$$= \binom{15}{0}30^{15}\cdot 1^0 + \binom{15}{1}30^{14}\cdot 1^1 + \binom{15}{2}30^{13}\cdot 1^2 + \binom{15}{3}30^{12}\cdot 1^3 + \cdots$$

$$= 30^{15} + 15\cdot 30^{14} + \binom{15}{2}30^{13} + \binom{15}{3}30^{12} + \cdots$$

Similarly, $29^{15} = (30-1)^{15}$

$$= \binom{15}{0}30^{15}(-1)^0 + \binom{15}{1}30^{14}(-1)^1 + \binom{15}{2}30^{13}(-1)^2 + \binom{15}{3}30^{12}(-1)^3 + \cdots$$

$$= 30^{15} - 15\cdot 30^{14} + \binom{15}{2}30^{13} - \binom{15}{3}30^{12} + \cdots$$

By subtracting, we get

$$31^{15} - 29^{15} = 2(15\cdot 30^{14}) + 2\binom{15}{3}3^{12} + \cdots$$

$= 30\cdot 30^{14} + k$, where is some positive constant.

$$31^{15} - 29^{15} = 30^{15} + k$$

$$\Rightarrow \quad 31^{15} - 29^{15} > 30^{15}$$

$$\Rightarrow \quad 31^{15} > 29^{15} + 30^{15}$$

Hence 31^{15} is greater than $29^{15} + 30^{15}$.

5. Using the binomial theorem, show that:

(i) $5^7 + 7^5$ is divisible by 36.

Solution:

$$5^7 = (6-1)^7$$

$$= \binom{7}{0}6^7 - \binom{7}{1}6^6 + \binom{7}{2}6^5 - \binom{7}{3}6^4 + \binom{7}{4}6^3 - \binom{7}{5}6^2 + \binom{7}{6}6^1 - \binom{7}{7}6^0$$

$$= 6^7 - 7\cdot 6^6 + 21\cdot 6^5 - 35\cdot 6^4 + 35\cdot 6^3 - 21\cdot 6^2 + 7\cdot 6 - 1$$

Similarly, $7^5 = (6+1)^5$

$$= \binom{5}{0}6^5 + \binom{5}{1}6^4 + \binom{5}{2}6^3 + \binom{5}{3}6^2 + \binom{5}{4}6^1 + \binom{5}{5}6^0$$

$$= 6^5 + 5\cdot 6^4 + 10\cdot 6^3 + 10\cdot 6^2 + 5\cdot 6 + 1$$

By adding, we get

$$5^7 + 7^5 = 6^7 - 7\cdot 6^6 + (21+1)\cdot 6^5 + (-35+5)\cdot 6^4 + (35+10)\cdot 6^3 + (-21+10)\cdot 6^2 + (7+5)\cdot 6$$

$$= 6^7 - 7\cdot 6^6 + 22\cdot 6^5 - 30\cdot 6^4 + 45\cdot 6^3 - 11\cdot 6^2 + 2\cdot 6^2$$

$$= 6^2\left(6^5 - 7\cdot 6^4 + 22\cdot 6^3 - 30\cdot 6^2 + 45\cdot 6 - 11 + 2\right)$$

$5^7 + 7^5 = 36k$, where $k \in Z$

Hence, $5^7 + 7^5$ is divisible by 36.

(ii) $(17)^{15} + (13)^7$ is divisible by 6

Solution:

$$(17)^{15} = (18-1)^{15}$$

$$= \binom{15}{0}(18)^{15}(-1)^0 + \binom{15}{1}(18)^{14}(-1)^1 + \binom{15}{2}(18)^{13}(-1)^2 + \cdots + \binom{15}{14}(18)^1(-1)^{14} + \binom{15}{15}(18)^0(-1)^{15}$$

$$= (18)^{15} - 15(18)^{14} + \binom{15}{2}(18)^{13} - \cdots + \binom{15}{14}\cdot 18 - 1$$

Similarly, $(13)^7 = (12+1)^7$

$$= \binom{7}{0}(12)^7 + \binom{7}{1}(12)^6 + \binom{7}{2}(12)^5 + \cdots + \binom{7}{6}(12)^1 + \binom{7}{7}(12)^0$$

$$= (12)^7 + 7(12)^6 + \binom{7}{2}(12)^5 + \cdots + \binom{7}{6}12 + 1$$

By adding, we get

$$(17)^{15} + (13)^7 = 18\left\{(18)^{14} - 15(18)^{13} + \binom{15}{2}(18)^{12} - \cdots + \binom{15}{14}\right\} + 12\left\{(12)^6 + 7(12)^5 + \binom{7}{2}(12)^4 + \cdots + \binom{7}{6}\right\}$$

$= 18k_1 + 12k_2$, where $k_1, k_2 \in Z$

$$= 6(3k_1 + 2k_2)$$

Hence $(17)^{15} + (13)^7$ is divisible by 6.

(iii) $(21)^9 + (19)^{11}$ is divisible by 20

Solution:

$$(21)^9 = (20+1)^9$$

$$= \binom{9}{0}20^9 + (9)20^8 + \binom{9}{2}20^7 + \cdots + \binom{9}{8}20^1 + \binom{9}{9}20^0$$

$$= 20^9 + 9\cdot 20^8 + \binom{9}{2}20^7 + \cdots + 9\cdot 20 + 1$$

Similarly, $(19)^{11} = (20-1)^{11}$

$$= \binom{11}{0}20^{11} - \binom{11}{1}20^{10} + \binom{11}{2}20^9 \cdots + \binom{11}{10}20^1 - \binom{11}{11}20^0$$

$$= 20^{11} - 11\cdot 20^{10} + \binom{11}{2}20^9 \cdots + 11\cdot 20 - 1$$

By adding, we get

$$(21)^9 + (19)^{11} = 20\left[\left(20^8 + 9\cdot 20^7 + 36\cdot 20^6 + \cdots + 9\right) + \left(20^{10} - 11\cdot 20^9 + 55\cdot 20^8 - \cdots + 11\right)\right]$$

$(21)^9 + (19)^{11} = 20k$, where $k \in Z$

Hence $(21)^9 + (19)^{11}$ is divisible by 20.

(iv) $(31)^4 + (29)^6$ is divisible by 30

Solution:

$$(31)^4 = (30+1)^4$$

$$= \binom{4}{0}30^4 + \binom{4}{1}30^3 + \binom{4}{2}30^2 + \binom{4}{3}30^1 + \binom{4}{4}30^0$$

$$= 30\left(30^3 + 4\cdot 30^2 + 6\cdot 30 + 4\right) + 1$$

$$(29)^6 = (30-1)^6$$

$$= \binom{6}{0}30^6 - \binom{6}{1}30^5 + \binom{6}{2}30^4 \cdots + \binom{6}{5}30^1 + \binom{6}{6}30^0$$

$$= 30\left(30^5 - 6\cdot 30^4 + 15\cdot 30^3 \cdots + 6\right) - 1$$

By adding, we get

$$(31)^4 + (29)^6 = 30\left[\left(30^3 + 4\cdot 30^2 + 6\cdot 30 + 4\right) + \left(30^5 - 6\cdot 30^4 + 15\cdot 30^3 - \cdots + 6\right)\right]$$

$= 30\,k$, where $k \in Z$

Hence, $(31)^4 + (29)^6$ is divisible by 30.

(v) $(101)^5 + (99)^7$ **is divisible by 100**

Solution:

$$\begin{aligned}(101)^5 &= (100+1)^5\\ &= \binom{5}{0}100^5 + \binom{5}{1}100^4 + \binom{5}{2}100^3 + \cdots + \binom{5}{4}100^1 + \binom{5}{5}100^0\\ &= 100^5 + 5\cdot 100^4 + 10\cdot 100^3 + \cdots + 5\cdot 100 + 1\\ &= 100\left(100^4 + 5\cdot 100^3 + 10\cdot 100^2 + \cdots + 5\right) + 1\end{aligned}$$

Similarly, $$\begin{aligned}(99)^7 &= (100-1)^7\\ &= \binom{7}{0}100^7 - \binom{7}{1}100^6 + \binom{7}{2}100^5 - \cdots + \binom{7}{6}100^1 - \binom{7}{7}100^0\\ &= 100^7 - 7\cdot 100^6 + 21\cdot 100^5 - \cdots + 7\cdot 100 - 1\\ &= 100\left(100^6 - 7\cdot 100^5 + 21\cdot 100^4 - \cdots + 7\right) - 1\end{aligned}$$

By adding, we get

$$\begin{aligned}(101)^5 + (99)^7 &= 100\left\{\left(100^4 + 5\cdot 100^3 + 10\cdot 100^2 + \cdots + 5\right) + \left(100^6 - 7\cdot 100^5 + 21\cdot 100^4 - \cdots + 7\right)\right\}\\ &= 100k, \text{ where } k \in Z\end{aligned}$$

Hence $(101)^5 + (99)^7$ is divisible by 100 .

6. Using the binomial theorem, find the remainder when 3^{101} is divided by 8 .

Solution:

$$\begin{aligned}3^{101} &= 3\cdot 3^{100}\\ &= 3\cdot(3^2)^{50} = 3(9)^{50} = 3(1+8)^{50}\\ &= 3\left\{\binom{50}{0}8^0 + \binom{50}{1}8^1 + \binom{50}{2}8^2 + \cdots + \binom{50}{50}8^{50}\right\}\\ &= 3\left\{1 + \binom{50}{1}8 + \binom{50}{2}8^2 + \cdots + 8^{50}\right\} = 3 + 3\left\{\binom{50}{1}\cdot 8 + \binom{50}{2}\cdot 8^2 + \cdots + 8^{50}\right\}\\ &= 3 + 8\left\{3\binom{50}{1} + 3\binom{50}{2}\cdot 8 + \cdots + 3\cdot 8^{49}\right\} = 3 + 8k, \text{where } k \in Z\\ &= 8k + 3\end{aligned}$$

Hence, 3 is the remainder, when 3^{101} is divided by 8 .

7. Using the binomial theorem, find the last digit of the number $(32)^{32}$.

Solution:

$$\begin{aligned}(32)^{32} &= (30+2)^{32}\\ &= \binom{32}{0}30^{32}\cdot 2^0 + \binom{32}{1}30^{31}\cdot 2^1 + \ldots + \binom{32}{31}30^1\cdot 2^{31} + \binom{32}{32}30^0\cdot 2^{32}\\ &= (1)\cdot 32^{32}\cdot(1) + 32\cdot 30^{31}\cdot 2 + \ldots + 32(30)\cdot 2^{31} + (1)(1)\cdot 2^{32}\\ &= (30^{32} + 64\cdot 30^{31} + \ldots + 32\cdot 30\cdot 2^{31}) + 2^{32}\\ &= 10k_1 + 2^{32}, \text{ where } k_1 \in Z\end{aligned}$$

$\Rightarrow$ Last digit in $(32)^{32}$ = Last digit in $(2)^{32}$

Now, $$\begin{aligned}2^{32} &= 2^2\cdot 2^{30} = 4\cdot(2^5)^6 = 4(32)^6\\ &= 4(30+2)^6\\ &= 4\left\{\binom{6}{0}30^6\cdot 2^0 + \binom{6}{1}30^5\cdot 2^1 + \ldots + \binom{6}{5}30^1\cdot 2^5 + \binom{6}{6}30^0\cdot 2^6\right\}\end{aligned}$$

$$\begin{aligned}&= 4\{(30^6 + 6\cdot 30^5\cdot 2 + \ldots + 6\cdot 30\cdot 2^5) + 2^6\} = 4(10k_2 + 2^6), \text{ where } k_2 \in Z\\ &= 40k_2 + 4\cdot 2^6\end{aligned}$$

$\Rightarrow$ Last digit in 2^{32} = Last digit in $4\cdot 2^6$
= Last digit in 256
= 6

Therefore, last digit in $(32)^{32} = 6$.

8. Using the binomial theorem, show that $7^n - 6n$ leaves remainder 1 when divided by 6 for all positive integers n.

Solution:

$$\begin{aligned}7^n &= (1+6)^n\\ 7^n &= \binom{n}{0}1^n\cdot 6^0 + \binom{n}{1}1^{n-1}\cdot 6^1 + \binom{n}{2}1^{n-2}\cdot 6^2 + \binom{n}{3}1^{n-3}\cdot 6^3 + \ldots + \binom{n}{n}1^0\cdot 6^n\\ 7^n &= 1 + n(1)(6) + \binom{n}{2}6^2 + \binom{n}{3}6^3 + \ldots + 6^n\\ 7^n - 6n &= 1 + \binom{n}{2}6^2 + \binom{n}{3}6^3 + \ldots + 6^n\\ 7^n - 6n &= 1 + 6\left[\binom{n}{2}6 + \binom{n}{3}6^2 + \ldots + 6^{n-1}\right]\\ 7^n - 6n &= 1 + 6k, \text{ where } k \in Z\end{aligned}$$

This shows that $7^n - 6n$ leaves a remainder 1, when divided by 6.

9. By using Binomial Theorem show that for each $n \in N, 5^n - 1$ is divisible by 4 .

Solution:

$$\begin{aligned}5^n &= (1+4)^n\\ &= \binom{n}{0}1^n\cdot 4^0 + \binom{n}{1}1^{n-1}\cdot 4^1 + \binom{n}{2}1^{n-2}\cdot 4^2 + \binom{n}{3}1^{n-3}\cdot 4^3 + \ldots + \binom{n}{n}1^0\cdot 4^n\\ 5^n &= 1 + n(1)(4) + \binom{n}{2}4^2 + \binom{n}{3}4^3 + \ldots + 4^n\\ 5^n - 1 &= 4\left[n + \binom{n}{2}4 + \binom{n}{3}4^2 + \ldots + 4^{n-1}\right]\\ 5^n - 1 &= 4k, \text{ where } k \in Z\\ &= \text{Multiples of 4}\end{aligned}$$

Hence, $5^n - 1$ is divisible by 4 for all $n \in N$.

10. By using Binomial Theorem show that for each $n \in N, 5^n - 2^n$ is divisible by 3 .

Solution:

$$\begin{aligned}5^n &= (2+3)^n\\ 5^n &= \binom{n}{0}2^n\cdot 3^0 + \binom{n}{1}2^{n-1}\cdot 3^1 + \binom{n}{2}2^{n-2}\cdot 3^2 + \binom{n}{2}2^{n-3}\cdot 3^3 + \ldots + \binom{n}{n}2^0\cdot 3^n\\ 5^n &= 2^n + n\cdot 2^{n-1}\cdot 3 + \binom{n}{2}\cdot 2^{n-2}\cdot 3^2 + \binom{n}{2}2^{n-3}\cdot 3^3 + \ldots + 3^n\\ 5^n - 2^n &= 3n\cdot 2^{n-1} + \binom{n}{2}\cdot 2^{n-2}\cdot 3^2 + \binom{n}{2}\cdot 2^{n-3}\cdot 3^3 + \ldots + 3^n\\ &= 3\left[n\cdot 2^{n-1} + \binom{n}{2}\cdot 2^{n-2}\cdot 3 + \binom{n}{2}\cdot 2^{n-3}\cdot 3^2 + \ldots + 3^{n-1}\right]\\ &= 3k, \text{ where } k \in Z\\ &= \text{Multiples of 3}\end{aligned}$$

Hence, $5^n - 2^n$ is divisible by 3 for all $n \in N$.

11. Show that $a^2+(a+2)^2+(a+4)^2+1$ is divisible by 12, whenever "a" is an odd integer.

Solution:

Since 'a' is an odd integer, so $a = 2n + 1$, where $n \in Z$

Consider $a^2+(a+2)^2+(a+4)^2+1 = (2n+1)^2+(2n+1+2)^2+(2n+1+4)^2+1$ $\because a = 2n+1$

$= (2n+1)^2+(2n+3)^2+(2n+5)^2+1$

$= 4n^2+1+4n+4n^2+9+12n+4n^2+25+20n+1$

$= 12n^2+36n+36$

$= 12(n^2+3n+3)$

$\Rightarrow a^2+(a+2)^2+(a+4)^2+1 = 12k$, where $k \in Z$

= Multiples of 12

Hence, $a^2+(a+2)^2+(a+4)^2+1$ is divisible by 12.

12. A company expects its annual revenue to grow at a fixed rate of 6% per year. The revenue in year 1 is Rs. 10,000,000. Estimate the company's revenue after 4 years using the binomial theorem for small growth rates.

Solution:

Initial revenue = P = 10,000,000

Annual growth rate = r = 6% = $\frac{6}{100} = 0.06$

Number of years = n = 4

Revenue after n years = $P(1+r)^n$

Revenue after 4 years = $10,000,000(1+0.06)^4$ ...(1)

Using Binomial theorem

$(1+r)^n \approx 1+nr+\frac{n(n-1)}{2!}r^2$ Neglecting r^3 & higher powers due to small growth rates.

$(1+0.06)^4 \approx 1+4(0.06)+\frac{4(4-1)}{2!}(0.06)^2$

$= 1 + 0.24 + 0.0216$

$= 1.2616$ Put in (1)

Revenue after 4 years $= 10,000,000 + 1.2616$

$= 12616000$

13. A bank offers a compound interest rate of 10% per year. Zafar invests Rs. 2,000,000 for 4 years. How much will his investment be worth at the end of 4 years?

Solution:

Initial investment = P = Rs. 2,000,000

Annual interest rate = 10% = $\frac{10}{100} = 0.1$

Number of years = n = 4

Investment worth after n years = $P(1+r)^n$

Investment worth after 4 years = $2,000,000(1+0.1)^4$

Using Binomial theorem

$= 2000000\left[\binom{4}{0}(1)^4(0.1)^0+\binom{4}{1}(1)^3(0.1)^1+\binom{4}{2}(1)^2(0.1)^2+\binom{4}{3}(1)^1(0.1)^3+\binom{4}{4}(1)^0(0.1)^4\right]$

$= 2000000[(1)(1)(1)+(4)(1)(0.1)+6(1)(0.01)+4(1)(0.001)+(1)(1)(0.0001)]$

$= 2000000(1.4641)$

= Rs. 2928200

14. Zaid is organizing a sports competition with 8 teams. Every team plays against every other team exactly once. How many matches will be played in total? Use Pascal's triangle to solve this.

Solution:

Total no. of terms = 8

Since each team plays against every other team exactly once, therefore

Number of matches played = $\binom{8}{2} = 28$

Using Pascal's Triangle

Row 0: 1

Row 1: 1 1

Row 2: 1 2 1

Row 3: 1 3 3 1

Row 4: 1 4 6 4 1

Row 5: 1 5 10 10 5 1

Row 6: 1 6 15 20 15 6 1

Row 7: 1 7 21 35 35 21 7 1

Row 8: 1 8 (28) 56 70 56 28 8 1

Required value is at row 8, 3rd element

Number of matches played = $\binom{8}{2} = 28$

Formula Sheet

1. Binomial Theorem: For any positive integer n,

$(a+x)^n = \binom{n}{0}a^n+\binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\cdots+\binom{n}{r}a^{n-r}x^r+\cdots+\binom{n}{n-1}ax^{n}+\binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}x^r$

Where the general term is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$

2. The Middle Term in the Expansion of $(a+b)^n$:

Case I: If n is even, then middle term is: $\left(\frac{n}{2}+1\right)^{th}$ term

Case II: If n is odd, then middle terms are: $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms

3. $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n} = 2^n$

4. $\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{n} = \binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1} = 2^{n-1}$

5. Binomial Theorem: (When n negative integer or a fraction)

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots+\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r+\cdots$, provided $|x|<1$

Where the general term is $T_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$

6. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $0 \le r \le n$.

7. Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1}+\binom{n-1}{k}$, for $0<k<n$

Multiple Choice Questions (MCQs)

Exercise 8.1

1. A case or exception which fails the mathematical formula or statement is called a:
   (A) factorial (B) permutation (C) combination (D) counter example
2. $n^2 - n + 41$ represent a prime number for $n \in N$ where -----
   (A) $n \le 10$ (B) $n \le 20$ (C) $n \le 40$ (D) $n \le 5$
3. There is no natural number n for which 3^n is -----
   (A) rational (B) odd (C) even (D) none of these
4. $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{n+2}{3}$ equal to -----
   (A) $\binom{n+3}{2}$ (B) $\binom{n+4}{2}$ (C) $\binom{n+3}{4}$ (D) $\binom{n+4}{4}$
5. The inequality $n! > 2^n + 1$ is valid if -----
   (A) $n < 4$ (B) $n \le 3$ (C) $n = 3$ (D) $n \ge 4$

Exercise 8.2

6. ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called -----
   (A) factors (B) roots (C) binomial coefficients (D) none of these
7. If n is an even positive integer then $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}=?$
   (A) 2^{n-1} (B) 2^n (C) 2^{n+1} (D) 2^n
8. The sum of even coefficients in expansion $(a+b)^4$ is -----
   (A) 16 (B) 12 (C) 10 (D) 8
9. The coefficients of the terms equidistant from beginning and end of the binomial expansion of $(a+x)^n$; $n \in N$ are equal as:
   (A) $\binom{n}{r}=\binom{n}{n-r}$ (B) $\binom{n}{r}=\binom{n}{n+r}$ (C) $\binom{n}{r}=\binom{n}{r-1}$ (D) $\binom{n}{r}=\binom{n-1}{n-1}$
10. The number of terms in the expansion of $(a+x)^{10}$ is -----
    (A) 10 (B) 11 (C) 12 (D) 9
11. General term of the expansion $(a+b)^n$ is -----
    (A) $\binom{n}{r}a^{n-r}b^{n-r}$ (B) $\binom{n}{r}a^{n-r}b^r$ (C) $\binom{n}{r}a^n b^{n-r}$ (D) $\binom{n}{r}a^n b^r$
12. Middle terms in the expansion of $(a+b)^{11}$ are:
    (A) T_6, T_7 (B) T_5, T_6 (C) T_7, T_8 (D) T_8, T_9

Exercise 8.3

13. The symbols $\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n}$ are meaning less if n is a -----
    (A) negative integer (B) +ive integer (C) 7^{th} term (D) none of these
14. If $n \notin Z^+$ and $|x| < 1$, then the expansion $1 + nx + \frac{(n)(n-1)}{2!}x^2 + \ldots$ is:
    (A) arithmetic series (B) geometric series (C) harmonic series (D) binomial series
15. In the expansion of $(1+x)^{-3}$ the 4th term is:
    (A) $-3x$ (B) $-10x^3$ (C) $6x^2$ (D) $10x^3$
16. Number of terms in expansion of $(1+2x)^{\frac{1}{2}}$ are -----
    (A) n (B) n! (C) 1 (D) ∞

17. The expansion of $(8-2x)^{-1}$ is valid if -----
    (A) $|x| > 4$ (B) $|x| < 4$ (C) $|x| = 4$ (D) $|x| = 0$

Exercise 8.4

18. In Pascal's triangle: For $n = 4$, the binomial coefficients are -----
    (A) 1 3 4 3 1 (B) 1 4 6 4 1 (C) 1 5 6 5 1 (D) 2 4 6 4 2
19. The 5^{th} row of Pascal's triangle corresponds to coefficients of which expansion?
    (A) $(x+y)^5$ (B) $(x+y)^4$ (C) $(x+y)^6$ (D) $(x+y)^3$
20. The last two digits of the number 11^{12} are -----
    (A) 1, 2 (B) 2, 1 (C) 1, 3 (D) 3, 1

ANSWER KEY

1.	D	2.	C	3.	C	4.	C	5.	D	6.	C	7.	A	8.	D	9.	A	10.	B
11.	B	12.	A	13.	A	14.	D	15.	B	16.	D	17.	B	18.	B	19.	B	20.	B

Unit 9 Division of Polynomials

Introduction

Polynomials play a fundamental role in algebra and have wide-ranging applications in various fields, including engineering, data science and digital communication. This unit explores polynomial division to determine the quotient and remainder. The remainder theorem is introduced as a powerful tool for evaluating polynomials efficiently, while the factor theorem is applied to factorize cubic polynomials. These concepts extend beyond theoretical mathematics, finding practical applications in polynomial regression, signal processing and coding theory. By mastering these techniques, students will develop a deeper understanding of polynomials and their significance in solving real-world problems.

Polynomial Function

A polynomial in x is an expression of the form:

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

where n is a non-negative integer and the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$ and a_0 are real numbers. It can be considered as a polynomial function of x. If $a_n \neq 0$ then it is a polynomial of degree n.

Degree of the polynomial: The highest power of x in a polynomial is called the degree of the polynomial. e.g., the polynomials $x^2 - 2x + 3$, $3x^3 + 2x^2 - 5x + 4$ are of degree 2 and 3 respectively.

Example 1: Divide the cubic polynomial $3x^3 - 10x^2 + 13x - 6$ by the linear polynomial $x - 2$. Also find quotient and remainder.

Solution:

$$\begin{array}{r l}
 & 3x^2 - 4x + 5 \\
x - 2 \;) & 3x^3 - 10x^2 + 13x - 6 \\
 & \underline{\underset{-}{}3x^3 \underset{+}{-} 6x^2} \\
 & -4x^2 + 13x \\
 & \underline{\underset{+}{-}4x^2 \underset{-}{+} 8x} \\
 & 5x - 6 \\
 & \underline{\underset{-}{}5x \underset{+}{-} 10} \\
 & 4
\end{array}$$

Hence, we can write: $3x^3 - 10x^2 + 13x - 6 = (x - 2)(3x^2 - 4x + 5) + 4$

So, quotient $= 3x^2 - 4x + 5$ and remainder $= 4$.

Example 2: Divide the polynomial $x^4 - 3x^3 + 5x^2 - 7x + 2$ by $x^2 - x + 1$. Also find quotient and remainder.

Solution:

$$\begin{array}{r l}
 & x^2 - 2x + 2 \\
x^2 - x + 1 \;) & x^4 - 3x^3 + 5x^2 - 7x + 2 \\
 & \underline{\underset{-}{}x^4 \underset{+}{-} x^3 \underset{-}{+} x^2} \\
 & -2x^3 + 4x^2 - 7x \\
 & \underline{\underset{+}{-}2x^3 \underset{-}{+} 2x^2 \underset{+}{-} 2x} \\
 & 2x^2 - 5x + 2 \\
 & \underline{\underset{-}{}2x^2 \underset{+}{-} 2x \underset{-}{+} 2} \\
 & -3x
\end{array}$$

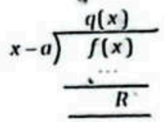
So, quotient $= x^2 - 2x + 2$ and remainder $= -3x$

Remainder Theorem:

Statement: If a polynomial $f(x)$ of degree $n \geq 1$ is divided by $x - a$ till no x-term exists in the remainder, then $f(a)$ is the remainder.

Proof: Suppose we divide the polynomial $f(x)$ by $(x - a)$. Then there exists a unique quotient $q(x)$ and a unique remainder R such that

$$f(x) = (x - a)q(x) + R \quad \ldots(i)$$



Substituting $x = a$ in equation (i), we get

$$f(x) = (a - a)q(a) + R$$
$$f(a) = R$$

Hence remainder $= f(a)$

Example 3: Find the remainder without performing division when $f(x) = x^4 + x^3 + x^2 + 1$ is divided by $x + 1$

Solution: Here $f(x) = x^4 + x^3 + x^2 + 1$ and $x - a = x + 1 \Rightarrow a = -1$

Remainder $= f(-1)$ (By remainder theorem)

$= (-1)^4 + (-1)^3 + (-1)^2 + 1$

$= 1 + (-1) + 1 + 1 = 2$

Example 4: Find the value of k if the polynomial $x^3 + kx^2 - 7x + 6$ has a remainder -4, when divided by $x + 2$

Solution: Let $f(x) = x^3 + kx^2 - 7x + 6$ and $x - a = x + 2$, we have, $a = -2$

Remainder $= f(-2)$ (By remainder theorem)

$= (-2)^3 + k(-2)^2 - 7(-2) + 6$

$= -8 + 4k + 14 + 6$

$= 4k + 12$

Given that remainder $= -4$

$4k + 12 = 4$

$\Rightarrow \quad 4k = -16$

$\Rightarrow \quad k = -4$

Factor Theorem:

Statement: The polynomial $x - a$ is a factor of the polynomial $f(x)$ iff $f(a) = 0$.

Or

In other words $x - a$ is a factor of $f(x)$ if and only if $x = a$ is the root of the polynomial equation $f(x) = 0$.

Proof: Suppose $q(x)$ is the quotient and R is the remainder when the polynomial $f(x)$ is divided by $x - a$, till no x-term exists in the remainder, then:

$$f(x) = (x - a)q(x) + R$$

Suppose $f(a) = 0 \Rightarrow R = 0$

$$f(x) = (x - a)q(x)$$

$(x - a)$ is a factor of $f(x)$

Conversely, if $(x - a)$ is a factor of $f(x)$, then $f(x) = (x - a)q(x)$ for some polynomial $q(x)$

$$f(a) = 0$$

which proves the theorem.

Example 5: Show that $x - 2$ is a factor of $f(x) = x^3 - 7x + 6$ without factorizing.

Solution: Here, $f(x) = x^3 - 7x + 6$ and $a = 2$

$f(2) = 2^3 - 7(2) + 6$

$= 8 - 14 + 6 = 0$ (Remainder)

So, by factor theorem $x - 2$ is a factor of $f(x)$.

Note:
To determine if a given linear polynomial $x - a$ is a factor of $f(x)$, we need to check whether $f(a) = 0$.

Example 6: If $x + 1$ and $x - 2$ are factors of $x^3 + px^2 + qx + 2$. Find the values of p and q.

Solution: Let $(x) = x^3 + px^2 + qx + 2$, $\quad x + 1 = 0 \Rightarrow x = -1$ and $x - 2 = 0 \Rightarrow x = 2$

Since, $x + 1$ and $x - 2$ are factors of $f(x)$, therefore remainder $= 0$

So, $f(-1) = 0$

$\Rightarrow (-1)^3 + p(-1)^2 + q(-1) + 2 = 0$

$-1 + p - q + 2 = 0$

$p - q + 1 = 0 \quad \ldots(i)$

and $f(2) = 0$

$\Rightarrow (2)^3 + p(2)^2 + q(2) + 2 = 0$

$8 + 4p + 2q + 2 = 0$

$2p + q + 5 = 0 \quad \ldots(ii)$

By adding (i) and (ii), we have

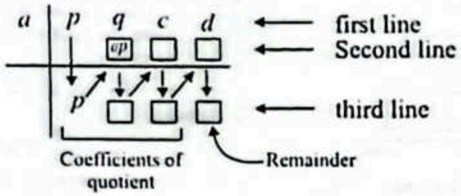
$3p + 6 = 0 \quad \Rightarrow 3p = -6 \quad \Rightarrow \boxed{p = -2}$ Put this in eq (i), we have

$q = p + 1 \quad \Rightarrow q = -2 + 1 \quad \Rightarrow \boxed{q = -1}$

Synthetic Division:

There is a nice shortcut method for long division of a polynomial $f(x)$ by a polynomial of the form $x - a$. This process of division is called Synthetic Division.

To divide the polynomial $px^3 + qx^2 + cx + d$ by $x - a$



Outline of the Method:

(i) Write down the coefficients of the dividend $f(x)$ from left to right in decreasing order of powers of x. Insert 0 for any missing term.

(ii) To the left of the first line, write a of the divisor $(x - a)$.

(iii) Use the following patterns to write the second and third lines:

Vertical pattern (↓) Add terms

Diagonal pattern (↗) Multiply by a.

Example 7: If $(x - 2)$ and $(x + 2)$ are factors of $x^4 - 13x^2 + 36$. Using synthetic division, find the other two factors.

Solution:

Let $f(x) = x^4 - 13x^2 + 36$

$= x^4 + 0x^3 - 13x^2 - 0x + 36$

Here $x - a = x - 2 \Rightarrow a = 2$ and $x - a = x + 2 = x - (-2) \Rightarrow a = -2$

By synthetic Division:

2	1	0	−13	0	36
		2	4	−18	−36
−2	1	2	−9	−18	0 ← Remainder
		−2	0	18	
	1	0	−9	0 ← Remainder	

$\therefore$ Quotient $= x^2 + 0x - 9 = x^2 - 9 = (x + 3)(x - 3)$

Therefore, other two factors are $(x + 3)$ and $(x - 3)$.

Exercise 9.1

1. Find remainder and quotient by simplifying the following:

(i) $(3x^2 - x + 2) \div (x - 1)$

Solution:

$$\begin{array}{r} 3x+2 \\ x-1\,)\overline{\,3x^2 - x + 2\,} \\ \underline{\pm 3x^2 \mp 3x} \\ 2x+2 \\ \underline{\pm 2x \mp 2} \\ 4 \end{array}$$

So, quotient $= 3x + 2$ and remainder $= 4$

(ii) $(x^3 + 12x^2 - 3x + 4) \div (x - 2)$

Solution:

$$\begin{array}{r} x^2+14x+25 \\ x-2\,)\overline{\,x^3+12x^2-3x+4\,} \\ \underline{\pm x^3 \mp 2x^2} \\ 14x^2 - 3x + 4 \\ \underline{\pm 14x^2 \mp 28x} \\ 25x + 4 \\ \underline{\pm 25x \mp 50} \\ 54 \end{array}$$

So, quotient $= x^2 + 14x + 25$ and remainder $= 54$

(iii) $(x^4 - 5x^3 - 8x^2 + 13x + 12) \div (x - 6)$

Solution:

$$\begin{array}{r} x^3+x^2-2x+1 \\ x-6\,)\overline{\,x^4-5x^3-8x^2+13x+12\,} \\ \underline{\pm x^4 \mp 6x^3} \\ x^3-8x^2+13x+12 \\ \underline{\pm x^3 \mp 6x^2} \\ -2x^2+13x+12 \\ \underline{\mp 2x^2 \pm 12x} \\ x+12 \\ \underline{\pm x \mp 6} \\ 18 \end{array}$$

So, quotient $= x^3 + x^2 - 2x + 1$ and remainder $= 18$

(iv) $(5x^4 - 3x^3 + 2x^2 - 1) \div (x^2 + 4)$

Solution:

$$\begin{array}{r} 5x^2-3x-18 \\ x^2+4\,)\overline{\,5x^4-3x^3+2x^2+0x-1\,} \\ \underline{\pm 5x^4 \qquad \pm 20x^2} \\ -3x^3-18x^2+0x-1 \\ \underline{\mp 3x^3 \qquad \mp 12x} \\ -18x^2+12x-1 \\ \underline{\pm 18x^2 \qquad \mp 72} \\ 12x+71 \end{array}$$

So, quotient $= 5x^2 - 3x - 18$ and remainder $= 12x + 71$

(v) $(3x^4 - 5x^3 + 4x - 6) \div (x^2 - 3x + 5)$

Solution:

$$\begin{array}{r} 3x^2+4x-3 \\ x^2-3x+5\,)\overline{\,3x^4-5x^3+0x^2+4x-6\,} \\ \underline{\pm 3x^4 \mp 9x^3 \pm 15x^2} \\ 4x^3-15x^2+4x-6 \\ \underline{\pm 4x^3 \mp 12x^2 \pm 20x} \\ -3x^2-16x-6 \\ \underline{\mp 3x^2 \pm 9x \mp 15} \\ -25x+9 \end{array}$$

So, quotient $= 3x^2 + 4x - 3$ and remainder $= -25x + 9$

2. Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial.

(i) $x^2 + 5x + 6, x - 2$

Solution:

Let $f(x) = x^2 + 5x + 6$ and $x - 2 = 0 \Rightarrow x = 0$

By remainder theorem, we have

Remainder $= f(2)$
$= (2)^2 + 5(2) + 6$
$= 4 + 10 + 6$
$= 20$

(ii) $x^3 + 5x^2 + 6, x + 1$

Solution:

Let $f(x) = x^3 + 5x^2 + 6$ and $x + 1 = 0 \Rightarrow x = -1$

By remainder theorem, we have

Remainder $= f(-1)$
$= (-1)^3 + 5(-1)^2 + 6$
$= -1 + 5 + 6$
$= 10$

(iii) $x^4 + x^3 + x^2 + x + 1, x - 1$

Solution:

Let $f(x) = x^4 + x^3 + x^2 + x + 1$ and $x - 1 = 0 \Rightarrow x = 1$

By remainder theorem, we have

Remainder $= f(1)$
$= (1)^4 + (1)^3 + (1)^2 + 1 + 1$
$= 1 + 1 + 1 + 1 + 1$
$= 5$

(iv) $x^4 + x^2 + 1, x + 3$

Solution:

Let $f(x) = x^4 + x^2 + 1$ and $x + 3 = 0 \Rightarrow x = -3$

By remainder theorem, we have

Remainder $= f(-3)$
$= (-3)^4 + (-3)^2 + 1$
$= 81 + 9 + 1$
$= 91$

(v) $x^4 + x^3 + 2, x + 2$

Solution:

Let $f(x) = x^4 + x^3 + 2$ and $x + 2 = 0 \Rightarrow x = -2$

By remainder theorem, we have

Remainder $= f(-2)$
$= (-2)^4 + (-2)^3 + 2$
$= 16 - 8 + 2$
$= 10$

3. Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.

(i) $x + 1, x^2 - 1$

Solution:

Let $f(x) = x^2 - 1$ and $x + 1 = 0 \Rightarrow x = -1$

Remainder $= f(-1)$
$= (-1)^2 - 1$
$= 1 - 1 = 0$

Hence by factor theorem $(x + 1)$ is a factor of $x^2 - 1$.

(ii) $x - 2, x^2 - 5x + 6$

Solution:

Let $f(x) = x^2 - 5x + 6$ and $x - 2 = 0 \Rightarrow x = 2$

Remainder $= f(2)$
$= (2)^2 - 5(2) + 6$
$= 4 - 10 + 6 = 0$

Hence by factor theorem $(x - 2)$ is a factor of $x^2 - 5x + 6$.

(iii) $x + 1, x^3 + x^2 + x - 3$

Solution:

Let $f(x) = x^3 + x^2 + x - 3$ and $x + 1 = 0 \Rightarrow x = -1$

Remainder $= f(-1)$
$= (-1)^3 + (-1)^2 + (-1) - 3$
$= -1 + 1 - 1 - 3 = -4 \neq 0$

Hence by factor theorem $(x + 1)$ is not a factor of $x^3 + x^2 + x - 3$.

(iv) $x - 2, x^3 + x^2 - 7x + 2$

Solution:

Let $f(x) = x^3 + x^2 - 7x + 2$ and $x - 2 = 0 \Rightarrow x = 2$

Remainder $= f(2)$
$= (2)^3 + (2)^2 - 7(2) + 2$
$= 8 + 4 - 14 + 2$
$= 14 - 14 = 0$

Hence by factor theorem $(x - 2)$ is a factor of $x^3 + x^2 - 7x + 2$.

(v) $x - 3, x^4 - 3x^3 + x^2 - x + 1$

Solution:

Let $f(x) = x^4 - 3x^3 + x^2 - x + 1$ and $x - 3 = 0 \Rightarrow x = 3$

Remainder $= f(3)$
$= (3)^4 - 3(3)^3 + (3)^2 - 3 + 1$
$= 81 - 81 + 9 - 3 + 1$
$= 7 \neq 0$

Hence by factor theorem $(x - 3)$ is not a factor of $x^4 - 3x^3 + x^2 - x + 1$.

4. Use synthetic division to show that x is the zero of the polynomial and use the result to factorize the polynomial completely.

(i) $x^3 - 7x + 6, x = 2$

Solution:

Let $P(x) = x^3 + 0x^2 - 7x + 6$; $x = 2$

By using synthetic division

2	1	0	−7	6
		2	4	−6
	1	2	−3	0 = remainder

As remainder = 0

So $x = 2$ is a zero of $P(x) = 0$

or

$x - 2$ is a factor of $P(x)$

Quotient $= x^2 + 2x - 3$
$= x^2 + 3x - x - 3$
$= x(x + 3) - 1(x + 3)$
$= (x + 3)(x - 1)$

Hence, $x^3 - 7x + 6 = (x - 2)(x + 3)(x - 1)$

(ii) $x^3 - 28x - 48, x = -4$

Solution:

Let $P(x) = x^3 + 0x^2 - 28x - 48 = 0$; $x = -4$

By using synthetic division

−4	1	0	−28	−48
		−4	16	48
	1	−4	−12	0 = remainder

As remainder = 0

So $x = -4$ is a zero of $P(x) = 0$

or

$x + 4$ is a factor of $P(x)$

Quotient $= x^2 - 4x - 12$
$= x^2 - 6x + 2x - 12$
$= x(x - 6) + 2(x - 6)$
$= (x - 6)(x + 2)$

Hence, $x^3 - 28x + 48 = (x + 4)(x + 2)(x - 6)$

(iii) $2x^4 + 7x^3 - 4x^2 - 27x - 18, x = 2, x = -3$

Solution:

Let $P(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$; $x = 2$ and $x = -3$

By using synthetic division

2	2	7	-4	-27	-18
		4	22	36	+18
-3	2	11	18	9	0 = remainder
		-6	-15	-9	
	2	5	3	0 = remainder	

As remainder = 0

So $x = 2, x = -3$ are zeros of $P(x) = 0$

or

$x - 2, x + 3$ are factors of $P(x)$

Quotient $= 2x^2 + 5x + 3$

$= 2x^2 + 2x + 3x + 3$

$= 2x(x+1) + 3(x+1)$

$= (x+1)(2x+3)$

Hence, $2x^4 + 7x^3 - 4x^2 - 27x - 18 = (x+1)(2x+3)(x+3)(x-2)$

5. Use synthetic division to find the quotient and the remainder when the polynomial $x^4 - 10x^2 - 2x + 4$ is divided by $x + 3$.

Solution:

Let $P(x) = x^4 - 10x^2 - 2x + 4$; $x + 3 = 0$

$= x^4 + 0x^3 - 10x^2 - 2x + 4$; $x = -3$

By using synthetic division

-3	1	0	-10	-2	4
		-3	9	3	-3
	1	-3	-1	1	1 = remainder

So, Quotient $= x^3 - 3x^2 - x + 1$

Remainder = 1

6. If $x + 1$ and $x - 2$ are factors of $x^3 - px^2 + qx + 2$. Using synthetic division, find the values of p and q.

Solution:

Let $f(x) = x^3 - px^2 + qx + 2$, $x + 1 = 0 \Rightarrow x = -1$

$x - 2 = 0 \Rightarrow x = 2$

By using synthetic division

-1	1	-p	q	2
	↓	-1	p + 1	-p - q - 1
2	1	-p - 1	p + q + 1	-p - q + 1 = remainder
		2	-2p + 2	
	1	-p + 1	-p + q + 3 = remainder	

As $x + 1$ and $x - 2$ are the factors of $f(x)$, so remainder = 0

$\Rightarrow \quad -p - q + 1 = 0$ ...(1)

and $\quad -p + q + 3 = 0$ ...(2)

Equation (1) + Equation (2)

$-2p + 4 = 0 \Rightarrow 2p = 4 \Rightarrow \boxed{p = 2}$

Put $p = 2$ in equation (2), we have

$-2 + q + 3 = 0 \Rightarrow q + 1 = 0 \Rightarrow \boxed{q = -1}$

Hence $p = 2$ and $q = -1$

7. When the polynomial $4x^4 + 2x^3 + kx^2 + 13$ is divided by $x + 1$, the remainder is 16. Find the value of k.

Solution:

Let $f(x) = 4x^4 + 2x^3 + kx^2 + 13$ and $x + 1 = 0 \Rightarrow x = -1$

By remainder theorem, we have

Remainder $= f(-1)$

$16 = 4(-1)^4 + 2(-1)^3 + k(-1)^2 + 13$

$16 = 4 - 2 + k + 13$

$16 = 15 + k$

$k = 16 - 15$

$\boxed{k = 1}$

8. When the polynomial $x^3 + x^2 + x + k$ is divided by $x + 1$, the reminder is 7. Find the value of k.

Solution:

Let $f(x) = x^3 + x^2 + x + k$ and $x + 1 = 0 \Rightarrow x = -1$

By remainder theorem, we have

Remainder $= f(-1)$

$7 = (-1)^3 + (-1)^2 - 1 + k$

$7 = -1 + 1 - 1 + k$

$7 = -1 + k$

$k = 7 + 1$

$\boxed{k = 8}$

9. Use factor theorem to find the values of p and q if $x + 1$ and $x - 2$ are the factors of the polynomial $x^3 + px^2 + qx + 3$.

Solution:

Let $f(x) = x^3 + px^2 + qx + 3$; $x + 1 = 0 \Rightarrow x = -1$

$x - 2 = 0 \Rightarrow x = 2$

By using synthetic division

-1	1	p	q	3
		-1	-p + 1	p - q - 1
2	1	p - 1	-p + q + 1	p - q + 2 = remainder
		2	2p + 2	
	1	p + 1	p + q + 3 = remainder	

As $x + 1$ and $x - 2$ are factors of $f(x)$, so remainder = 0

$\Rightarrow \quad p - q + 2 = 0$ ...(1)

and $\quad p + q + 3 = 0$ ...(2)

Equation (1) + Equation (2)

$2p + 5 = 0 \Rightarrow p = \frac{-5}{2}$

Put $p = \frac{-5}{2}$ in eq. (2), we have

$\frac{-5}{2} + q + 3 = 0$

$q = \frac{5}{2} - 3 \Rightarrow q = \frac{-1}{2}$

Hence $p = \frac{-5}{2}$ and $q = \frac{-1}{2}$

10. Use factor theorem to find the values of a and b if -2 and 2 are the roots of the polynomial $2x^3 + 4x^2 + ax + b$

Solution:

Let $f(x) = 2x^3 + 4x^2 + ax + b$; $x = -2$ and $x = 2$

By using synthetic division

-2	2	4	a	b
		-4	0	-2a
2	2	0	a	-2a + b = remainder
		4	8	
	2	4	a + 8 = remainder	

As $x = -2$ and $x = 2$ are the roots of $f(x) = 0$, so remainder = 0

$\Rightarrow \quad -2a + b = 0$ ...(1)

and $\quad a + 8 = 0$

$a = -8$

Put $a = -8$ in equation (1), we have

$-2(-8) + b = 0$

$16 + b = 0$

$b = -16$

Hence $a = -8$ and $b = -16$

Real Life Applications of Remainder and Factor Theorems:

In this article, we shall demonstrate how remainder and factor theorems are applied in different areas such as **polynomial regression** (used in statistical modeling), **signal processing** (used for filtering and error detection) and **coding theory** (used in data encryption and error correction in communication systems). These applications highlight the significance of polynomial analysis beyond theoretical mathematics.

Regression Analysis: It is a statistical method used to model the relationship between a dependent variable and one or more independent variables.

Polynomial Regression: It is a type of regression analysis where the relationship between the independent and dependent variables is modeled as an n^{th}-degree polynomial. It is used when the data shows a curved (non-linear) relationship, but we still want to fit a smooth, continuous function. Factor theorem is useful for reducing polynomial regression degree and remainder theorem helps in evaluating polynomials at given points.

Example 8: Consider a data set of monthly sales figures. A polynomial regression model $P(x) = x^3 + x^2 + 2x + 1$ is fitted to this data. If the observed sales in the 3^{rd} month are 40 units, find the percentage error.

Solution:

Polynomial regression model: $P(x) = x^3 + x^2 + 2x + 1$

Observed Sale = 240 units

Month Number $= x = 3$

Predicted Sale $= P(3) = 3^3 + 3^2 + 2(3) + 1 = 27 + 9 + 6 + 1 = 43$

As we know: Error = Observed − Predicted

$= 40 - P(3)$

$= 40 - 43 = -3$

So, Percentage Error $= \left|\frac{-3}{40}\right| \times 100\% = 7.5\%$

Example 9: Suppose a polynomial regression model $P(x) = 3x^3 - 4x^2 + 2x - 5$. If a data point at $x = -1$ is missing. What should be its predicted value?

Solution:

Polynomial regression model: $P(x) = 3x^3 - 4x^2 + 2x - 5$

Predicted Value $= P(-1)$

$= 3(-1)^3 - 4(-1)^2 + 2(-1) - 5$

$= -3 - 4 - 2 - 5$

$= -14$

So, the predicted value of given polynomial regression model at $x = -1$ is -14.

Digital Signal Processing (DSP): It is the used in computers or digital devices to analyze, change or improve signals like sound, images or sensor data. In the context of DSP, we often deal with systems represented by transfer functions in the z-domain, denoted as $H(z)$. These transfer functions are rational functions, meaning they are ratios of two polynomials in z i.e., $H(z) = \frac{B(z)}{A(z)}$, where $B(z)$ represents the numerator polynomial (related to the system's zeros) and $A(z)$ represents the denominator polynomial (related to the system's poles).

In signal processing, finding the roots of the numerator polynomial $B(z)$ provides the zeros of the system. If $B(z_0) = 0$, then $(z - z_0)$ is a factor of $B(z)$. If $|z_0| = 1$, this corresponds to a frequency that the system blocks. Similarly, finding the roots of the denominator polynomial $A(z)$ provides the poles of the system. If $A(p_0) = 0$, then $(z - p_0)$ is a factor of $A(z)$. The locations of these poles in the complex z-plane are crucial for determining the stability of the system. For a stable system, all poles must lie inside the unit circle ($|p_0| < 1$).

Example 10: A signal processing system has a transfer function with a denominator $A(z) = z^2 - 0.25$. Use factor theorem to find the poles of the system and determine if the system is stable.

Solution

Denominator Polynomial: $A(z) = z^2 - 0.25$

The poles occur when $A(z) = 0$

$$z^2 - 0.25 = 0$$
$$z^2 - (0.5)^2 = 0$$
$$(z - 0.5)(z + 0.5) = 0$$
$$z - 0.5 = 0 \text{ or } z + 0.5 = 0$$
$$z = 0.5$$
$$\text{or } z = -0.5$$

Thus, the system has poles at $z = 0.5$ and $z = -0.5$.

Here, $|0.5| = 0.5 < 1$ and $|-0.5| = 0.5 < 1$. Since both poles are inside the unit circle, the system is stable.

Exercise 9.2

1. Consider a data set at monthly sales figures. A polynomial regression model $P(x) = x^3 + 2x^2 + x - 3$ is fitted to this data. If the observed sales in the 5th month are 240 units, find the percentage error.

Solution:

Polynomial regression model:

$$P(x) = x^3 + 2x^2 + x - 3$$

Observed sale = 240 units

Month number = $x = 5$

Predicted sale = $P(5)$
$= (5)^3 + 2(5)^2 + 5 - 3$
$= 125 + 50 + 5 - 3 = 177$

As we know

Error = Observed − Predicted
$= 240 - P(5)$
$= 240 - 177 = 63$

So, percentage error $= \left|\dfrac{63}{240}\right| \times 100 = 26.25\%$

2. A retailer company has developed a polynomial regression model to predict weekly product demand: $D(w) = w^3 - 2w^2 + 5w - 4$, where $D(w)$ represents predicted demand(in units) and w is the week number. Use the remainder theorem to predict demand for 3rd week. If the observe demand is 22 units, calculate the prediction error.

Solution:

Polynomial regression model:

$$D(w) = w^3 - 2w^2 + 5w - 4$$

where $D(w)$ = Predicted demand (in units)

and w = Week number

Observed demand = 22 units

Week number: $w = 3$

Predicted demand = $D(3)$
$= (3)^3 - 2(3)^2 + 5(3) - 4$
$= 27 - 18 + 15 - 4$
$= 42 - 22 = 20$ units

As we know

Prediction error = Observed − Predicted
$= 22 - 20 = 2$ units

3. A digital signal processing system has a transfer function with a numerator $B(z) = z^2 - z - 2$. Use the factor theorem to find the zeros of the system.

Solution:

Numerator polynomial: $B(z) = z^2 - z - 2$

$$B(z) = z^2 - z - z$$
$$= z^2 - 2z + z - 2$$
$$= z(z - 2) + 1(z - 2)$$
$$= (z - 2)(z + 1)$$

Put $B(z) = 0$, we have

$$(z-2)(z+1) = 0$$

$\Rightarrow \quad z - 2 = 0 \quad$ and $\quad z + 1 = 0$

$z = 2 \quad$ and $\quad z = -1$

Hence $z = 2$ and $z = -1$ are the zeros of the system.

4. A signal process system has a transfer function $H(z) = \dfrac{z^2 + 3z + 2}{z^2 - 0.2z + 0.9}$. Find the zero(s) of the transfer function by using factor theorem.

Solution:

Transfer function: $H(z) = \dfrac{z^2 + 3z + 2}{z^2 - 0.2z + 0.9}$

where $B(z) = z^2 + 3z + 2$, $A(z) = z^2 - 0.2z + 0.9$

Zeros of the transfer function are the values of z for which $B(z) = 0$

$$B(z) = z^2 + 3z + 2$$
$$= z^2 + 2z + z + 2$$
$$= z(z + 2) + 1(z + 2)$$
$$= (z + 2)(z + 1)$$

Put $B(z) = 0$, we have

$$(z+2)(z+1) = 0$$

$z + 2 = 0 \quad$ and $\quad z + 1 = 0$

$\Rightarrow \quad z = -2 \quad$ and $\quad z = -1$

Hence $z = -2$ and $z = -1$ are the zeros of the given transfer function.

5. A signal process system has a transfer function $H(z) = \dfrac{z^2 - 0.5z - 0.5}{z^3 + 1}$. Find the zero(s) of the transfer function by using factor theorem.

Solution:

Transfer function: $H(z) = \dfrac{z^2 - 0.5z - 0.5}{z^3 + 1}$

where $B(z) = z^2 - 0.5z - 0.5$, $A(z) = z^3 + 1$

Zeros of the transfer function are the values of z for which $B(z) = 0$

$$B(z) = z^2 - 0.5z - 0.5$$
$$= z^2 - \frac{1}{2}z - \frac{1}{2}$$
$$= \frac{2z^2 - z - 1}{2}$$
$$= \frac{1}{2}(2z^2 - 2z + z - 1)$$
$$= \frac{1}{2}(2z(z-1) + 1(z-1))$$
$$= \frac{1}{2}(z-1)(2z+1)$$

Put $B(z) = 0$, we have

$$\frac{1}{2}(z-1)(2z+1) = 0$$

$\Rightarrow \quad z - 1 = 0 \quad$ and $\quad 2z + 1 = 0$

$z = 1 \quad$ and $\quad z = \dfrac{-1}{2} = -0.5$

Hence $z = 1$ and $z = -0.5$ are the zeros of the given transfer function.

6. A signal processing system has a transfer function with a denominator $A(z) = z^2 - 0.3z - 0.4$. Use factor theorem to find the poles of the system and determine if the system is stable.

Solution:

Denominator polynomial: $A(z) = z^2 - 0.3z - 0.4$

The poles occur when $A(z) = 0$

$$\Rightarrow \quad z^2 - 0.3z - 0.4 = 0$$
$$z^2 - \frac{3}{10}z - \frac{4}{10} = 0$$

Multiply by '10'

$$10z^2 - 3z - 4 = 0$$
$$10z^2 - 8z + 5z - 4 = 0$$
$$2z(5z - 4) + 1(5z - 4) = 0$$
$$(5z - 4)(2z + 1) = 0$$

$\Rightarrow \quad 5z - 4 = 0 \quad$ and $\quad 2z + 1 = 0$

$z = \dfrac{4}{5} \quad$ and $\quad z = \dfrac{-1}{2}$

$z = 0.8 \quad$ and $\quad z = -0.5$

Thus, the system has poles at $z = 0.8$ and $z = -0.5$

Here $|0.8| = 0.8 < 1$ and $|-0.5| = 0.5 < 1$

Since both the poles are inside the unit circle, so the system is stable.

7. The denominator of signal processing system's transfer function equal to $A(z) = z^2 + 1.2z + 0.35$ Use factor theorem to determine the location of the corresponding poles and assess the stability of the system.

Solution:

Denominator polynomial: $A(z) = z^2 + 1.2z + 0.35$

The poles occur when $A(z) = 0$

$$\Rightarrow \quad z^2 + 1.2z + 0.35 = 0$$
$$z^2 + \frac{12}{10}z + \frac{35}{100} = 0$$
$$z^2 + \frac{6}{5}z + \frac{7}{20} = 0$$

Multiply by '20'

$$20z^2 + 24z + 7 = 0$$
$$20z^2 + 14z + 10z + 7 = 0$$
$$2z(10z + 7) + 1(10z + 7) = 0$$
$$(10z + 7)(2z + 1) = 0$$

$\Rightarrow \quad 10z + 7 = 0 \quad$ and $\quad 2z + 1 = 0$

$z = \dfrac{-7}{10} \quad$ and $\quad z = \dfrac{-1}{2}$

$z = -0.7 \quad$ and $\quad z = -0.5$

Thus, the system has poles at $z = -0.7$ and $z = -0.5$

Here $|-0.7| = 0.7 < 1$ and $|-0.5| = 0.5 < 1$

Since both the poles are inside the unit circle, so the system is stable.

Multiple Choice Questions (MCQs)

Exercise 9.1

1. Leading coefficient of the equation $x^3 + 2x^2 + 3x + 4 = 0$ is -----
   (A) 1 (B) 2 (C) 3 (D) 4
2. What is the degree of $x^3 - 2x^4y^2 + 7y^5$ -----
   (A) 3 (B) 4 (C) 5 (D) 6
3. If a polynomial $P(x) = x^3 + 4x^2 - 2x + 5$ is divided by $x - 1$, then the remainder is:
   (A) 4 (B) −2 (C) 5 (D) 8
4. If $x - a$ is a factor of a polynomial $f(x)$ then $f(a)$ = -----
   (A) 1 (B) 0 (C) 2 (D) −1
5. If '$x - 2$' is a factor of polynomial $x^3 + 2x^2 + kx + 4$ then k equals -----
   (A) 10 (B) −10 (C) 2 (D) 4
6. Synthetic division is a process of -----
   (A) addition (B) multiplication (C) subtraction (D) division
7. $x^3 - 3x^2 + 2x - 6$ has factor:
   (A) $x - 4$ (B) $x - 3$ (C) $x + 3$ (D) $x + 2$

Exercise 9.2

8. A statistical method used to model the relationship between a dependent variable and one or more independent variables is called -----
   (A) regression analysis (B) correlation analysis (C) analysis (D) real analysis
9. Polynomial regression is a type of regression analysis where the relationship between the independent and dependent variables is modeled as an ----- degree polynomial.
   (A) (n −1)th (B) nth (C) (n +1)th (D) (n − 2)th
10. Suppose a polynomial regression model $P(x) = 3x^3 - 4x^2 + 2x - 5$. If a data point at $x = -1$ is missing. What should be its predicted value?
   (A) −12 (B) 13 (C) 14 (D) −14

ANSWER KEY

1.	A	2.	D	3.	D	4.	B	5.	B	6.	D	7.	B	8.	A	9.	B	10.	D

Unit 10 Trigonometric Identities

Introduction

In this unit, we shall first establish the fundamental law of trigonometry before discussing the Trigonometric Identities. For this we should know the formula to find the distance between two points in a plane.

Distance Formula: (Recall)

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. If "d" denotes the distance between them,

then $$d = |\overline{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

or $$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1: Find distance between the following points:
(i) $A(3, 8), B(5, 6)$ (ii) $P(\cos x, \cos y), Q(\sin x, \sin y)$

Solution:

(i) Distance $= |\overline{AB}| = \sqrt{(3-5)^2 + (8-6)^2} = \sqrt{4+4} = 2\sqrt{2}$

(ii) Distance $= |\overline{PQ}| = \sqrt{(\cos x - \sin x)^2 + (\cos y - \sin y)^2}$

$= \sqrt{\cos^2 x + \sin^2 x - 2\cos x \sin x + \cos^2 y + \sin^2 y - 2\cos y \sin y}$

$= \sqrt{1 - 2\cos x \sin x + 1 - 2\cos y \sin y}$ $\quad \because \sin^2\theta + \cos^2\theta = 1$

$= \sqrt{2 - 2\cos x \sin x - 2\cos y \sin y}$

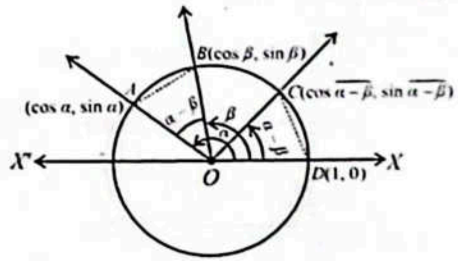
$= \sqrt{2 - 2(\cos x \sin x + \cos y \sin y)}$

Fundamental Law of Trigonometry:

Let α and β be any two angles (real numbers), then

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

which is called the **Fundamental Law of Trigonometry**.



Proof: For our convenience, let us assume that $\alpha > \beta > 0$.
Consider a unit circle with centre at origin O.
Let terminal sides of angles α and β cut the unit circle at A and B respectively, such that

$$m\angle AOB = \alpha - \beta$$

Take a point C on the unit circle such that

$$m\angle DOC = m\angle AOB = \alpha - \beta$$

Join A, B and C, D.
Now angles α, β and α − β are in standard position.
∴ The coordinates of A are $(\cos\alpha, \sin\alpha)$.
The coordinates of B are $(\cos\beta, \sin\beta)$
The coordinates of C are $(\cos\overline{\alpha-\beta}, \sin\overline{\alpha-\beta})$
and the coordinates of D are $(1, 0)$.