$6xy$	$6x^2y^2$	$6xy^2$	$24x^2y^2$	☐	☐	☒	☐
7. What should be the value of k if $x \sim y$ shows $kx > ky^2$	$k < 0$	$k > 0$	$k = 0$	$k \geq 0$	☐	☐	☒	☐
8. If $\sin \theta = \frac{1}{4}$, What is the value of $\cos \theta$	$\frac{\sqrt{15}}{4}$	$\frac{3}{4}$	$\frac{\sqrt{17}}{4}$	$\frac{15}{4}$	☒	☐	☐	☐
9. In which quadrant do the angles between 90° and 180° lie?	I st	II nd	III rd	IV th	☐	☒	☐	☐
10. If $x < 0$ and $y > 0$ then $(-x, y)$ lies in quadrant	I	II	III	IV	☐	☒	☐	☐
11. Point of intersection of lines $3x - 7y = 9$ and $8x + 11y = 5$ is	(0, 0)	(0, 7)	(0, 0)	(7, 0)	☒	☐	☐	☐
12. If two triangles are similar then their sides are	Proportional	Equal	Congruent	Parallel	☒	☐	☐	☐
13. Medians of the triangles are	Collinear	Concurrent	Perpendicular	Parallel	☐	☒	☐	☐
14. The probability of an event cannot take the value	0	1	2	All of these	☐	☐	☒	☐
15. A line passing through mid-point and perpendicular to a side of triangle is called	Right bisector	Angle bisector	Mid-point	Altitude	☒	☐	☐	☐



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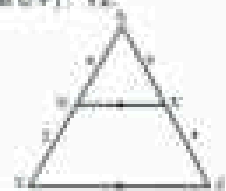
SUBJECTIVE

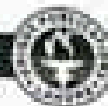
Subject: Mathematics
Time Allowed: 2:40 Hours

Class: 9th
Marks: 60

SECTION - B (Marks:46)

Q.2 Attempt all parts from the following. All parts carry equal marks. (9+4+36)

i.	Simplify by using laws of exponents: $\left[\frac{2163a^3b^2}{a^{12}b^2} \right]^{\frac{1}{3}}$	1+4	OR	Factorize $m^2 - m^2 - 1$	1+4												
ii.	Salomon bought 3.2 yard of fabric for a total price of Rs. 139.2. How much did the fabric cost per yard.	1+4	OR	Find HCF by factorization: $x^2 - 48, x^2 - 4x + 21$	1+4												
iii.	Find the value of 'x' $\log_2 (x^2 - 1) = 3$	1+4	OR	Solve: $\sqrt{\frac{x+6}{x+2}} = \sqrt{\frac{x+2}{x-1}}$	1+4												
iv.	Reduce the equation $6x - 3y + 12 = 0$ into (a) Slope-Intercept form (b) Two Intercept form	1+4	OR	Find median and mode for discrete frequency distribution. <table border="1"> <tr> <td>x</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> </tr> <tr> <td>f</td> <td>2</td> <td>4</td> <td>5</td> <td>2</td> <td>2</td> </tr> </table>	x	1	2	3	4	5	f	2	4	5	2	2	1+4
x	1	2	3	4	5												
f	2	4	5	2	2												
v.	Find the value of unknown from figure such that $\overline{UV} \parallel \overline{YZ}$ 		OR	Aftab is learning to play dart. He plays 20 times each day and probability of hitting bulls eye is $\frac{1}{10}$. Find (a) Expected number of misses in 25 days. (b) Expected number of hits in 25 days.	1+4												
vi.	Construct the triangle ABC when $m\angle A = 5.8^\circ, m\angle C = 4.2^\circ, m\angle A = 90^\circ$	1+4	OR	A model bus is built with a scale of 1:10. The model bus has a volume of $30m^3$. What is the volume of the actual bus?	1+4												
vii.	x-intercept of a line is double of y-intercept. Find the equation of line if it passes through (2, 1).	1+4	OR	Prove that the points (3, 4), (3, 1) and (8, 4) are the vertices of right angle triangle.													
viii.	Prove that: $\frac{1 - \sin \theta}{1 + \sin \theta} = \left(\frac{\sec \theta - \tan \theta}{\sec \theta + \tan \theta} \right)^2$	1+4	OR	Simplify: $\frac{(-2)^2 \cdot (-2)^2 \cdot (-2)}{(-2)^3}$	1+4												



ix	Evaluate: $\sec 45^\circ \operatorname{cosec} 45^\circ - \sin 30^\circ \cos 60^\circ$	1×4	OR	A (3,4), B (-1, 7) and C (x, y) are three collinear points such that B is mid-point of $\overline{AC}$. Find value of x and y.	1×4
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Section – C (Marks 24)

Note: Attempt the following questions. (8×3=24)

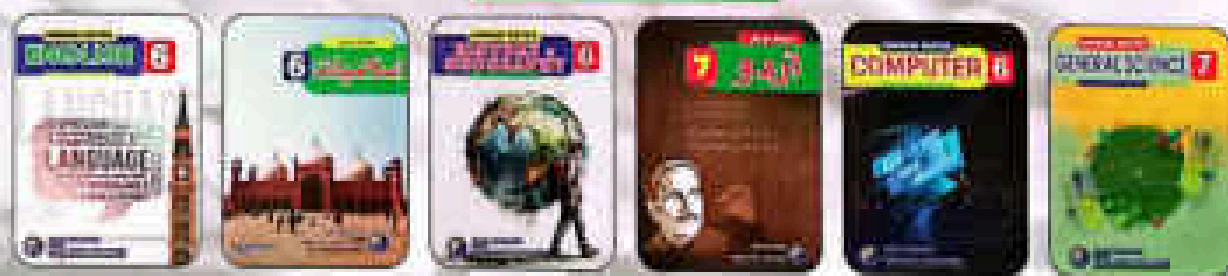
Q.3	Use suitable laws of exponents to show that $\left(\frac{x^p}{x^q}\right)^{r+s} \times \left(\frac{y^s}{y^t}\right)^{r+t} \times \left(\frac{z^t}{z^p}\right)^{r+p}$ $x^{q^2} \times y^{t^2} \times z^{p^2} = x^{p^2} \times y^{s^2} \times z^{t^2}$		OR	Find the equation of the passing through points of intersection of lines. $2x-5y+4=0$, $6x-4y+5=0$ and (a) Parallel to x-axis (b) Parallel to y-axis	
Q.4	Find square root by division method: $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$		OR	Let $E = \{1, 3\}$ and $F = \{4, 6, 8\}$, express $F \times F$, $F \times E$, $E \times E$ graphically.	
Q.5	If terminal Ray of θ is in 2 nd quadrant and $\operatorname{cosec} \theta = 2$ then find the value of $\cos \theta \left[\frac{\sin^2 \theta - \cos^2 \theta}{\sin^3 \theta + \cos^3 \theta} \right]$	1×4	OR	Show that the medians of triangle $\triangle ABC$ are concurrent when: $\overline{AB}=5.8\text{cm}$, $\overline{BC}=5\text{cm}$, $m\angle B=45^\circ$	1×4

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