Exercise 7.4

14. ^nC_r is equal to - - - - -

(A) $\dfrac{n!}{(n-r)!}$ (B) $\dfrac{n!}{r!\,(n-r)!}$ (C) $\dfrac{n!}{n!\,(n-r)!}$ (D) $\dfrac{(n-r)!}{(n-1)!}$

15. $r! \cdot {}^nC_r$ = - - - - -

(A) ^nP_r (B) $^{n-1}P_r$ (C) $^{n+1}P_r$ (D) $^{n-1}P_{r-1}$

16. The complementary combination $^nC_r = {}^nC_{n-r}$ is useful when:

(A) $n = r$ (B) $n < r$ (C) $r < \dfrac{n}{2}$ (D) $r > \dfrac{n}{2}$

17. $^nC_{r-1} + {}^nC_{r-2}$ = - - - - -

(A) $^nC_{r-1}$ (B) $^{n+1}C_{r-1}$ (C) $^{n+1}C_{r-2}$ (D) $^nC_{r-2}$

ANSWER KEY

1.	A	2.	B	3.	D	4.	D	5.	A	6.	A	7.	B	8.	C	9.	B	10.	D
11.	B	12.	C	13.	B	14.	B	15.	A	16.	D	17.	B						

Unit 8 Mathematical Inductions and Binomial Theorem

Introduction:

Francesco Mourolico (1494-1575) devised the method of induction and applied this device first to prove that the sum of the first n odd positive integers equals n^2. He presented many properties of integers and proved some of these properties using the method of mathematical induction. In theoretical computer science, it bears the pivotal role of developing the appropriate cognitive skills necessary for the effective design and implementation of algorithms, assessing for both their correctness and complexity.

Counter Example: We are aware of the fact that even one exception or case to a mathematical formula is enough to prove it to be false. Such a case or exception which fails the mathematical formula or statement is called a counter example.

- The validity of a formula or statement depending on a variable belonging to a certain set is established if it is true for each element of the set under consideration.

For example,

We consider the statement $S(n) = n^2 - n + 41$ is a prime number for every natural number n.

The values of the expression $n^2 - n + 41$ for some first natural numbers are given in the table as shown below:

n	1	2	3	4	5	6	7	8	9	10	11
$S(n)$	41	43	47	53	61	71	83	97	113	131	151

From the table, it appears that the statement $S(n)$ has enough chance of being true. If we go on trying for the next natural numbers, we find $n = 41$ as a counter example which fails the claim of the above statement. So we conclude that to derive a general formula without proof from some special cases is not a wise step. This example was discovered by Euler (1707-1783).

Another Example,

In this example we will try to formulate the result. Our task is to find the sum of the first n odd natural numbers. We write first few sums to see the pattern of sums.

n (The number of terms)	Sum
1	$1 = 1^2$
2	$1 + 3 = 4 = 2^2$
3	$1 + 3 + 5 = 9 = 3^2$
4	$1 + 3 + 5 + 7 = 16 = 4^2$
5	$1 + 3 + 5 + 7 + 9 = 25 = 5^2$
6	$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$

The sequence of sums is $(1)^2, (2)^2, (3)^2, (4)^2, \ldots$

We see that each sum is the square of the number of terms in the sum. So the following statement seems to be true.

For each natural number n,

$$1 + 3 + 5 + \cdots + (2n-1) = n^2 \ldots \text{(i)} \qquad \because n^{th} \text{ term} = a_1 + (n-1)d = 1 + 2(n-1)$$

But it is not possible to verify the statement (i) for each positive integer n, because it involves infinitely many calculations which never end.

The method of mathematical induction is used to avoid such situations. Usually it is used to prove the statements or formulae relating to the set $\{1, 2, 3, \ldots\}$ but in some cases, it is also used to prove the statements relating to the set $\{0, 1, 2, 3, \ldots\}$.

Hypothesis: A hypothesis is an educated guess or proposed explanation for a statement based on limited evidence.

- It serves as a starting point for further investigation and can be tested through experiments and observations. In scientific research, a hypothesis is usually framed as a statement that can be tested and either supported or rejected by data.

Induction of Hypothesis: It refers to the process of formulating a general statement or hypothesis based on specific examples or patterns observed in particular cases.

- This technique is often employed in mathematical reasoning to propose conjectures that can later be proven rigorously using deductive methods.

Principle of Mathematical Induction:

The principle of mathematical induction is stated as follows:

If a proposition or statement $S(n)$ for each positive integer n is such that

1. **Base Case:** $S(1)$ is true i.e., $S(n)$ is true for $n = 1$.
2. **Induction of Hypothesis:** $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k.
3. **Conclusion:** $S(n)$ is true for all positive integers.

Procedure for Induction of Hypothesis:

- Substituting $n = 1$, show that the statement is true for $n = 1$.
- Assuming that the statement is true for any positive integer k, then show that it is true for the next higher integer.
- For the second condition, one of the following two methods can be used:
- $S(k+1)$ is proved using $S(k)$.
- $S(k+1)$ is established by performing algebraic operations on $S(k)$.

Example 1: Use mathematical induction to prove that $3+6+9+\ldots+3n = \frac{3n(n+1)}{2}$ for every positive integer n

Solution: Let $S(n)$ be the given statement, that is,

$$S(n):\quad 3+6+9+\ldots+3n = \frac{3n(n+1)}{2}$$

Base Case: When $n = 1$

$$S(1):\quad 3 = \frac{3(1)(1+1)}{2} = 3 \text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis: Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$S(k): 3+6+9+\ldots+3k = \frac{3k(k+1)}{2} \quad \ldots (A)$$

The statement for $n = k+1$ becomes

$$S(k+1): 3+6+9+\ldots+3k+3(k+1) = \frac{3(k+1)[(k+1)+1]}{2}$$

$$= \frac{3(k+1)(k+2)}{2} \quad \ldots (B)$$

Adding $3(k+1)$ on both the sides of (A) gives

$$3+6+9+\ldots+3k+3(k+1) = \frac{3k(k+1)}{2}+3(k+1)$$

$$= 3(k+1)\left(\frac{k}{2}+1\right)$$

$$= \frac{3(k+1)(k+2)}{2} \text{ [As required in eq. (B)]}$$

Thus $S(k+1)$ is true if $S(k)$ is true.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

Example 2: Use mathematical induction to prove that for any positive integer n, $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution: Let $S(n)$ be the given statement, that is,

$$S(n):\quad 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Base Case: When $n = 1$

$$S(1):\quad (1)^2 = \frac{1(1+1)(2\times1+1)}{6} = \frac{1\times2\times3}{6} = 1 \text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis: Let us assume that $S(k)$ is true for any $n = k \in N$, that is,

$$S(k): 1^2+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6} \quad \ldots (A)$$

$$S(k+1): 1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \frac{(k+1)(\overline{k+1}+1)(2\overline{k+1}+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \quad \ldots (B)$$

Adding $(k+1)^2$ to both the sides of (A) gives

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} = \frac{(k+1)(2k^2+k+6k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \text{ [As required in eq. (B)]}$$

Thus, formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, by mathematical induction, the given statement holds for all positive integers.

Example 3: Show that $\frac{n^3+2n}{3}$ represents an integer $\forall n \in N$.

Solution: Let $S(n) = \frac{n^3+2n}{3} \in Z \quad \forall n \in N$

Base Case: When $n = 1$.

$$S(1) = \frac{1^3+2(1)}{3} = \frac{3}{3} = 1 \in Z \text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis: Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$S(k) = \frac{k^3+2k}{3}$ represents an integer.

Now we want to show that $S(k+1)$ is also an integer. For $n = k+1$, the statement becomes

$$S(k+1) = \frac{(k+1)^3+2(k+1)}{3}$$

$$= \frac{k^3+3k^2+3k+1+2k+2}{3} = \frac{(k^3+2k)+(3k^2+3k+3)}{3}$$

$$S(k+1)=\frac{(k^3+2k)+3(k^2+k+1)}{3}=\frac{k^3+2k}{3}+\frac{+3(k^2+k+1)}{3}$$

$$=\frac{k^3+2k}{3}+(k^2+k+1)\quad \text{[As required in eq. (B)]}$$

As $\frac{k^3+2k}{3}$ is an integer by assumption (A) and we know that (k^2+k+1) is an integer as $k \in N$. $S(k+1)$ being sum of integers is an integer. Thus statements holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, we conclude by mathematical induction that $\frac{n^3+2n}{3}$ represents an integer for all positive integral values of n.

Example 4: Use mathematical induction to prove that $3+3\cdot5+3\cdot5^2+\cdots+3\cdot5^n=\frac{3(5^{n+1}-1)}{4}$, whenever n is non-negative integer.

Solution: Let $S(n)$ be the given statement, that is,

> The dot (.) between two numbers stands for multiplication symbol.

$$S(n):\quad 3+3\cdot5+3\cdot5^2+\cdots+3\cdot5^n=\frac{3(5^{n+1}-1)}{4}$$

Base Case: For $n=0$

$$S(0):\quad 3\cdot5^0=\frac{3(5^{0+1}-1)}{4}=\frac{3(5-1)}{4}$$

or $\quad 3=3$ (True)

The base case is satisfied.

Induction of Hypothesis: Let us assume that $S(k)$ is true for any $n=k\in W$, that is,

$$S(k):3+3\cdot5+3\cdot5^2+\cdots+3.5^k=\frac{3(5^{k+1}-1)}{4}\quad\text{...(A)}$$

Here $S(k+1)$ becomes

$$S(k+1):3+3\cdot5+3\cdot5^2+\cdots+3\cdot5^k+3\cdot5^{k+1}=\frac{3(5^{(k+1)+1}-1)}{4}$$

$$=\frac{3(5^{k+2}-1)}{4}\quad\text{...(B)}$$

Adding 3.5^{k+1} on both sides of (A), we get

$$3+3\cdot5+3\cdot5^2+\cdots+3\cdot5^k+3\cdot5^{k+1}=\frac{3(5^{k+1}-1)}{4}+3.5^{k+1}$$

$$=\frac{3(5^{k+1}-1)+12.5^{k+1}}{4}=\frac{3(5^{k+1}-1+4.5^{k+1})}{4}$$

$$=\frac{3[5^{k+1}(1+4)-1]}{4}=\frac{3(5^{k+2}-1)}{4}\quad\text{[As required in eq. (B)]}$$

This shows that $S(k+1)$ is true when $S(k)$ is true.

Conclusion: Since both the conditions are satisfied, therefore, by the principle of mathematical induction, $S(n)$ is true for each $n\in W$.

Example 5: Prove that 4^n+6n-1 is divisible by 9 for all $n\in N$

Solution: Let $S(n)$ be the given statement,

$S(n)=4^n+6n-1$ is divisible by 9 for all $n\in N$

Base Case: When $n=1$

$S(1)=4^1+6(1)-1=4+6-1=9$ (which is divisible by 9)

Hence it is true for $n=1$. The base case is satisfied.

Induction of Hypothesis: Suppose the statement is true for $n=k\in N$. i.e.,

$S(k)=4^k+6k-1$ is divisible by 9

$\Rightarrow\quad S(k)=4^k+6k-1=9k_1$, where $k_1\in Z\quad$ ...(A)

The statement for $n=k+1$ becomes

$$S(k+1)=4^{k+1}+6(k+1)-1=4\cdot4^k+6k+6-1$$

$$=4(9k_1-6k+1)+6k+6-1\quad \because 4^k=9k_1-6k+1\quad\text{Using (A)}$$

$$=36k_1-24k+4+6k+5$$

$$=36k_1-18k+9$$

$$=9(4k_1-2k+1)$$

Which is divisible by 9.

Thus $S(k)$ is true for $n=k+1$.

Conclusion: Since both the conditions are satisfied, therefore, by the principle of mathematical induction, the given statement is true for all integers $n\geq1$.

Example 6: Use mathematical induction to prove that $\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$, whenever n is a positive integer.

Solution: Let $S(n)$ be the given statement, that is,

$$S(n):\quad\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$$

Base Case: When $n=1$,

$$S(1):\quad\sum_{k=1}^{1}\frac{1}{(2k-1)(2k+1)}=\frac{1}{2.1+1}$$

$$\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{2\cdot1+1}$$

$$\frac{1}{1\cdot3}=\frac{1}{3}\Rightarrow\frac{1}{3}=\frac{1}{3}\text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis: Let us assume that $S(n)$ is true for $n=m$, that is,

$$S(m):\sum_{k=1}^{m}\frac{1}{(2k-1)(2k+1)}=\frac{m}{2m+1}\quad\text{...(A)}$$

The statement for $n=k+1$ becomes

$$S(m+1):\sum_{k=1}^{m+1}\frac{1}{(2k-1)(2k+1)}=\frac{m+1}{2(m+1)+1}\quad\text{...(B)}$$

Adding $\frac{1}{(2(m+1)-1)(2(m+1)+1)}$ to both the sides of (A) gives

$$\sum_{k=1}^{m}\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(m+1)-1)(2(m+1)+1)}=\frac{m}{2m+1}+\frac{1}{(2(m+1)-1)(2(m+1)+1)}$$

$$\sum_{k=1}^{m+1}\frac{1}{(2k-1)(2k+1)}=\frac{m}{2m+1}+\frac{1}{(2m+1)(2m+3)}$$

$$= \frac{m(2m+3)+1}{(2m+1)(2m+3)} = \frac{2m^2+3m+1}{(2m+1)(2m+3)} = \frac{(m+1)(2m+1)}{(2m+1)(2m+3)}$$

$$= \frac{m+1}{2m+3} = \frac{m+1}{2m+2+1} = \frac{m+1}{2(m+1)+1} \quad \text{[As required in eq. (B)]}$$

This shows that $S(k+1)$ is true when $S(k)$ is true.

Conclusion: Since both the conditions are satisfied, therefore, by the principle of mathematical induction, $S(n)$ is true for each $n \in N$.

Principle of Extended Mathematical Induction:

Let i be an integer. A formula or identity or statement $S(n)$ for $n \geq i$ is such that

1. **Base Case:** $S(i)$ is true and
2. **Induction of Hypothesis:** $S(k+1)$ is true whenever $S(k)$ is true for any integer $n \geq i$.
3. **Conclusion:** $S(n)$ is true for all integers $n \geq i$.

Example 7: Show that $1+3+5+\cdots+(2n+5)=(n+3)^2$ for integral values of $n \geq -2$.

Solution: Let $S(n)$ be the given statement, that is,

$S(n)$: $1+3+5+\cdots+(2n+5)=(n+3)^2$

Base Case: When $n=-2$

$S(-2)$: $1=(-2+3)^2$

$1=(1)^2$ (True)

The base case is satisfied.

Induction of Hypothesis: Let us assume that $S(n)$ is true for any $n=k(\geq -2) \in Z$, so that

$$S(k): 1+3+5+\cdots+(2k+5)=(k+3)^2 \quad \text{...(A)}$$

The statement for $n=k+1$ becomes

$$S(k+1): 1+3+5+\ldots+(2k+5)+\left(2\overline{k+1}+5\right)=(\overline{k+1}+3)^2$$

$$=(k+4)^2 \quad \text{...(B)}$$

Adding $\left(2\overline{k+1}+5\right)=(2k+7)$ on both sides of equation (A) we get,

$$1+3+5+\cdots+(2k+5)+(2k+7) = (k+3)^2+(2k+7)$$

$$= k^2+6k+9+2k+7$$

$$= k^2+8k+16=(k+4)^2 \quad \text{[As required in eq. (B)]}$$

The formula holds for $k+1$.

Conclusion: As both the conditions are satisfied, so we conclude that the $S(n)$ is true for all integers $n \geq -2$

Example 8: Show that the inequality $4^n > 3^n + 4$ is true, for integral values of $n \geq 2$.

Solution: Let $S(n)$ represents the given statement i.e.,

$S(n): 4^n > 3^n + 4$ for integral values of $n \geq 2$

Base Case: For $n=2$

$S(2): 4^2 > 3^2 + 4$

$16 > 13$ (True)

The base case is satisfied.

Induction of Hypothesis: Let the statement be true for any $n=k(\geq 2)\in Z$, that is

$$S(k): 4^k > 3^k + 4 \quad \text{...(A)}$$

The statement for $n=k+1$ becomes

$$S(k+1): 4^{k+1} > 3^{k+1} + 4 \quad \text{...(B)}$$

Multiplying both sides of inequality (A) by 4, we get

$$4 \cdot 4^k > 4(3^k+4)$$

$$4^{k+1} > (3+1)3^k + 16$$

$$4^{k+1} > 3^{k+1} + 4 + 3^k + 12$$

$$4^{k+1} > 3^{k+1} + 4 \quad (\because 3^k + 12 > 0) \quad \text{[As required in eq. (B)]}$$

The formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, by the principle of extended mathematical induction, the given inequality is true for all integers $n \geq 2$.

Example 9: If $a_n = 2^{2^n} + 1$, then for $n > 1$, show that last digit of a_n is 7.

Solution: We will prove the statement by mathematical induction.

Base case: When $n = 2$

$a_2 = 2^{2^2} + 1 = 2^4 + 1 = 17$. Clearly unit digit is 7.

Inductive Hypothesis: Assume that $a_k = 2^{2^k} + 1 = 10m + 7$ ... (A)

where $k > 1$ and m is some positive integer.

Now, $a_{k+1} = 2^{2^{k+1}} + 1 = 2^{2^k \cdot 2} + 1$

$= (2^{2^k})^2 + 1 = (10m+6)^2 + 1$ $\quad (\because 2^{2^k} = 10m+6)$ From (A)

$= 100m^2 + 120m + 36 + 1$

$= 100m^2 + 120m + 30 + 7$

$= 10(10m^2 + 12m + 3) + 7$

Thus, last digit of a_n is 7 for all $n > 1$.

Conclusion: Since both the conditions are satisfied, therefore, by the principle of mathematical induction, the given statement is true for all integers $n > 1$.

Real Life Application of Mathematical Induction:

Mathematical induction is a powerful method used to prove statements that are formulated for natural numbers. It is often used in mathematics to justify conclusions about sequences, series, and other constructs that involve integer values.

Example 10: Faris starts a savings plan where he deposits 1,000 rupees into his bank account every month. Using mathematical induction, prove that the total amount saved after n months is given by:

$$S(n) = 1000 \times n \text{ rupees}$$

where n is a positive integer representing the number of months.

Solution:

Given statement: $S(n) = 1000 \times n$

Base Case: For = 1 :

After the first month, Faris save 1000 rupees. Therefore, the total savings after one month is $1000 \times 1 = 1000$ rupees. The base case $S(1)$ holds true.

Induction of Hypothesis: Assume the statement is true for some positive integer k, i.e.,

After k months, the total savings is $S(k) = 1000 \times k$ rupees.

Now, prove that the statement holds for $k+1$ months:

After $k+1$ months, you would save an additional Rs. 1000, so the total savings becomes:

$S(k+1) = 1000 \times k + 1000 = 1000 \times (k+1)$ rupees.

Thus, if the statement holds for k, it also holds for $k+1$.

Justification and Communication: Using mathematical induction, we prove that saving Rs.1000 monthly for n months totals $1000 \times n$ rupees.

The base case ($n = 1$) holds, and assuming it's true for k months, we show it for $k+1$. Thus, the statement is valid for all natural numbers n, making it reliable for real-life applications.

Example 11: Ali starts a daily exercise routine where each day he increases the number of push-ups he does by 2. On the first day, he does 10 push-ups. Prove that after n^{th} day, the total number of push-ups Ali has done is $n^2 + 9n$

Solution: Let $S(n)$ be the given statement, that is,

$S(n)$: $n^2 + 9n$

Base Case: For = 1 : On the first day, Ali do 10 push-ups.

Total push-ups $S(1)$: $(1)^2 + 9(1) = 10$.

The base case $S(1)$ holds true.

Induction of Hypothesis: Assume the statement is true for some positive integer k, i.e.,

The total number of push-ups after k days is $S(k) = k^2 + 9k$.

Now, prove it for $k + 1$ days: On the $(k + 1)$th day, you do $10 + 2k$ push-ups.

The total after $k + 1$ days becomes: $k^2 + 9k + (10 + 2k) = k^2 + 2k + 1 + 9k + 9$

$$= (k+1)^2 + 9(k+1)$$

The formula holds for $S(k + 1)$.

Conclusion: By mathematical induction, the total number of push-ups after n days is $n^2 + 9n$.

Example 12: Suppose you aim to lose weight by reducing your calories intake by 50 calories each week. If you start at 2500 calories, prove that after n weeks, your daily intake is $2500 - 50n$ calories.

Solution: Let $S(n)$ be the given statement, that is,

$S(n)$: $2500 - 50n$

Base Case: For = 1 :

After 1 week, your intake is $S(1) = 2500 - 50(1) = 2450$ calories.

The base case $S(1)$ holds true.

Induction of Hypothesis: Assume the statement is true for some positive integer k, i.e.,

After k weeks, your intake is $S(k)$: $2500 - 50k$ calories.

Now, prove it for $k + 1$ weeks:

After $k + 1$ weeks, your intake will be: $2500 - 50k - 50 = 2500 - 50(k + 1)$ calories.

The formula holds for $k + 1$.

Conclusion: By mathematical induction, your daily intake after n weeks is $2500 - 50n$ calories.

Exercise 8.1

1. Use mathematical induction to prove the following formulae for every positive integer n.

(i) $\log x^n = n\log x$, **where x is positive**

Solution:

Let $S(n): \log x^n = n\log x$

Base case: When $n = 1$

$S(1): \log x^1 = (1)\log x$

$\log x = \log x$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k \in N$, that is

$S(k): \log x^k = k\log x$ ...(A)

The statement for $n = k + 1$ becomes

$S(k+1): \log x^{k+1} = (k+1)\log x$ ...(B)

Adding $\log x$ on both sides of (A) gives

$\log x^k + \log x = k\log x + \log x$

$\log(x^k \cdot x) = (k+1)\log x$ $\quad \because \log a + \log b = \log(ab)$

$\log x^{k+1} = (k+1)\log x$ (As required in (B))

Thus, formula holds for $k + 1$

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(ii) $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n}{2}(3n + 1)$

Solution:

Let $S(n): 2 + 5 + 8 + \cdots + (3n - 1) = \frac{n}{2}(3n + 1)$

Base case: When $n = 1$

$S(1): 2 = \frac{1}{2}(3(1) + 1) \Rightarrow 2 = \frac{1}{2}(4) \Rightarrow 2 = 2$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume That $S(n)$ is true for any $n = k \in N$, that is

$S(k): 2 + 5 + 8 + \cdots + (3k - 1) = \frac{k}{2}(3k + 1)$ ...(A)

The statement for $n = k + 1$ becomes

$S(k+1): 2 + 5 + 8 + \cdots + (3k - 1) + (3(k+1) - 1) = \frac{k+1}{2}(3(k+1) + 1)$

$= \frac{k+1}{2}(3k + 4)$ ...(B)

Adding $(3(k+1) - 1)$ on both sides of (A) gives

$2 + 5 + 8 + \cdots + (3k - 1) + (3(k+1) - 1) = \frac{k}{2}(3k + 1) + 3(k+1) - 1$

$= \frac{3k^2 + k + 6k + 6 - 2}{2} = \frac{1}{2}(3k^2 + 7k + 4) = \frac{1}{2}(3k^2 + 4k + 3k + 4)$

$= \frac{1}{2}(k(3k + 4) + 1(3k + 4))$

$2 + 5 + 8 + \cdots + (3k - 1) + (3(k+1) - 1) = \frac{(k+1)}{2}(3k + 4)$ (As required in (B))

Thus, formula holds for $k + 1$.

Conclusion: since both the conditions are satisfied, Therefore, $S(n)$ is true for each positive integer n.

(iii) $2 + (2 + 5) + (2 + 5 + 8) + \cdots + \frac{n}{2}(3n + 1) = \frac{n}{4}(n + 1)^2$

Solution:

Let $S(n): 2 + (2 + 5) + (2 + 5 + 8) + \cdots + \frac{n}{2}(3n + 1) = \frac{n}{2}(n + 1)^2$

Base case: When $n = 1$

$S(1): 2 = \frac{1}{2}(1 + 1)^2 \Rightarrow 2 = \frac{1}{2}(4) \Rightarrow 2 = 2$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k \in N$, that is

$S(k): 2 + (2 + 5) + (2 + 5 + 8) + \cdots + \frac{k}{2}(3k + 1) = \frac{k}{2}(k + 1)^2$ ...(A)

The statement for $n = k+1$ becomes

$$\begin{aligned} S(k+1): 2+(2+5)+(2+5+8)+\cdots+\frac{k}{2}(3k+1)+\frac{k+1}{2}(3(k+1)+1) &= \frac{k+1}{2}(k+1+1) \\ &= \frac{k+1}{2}(k+2)^2 \quad \ldots(B) \end{aligned}$$

Adding $\frac{k+1}{2}(3(k+1)+1)$ on both sides of (A) gives

$$\begin{aligned} 2+(2+5)+(2+5+8)+\cdots+\frac{k}{2}(3k+1)+\frac{k+1}{2}(3(k+1)+1) &= \frac{k}{2}(k+1)^2+\frac{k+1}{2}(3(k+1)+1) \\ &= \frac{k}{2}(k+1)^2+\frac{(k+1)}{2}\cdot(3k+4) \\ &= \frac{k+1}{2}\{k(k+1)+3k+4\} \\ &= \frac{k+1}{2}(k^2+k+3k+4) = \frac{k+1}{2}(k^2+4k+4) \end{aligned}$$

$$2+(2+5)+(2+5+8)+\cdots+\frac{k}{2}(3k+1)+\frac{k+1}{2}(3(k+1)+1) = \frac{k+1}{2}\cdot(k+2)^2 \text{ (As required in (B))}$$

Thus, formula holds for $k+1$

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(iv) $2+6+18+\cdots+2\times 3^{n-1} = 3^n - 1$

Solution:

Let $S(n): 2+6+18+\cdots+2\times 3^{n-1} = 3^n - 1$

Base Case: When $n = 1$

$S(1): 2 = 3^1 - 1 \quad \Rightarrow 2 = 2$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for $n = k \in N$, that is

$$S(k): 2+6+18+\cdots+2\times 3^{k-1} = 3^k - 1 \quad \ldots(A)$$

The statement for $n = k+1$ becomes

$$S(k+1): 2+6+18+\cdots+2\times 3^{k-1}+2\times 3^{k+1-1} = 3^{k+1} - 1 \quad \ldots(B)$$

Adding $2\times 3^{k+1-1}$ on both sides of (A) gives

$$\begin{aligned} 2+6+18+\cdots+2\times 3^{k-1}+2\times 3^{k+1-1} &= 3^k - 1 + (2\times 3^{k+1-1}) \\ &= 3^k - 1 + (2\times 3^k) = 3^k + 2\cdot 3^k - 1 \\ &= 3^k(1+2) - 1 = 3^k\cdot 3 - 1 \end{aligned}$$

$$2+6+18+\cdots+2\times 3^{k-1}+2\times 3^{k+1-1} = 3^{k+1} - 1 \text{ (As required in (B))}$$

Thus, formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(v) $1\times 3+2\times 5+3\times 7+\cdots+n\times(2n+1) = \frac{n(n+1)(4n+5)}{6}$

Solution:

Let $S(n): 1\times 3+2\times 5+3\times 7+\cdots+n\times(2n+1) = \frac{n(n+1)(4n+5)}{6}$

Base Case: When $n = 1$

$S(1): 1\times 3 = \frac{1(1+1)(4(1)+5)}{6} \quad \Rightarrow 3 = \frac{(2)(9)}{6} \quad \Rightarrow 3 = 3$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for $n = k \in N$, that is

$$S(k): 1\times 3+2\times 5+3\times 7+\cdots+k\times(2k+1) = \frac{k(k+1)(4k+5)}{6} \quad \ldots(A)$$

the statement for $n = k+1$ becomes

$$\begin{aligned} S(k+1): 1\times 3+2\times 5+3\times 7+\cdots+k\times(2k+1)+(k+1)\times(2(k+1)+1) &= \frac{(k+1)(k+1+1)(4(k+1)+5)}{6} \\ &= \frac{(k+1)(k+2)(4k+9)}{6} \quad \ldots(B) \end{aligned}$$

Adding $(k+1)(2(k+1)+1)$ on both sides of (A) gives

$$\begin{aligned} 1\times 3+2\times 5+3\times 7+\cdots+k\times(2k+1)+(k+1)(2(k+1)+1) &= \frac{k(k+1)(4k+5)}{6}+(k+1)(2(k+1)+1) \\ &= \frac{k(k+1)(4k+5)}{6}+(k+1)(2k+3) \\ &= (k+1)\left\{\frac{k(4k+5)}{6}+(2k+3)\right\} \\ &= (k+1)\left\{\frac{4k^2+5k+12k+18}{6}\right\} = \frac{k+1}{6}\cdot(4k^2+17k+18) \\ &= \frac{k+1}{6}\cdot(4k^2+9k+8k+18) \\ &= \frac{k+1}{6}(k(4k+9)+2(4k+9)) \end{aligned}$$

$$1\times 3+2\times 5+3\times 7+\cdots+k\times(2k+1)+(k+1)(2(k+1)+1) = \frac{(k+1)(k+2)(4k+9)}{6} \text{ (As required in (B))}$$

Thus, formula holds for $k+1$

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(vi) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$

Solution:

Let $S(n): \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$

Base Case: When $n = 1$

$S(1): \frac{1}{1\times 2} = 1-\frac{1}{1+1} \quad \Rightarrow \frac{1}{2} = 1-\frac{1}{2} \quad \Rightarrow \frac{1}{2} = \frac{1}{2}$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$S(k): \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{k(k+1)} = 1-\frac{1}{k+1} \quad \ldots(A)$$

The statement for $n = k+1$ becomes

$$S(k+1): \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} = 1-\frac{1}{k+2} \quad \ldots(B)$$

Adding $\frac{1}{(k+1)(k+2)}$ on both sides of (A) gives

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=1-\frac{1}{k+1}\left(1-\frac{1}{k+2}\right)=1-\frac{1}{k+1}\left(\frac{k+2-1}{k+2}\right)$$

$$=1-\frac{1}{k+1}\left(\frac{k+1}{k+2}\right)$$

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+2} \text{ (As required in (B))}$$

Thus, formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefor, $S(n)$ is true for each positive integer n.

(vii) $r+r^2+r^3+\cdots+r^n=\frac{r(1-r^n)}{1-r},(r\neq1)$

Solution:

Let $S(n)$: $r+r^2+r^3+\cdots+r^n=\frac{r(1-r^n)}{1-r},(r\neq1)$

Base Case: When $n=1$

$$S(1): r=\frac{r(1-r^1)}{1-r} \quad \Rightarrow r=r\text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n=K\in N$, that is,

$$S(k): r+r^2+r^3+\cdots+r^k=\frac{r(1-r^k)}{1-r} \quad ...(A)$$

The statement for $n=k+1$ becomes

$$S(k+1): r+r^2+r^3+\cdots+r^k+r^{k+1}=\frac{r(1-r^{k+1})}{1-r} \quad ...(B)$$

Adding r^{k+1} on both sides of (A) gives

$$r+r^2+r^3+\cdots+r^k+r^{k+1}=\frac{r(1-r^k)}{1-r}+r^{k+1}$$

$$=\frac{r-r^{k+1}+r^{k+1}(1-r)}{1-r}=\frac{r-r^{k+1}+r^{k+1}-r^{k+2}}{1-r}=\frac{r-r^{k+2}}{1-r}$$

$$r+r^2+r^3+\cdots+r^k+r^{k+1}=\frac{r(1-r^{k+1})}{1-r} \text{ (As required in (B))}$$

Thus, formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer, n.

(viii) $a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$

Solution:

Let $S(n): a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$

Base Case: When $n=1$

$$S(1): a=\frac{1}{2}[2a+(1-1)d] \quad \Rightarrow a=\frac{1}{2}[2a-0] \quad \Rightarrow a=a\text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n=k\in N$, that is,

$$S(k): a+(a+d)+(a+2d)+\cdots+[a+(k-1)d]=\frac{k}{2}[2a+(k-1)d] \quad ...(A)$$

The statement for $n=k+1$ becomes

$$S(k+1): a+(a+d)+(a+2d)+\cdots+[a+(k-1)d]+(a+kd)=\frac{k+1}{2}[2a+kd] \quad ...(B)$$

Adding $(a+kd)$ on both sides of (A) gives

$$a+(a+d)+(a+2d)+\cdots+[a+(k+1)d]+(a+kd)=\frac{k}{2}[2a+(k-1)d]+(a+kd)$$

$$=\frac{2ak+k^2d-kd+2a+2kd}{2}=\frac{k^2d+kd+2ak+2a}{2}$$

$$=\frac{1}{2}[kd(k+1)+2a(k+1)]$$

$$a+(a+d)+(a+2d)+\cdots+[a+(k+1)d]+(a+kd)=\frac{k+1}{2}[2a+kd] \text{ (As required in (B))}$$

Thus, formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(ix) $a_n=a_1+(n-1)d$ **when,** $a_1, a_1+d, a_1+2d,\ldots$ **form an A.P.**

Solution:

Let $S(n)$: $a_n=a_1+(n-1)d$

Base case: When $n=1$

$$S(1): a_1=a_1+(1-1)d \quad \Rightarrow a_1=a_1+0d \quad \Rightarrow a_1=a_1\text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n=k\in N$, that is,

$$S(k): a_k=a_1+(k-1)d \quad ...(A)$$

The statement for $n=k+1$ becomes

$$S(k+1): a_{k+1}=a_1+(k+1-1)d$$

$$=a_1+kd \quad ...(B)$$

Adding d on both sides of (A) gives

$$a_k+d=a_1+(k-1)d+d$$

$$=a_1+(k-1+1)d$$

$$a_{k+1}=a_1+kd \text{ (As required in (B))}$$

Thus, formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(x) $a_n=a_1r^{n-1}$ **when** $a_1, a_1r, a_1r^2,\ldots$ **form a G.P.**

Solution:

Let $S(n): a_n=a_1r^{n-1}$

Base Case: When $n=1$

$$S(1): a_1=a_1r^{1-1} \quad \Rightarrow a_1=a_1r^0 \quad \Rightarrow a_1=a_1 \quad \because r^0=1$$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$S(k): a_k = a_1 \cdot r^{k-1} \quad \text{...(A)}$$

The statement for $n = k+1$ becomes

$$S(k+1): a_{k+1} = a_1 r^{k+1-1}$$
$$a_{k+1} = a_1 r^k \quad \text{...(B)}$$

Multiplying both sides of (A) by k, we get

$$ra_k = a_1 r^{k-1} \cdot r$$
$$a_{k+1} = a_1 r^{k-1+1}$$
$$a_{k+1} = a_1 r^k \text{ (As required in (B))}$$

Thus, formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(xi) $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{n+2}{3}=\binom{n+3}{4}$

Solution:

Let $S(n): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{n+2}{3}=\binom{n+3}{4}$

Base Case: When n = 1

$$\binom{3}{3} = \binom{1+3}{4} \Rightarrow {}^3c_3 = {}^4c_4 \Rightarrow 1 = 1 \text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$S(k): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+2}{3}=\binom{k+3}{4} \quad \text{...(A)}$$

The statement for $n = k+1$ becomes

$$S(k+1): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+2}{3}+\binom{k+3}{3}=\binom{k+4}{4} \quad \text{...(B)}$$

Adding $\binom{k+3}{3}$ on both sides of (A) gives

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+2}{3}+\binom{k+3}{3} = \binom{k+3}{4}+\binom{k+3}{3}$$
$$= \binom{k+3+1}{4} \quad \text{using formula: } {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$
$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+2}{3}+\binom{k+3}{3} = \binom{k+4}{4} \text{ (As required in (B))}$$

Thus, formula holds for $k+1$

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(xii) The sum of first n odd natural numbers is n^2.

Solution:

Let $S(n)$: The sum of first odd natural numbers is n^2. $\because S(n): 1+3+5+\ldots$ to n terms $= n^2$

$$S(n): 1+3+5+\cdots+(2n-1) = n^2 \quad \because n \text{ term} = 1+(n-1)(2) = 2n-1$$

Base case: When $n = 1$

$$S(1): 1 = (1)^2 \Rightarrow 1 = 1 \text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$S(k): 1+3+5+\cdots+(2k-1) = k^2 \quad \text{...(A)}$$

The statement for $n = k+1$ becomes

$$S(k+1): 1+3+5+\cdots+(2k-1)+(2(k+1)-1) = (k+1)^2 \quad \text{...(B)}$$

Adding $(2(k+1)-1)$ on both sides of (A) gives

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1) = k^2+2(k+1)-1$$
$$= k^2+2k+2-1 = k^2+2k+1$$
$$1+3+5+\cdots+(2k-1)+(2(k+1)-1) = (k+1)^2 \text{ (As required in (B))}$$

Thus, formula holds for $k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

2. Prove by mathematical induction that for all positive integral values of n

(i) $n^2 + n$ is divisible by 2

Solution:

Let $S(n) = n^2 + n$ is divisible by 2 for all $n \in N$

Base Case: When $n = 1$.

$$S(1) = 1^2 + 1 = 2 \quad \text{which is divisible by '2'.}$$

The base case is satisfied.

Induction of Hypothesis:

Suppose the statement is true for $n = k \in N$. i.e.,

$S(k) = k^2 + k$ is divisible by 2

$\Rightarrow S(k) = k^2 + k = 2k_1$, where $k_1 \in Z$ ...(A)

The statement for $n = k+1$ becomes

$$S(k+1) = (k+1)^2 + (k+1)$$
$$= k^2+1+2k+k+1 = (k^2+k)+2k+2 = 2k_1+2k+2$$
$$= 2(k_1+k+1) \quad \text{which is divisible by 2.}$$

Thus, $S(n)$ is true for $n = k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(ii) $5^n - 2^n$ is divisible by 3

Solution:

Let $S(n) = 5^n - 2^n$ is divisible by 3 for all $n \in N$.

Base case: When $n = 1$

$$S(1) = 5^1 - 2^1 = 3 \quad \text{which is divisible by '3'.}$$

The base case is satisfied.

Induction of Hypothesis:

Suppose the statement is true for $n = k \in N$ i.e.,

$S(k) = 5^k - 2^k$ is divisible by 3.

$\Rightarrow S(k) = 5^k - 2^k = 3k_1$, where $k_1 \in z$ ...(A)

The statement for $n = k+1$ becomes

$$S(k+1) = 5^{k+1} - 2^{k+1}$$
$$= 5^k \cdot 5 - 2^k \cdot 2$$

Add and subtract $5 \cdot 2^k$, we have

$$= 5^k \cdot 5 - 5 \cdot 2^k + 5 \cdot 2^k - 2^k \cdot 2 = 5(5^k - 2^k) + 2^k(5-2) = 5 \cdot (3k_1) + 2^k \cdot 3$$

$$= 3(5k_1 + 2^k) \quad \text{which is divisible by 3 .}$$

Thus, $S(n)$ is true for $n = k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

(iii) $8 \times 10^n - 2$ is divisible by 6

Solution:

Let $S(n) = 8 \times 10^n - 2$ is divisible by 6 for $n \in N$.

Base Case: when $n = 1$

$S(1) = 8 \times 10^1 - 2 = 78$ which is divisible by '6'.

The base case is satisfied.

Induction of Hypothesis:

Suppose the statement is true for $n = k \in N$. i.e.,

$S(k) = 8 \times 10^k - 2$ is divisible by 2.

$\Rightarrow S(k) = 8 \times 10^k - 2 = 6k_1$, where $k_1 \in Z$ ...(A)

The statement for $n = k+1$ becomes

$$S(k+1) = 8 \times 10^{k+1} - 2$$

$$= 8 \times 10^k \cdot 10 - 2$$

Add and subtract 10.2, we have

$$= 8 \times 10^k \cdot 10 - 10 \cdot 2 + 10 \cdot 2 - 2 = 10(8 \times 10^k - 2) + 20 - 2 = 10(6k_1) + 18 \quad \text{using (A)}$$

$$= 6(10k_1 + 3) \quad \text{which is divisible by 6 .}$$

Thus, $S(n)$ is true for $n = k+1$

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

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3. Prove that $\sum_{k=1}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$, whenever n is a positive integer.

Solution:

Let $S(n)$: $\sum_{k=1}^{n} r^k = \frac{r^{n+1} - r}{r - 1}$

Base Case: When $n = 1$

$$\sum_{k=1}^{1} r^k = \frac{r^{1+1} - r}{r - 1} \Rightarrow r^1 = \frac{r^2 - r}{r - 1} \Rightarrow r = \frac{r(r-1)}{r-1} \Rightarrow r = r \text{ (True)}$$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $s(n)$ is true for any $n = m \in N$. i.e.,

$$S(m): \sum_{k=1}^{m} r^k = \frac{r^{m+1} - r}{r - 1} \quad \text{...(A)}$$

The statement for $n = m+1$ becomes.

$$S(m+1): \sum_{k=1}^{m+1} r^k = \frac{r^{m+2} - r}{r - 1} \quad \text{...(B)}$$

Adding r^{m+1} on both sides of (A) gives

$$\sum_{k=1}^{m} r^k + r^{m+1} = \frac{r^{m+1} - r}{r - 1} + r^{m+1}$$

$$\sum_{k=1}^{m+1} r^k = \frac{r^{m+1} - r + r^{m+1}(r-1)}{r-1} = \frac{r^{m+1} - r + r^{m+2} - r^{m+1}}{r-1}$$

$$= \frac{r^{m+2} - r}{r - 1}$$

Thus, formula holds for $(m+1)$

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n

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4. $x - y$ is a factor of $x^n - y^n$ for all positive integral values of $n, (x \neq y)$.

Solution:

Let $S(n)$: $x - y$ is a factor of $x^n - y^n$ for all $n \in N, (x \neq y)$

Base case: When $n = 1$

$S(1)$: $x^1 - y^1 = 1 \cdot (x - y)$

$\Rightarrow x - y$ is a factor of $(x^1 - y^1)$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n \in N$. i.e.,

$S(k)$: $x - y$ is a factor of $x^k - y^k$

$\Rightarrow x^k - y^k = (x - y) \cdot p(x, y)$ ...(A)

Where $p(x, y)$ is some polymial in x and y.

The statement for $n = k+1$ becomes

$\Rightarrow S(k+1)$: $x - y$ is a factor of $x^{k+1} - y^{k+1}$

Consider $x^{k+1} - y^{k+1} = x^k \cdot x - y^k \cdot y$

Add and subtract $x \cdot y^k$, we get

$$= x^k \cdot x - xy^k + xy^k - y^k \cdot y = x(x^k - y^k) + y^k(x - y)$$

$$= x \cdot (x-y)p(x,y) + y^k(x-y) = (x-y)\{xp(x,y) + y^k\}$$

$$= (x-y) \cdot Q(x,y) \quad \text{where } Q(x,y) \text{ is some polynomial in } x \text{ and } y.$$

$\Rightarrow (x - y)$ is a factor of $x^{k+1} - y^{k+1}$

Thus, $S(n)$ is true for $n = k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

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5. $n! > 2^n - 1$ for integral values of $n \geq 4$.

Solution:

Let $S(n)$: $n! > 2^n - 1$ for integral values of $n \geq 4$.

Base Case: When $n = 4$

$S(4)$: $4! > 2^4 - 1 \Rightarrow 24 > 16 - 1 \Rightarrow 24 > 15$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k (\geq 4) \in Z$, that is

$S(k)$: $k! > 2^k - 1$ ...(A)

The statement for $n = k+1$ becomes

$S(k+1)$: $(k+1)! > 2^{k+1} - 1$ ...(B)

Multiplying both sides of (A) by $(k+1)$, we get

$$(k+1)k! > (k+1)(2^k - 1)$$

$$(k+1)! > k \cdot 2^k - k + 2^k - 1$$

$(k+1)! > k\cdot 2^k - 1 + (2^k - k)$

$(k+1)! > k\cdot 2^k - 1 \qquad \because 2^k - k > 0$

$(k+1)! > 2\cdot 2^k - 1 \qquad \because 2 < k$

$(k+1)! > 2^{k+1} - 1$ (As required in (B))

Thus, $S(n)$ is true for $n = k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for all integers $n \geq 4$.

6. $4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$.

Solution:

Let $S(n): 4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$

Base Case: When $n = 2$

$S(2): 4^2 > 3^2 + 2^{2-1} \quad \Rightarrow 16 > 9 + 2 \quad \Rightarrow 16 > 11$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k (\geq 2) \in Z$, that is

$S(k): 4^k > 3^k + 2^{k-1}$ ...(A)

the statement for $n = k+1$ becomes

$S(k+1): 4^{k+1} > 3^{k+1} + 2^k$ ...(B)

Multiplying both sides of (A) by 4, we get

$4\cdot 4^k > 4\cdot 3^k + 4\cdot 2^{k-1}$

$4^{k+1} > 3^k(3+1) + (2+2)\cdot 2^{k-1}$

$4^{k+1} > 3^{k+1} + 3^k + 2^k + 2^k$

$4^{k+1} > (3^{k+1} + 2^k) + (3^k + 2^k)$

$4^{k+1} > 3^{k+1} + 2^k \qquad \because (3^k + 2^k) > 0$ (As required in (B))

Thus, $S(n)$ is true for $n = k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for all integers $n \geq 2$.

7. $1 + nx \leq (1+x)^n$ for $n \geq 2$ and $x > -1$.

Solution:

Let $S(n): (1+x)^n \geq 1 + nx$ for $n \geq 2$ and $x > -1$

Base case: When $n = 2$

$S(2): (1+x)^2 \geq 1 + 2x \quad \Rightarrow 1 + 2x + x^2 \geq 1 + 2x$ (True) $\quad \because x > -1$

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for any $n = k (\geq 2) \in Z$, that is

$S(k): (1+x)^k \geq 1 + kx$ ...(A)

The statement for $n = k+1$ becomes

$S(k+1): (1+x)^{k+1} \geq 1 + (k+1)x$

Multiplying both sides of (A) by $(1+x)$, we get

$(1+x)^k\cdot(1+x) \geq (1+kx)(1+x)$

$(1+x)^{k+1} \geq 1 + x + kx + kx^2$

$(1+x)^{k+1} \geq 1 + (k+1)x + kx^2$

$(1+x)^{k+1} \geq 1 + (k+1)x \qquad \because kx^2 > 0$ (As required in (B))

thus, $S(n)$ is true for $n = k+1$.

Conclusion: Since both the conditions are satisfied, therefore, $S(n)$ is true for all integers $n \geq 2$.

8. Aliza invests Rs. 1,000,000 in a business that promises a 6% return compounded annually. Prove by mathematical induction that the amount of money after n years is $1,000,000(1.06)^n$.

Solution:

Let $S(n): A_n = 1,000,000(1.06)^n$ where A_n is the amount after n years.

Base Case: when $n = 1$

Amount after 1 year $= 1000000 + 6\%(1000000) = 1000000 + 60,000 = 1060000$

and $S(1): A_1 = 1000000(1.06)^1 = 1060000$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for some positive integer k. i.e.,

$S(k): A_k = 1000000(1.06)^k$ ...(A)

The statement for $n = k+1$ becomes

$S(k+1): A_{k+1} = 1000000(1.06)^{k+1}$ ...(B)

Adding 6% (A_k) on both sides of (A) gives

$A_k + 6\%(A_k) = 1000000(1.06)^k + 6\%(A_k)$

$A_{k+1} = 1000000(1.06)^k + 0.06(1000000(1.06)^k) = 1000000(1.06)^k(1 + 0.06)$

$= 1000000(1.06)^k(1.06)$

$A_{k+1} = 1000000(1.06)^{k+1}$ (As required in (B))

Thus, $S(n)$ is true for $n = k+1$.

Conclusion: By mathematical induction, the amount of money after n years is $1,000,000(1.06)^n$.

9. A bank offers an investment with an annual interest rate r. If P rupees are invested, the amount after n years is given by: $A(n) = P(1+r)$

Prove by induction that this formula holds for all $n \geq 0$.

Solution:

Let $S(n): A(n) = P(1+r)^n$, $n \geq 0$ where, P is the initial investment and r is the annual interest rate.

Base case: When $n = 0$

Initial investment $= P$

and $A(0) = P(1+r)^0 = P(1) = P$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for same integer $k \geq 0$, i.e.,

$S(k): A(k) = p(1+r)^k$ ...(A)

The statement for $n = k+1$ becomes

$S(k+1): A(k+1) = P(1+r)^{k+1}$ ...(B)

Multiplying $(1+r)$ on both sides of (A) gives

$(1+r)\cdot A(k) = P(1+r)^k\cdot(1+r)$

$A(k+1) = P(1+r)^{k+1}$ (As required in (B))

Thus, $S(n)$ is true for $n = k+1$

Conclusion: By mathematical induction. The amount of money after n years is $A(n) = P(1+r)^n$, $n \geq 0$.

10. Sikander saves Rs. 500 in the first month and increases his savings by Rs. 500 every subsequent month. Using mathematical induction, determine whether his total savings will reach at least Rs. 12,000 after 24 months.

Solution:

Sequence: $500, 1000, 15000, \ldots, a_n$ (A.P)

Here $a_1 = 500, d = 1000 - 500 = 500$

As we know that

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[2(500) + (n-1)500] = \frac{n}{2}\,2(500 + 250n - 250) = n(250 + 250n)$$

$S_n = 250n^2 + 250n$ where S_n is saving after n months.

Base case: (when $n = 1$)

Saving after 1 year = 500

and $S_1 = 250(1)^2 + 250(1) = 500$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that statement is true for some positive integer k. i.e.,

$$S_k = 250k^2 + 250k \qquad ...(A)$$

The statement for $n = k+1$ becomes

$$S_{k+1} = 250(k+1)^2 + 250(k+1) \qquad ...(B)$$

By adding a_{k+1} on both sides of (A) gives

$$S_k + a_{k+1} = 250k^2 + 250k + a_{k+1}$$

Put $a_{k+1} = a_1 + kd = 500 + 500k$ on R.H.S

$$S_{k+1} = 250k^2 + 250k + 500 + 500k = \left(250k^2 + 500k + 250\right) + 250k + 250$$

$$= 250\left(k^2 + 2k + 1\right) + 250(k+1) = 250(k+1)^2 + 250(k+1) \text{ (As required in (B))}$$

Thus, the statement is true for $n = k+1$.

Conclusion: By mathematical induction, the formula is true for all positive integers n.

Therefore, $S_n = 250n^2 + 250n$

Calculation of total amount after 24 months:

$$S_{24} = 250(24)^2 + 250(24) = 144{,}000 + 6000 = 150{,}000 \geq 12000$$

Hence, the total savings after 24 months will reach at least Rs. 12,000

11. Prove by mathematical induction that if Ali takes a loan of Rs.2,000,000 and pay Rs. 50,000 at the end of each year, the remaining balance after n years is $R_n = 2{,}000{,}000 - 50{,}000n$.

Solution:

Let $S(n): R_n = 2{,}000{,}000 - 50{,}000n$ where R_n is the remaining balance after n years.

Base case: When $n = 1$.

Remaining amount after 1 year $= 2{,}000{,}000 - 50{,}000 = 1{,}950{,}000$

and $S(1): R_1 = 2{,}000{,}000 - 50{,}000(1) = 1{,}950{,}000$ (True)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for some positive integer k. i.e., $S(k): R_k = 2{,}000{,}000 - 50{,}000k$ ...(A)

The statement for $n = k+1$ becomes.

$$S(k+1): R_{k+1} = 2{,}000{,}000 - 50{,}000(k+1) \qquad ...(B)$$

Subtracting 50,000 on both sides of (A) gives

$$R_k - 50{,}000 = 2{,}000{,}000 - 50{,}000k - 50{,}000$$

$\Rightarrow$ $R_{k+1} = 2{,}000{,}000 - 50{,}000(k+1)$ (As required in (B))

Thus, $S(n)$ is true for $n = k+1$.

Conclusion: By mathematical induction, the remaining balance after n years is $R_n = 2{,}000{,}000 - 50{,}000n$.

12. If Salman start savings with Rs. 5,000 and saves an additional Rs. 1,000 at the end of every month, derive a formula $S(n)$ for his total savings after n months. Prove the correctness of year formula using mathematical induction.

Solution:

Given that: Initial saving = Rs. 5000

Monthly saving = Rs. 1000

After n months, the total saving will be:

$$S(n) = 5000 + \{1000 + 1000 + 1000 + \cdots + 1000(n\text{-terms})\}$$

$$S(n) = 5000 + 1000n$$

Base case: (When $n = 1$)

Saving after 1 month = 5000 + 1000 = 6000

and $S(1) = 5000 + 1000(1) = 6000$ (true)

The base case is satisfied.

Induction of Hypothesis:

Let us assume that $S(n)$ is true for some positive integer k, i.e.,

$$S(k) = 5000 + 1000k \qquad ...(A)$$

The statement for $n = k+1$ becomes

$$S(k+1) = 5000 + 1000(k+1) \qquad ...(B)$$

Adding 1000 on both sides of (A) gives

$$S(k) + 1000 = 5000 + 1000k + 1000$$

$S(k+1) = 5000 + 1000(k+1)$ (As required in (B))

Thus, $S(n)$ is true for $n = k+1$.

Conclusion: By mathematical induction, the total savings after the n months is $S(n) = 5000 + 1000n$.

Binomial Theorem:

Binomial Expression: An algebraic expression consisting of two terms such as $a + x$, $x - 2y$, $ax + b$ etc., is called a binomial or a binomial expression.

We know by actual multiplication that

$$(a+x)^2 = a^2 + 2ax + x^2 \qquad ...(i)$$

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3 \qquad ...(ii)$$

The right sides of (i) and (ii) are called binomial expansions of the binomial $a + x$ for the indices 2 and 3 respectively.

Binomial Theorem: For any positive integer n,

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots + \binom{n}{r}a^{n-r}x^r + \cdots + \binom{n}{n-1}ax^n + \binom{n}{n}x^n$$

or briefly $(a+x)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}x^r$, where a and x are real numbers.

The rule of expansion given above is called the binomial theorem and it also holds if a or x is complex.

In general, the rule or formula for expansion of a binomial raised to any positive integral power n is called the binomial theorem for positive integral index n.

Question: State and prove the binomial theorem for any positive integer n.

Statement: For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

Now we prove the binomial theorem for any positive integer n, using the principle of mathematical induction.

Proof: Let $S(n)$ be the statement given above as (A).

Base Case: If $n = 1$,

$$S(1): \quad (a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}b$$

$$a + b = a + b \text{ (True)} \qquad \because \binom{1}{0} = \binom{1}{1} = 1$$

The base case is satisfied.

Induction of Hypothesis: Let us assume that the statement is true for any $n = k \in N$, then

$$S(k): (a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k}b^k \quad \text{...(A)}$$

$$S(k+1): (a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k+1}b^{k+1} \quad \text{...(B)}$$

Multiplying both sides of equation (B) by $(a+b)$, we have

$$(a+b)(a+b)^k = (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k}b^k\right]$$

$$(a+b)^{k+1} = \left[\binom{k}{o}a^{k+1} + \binom{k}{1}a^k b + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k}ab^k\right] +$$

$$\left[\binom{k}{0}a^k b + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k}b^{k+1}\right]$$

$$(a+b)^{k+1} = \binom{k}{0}a^{k+1} + \left[\binom{k}{1} + \binom{k}{0}\right]a^k b + \left[\binom{k}{2} + \binom{k}{1}\right]a^{k-1}b^2 + \cdots + \binom{k}{k}b^{k+1}$$

As we know $\binom{k}{0} = \binom{k+1}{0}$, $\binom{k}{k} = \binom{k+1}{k+1}$ and $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$ for $0 \le r \le k$

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k+1}b^{k+1} \quad \text{[As required in eq. (B)]}$$

We find that if the statement is true for $n = k$, then it is also true for $n = k + 1$.

Conclusion: Hence, we conclude that the statement is true for all positive integral values of n.

> **Note:** $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ are called the binomial coefficients.

The following points can be observed in the expansion of $(a + b)^n$

(i) The number of terms in the expansion is one greater than its index.

(ii) The sum of exponents of a and b in each term of the expansion is equal to its index.

(iii) The exponent of a decreases from index to zero.

(iv) The exponent of b increases from zero to index.

(v) The coefficients of the terms equidistant from beginning and end of the expansion are equal as $\binom{n}{r} = \binom{n}{n-r}$

(vi) The $(r+1)^{th}$ term in the expansion is $\binom{n}{r}a^{n-r}b^r$ and we denote it as T_{r+1} i.e., $T_{r+1} = \binom{n}{r}a^{n-r}b^r$

As all the terms of the expansion can be found from it by putting $r = 0, 1, 2, \cdots, n$, so we call it as the general term of the expansion.

Example 13: Expand $\left(\frac{a}{2} - \frac{2}{a}\right)^6$ and also find its general term.

Solution: $\left(\frac{a}{2} - \frac{2}{a}\right)^6 = \left(\frac{a}{2} + \left(\frac{-2}{a}\right)\right)^6$

$$= \binom{6}{1}\left(\frac{a}{2}\right)^6\left(-\frac{2}{a}\right)^0 + \binom{6}{1}\left(\frac{a}{2}\right)^5\left(-\frac{2}{a}\right) + \binom{6}{2}\left(\frac{a}{2}\right)^4\left(-\frac{2}{a}\right)^2 + \binom{6}{3}\left(\frac{a}{2}\right)^3\left(-\frac{2}{a}\right)^3 +$$

$$\binom{6}{4}\left(\frac{a}{2}\right)^2\left(-\frac{2}{a}\right)^4 + \binom{6}{5}\left(\frac{a}{2}\right)\left(-\frac{2}{a}\right)^5 + \binom{6}{6}\left(\frac{a}{2}\right)^0\left(-\frac{2}{a}\right)^6$$

$$= \frac{a^6}{64} + 6\left(\frac{a^5}{32}\right)\left(-\frac{2}{a}\right) + 15\frac{a^4}{16}\frac{4}{a^2} + 20\frac{a^3}{8}\left(-\frac{8}{a^3}\right) + 15\frac{a^2}{4}\frac{16}{a^4} + 6\frac{a}{2}\left(-\frac{32}{a^5}\right) + \frac{64}{a^6}$$

$$\left(\frac{a}{2} - \frac{2}{a}\right)^6 = \frac{a^6}{64} - \frac{3}{8}a^4 + \frac{15}{4}a^2 - 20 + \frac{60}{a^2} - \frac{96}{a^4} + \frac{64}{a^6}$$

As $\left(\frac{a}{2} - \frac{2}{a}\right)^6 = \left(\frac{a}{2} + \left(\frac{-2}{a}\right)\right)^6$

Here $a = \frac{a}{2}$, $b = -\frac{2}{a}$, $n = 6$

T_{r+1}, the general term is given by

$$T_{r+1} = \binom{n}{r}a^{n-r}b^r$$

$$T_{r+1} = \binom{6}{r}\left(\frac{a}{2}\right)^{6-r}\left(-\frac{2}{a}\right)^r = \binom{6}{r}\frac{a^{6-r}}{2^{6-r}}(-1)^r\frac{2^r}{a^r}$$

$$= (-1)^r\binom{6}{r}\frac{a^{6-r}a^{-r}}{2^{6-r}2^{-r}} = (-1)^r\binom{6}{r}\frac{a^{6-2r}}{2^{6-2r}} = (-1)^r\binom{6}{r}\left(\frac{a}{2}\right)^{6-2r}$$

Example 14: Evaluate $(9.9)^5$ using binomial theorem.

Solution: $(9.9)^5 = (10 - 0.1)^5$

$$= (10)^5 + \binom{5}{1}(10)^4(-0.1) + \binom{5}{2}(10)^3(-0.1)^2 + \binom{5}{3}(10)^2(-0.1)^3 + \binom{5}{4}(10)(-0.1)^4 + (-0.1)^5$$

$$= (10)^5 + (5)(10)^4(-0.1) + 10(10)^3(-0.1)^2 + 10(10)^2(-0.1)^3 + 5(10)(-0.1)^4 + (-0.1)^5$$

$$= 100000 - (0.5)(10000) + (10000)(0.01) + 1000(-0.001) + (50)(0.0001) - 0.00001$$

$$= 100000 - 5000 + 100 - 1 + 0.005 - 0.000001$$

$$= 100100.005 - 5001.00001$$

$$= 95099.00499$$

Example 15: Find the specified term in the expansion of $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$;

(i) the term involving x^5 (ii) the fifth term

(iii) the sixth term from the end (iv) coefficient of the term involving x^{-1}.

Solution:

(i)

$$\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$$

Here $a = \frac{3}{2}x$, $b = -\frac{1}{3x}$, $n = 11$

By using general term formula

$$T_{r+1} = \binom{n}{r}a^{n-r}b^r$$

$$T_{r+1} = \binom{11}{r}\left(\frac{3}{2}x\right)^{11-r}\left(-\frac{1}{3x}\right)^r = \binom{11}{r}\frac{3^{11-r}}{2^{11-r}}x^{11-r}\cdot(-1)^r\cdot 3^{-r}\cdot x^{-r}$$

$$T_{r+1} = (-1)^r\cdot\binom{11}{r}\frac{3^{11-2r}}{2^{11-r}}\cdot x^{11-2r} \quad \text{...(1)}$$

As this term involves x^5, so the exponent of x is 5, that is,

$$11-2r=5 \quad \Rightarrow -2r=5-11 \quad \Rightarrow -2r=-6 \quad \Rightarrow r=3 \text{ put this in eq. (1)}$$

$$\therefore \quad T_{3+1}=(-1)^3\binom{11}{3}\frac{3^{11-6}}{2^{11-3}}\cdot x^{11-6}=(-1)(165)\cdot\frac{3^5}{2^8}x^5$$

$$T_4=-\frac{165\cdot 243}{256}x^5=-\frac{40095}{256}x^5$$

(ii) For the 5^{th} term, put $r+1=5 \Rightarrow r=4$ in eq. (1), we get

$$\therefore \quad T_{4+1}=(-1)^4\binom{11}{4}\frac{3^{11-8}}{2^{11-4}}x^{11-8}=(1)(330)\cdot\frac{3^3}{2^7}x^3$$

$$T_5=\frac{330\times 27}{128}x^3=\frac{4455}{64}x^3$$

(iii) The 6th term from the end term will have $(11+1)-6$ that is, 6 terms before it,

$\therefore$ It will be $(6+1)^{th}$ term, that is the 7^{th} term of the expansion.

Thus $$T_7=(-1)^6\binom{11}{6}\frac{3^{11-12}}{2^{11-6}}x^{11-12}=462\cdot\frac{3^{-1}}{2^5}x^{-1}$$

$$=\frac{462}{3\times 32}\cdot\frac{1}{x}=\frac{77}{16x}$$

Alternate Method:

$$\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$$

Here $a=-\frac{1}{3x},\ b=\frac{3}{2}x,\ n=11,\ r+1=6 \Rightarrow r=5$

By using general term formula

$$T_{r+1}=\binom{n}{r}a^{n-r}b^r$$

$$T_{5+1}=\binom{11}{5}\left(-\frac{1}{3x}\right)^{11-5}\left(\frac{3}{2}x\right)^5=462\left(-\frac{1}{3x}\right)^6\left(\frac{3^5.x^5}{2^5}\right)=462\left(\frac{1}{3^6x^6}\right)\left(\frac{3^5.x^5}{2^5}\right)$$

$$T_6=462\left(\frac{1}{3^6x^6}\right)\left(\frac{3^5.x^5}{2^5}\right)=462\left(\frac{1}{3x}\right)\left(\frac{1}{32}\right)=\frac{77}{16x}$$

(iv) For the term involving x^{-1}, put $11-2r=-1 \Rightarrow -2r=-12 \Rightarrow r=6$ in eq. (1)

$$T_{6+1}=(-1)^6\cdot\binom{11}{6}\frac{3^{11-12}}{2^{11-6}}\cdot x^{11-12}=(1)\cdot(462)\frac{3^{-1}}{2^5}\cdot x^{-1}=\frac{462}{3.32}\cdot x^{-1}$$

$$T_7=\frac{77}{16}\cdot x^{-1}$$

Thus, th coefficient of the term involving x^{-1} is $\frac{77}{16}$.

The Middle Term in the Expansion of:

In the expansion of $(a+b)^n$, the total number of terms are $n+1$

Case I: (n is even)

If n is even then $n+1$ is odd, so $\left(\frac{n}{2}+1\right)^{th}$ term will be the only one middle term in the expansion.

Case II: (n is odd)

If n is odd then $n+1$ is even so $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms of the expansion will be the two middle terms.

Example 16: Find the following in the expansion of $\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$;

(i) the term independent of x (ii) the middle term

Solution:

(i) $$\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$$

Here $a=\frac{x}{2},\ b=\frac{2}{x^2},\ n=12$

By using general term formula

$$T_{r+1}=\binom{n}{r}a^{n-r}b^r$$

$$T_{r+1}=\binom{12}{r}\left(\frac{x}{2}\right)^{12-r}\left(\frac{2}{x^2}\right)^r \quad \dots (1)$$

$$=\binom{12}{r}\frac{x^{12-r}}{2^{12-r}}\cdot 2^r\cdot x^{-2r}=\binom{12}{r}2^{2r-12}\cdot x^{12-3r}$$

As the term is independent of x, so exponent of x, will be zero.

That is, $12-3r=0 \Rightarrow r=4$.

$$T_{4+1}=\binom{12}{4}2^{8-12}x^{12-12}$$

$$T_5=495\cdot 2^{-4}x^0=\frac{495}{2^4}=\frac{495}{16}$$

(ii) In this case, $n=12$ which is even, so $\left(\frac{12}{2}+1\right)^{th}=7^{th}$ term is the middle term.

For the 7^{th} term, put $r+1=7 \Rightarrow r=6$ in eq. (1), we get

$$T_{6+1}=\binom{12}{6}\left(\frac{x}{2}\right)^{12-6}\cdot\left(\frac{2}{x^2}\right)^6$$

$$=(924)\frac{x^6}{2^6}\cdot\frac{2^6}{x^{12}}=924\cdot x^{6-12}=924\cdot x^{-6}$$

$$T_7=\frac{924}{x^6}$$

Some Deductions from the Binomial Expansion of $(a+x)^n$

(i) Prove that the sum of coefficients in the binomial expansion equals to 2^n.

Or

Prove that $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n-1}+\binom{n}{n}=2^n$

Proof:

As we know:

$$(a+b)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\dots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n$$

Put $a = 1$ and $b = 1$, in (A), then we have;

$$(1+1)^n = \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}(1) + \binom{n}{2}(1)^{n-2}(1)^2 + \cdots + \binom{n}{n-1}(1)(1)^{n-1} + \binom{n}{n}(1)^n$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Thus, the sum of coefficients in the binomial expansion equals to 2^n.

(ii) Prove that the sum of odd coefficients of a binomial expansion equals to the sum of its even coefficients

Proof:

As we know:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \quad \ldots (A)$$

Putting $a = 1$ and $b = -1$, in (A), we have

$$(1-1)^n = \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}(-1) + \binom{n}{2}(1)^{n-2}(-1)^2 + \binom{n}{3}(1)^{n-3}(-1)^3 + \cdots + \binom{n}{n-1}(1)(-1)^{n-1} + \binom{n}{n}(-1)^n$$

$$(0)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0 \quad \ldots (1)$$

If n is odd positive integer, then eq. (1) becomes .

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n-1} - \binom{n}{n} = 0$$

$$\Rightarrow \quad \boxed{\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-1} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n}}$$

If n is even positive integer, then eq. (1) becomes

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots - \binom{n}{n-1} + \binom{n}{n} = 0$$

$$\Rightarrow \quad \boxed{\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1}}$$

Thus, sum of odd coefficients of a binomial expansion equals to the sum of its even coefficients.

Example 17: Show that: $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$

Solution:

$$\text{L.H.S} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$$

$$= \frac{n!}{1!(n-1)!} + 2.\frac{n!}{2!(n-2)!} + 3.\frac{n!}{3!(n-3)!} + \cdots + n.\frac{n!}{n!(n-n)!}$$

$$= \frac{n(n-1)!}{(n-1)!} + 2.\frac{n(n-1)(n-2)!}{2(n-2)!} + 3.\frac{n(n-1)(n-2)(n-3)!}{6(n-3)!} + \cdots + n.\frac{1}{0!}$$

$$= n + n(n-1) + \frac{n(n-1)(n-2)}{2!} + \cdots + n$$

$$= n\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \cdots + 1\right]$$

$$= n\left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1}\right]$$

As $\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1} = 2^{n-1}$, so

$$= n \cdot 2^{n-1} = \text{R.H.S (Proved)}$$

Exercise 8.2

1. Using binomial theorem, expand the following:

(i) $\left(\frac{x}{2} - \frac{2}{x^2}\right)^6$

Solution:

$$\left(\frac{x}{2} - \frac{2}{x^2}\right)^6 = \binom{6}{0}\left(\frac{x}{2}\right)^6\left(-\frac{2}{x^2}\right)^0 + \binom{6}{1}\left(\frac{x}{2}\right)^5\left(-\frac{2}{x^2}\right)^1 + \binom{6}{2}\left(\frac{x}{2}\right)^4\left(-\frac{2}{x^2}\right)^2 + \binom{6}{3}\left(\frac{x}{2}\right)^3\left(-\frac{2}{x^2}\right)^3 + \binom{6}{4}\left(\frac{x}{2}\right)^2\left(-\frac{2}{x^2}\right)^4 +$$

$$\binom{6}{5}\left(\frac{x}{2}\right)^1\left(-\frac{2}{x^2}\right)^5 + \binom{6}{6}\left(\frac{x}{2}\right)^0\left(-\frac{2}{x^2}\right)^6$$

$$= 1\times\frac{x^6}{64}\times 1 - 6\times\frac{x^5}{32}\times\frac{2}{x^2} + 15\times\frac{x^4}{16}\times\frac{4}{x^4} - 20\times\frac{x^3}{8}\times\frac{8}{x^6} + 15\times\frac{x^2}{4}\times\frac{16}{x^8} - 6\times\frac{x^1}{2}\times\frac{32}{x^{10}} + 1\times\frac{64}{x^{12}}$$

$$= \frac{x^6}{64} - \frac{3}{8}x^3 + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}$$

(ii) $\left(2a - \frac{x}{a}\right)^7$

Solution:

$$\left(2a - \frac{x}{a}\right)^7 = \binom{7}{0}(2a)^7\left(-\frac{x}{a}\right)^0 + \binom{7}{1}(2a)^6\left(-\frac{x}{a}\right)^1 + \binom{7}{2}(2a)^5\left(-\frac{x}{a}\right)^2 + \binom{7}{3}(2a)^4\left(-\frac{x}{a}\right)^3 + \binom{7}{4}(2a)^3\left(-\frac{x}{a}\right)^4 +$$

$$\binom{7}{5}(2a)^2\left(-\frac{x}{a}\right)^5 + \binom{7}{6}(2a)^1\left(-\frac{x}{a}\right)^6 + \binom{7}{7}(2a)^0\left(-\frac{x}{a}\right)^7$$

$$= 1\times 128a^7\times 1 - 7\times 64a^6\times\frac{x}{a} + 21\times 32a^5\times\frac{x^2}{a^2} - 35\times 16a^4\times\frac{x^3}{a^3} + 35\times 8a^3\times\frac{x^4}{a^4} - 21\times 4a^2\times\frac{x^5}{a^5} + 7\times 2a^1\times\frac{x^6}{a^6} - 1\times 1\times\frac{x^7}{a^7}$$

$$= 128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + 280\frac{x^4}{a} - 84\frac{x^5}{a^3} + 14\frac{x^6}{a^5} - \frac{x^7}{a^7}$$

(iii) $\left(\sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}}\right)^6$

Solution:

$$\left(\sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}}\right)^6 = \binom{6}{0}\left(\sqrt{\frac{a}{x}}\right)^6\left(-\sqrt{\frac{x}{a}}\right)^0 + \binom{6}{1}\left(\sqrt{\frac{a}{x}}\right)^5\left(-\sqrt{\frac{x}{a}}\right)^1 + \binom{6}{2}\left(\sqrt{\frac{a}{x}}\right)^4\left(-\sqrt{\frac{x}{a}}\right)^2 + \binom{6}{3}\left(\sqrt{\frac{a}{x}}\right)^3\left(-\sqrt{\frac{x}{a}}\right)^3 + \binom{6}{4}\left(\sqrt{\frac{a}{x}}\right)^2\left(-\sqrt{\frac{x}{a}}\right)^4 +$$

$$\binom{6}{5}\left(\sqrt{\frac{a}{x}}\right)^1\left(-\sqrt{\frac{x}{a}}\right)^5 + \binom{6}{6}\left(\sqrt{\frac{a}{x}}\right)^0\left(-\sqrt{\frac{x}{a}}\right)^6$$

$$= \frac{a^3}{x^3} - 6\frac{a^2}{x^2} + 15\frac{a}{x} - 20 + 15\frac{x}{a} - 6\frac{x^2}{a^2} + \frac{x^3}{a^3}$$

2. Calculate the following by means of binomial theorem:

(i) $(0.97)^3$

Solution:

$(0.97)^3 = (1-0.03)^3$

$= \binom{3}{0}(1)^3(-0.03)^0 + \binom{3}{1}(1)^2(-0.03)^1 + \binom{3}{2}(1)^1(-0.03)^2 + \binom{3}{3}(1)^0(-0.03)^3$

$= 1\times1\times1 - 3\times1\times0.03 + 3\times1\times0.0009 - 1\times1\times0.000027$

$= 1-0.09+0.0027-0.000027 = 0.91267$

(ii) $(2.02)^4$

Solution:

$(2.02)^4 = (2+0.02)^4$

$= \binom{4}{0}(2)^4(0.02)^0 + \binom{4}{1}(2)^3(0.02)^1 + \binom{4}{2}(2)^2(0.02)^2 + \binom{4}{3}(2)^1(0.02)^3 + \binom{4}{4}(2)^0(0.02)^4$

$= 1\times16\times1 + 4\times8\times0.02 + 6\times4\times0.0004 + 4\times2\times0.000008 + 1\times1\times0.00000016$

$= 16+0.64+0.0096+0.000064+0.00000016$

$= 16.64966416$

(iii) $(9.98)^4$

Solution:

$(9.98)^4 = (10-0.02)^4$

$= \binom{4}{0}(10)^4(-0.02)^0 + \binom{4}{1}(10)^3(-0.02)^1 + \binom{4}{2}(10)^2(-0.02)^2 + \binom{4}{3}(10)^1(-0.02)^3 + \binom{4}{4}(10)^0(-0.02)^4$

$= 1\times10000\times1 - 4\times1000\times0.02 + 6\times100\times0.0004 - 4\times10\times0.000008 + 1\times1\times0.00000016$

$= 10000-80+0.24-0.00032+0.00000016$

$= 9920.23968016$

(iv) $(2.1)^5$

Solution:

$(2.1)^5 = (2+0.1)^5$

$= \binom{5}{0}(2)^5(0.1)^0 + \binom{5}{1}(2)^4(0.1)^1 + \binom{5}{2}(2)^3(0.1)^2 + \binom{5}{3}(2)^2(0.1)^3 + \binom{5}{4}(2)^1(0.1)^4 + \binom{5}{5}(2)^0(0.1)^5$

$= 1.32.1 + 5.16.(0.1) + 10.8.(0.01) + 10.4.(0.001) + 5.2.(0.0001) + 1.1.(0.00001)$

$= 32 + 8 + 0.8 + 0.04 + 0.001 + 0.00001$

$= 40.84101$

3. Expand and simplify the following:

(i) $(a+\sqrt{2}x)^4 + (a-\sqrt{2}x)^4$

Solution:

Let $\sqrt{2}x = b$

$(a+b)^4 = \binom{4}{0}(a)^4(b)^0 + \binom{4}{1}(a)^3(b)^1 + \binom{4}{2}(a)^2(b)^2 + \binom{4}{3}(a)^1(b)^3 + \binom{4}{4}(a)^0(b)^4$

$= 1\times a^4\times1 + 4\times a^3\times b + 6\times a^2\times b^2 + 4\times a^1\times(b)^3 + 1\times1\times b^4$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$ ...(1)

Similarly, $(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4$ ...(2)

Equation (1) + Equation (2)

$(a+b)^4+(a-b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4+a^4-4a^3b+6a^2b^2-4ab^3+b^4$

$= 2a^4+12a^2b^2+2b^4$

$(a+\sqrt{2}x)^4+(a-\sqrt{2}x)^4 = 2a^4+12a^2(\sqrt{2}x)^2+2(\sqrt{2}x)^4$ $\because b=\sqrt{2}x$

$= 2a^4+24a^2x^2+8x^4$

(ii) $(2+\sqrt{3})^5+(2-\sqrt{3})^5$

Solution:

Let $\sqrt{3}=b$

$(2+b)^5 = \binom{5}{0}(2)^5(b)^0 + \binom{5}{1}(2)^4(b)^1 + \binom{5}{2}(2)^3(b)^2 + \binom{5}{3}(2)^2(b)^3 + \binom{5}{4}(2)^1(b)^4 + \binom{5}{5}(2)^0(b)^5$

$= 1\times32\times1+5\times16\times b+10\times8\times b^2+10\times4\times(b)^3+5\times2\times b^4+1\times1\times(b)^5$

$(2+b)^5 = 32+80b+80b^2+40(b)^3+10b^4+(b)^5$ ...(1)

Similarly, $(2-b)^5 = 32-80b+80b^2-40(b)^3+10b^4-(b)^5$ ...(2)

Equation (1) + Equation (2)

$(2+b)^5+(2-b)^5 = 32+80b+80b^2+40(b)^3+10b^4+(b)^5+32-80b+80b^2-40(b)^3+10b^4-(b)^5$

$= 64+160b^2+20b^4$

$(2+\sqrt{3})^5+(2-\sqrt{3})^5 = 64+160(\sqrt{3})^2+20(\sqrt{3})^4$ $\because b=\sqrt{3}$

$= 64+480+180 = 724$

4. Expand the following in ascending power of x:

(i) $(2+x-x^2)^4$

Solution:

$(2+(x-x^2))^4 = \binom{4}{0}(2)^4(x-x^2)^0 + \binom{4}{1}(2)^3(x-x^2)^1 + \binom{4}{2}(2)^2(x-x^2)^2 + \binom{4}{3}(2)^1(x-x^2)^3 + \binom{4}{4}(2)^0(x-x^2)^4$

$= 1\times16\times1+4\times8\times(x-x^2)+6\times4\times(x^2-2x^3+x^4)+4\times2\times(x^3-3x^4+3x^5-x^6)+1\times1\times(x^4-4x^5+6x^6-4x^7+x^8)$

$= 16+32x-32x^2+24x^2-48x^3+24x^4+8x^3-24x^4+24x^5-8x^6+x^4-4x^5+6x^6-4x^7+x^8$

$= 16+32x-8x^2-40x^3+x^4+20x^5-2x^6-4x^7+x^8$

(ii) $(1-x+x^2)^4$

Solution:

$(1-(x-x^2))^4 = \binom{4}{0}(1)^4(x-x^2)^0 - \binom{4}{1}(1)^3(x-x^2)^1 + \binom{4}{2}(1)^2(x-x^2)^2 - \binom{4}{3}(1)^1(x-x^2)^3 + \binom{4}{4}(1)^0(x-x^2)^4$

$= 1\times1\times1-4\times1\times(x-x^2)+6\times1\times(x^2-2x^3+x^4)-4\times1\times(x^3-3x^4+3x^5-x^6)+1\times1\times(x^4-4x^5+6x^6-4x^7+x^8)$

$= 1-4x+4x^2+6x^2-12x^3+6x^4-4x^3+12x^4-12x^5+4x^6+x^4-4x^5+6x^6-4x^7+x^8$

$= 1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$

5. Find the term involving:

(i) x^4 in the expansion of $(3-2x)^7$

Solution:

Here $a=3, b=-2x, n=7, r=?$

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$

$$=\binom{7}{r}3^{7-r}(-2x)^r=\binom{7}{r}3^{7-r}(-2)^r(x)^r$$

For the term involving x^4 put $r=4$

$$T_{4+1}=\binom{7}{4}3^{7-4}(-2)^4(x)^4$$

$$T_5=35\times3^3\times16\times x^4=35\times27\times16\times x^4=15120x^4$$

(ii) x^{-2} in the expansion of $\left(x-\frac{2}{x^2}\right)^{13}$

Solution:

Here $a=x, b=-\frac{2}{x^2}, n=13, r=?$

$$T_{r+1}=\binom{n}{r}a^{n-r}b^r$$

$$=\binom{13}{r}x^{13-r}\left(-\frac{2}{x^2}\right)^r=\binom{13}{r}x^{13-r}\frac{(-2)^r}{x^{2r}}=\binom{13}{r}x^{13-r-2r}(-2)^r$$

$$=\binom{13}{r}x^{13-3r}(-2)^r$$

For the term involving x^{-2} put $13-3r=-2$

$\Rightarrow\quad 13+2=3r\quad\Rightarrow 15=3r\quad\Rightarrow r=5$

$$T_{5+1}=\binom{13}{5}x^{-2}(-2)^5$$

$$T_6=1287\times x^{-2}\times-32=-41184x^{-2}$$

(iii) a^4 in the expansion of $\left(\frac{2}{x}-a\right)^9$

Solution:

Here $a=\frac{2}{x}, b=-a, n=9, r=?$

$$T_{r+1}=\binom{n}{r}a^{n-r}b^r$$

$$=\binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r=\binom{9}{r}\frac{(2)^{9-r}}{x^{9-r}}(-1)^r(a)^r$$

$$=\binom{9}{r}(2)^{9-r}.x^{-9+r}(-1)^r(a)^r$$

For the term involving a^4, put $r=4$

$$T_{4+1}=\binom{9}{4}(2)^{9-4}x^{-9+4}(-1)^4(a)^4$$

$$T_5=126\times(2)^5x^{-5}(a)^4=126\times32x^{-5}a^4$$

$$T_5=4032\frac{a^4}{x^5}$$

(iv) y^3 in the expansion of $(x-\sqrt{y})^{11}$

Solution:

Here $a=x, b=-\sqrt{y}, n=11, r=?$

$$T_{r+1}=\binom{n}{r}a^{n-r}b^r$$

$$=\binom{11}{r}(x)^{11-r}(-\sqrt{y})^r=\binom{11}{r}x^{11-r}(-1)^r(y)^{\frac{r}{2}}$$

For the term involving y^3 put $\frac{r}{2}=3\quad\Rightarrow r=6$

$$T_{6+1}=\binom{11}{6}x^{11-6}(-1)^6(y)^3$$

$$T_7=462x^5y^3$$

6. Find the coefficient of; x^5 in the expansion of:

(i) $\left(x^2-\frac{3}{2x}\right)^{10}$

Solution:

Here $a=x^2, b=\frac{-3}{2x}, n=10, r=?$

$$T_{r+1}=\binom{n}{r}a^{n-r}b^r$$

$$=\binom{10}{r}(x^2)^{10-r}\left(\frac{-3}{2x}\right)^r=\binom{10}{r}x^{20-2r}\left(\frac{-3}{2}\right)^r.\frac{1}{x^r}=\binom{10}{r}x^{20-2r-r}\left(\frac{-3}{2}\right)^r$$

$$T_{r+1}=\binom{10}{r}x^{20-3r}\left(\frac{-3}{2}\right)^r$$

For the term involving x^5 put $20-3r=5\quad\Rightarrow 20-5=3r\quad\Rightarrow 3r=15\quad\Rightarrow r=5$

$$T_6=252.\frac{-243}{32}x^5=\frac{-15309}{8}x^5$$

Thus coefficient of $x^5=\frac{-15309}{8}$

(ii) x^n in the expansion of $\left(x^2-\frac{1}{x}\right)^{2n}$

Solution:

Here $a=x^2, b=\frac{-1}{x}, N=2n, r=?$

$$T_{r+1}=\binom{N}{r}a^{N-r}b^r$$

$$=\binom{2n}{r}(x^2)^{2n-r}\left(\frac{-1}{x}\right)^r=\binom{2n}{r}x^{4n-2r}\frac{(-1)^r}{x^r}=\binom{2n}{r}x^{4n-2r-r}(-1)^r$$

$$=\binom{2n}{r}x^{4n-3r}(-1)^r$$

For the term involving x^n put $4n-3r=n\quad\Rightarrow 4n-n=3r\Rightarrow 3r=3n\quad\Rightarrow r=n$

$$T_{n+1}=\binom{2n}{n}x^n(-1)^n$$

$$=\frac{2n!}{(2n-n)!.n!}(-1)^nx^n=(-1)^n\frac{2n!}{(n!)^2}x^n$$

Thus coefficient of $x^n=(-1)^n\frac{2n!}{(n!)^2}$

7. Find 6^{th} term in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$.

Solution:

Here $a=x^2, b=\frac{-3}{2x}, n=10, r+1=6 \Rightarrow r=5$

$$T_{r+1}=\binom{n}{r}a^{n-r}\,b^r$$

$$T_{5+1}=\binom{10}{5}(x^2)^{10-5}\left(\frac{-3}{2x}\right)^5$$

$$T_6=\binom{10}{5}x^{10}\left(\frac{-3}{2}\right)^5.x^{-5}$$

$$=\binom{10}{5}x^5\left(\frac{-3}{2}\right)^5=252\frac{-243}{32}x^5=\frac{-15309}{8}x^5$$

8. Find the term independent of x in the following expansions:

(i) $\left(x-\frac{2}{x}\right)^{10}$

Solution:

Here $a=x, b=\frac{-2}{x}, n=10, r=?$

$$T_{r+1}=\binom{n}{r}a^{n-r}\,b^r$$

$$=\binom{10}{r}(x)^{10-r}\left(\frac{-2}{x}\right)^r=\binom{10}{r}x^{10-r}\frac{(-2)^r}{x^r}$$

$$=\binom{10}{r}x^{10-r-r}(-2)^r=\binom{10}{r}x^{10-2r}(-2)^r$$

For the term independent of x put $10-2r=0 \Rightarrow 10=2r \quad \Rightarrow r=5$

$$T_{5+1}=\binom{10}{5}x^0(-2)^5$$

$$T_6=252(-32)=-8064$$

(ii) $\left(\sqrt{x}+\frac{1}{2x^2}\right)^{10}$

Solution:

Here $a=\sqrt{x}, b=\frac{1}{2x^2}, n=10, r=?$

$$T_{r+1}=\binom{n}{r}a^{n-r}\,b^r$$

$$=\binom{10}{r}(\sqrt{x})^{10-r}\left(\frac{1}{2x^2}\right)^r$$

$$=\binom{10}{r}x^{\frac{10-r}{2}}\left(\frac{1}{2}\right)^r.\frac{1}{x^{2r}}=\binom{10}{r}x^{\frac{10-r}{2}-2r}\left(\frac{1}{2}\right)^r$$

$$=\binom{10}{r}x^{\frac{10-r-4r}{2}}\left(\frac{1}{2}\right)^r$$

For the term independent of x put $\frac{10-r-4r}{2}=0 \Rightarrow 10-5r=0 \Rightarrow 5r=10 \Rightarrow r=2$

$$T_{2+1}=\binom{10}{2}x^0\left(\frac{1}{2}\right)^2$$

$$T_3=45\left(\frac{1}{4}\right)=\frac{45}{4}$$

(iii) $(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4$

Solution:

$$(1+x^2)^3\left[1+\frac{1}{x^2}\right]^4=(1+x^2)^3\left[\frac{x^2+1}{x^2}\right]^4=(1+x^2)^3\frac{(1+x^2)^4}{(x^2)^4}=\frac{(1+x^2)^7}{x^8}=\frac{(1+x^2)^7}{(x^{\frac{8}{7}})^7}$$

$$=\left(\frac{1+x^2}{x^{\frac{8}{7}}}\right)^7=\left(\frac{1}{x^{\frac{8}{7}}}+\frac{x^2}{x^{\frac{8}{7}}}\right)^7=\left(x^{-\frac{8}{7}}+x^{2-\frac{8}{7}}\right)^7=\left(x^{-\frac{8}{7}}+x^{\frac{14-8}{7}}\right)^7=\left(x^{-\frac{8}{7}}+x^{\frac{6}{7}}\right)^7$$

Here $a=x^{\frac{6}{7}}, b=x^{-\frac{8}{7}}, n=7, r=?$

$$T_{r+1}=\binom{n}{r}a^{n-r}\,b^r$$

$$=\binom{7}{r}\left(x^{\frac{6}{7}}\right)^{7-r}\left(x^{-\frac{8}{7}}\right)^r$$

$$=\binom{7}{r}x^{6-\frac{6r}{7}}x^{-\frac{8r}{7}}=\binom{7}{r}x^{6-\frac{6r}{7}-\frac{8r}{7}}=\binom{7}{r}x^{6-\frac{14r}{7}}$$

$$T_{r+1}=\binom{7}{r}x^{6-2r}$$

For the term independent of x put $6-2r=0 \quad \Rightarrow 2r=6 \quad \Rightarrow r=3$

$$T_{3+1}=\binom{7}{3}x^0$$

$$T_4=35$$

9. Determine the middle term in the following expansions:

(i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$

Solution:

Here $n=12$ is even then

Middle term $=\left(\frac{n}{2}+1\right)$th term $=\left(\frac{12}{2}+1\right)$th term $=$ 7th term

Here $a=\frac{1}{x}, b=-\frac{x^2}{2}, n=12, r+1=7 \quad \Rightarrow r=6$

$$T_{r+1}=\binom{n}{r}a^{n-r}\,b^r$$

$$T_7=\binom{10}{6}\left(\frac{1}{x}\right)^{12-6}\left(-\frac{x^2}{2}\right)^6$$

$$= \binom{10}{6} x^{-6} \cdot \frac{x^{12}}{64} = 924 \cdot \frac{x^{12-6}}{64}$$

$$T_7 = \frac{231}{16} x^6$$

(ii) $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$

Solution:

Here $n = 11$ is odd then

$$\text{Middle terms} = \left(\frac{n+1}{2}\right)\text{th and } \left(\frac{n+3}{2}\right)\text{th terms}$$

$$= \left(\frac{11+1}{2}\right)\text{th and } \left(\frac{11+3}{2}\right)\text{th terms}$$

Middle terms = 6th and 7th terms

Here $a = \frac{3x}{2}, b = -\frac{1}{3x}, n = 11$

For 6th term:

$r + 1 = 6 \Rightarrow r = 5$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_{5+1} = \binom{11}{5}\left(\frac{3x}{2}\right)^{11-5}\left(-\frac{1}{32x}\right)^5$$

$$= \binom{11}{5}\left(\frac{3x}{2}\right)^6\left(-\frac{1}{3}\right)^5 \cdot x^{-5} = 462 \cdot \frac{729}{64}x^6\left(-\frac{1}{243}\right) \cdot x^{-5} = -\frac{693}{32}x^{6-5}$$

$$T_6 = -\frac{693}{32}x$$

For 7th term:

$r + 1 = 7 \Rightarrow r = 6$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_{6+1} = \binom{11}{6}\left(\frac{3x}{2}\right)^{11-6}\left(-\frac{1}{3x}\right)^6$$

$$= \binom{11}{6}\left(\frac{3x}{2}\right)^5\left(-\frac{1}{3}\right)^6 \cdot x^{-6} = 462 \cdot \frac{243}{32}x^5\left(\frac{1}{729}\right) \cdot x^{-6} = \frac{77}{16}x^{-6+5} = \frac{77}{16}x^{-1}$$

$$T_7 = \frac{77}{16x}$$

(iii) $\left(2x - \frac{1}{2x}\right)^{2m+1}$

Solution:

Here $n = 2m + 1$ is odd then

$$\text{Middle terms} = \left(\frac{n+1}{2}\right)\text{th and } \left(\frac{n+3}{2}\right)\text{th terms}$$

$$= \left(\frac{2m+1+1}{2}\right)\text{th}, \left(\frac{2m+1+3}{2}\right)\text{th terms}$$

Middle terms $= (m + 1)$th and $(m + 2)$th terms

For $(m + 1)$th term:

$r + 1 = m + 1 \Rightarrow r = m$

Here $a = 2x, b = -\frac{1}{2x}, n = 2m + 1, r = m$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_{m+1} = \binom{2m+1}{m}(2x)^{2m+1-m}\left(-\frac{1}{2x}\right)^m$$

$$= \binom{2m+1}{m}(2x)^{m+1} \cdot \frac{(-1)^m}{(2x)^m} = \binom{2m+1}{m}(2x)^{m+1-m} \cdot (-1)^m = \frac{(2m+1)!}{m!(2m+1-m)!}(-1)^m \cdot (2x)^1$$

$$= \frac{(2m+1)!}{m!(m+1)!}(-1)^m \cdot 2x$$

$$T_{m+1} = 2(-1)^m \frac{(2m+1)!}{m!(m+1)!}x$$

For $(m + 2)$th term:

$r + 1 = m + 2 \Rightarrow r = m + 1$

Here $a = 2x, b = -\frac{1}{2x}, n = 2m + 1$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_{m+2} = \binom{2m+1}{m+1}(2x)^{2m+1-m-1}\left(-\frac{1}{2x}\right)^{m+1}$$

$$= \binom{2m+1}{m+1}(2x)^m \frac{(-1)^{m+1}}{(2x)^{m+1}} = \frac{(2m+1)!}{(m+1)!(2m+1-m-1)!}(-1)^{m+1} \cdot (2x)^{m-m-1} = \frac{(2m+1)!}{(m+1)!(m)!} \cdot (-1)^{m+1} \cdot (2x)^{-1}$$

$$T_{m+2} = \frac{1}{2}(-1)^{m+1} \frac{(2m+1)!}{(m+1)!(m)!} \cdot \frac{1}{x}$$

---

10. Show that: $\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} = 2^n - 1$

Solution:

As we know

$$(a+x)^n = \binom{n}{0}a^n x^0 + \binom{n}{1}a^{n-1}x^1 + \binom{n}{2}a^{n-2} \cdot x^2 + \binom{n}{3}a^{n-3} \cdot x^3 + \ldots + \binom{n}{n}a^0 \cdot x^n$$

Put $a = 1$ and $x = 1$

$$(1+1)^n = \binom{n}{0}(1)^n(1)^0 + \binom{n}{1}(1)^{n-1}(1) + \binom{n}{2}(1)^{n-2} \cdot (1)^2 + \binom{n}{3}(1)^{n-3}(1)^3 + \ldots + \binom{n}{n}(1)^0(1)^n$$

$$2n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}$$

$$2n = 1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} \quad \because \binom{n}{0} = 1$$

$$2n - 1 = \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}$$

$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} = 2^n - 1 \text{ (Proved)}$$

The Binomial Theorem (When the Index n is a Negative Integer or a Fraction):

When n is a negative integer or a fraction, then

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots+\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r+\cdots$$

provided $|x|<1$.

The series of the type $1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$ is called the binomial series.

Note:

- The proof of this theorem is beyond the scope of this book.
- Symbols $\binom{n}{0},\binom{n}{1},\binom{n}{2}$ etc. are meaningless when n is a negative integer or a fraction.
- The general term in the expansion is $T_{r+1}=\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$

Example 18: Find the general term in the expansion of $(1+x)^{-3}$ when $|x|<1$.

Solution:

$(1+x)^{-3}$

By using the general term formula

$$T_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

$$= \frac{-3(-3-1)(-3-2)\cdots(-3-r+1)}{r!}x^r \qquad \because n=-3$$

$$= \frac{(-3)(-4)(-5)\cdots(-r-2)}{r!}x^r$$

$$= \frac{(-1)^r\cdot 3\cdot 4\cdot 5\cdots(r+2)}{r!}x^r \qquad \text{(By taking } -1 \text{ common from each factor)}$$

$$= (-1)^r\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(r+2)}{1\cdot 2\cdot r!}x^r \qquad \text{(Multiply and divide by 1.2)}$$

$$= (-1)^r\frac{[1\cdot 2\cdot 3\cdot 4\cdot 5\cdots r](r+1)(r+2)}{1\cdot 2\cdot r!}x^r$$

$$= (-1)^r\frac{r!(r+1)(r+2)}{2\cdot r!}x^r = (-1)^r\frac{(r+1)(r+2)}{2}x^r$$

Some particular cases of the expansion of $(1+x)^n, n<0$.
(i) $(1+x)^{-1}=1-x+x^2-x^3+\cdots+(-1)^r x^r+\cdots$
(ii) $(1+x)^{-2}=1-2x+3x^2-4x^3+\ldots+(-1)^r(r+1)x^r+\ldots$
(iii) $(1+x)^{-3}=1-3x+6x^2-10x^3+\cdots+(-1)^r\frac{(r+1)(r+2)}{2}x^r+\cdots$
(iv) $(1-x)^{-1}=1+x+x^2+x^3+\cdots+x^r+\cdots$
(v) $(1-x)^{-2}=1+2x+3x^2+4x^3+\cdots+(r+1)x^r+\cdots$
(vi) $(1-x)^{-3}=1+3x+6x^2+10x^3+\cdots+\frac{(r+1)(r+2)}{2}x^r+\cdots$

Example 19: Find the coefficient of x^n in the expansion of $\frac{1-x}{(1+x)^2}$

Solution: $\frac{1-x}{(1+x)^2}=(1-x)(1+x)^{-2}$

$$=(-x+1)\left[1+(-2x)+\frac{(-2)(-2-1)}{2!}x^2+\cdots\right]$$

$$=(-x+1)\left[1+(-2x)+\frac{(-2)(-3)}{2!}x^2+\cdots\right]$$

$$=(-x+1)\left[1+(-1)^1 2x+(-1)^2 3x^2+\cdots+(-1)^{n-1}(n)x^{n-1}+(-1)^n(n+1)x^n+\cdots\right]$$

Coefficient of $x^n=(-1)(-1)^{n-1}n+(-1)^n(n+1)$

$=(-1)^n n+(-1)^n(n+1)=(-1)^n[n+(n+1)]=(-1)^n(2n+1)$

Example 20: If x is so small that its cube and higher power can be neglected, show that $\sqrt{\frac{1-x}{1+x}}\approx 1-x+\frac{1}{2}x^2$

Solution: $\sqrt{\frac{1-x}{1+x}}=(1-x)^{1/2}(1+x)^{-1/2}$

$$=\left[1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\ldots\right]\left[1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2+\ldots\right]$$

$$=\left[1-\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right]\left[1-\frac{1}{2}x+\frac{3}{8}x^2+\cdots\right]$$

$$=\left(1-\frac{1}{2}x+\frac{3}{8}x^2\right)+\left(-\frac{1}{2}x+\frac{1}{4}x^2\right)-\frac{1}{8}x^2+\cdots$$

$$=1-\left(\frac{1}{2}+\frac{1}{2}\right)x+\left(\frac{3}{8}+\frac{1}{4}-\frac{1}{8}\right)x^2+\cdots$$

$$=1-\left(\frac{1+1}{2}\right)x+\left(\frac{3+2-1}{8}\right)x^2+\cdots\approx 1-x+\frac{1}{2}x^2$$

Neglecting x^3 cube and higher powers of x.

Example 21: For $y=\frac{1}{2}\left(\frac{4}{9}\right)+\frac{1\cdot 3}{2^2 2!}\left(\frac{4}{9}\right)^2+\frac{1.3.5}{2^3 3!}\left(\frac{4}{9}\right)^3+\cdots$ show that $5y^2+10y-4=0$

Solution: $y=\frac{1}{2}\left(\frac{4}{9}\right)+\frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2+\frac{1.3.5}{8.3!}\left(\frac{4}{9}\right)^3+\cdots$

Adding 1 on both sides, we obtain

$$1+y=1+\frac{1}{2}\left(\frac{4}{9}\right)+\frac{1\cdot 3}{4\cdot 2!}\left(\frac{4}{9}\right)^2+\frac{1\cdot 3\cdot 5}{8\cdot 3!}\left(\frac{4}{9}\right)^3+\ldots$$

Comparing it with:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

$1+y=(1+x)^n \quad \ldots(1)$ | $nx=\frac{1}{2}\left(\frac{4}{9}\right) \quad \ldots(2)$ | $\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{4\cdot 2!}\left(\frac{4}{9}\right)^2 \quad \ldots(3)$

From eq. (2), we have

$x=\frac{2}{9n}$ Put this in eq. (3)

$\frac{n(n-1)}{2!}\left(\frac{2}{9n}\right)^2=\frac{1\cdot 3}{4\cdot 2!}\left(\frac{4}{9}\right)^2 \quad \Rightarrow \frac{n(n-1)}{2}\cdot\frac{4}{81n^2}=\frac{3}{8}\cdot\frac{16}{81} \quad \Rightarrow \frac{n-1}{1}\cdot\frac{2}{n}=\frac{3}{1}\cdot\frac{2}{1} \quad \Rightarrow 2n-2=6n$

$\Rightarrow -2=4n \quad \Rightarrow n=-\frac{1}{2}$ put this in eq. (2)

$-\frac{1}{2}x=\frac{1}{2}\left(\frac{4}{9}\right) \quad \Rightarrow x=-\frac{4}{9}$

Put $n=-\frac{1}{2}$ and $x=-\frac{4}{9}$ in eq. (1), we get

$1+y=\left(1-\frac{4}{9}\right)^{-1/2}=\left(\frac{5}{9}\right)^{-1/2}=\left(\frac{9}{5}\right)^{1/2}=\frac{3}{\sqrt{5}}$

or $\sqrt{5}(1+y)=3$

Squaring both the sides, we have

$5(1+2y+y^2)=9$ or $5y^2+10y-4=0$ (Proved)

Exercise 8.3

1. Expand the following upto 4 terms, taking the values of x such that the expansion in each case is valid:

(i) $(1+x)^{-1/3}$

Solution:

$$(1+x)^{\frac{-1}{3}}=1+\left(-\frac{1}{3}\right)(x)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(x)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}(x)^3+\dots$$

$$=1+\left(-\frac{1}{3}\right)(x)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2}x^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{6}x^3+\dots$$

$$=1+\left(-\frac{1}{3}\right)(x)+\frac{4}{9}\cdot\frac{1}{2}x^2-\frac{28}{27}\cdot\frac{1}{6}x^3+\dots$$

$$(1+x)^{\frac{-1}{3}}=1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+\dots \text{ is valid if } |x|<1$$

(ii) $(4-3x)^{1/2}$

Solution:

$$(4-3x)^{\frac{1}{2}}=4^{\frac{1}{2}}\left(1-\frac{3}{4}x\right)^{\frac{1}{2}}$$

$$=2\left\{1+\left(\frac{1}{2}\right)\left(\frac{-3}{4}x\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-3}{4}x\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{-3}{4}x\right)^3+\dots\right\}$$

$$=2\left\{1-\frac{3}{8}x+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2}\cdot\left(\frac{9}{16}x^2\right)-\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{6}\cdot\left(\frac{27}{64}x^3\right)+\dots\right\}$$

$$=2\left\{1-\frac{3}{8}x-\frac{1}{4}\cdot\frac{1}{2}\cdot\left(\frac{9}{16}x^2\right)-\frac{3}{8}\cdot\frac{1}{6}\cdot\left(\frac{27}{64}x^3\right)+\dots\right\}=2\left\{1-\frac{3}{8}x-\frac{9}{128}x^2-\frac{27}{1024}x^3+\dots\right\}$$

$$(4-3x)^{\frac{1}{2}}=2-\frac{3}{4}x-\frac{9}{64}x^2-\frac{27}{512}x^3+\dots \text{ is valid if } \left|\frac{3}{4}x\right|<1 \quad \Rightarrow |x|<\frac{4}{3}$$

(iii) $\frac{(1-x)^{-1}}{(1+x)^2}$

Solution:

$$\frac{(1-x)^{-1}}{(1-x)^2}=(1-x)^{-1}(1+x)^{-2}$$

$$=\left\{1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+\dots\right\}\times\left\{1+(-2)(x)+\frac{(-2)(-2-1)}{2!}(x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(x)^3+\dots\right\}$$

$$=\left\{1+x+\frac{(-1)(-2)}{2}x^2-\frac{(-1)(-2)(-3)}{6}x^3+\dots\right\}\times\left\{1-2x+\frac{(-2)(-3)}{2}x^2+\frac{(-2)(-3)(-4)}{6}x^3+\dots\right\}$$

$$=(1+x+x^2+x^3+\dots)\times(1-2x+3x^2-4x^3+\dots)$$

$$=1-2x+3x^2-4x^3+x-2x^2+3x^3+x^2-2x^3+x^3+\dots$$

$$\frac{(1-x)^{-1}}{(1+x)^2}=1-x+2x^2-2x^3+\dots$$

The expression $(1-x)^{-1}$ and $(1+x)^{-2}$ are valid if $|x|<1$

Thus the expression $\frac{(1-x)^{-1}}{(1+x)^2}$ is valid if $|x|<1$

(iv) $\frac{\sqrt{1+2x}}{1-x}$

Solution:

$$\frac{\sqrt{1+2x}}{1-x}=(1+2x)^{\frac{1}{2}}(1-x)^{-1}$$

$$=\left\{1+\left(\frac{1}{2}\right)(2x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(2x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(2x)^3+\dots\right\}\times\left\{1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+\dots\right\}$$

$$=\left\{1+x+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2}(4x^2)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{6}(8x^3)+\dots\right\}\times\left\{1+x+\frac{(-1)(-2)}{2}x^2-\frac{(-1)(-2)(-3)}{6}x^3+\dots\right\}$$

$$=\left\{1+x-\frac{1}{4}\cdot\frac{1}{2}\cdot(4x^2)+\frac{3}{8}\cdot\frac{1}{6}\cdot(8x^3)+\dots\right\}\times(1+x+x^2+x^3+\dots)$$

$$=\left\{1+x-\frac{1}{2}x^2+\frac{1}{2}x^2+\dots\right\}\times(1+x+x^2+x^3+\dots)$$

$$= 1+x+x^2+x^3+x+x^2+x^3-\frac{1}{2}x^2-\frac{1}{2}x^3+\frac{1}{2}x^3+\ldots$$

$$= 1+2x+\left(1+1-\frac{1}{2}\right)x^2+2x^3+\ldots$$

$$\frac{\sqrt{1+2x}}{1-x} = 1+2x+\frac{3}{2}x^2+2x^3+\ldots$$

The expression $\sqrt{1+2x}$ is valid if $|2x|<1 \Rightarrow |x|<\frac{1}{2}$ and the expression $(1-x)^{-1}$ is valid if $|x|<1$

Thus the expression $\frac{\sqrt{1+2x}}{1-x}$ is valid if $|x|<\frac{1}{2}$

2. Find the coefficient of x^n in the expansion of:

(i) $\dfrac{(1+x^2)}{(1+x)^2}$

Solution:

$$\frac{(1+x^2)}{(1+x)^2} = (1+x^2)(1+x)^{-2}$$

$$= (1+x^2)\left\{1+(-2)x+\frac{(-2)(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-2-2)}{3!}x^3+\ldots\right\}$$

$$= (1+x^2)(1-2x+3x^2-4x^3+\ldots)$$

$$= (1+x^2)(1-2x+3x^2-4x^3+\ldots+(-1)^{n-2}(n-1)x^{n-2}+(-1)^{n-1}nx^{n-1}+(-1)^n(n+1)x^n+\ldots)$$

The terms involving x^n

$$= (-1)^n(n+1)x^n+(-1)^{n-2}(n-1)x^n$$

$$= (-1)^n(n+1+(-1)^{-2}(n-1))x^n$$

$$= (-1)^n(n+1+n-1)x^n \qquad \because (-1)^{-2}=1$$

$$= (-1)^n(2n)x^n$$

Thus the coefficient of $x^n = (-1)^n \times (2n)$

(ii) $\dfrac{(1+x)^2}{(1-x)^2}$

Solution:

$$\frac{(1+x)^2}{(1-x)^2} = (1+x^2)(1-x)^{-2}$$

$$= (1+2x+x^2)\left\{1+(-2)(-x)+\frac{(-2)(-2-1)}{2!}(-x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3+\ldots\right\}$$

$$= (1+2x+x^2)(1+2x+3x^2+4x^3+\ldots)$$

$$= (1+2x+x^2)(1+2x+3x^2+4x^3+\ldots+(n-1)x^{n-2}+nx^{n-1}+(n+1)x^n+\ldots)$$

The terms involving x^n

$$= (n+1)x^n+2nx^n+(n-1)x^n$$

$$= (n+1+2n+n-1)x^n = (4n)x^n$$

Thus the coefficient of $x^n = 4n$

3. If x is so small that its square and higher powers can be neglected, then show that:

(i) $\dfrac{1-x}{\sqrt{1+x}} \approx 1-\dfrac{3}{2}x$

Solution:

$$\text{L.H.S} = \frac{1-x}{\sqrt{1+x}} = (1-x)(1+x)^{-\frac{1}{2}}$$

$$\approx (1-x)\left\{1+\left(-\frac{1}{2}\right)x\right\} \qquad \text{Neglecting } x^2 \text{ and higher power of } x$$

$$= (1-x)\left\{1-\frac{1}{2}x\right\}$$

$$\approx 1-\frac{1}{2}x-x \qquad \text{Neglecting } x^2$$

$$= 1-\frac{3}{2}x = \text{R.H.S}$$

Hence Proved L.H.S ≈ R.H.S

(ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1+\dfrac{3}{2}x$

Solution:

$$\text{L.H.S} = \frac{\sqrt{1+2x}}{\sqrt{1-x}} = (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$\approx \left\{1+\frac{1}{2}(2x)\right\}\left\{1+\left(-\frac{1}{2}\right)(-x)\right\} \qquad \text{Neglecting } x^2 \text{ and higher power of } x$$

$$\approx (1+x)\left(1+\frac{1}{2}x\right) = 1+\frac{1}{2}x+x \qquad \text{Neglecting } x^2$$

$$= 1+\frac{3}{2}x = \text{R.H.S}$$

Hence Proved L.H.S ≈ R.H.S

(iii) $\dfrac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x} \approx \dfrac{1}{4}-\dfrac{17}{384}x$

Solution:

$$\text{L.H.S} = \frac{(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}}{4+5x}$$

$$= \left[(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}\right](4+5x)^{-1}$$

$$= \left[9^{\frac{1}{2}}\left(1+\frac{7x}{9}\right)^{\frac{1}{2}}-16^{\frac{1}{4}}\left(1+\frac{3x}{16}\right)^{\frac{1}{4}}\right].4^{-1}\left(1+\frac{5x}{4}\right)^{-1} = \left[3\left(1+\frac{7x}{9}\right)^{\frac{1}{2}}-2\left(1+\frac{3x}{16}\right)^{\frac{1}{4}}\right].\frac{1}{4}\left(1+\frac{5x}{4}\right)^{-1}$$

$$\approx \frac{1}{4}\left[3\left(1+\frac{1}{2}\cdot\frac{7x}{9}\right)-2\left(1+\frac{1}{4}\cdot\frac{3x}{16}\right)\right]\left(1+(-1)\frac{5x}{4}\right) \qquad \text{Neglecting } x^2 \text{ and higher power of } x$$

$$= \frac{1}{4}\left[3\left(1+\frac{7x}{18}\right)-2\left(1+\frac{3x}{64}\right)\right]\left(1-\frac{5x}{4}\right) = \frac{1}{4}\left[3+\frac{7x}{6}-2-\frac{3x}{32}\right]\left(1-\frac{5x}{4}\right)$$

$$=\frac{1}{4}\left[1+\frac{7x}{6}-\frac{3x}{32}\right]\left(1-\frac{5x}{4}\right)=\frac{1}{4}\left[1-\frac{5x}{4}+\frac{7x}{6}-\frac{3x}{32}\right] \quad \text{Neglecting } x^2$$

$$=\frac{1}{4}\left[1-\frac{120x-112x+9x}{96}\right]=\frac{1}{4}-\frac{17}{384}x=\text{R.H.S}$$

Hence Proved L.H.S ≈ R.H.S

(iv) $\frac{\sqrt{4+x}}{(1-x)^3}\approx 2+\frac{25}{4}x$

Solution:

$$\text{L.H.S}=\frac{\sqrt{(4+x)}}{(1-x)^3}=(4+x)^{\frac{1}{2}}(1-x)^{-3}=4^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}.(1-x)^{-3}$$

$$=2\left\{1+\frac{1}{2}\cdot\frac{x}{4}\right\}.\{1+(-3)(-x)\} \quad \because \text{Neglecting } x^2 \text{ and higher power of } x$$

$$=2\left(1+\frac{x}{8}\right).(1+3x)=\left(2+\frac{x}{4}\right).(1+3x)\approx 2+6x+\frac{x}{4} \quad \text{Neglecting } x^2$$

$$=2+\frac{24x+x}{4}=2+\frac{25x}{4}=\text{R.H.S}$$

Hence Proved L.H.S ≈ R.H.S

4. If x is so small that its cube and higher power can be neglected, show that:

(i) $\sqrt{1-x-2x^2}\approx 1-\frac{1}{2}x-\frac{9}{8}x^2$

Solution:

$$\text{L.H.S}=\left[1-(x+2x^2)\right]^{\frac{1}{2}}$$

$$=1+\frac{1}{2}\left(-(x+2x^2)\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-(x+2x^2)\right)^2+\ldots$$

$$=1-\frac{1}{2}\left(x+2x^2\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(x^2+4x^4+4x^3\right)+\ldots$$

$$\approx 1-\frac{1}{2}x-x^2-\frac{1}{8}(x^2) \quad \text{Neglecting } x^3 \text{ and higher power of } x$$

$$=1-\frac{1}{2}x-x^2-\frac{x^2}{8}=1-\frac{1}{2}x-\frac{8x^2+x^2}{8}=1-\frac{1}{2}x-\frac{9x^2}{8}=\text{R.H.S}$$

Hence Proved L.H.S ≈ R.H.S

(ii) $\sqrt{\frac{1+x}{1-x}}\approx 1+x+\frac{1}{2}x^2$

Solution:

$$\text{L.H.S}=\sqrt{\frac{1+x}{1-x}}=(1+x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

$$\approx\left\{1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2\right\}.\left\{1-\frac{1}{2}(-x)+\frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)}{2!}(-x)^2\right\} \quad \text{Neglecting } x^3 \text{ and higher power of } x$$

$$=\left\{1+\frac{1}{2}x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2\right\}.\left\{1+\frac{1}{2}x+\frac{-\frac{1}{2}\left(\frac{-3}{2}\right)}{2}x^2\right\}=\left\{1+\frac{1}{2}x-\frac{x^2}{8}\right\}.\left\{1+\frac{1}{2}x+\frac{3x^2}{8}\right\}$$

$$\approx 1+\frac{1}{2}x+\frac{3x^2}{8}+\frac{1}{2}x+\frac{x^2}{4}-\frac{x^2}{8} \quad \text{Neglecting } x^3 \text{ and } x^4$$

$$=1+\frac{1}{2}x+\frac{1}{2}x+\frac{3x^2}{8}+\frac{x^2}{4}-\frac{x^2}{8}$$

$$=1+x+\frac{3x^2+2x^2-x^2}{8}=1+x+\frac{x^2}{2}=\text{R.H.S}$$

Hence Proved L.H.S ≈ R.H.S

5. If x is very nearly equal 1 , then prove that $px^p-qx^q\approx(p-q)x^{p+q}$

Solution:

$px^p-qx^q\approx(p-q)\ x^{p+q}$

As x is nearly equal to 1. So,

Let $x-1=h$, where h is very small so that its square and higher powers may be neglected.

$$\text{L.H.S}=px^p-qx^q$$

$$=p(1+h)^p-q(1+h)^q \quad \because x=1+h$$

$$\approx p(1+ph)-q(1+qh) \quad \text{Neglecting } h^2 \text{ and higher power of } h$$

$$=p+p^2h-q-q^2h=p-q+p^2h-q^2h$$

$$=(p-q)+h(p^2-q^2)=(p-q)+h(p+q)(p-q)$$

$$=(p-q)\{1+h(p+q)\}$$

$$\text{R.H.S}=(pq)x^{p+q}$$

$$=(p-q)(1+h)^{p+q} \quad \because x=1+h$$

$$\approx(p-q)\{1+(p+q)h\} \quad \text{Neglecting } h^2 \text{ and higher power of } h$$

Hence Proved L.H.S ≈ R.H.S

6. Identify the following series as binomial expansion and find the sum.

$$1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1\cdot3}{2!4}\left(\frac{1}{4}\right)^2-\frac{1\cdot3\cdot5}{3!8}\left(\frac{1}{4}\right)^3+\ldots$$

Solution:

Let $(1+x)^n=1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!.4}\left(\frac{1}{4}\right)^2+\frac{1.3.5}{3!.8}\left(\frac{1}{4}\right)^3+\ldots$

Comparing it with the series

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

$$nx=-\frac{1}{2}\left(\frac{1}{4}\right)\ \ldots(1) \quad \Big| \quad \frac{n(n-1)}{2}x^2=\frac{1.3}{2!.4}\left(\frac{1}{4}\right)^2\ \ldots(2)$$

From eq. (1)

$$x=-\frac{1}{2n}\left(\frac{1}{4}\right)\ \ldots(3) \quad \text{Put this in eq. (2)}$$

$$\frac{n(n-1)}{2}\left\{-\frac{1}{2n}\left(\frac{1}{4}\right)\right\}^2=\frac{1.3}{2!.4}\left(\frac{1}{4}\right)^2$$

$\Rightarrow \quad \frac{n(n-1)}{2}\left\{\frac{1}{4n^2}\cdot\frac{1}{16}\right\} = \frac{3}{2.4}\left(\frac{1}{16}\right)$

$\frac{n-1}{n}=3 \quad \Rightarrow 3n=n-1 \quad \Rightarrow 3n-n=-1 \quad \Rightarrow 2n=-1 \quad \Rightarrow n=\frac{-1}{2}$ put in eq. (3)

$x = \frac{1}{2\left(\frac{-1}{2}\right)}\left(\frac{1}{4}\right)=\frac{1}{4}$

Hence the sum of given series is

$(1+x)^n = \left(1+\frac{1}{4}\right)^{\frac{-1}{2}} = \left(\frac{5}{4}\right)^{\frac{-1}{2}} = \left(\frac{4}{5}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$

7. Use binomial theorem to show that $1+\frac{1}{4}+\frac{1\cdot3}{4\cdot8}+\frac{1\cdot3\cdot5}{4\cdot8\cdot12}+\cdots=\sqrt{2}$

Solution:

Let $\quad (1+x)^n = 1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots \quad$ ...(A)

Comparing it with the series

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots$

$nx=\frac{1}{4}$ ...(1) | $\frac{n(n-1)}{2}x^2=\frac{1.3}{4.8}$ ...(2)

From eq. (1)

$x=\frac{1}{4n}$ ...(3) Put in eq. (2)

$\frac{n(n-1)}{2}\left(\frac{1}{4n}\right)^2=\frac{1.3}{4.8} \quad \Rightarrow \frac{n(n-1)}{2}\left(\frac{1}{16n^2}\right)=\frac{3}{32}$

$\frac{(n-1)}{n}=3 \quad \Rightarrow 3n=n-1 \quad \Rightarrow 3n-n=-1 \quad \Rightarrow 2n=-1 \quad \Rightarrow n=\frac{-1}{2}$ put in eq. (3)

$x=\frac{1}{4\left(\frac{-1}{2}\right)}=-\frac{1}{2}$

Putting values of x and n in eq. (A)

$1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots = (1+x)^n$

$= \left(1-\frac{1}{2}\right)^{\frac{-1}{2}} = \left(\frac{1}{2}\right)^{\frac{-1}{2}} = \left(\frac{2}{1}\right)^{\frac{1}{2}} = (2)^{\frac{1}{2}}$

$1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots = \sqrt{2}$ (Proved)

8. If $y=\frac{1}{3}+\frac{1\cdot3}{2!}\left(\frac{1}{3}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{1}{3}\right)^3+\ldots$ prove that $y^2+2y-2=0$.

Solution:

$y=\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\ldots$

Adding 1 on both sides, we have

$1+y = 1+\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\ldots$

Comparing it with the series

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots$

$nx=\frac{1}{3}$ ...(1) | $\frac{n(n-1)}{2}x^2=\frac{1.3}{2!}\left(\frac{1}{3}\right)^2$ ...(2) | $1+y=(1+x)^n$ ...(3)

From eq. (1), we have

$x=\frac{1}{3n}$ ...(4) put this in eq. (2)

$\frac{n(n-1)}{2}\left(\frac{1}{3n}\right)^2=\frac{1.3}{2!}\left(\frac{1}{3}\right)^2 \quad \Rightarrow \frac{n(n-1)}{2}\left(\frac{1}{9n^2}\right)=\frac{3}{2}\left(\frac{1}{9}\right)$

$\frac{n-1}{n}=3 \quad \Rightarrow 3n=n-1 \quad \Rightarrow 3n-n=-1 \quad \Rightarrow 2n=-1 \quad \Rightarrow n=\frac{-1}{2}$ put in eq. (4)

$x=\frac{1}{3\left(\frac{-1}{2}\right)}=-\frac{2}{3}$

Put the values of x and n in eq. (3)

$1+y=(1+x)^n = \left(1-\frac{2}{3}\right)^{\frac{-1}{2}} = \left(\frac{1}{3}\right)^{\frac{-1}{2}} = \left(\frac{3}{1}\right)^{\frac{1}{2}} = \sqrt{3}$

$\Rightarrow \quad 1+y=\sqrt{3}$

Squaring both sides, we have

$1+y^2+2y=3 \quad \Rightarrow y^2+2y+1-3=0 \quad \Rightarrow y^2+2y-2=0$ (Proved)

9. If $2y=\frac{1}{2^2}+\frac{1\cdot3}{2!}\cdot\frac{1}{2^4}+\frac{1\cdot3\cdot5}{3!}\cdot\frac{1}{2^6}+\cdots$, prove that $4y^2+4y-1=0$

Solution:

$2y = \frac{1}{2^2}+\frac{1.3}{2!}\cdot\frac{1}{2^4}+\frac{1.3.5}{3!}\cdot\frac{1}{2^6}+\ldots$

Adding 1 on both sides, we have

$1+2y = 1+\frac{1}{2^2}+\frac{1.3}{2!}\cdot\frac{1}{2^4}+\frac{1.3.5}{3!}\cdot\frac{1}{2^6}+\ldots$

Comparing it with the series

$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$

$nx=\frac{1}{2^2}$ ...(1) | $\frac{n(n-1)}{2}x^2=\frac{1.3}{2!}\cdot\frac{1}{2^4}$ ...(2) | $1+2y=(1+x)^n$ ...(3)

From eq. (1), we have

$x=\frac{1}{4n}$ ...(4) put this in eq. (2)

$\frac{n(n-1)}{2}\left(\frac{1}{4n}\right)^2=\frac{1.3}{2!}\cdot\frac{1}{2^4} \quad \Rightarrow \frac{n(n-1)}{2}\left(\frac{1}{16n^2}\right)=\frac{3}{32}$

$\frac{n-1}{n}=3 \quad \Rightarrow 3n=n-1 \quad \Rightarrow 3n-n=-1 \quad \Rightarrow 2n=-1 \quad \Rightarrow n=\frac{-1}{2}$ put in eq. (4)

$$x=\frac{1}{4\left(\frac{-1}{2}\right)}=-\frac{1}{2}$$

Put the values of x and n in eq. (3)

$$1+2y=(1+x)^n$$

$$=\left(1-\frac{1}{2}\right)^{\frac{-1}{2}}=\left(\frac{1}{2}\right)^{\frac{-1}{2}}=\left(\frac{2}{1}\right)^{\frac{1}{2}}=\sqrt{2} \quad \Rightarrow 1+2y=\sqrt{2}$$

Squaring both sides, we have

$$1+4y^2+4y=2 \quad \Rightarrow 4y^2+4y+1-2=0 \quad \Rightarrow 4y^2+4y-1=0$$

10. Show that the coefficient of x^r in $\frac{x}{(1-px)(1-qx)}$ is $\frac{p^r-q^r}{p-q}$.

Solution:

Let $\frac{x}{(1-px)(1-qx)}=\frac{A}{1-px}+\frac{B}{1-qx}$

Multiply each term by L.C.M. $(1-px)(1-qx)$

$$x=A(1-qx)+B(1-px) \quad \ldots(2)$$

Put $1-px=0 \Rightarrow x=\frac{1}{p}$ in eq (2)

$$\frac{1}{p}=A\left(1-\frac{q}{p}\right)$$

$$\frac{1}{p}=A\left(\frac{p-q}{p}\right)$$

$$\Rightarrow \boxed{A=\frac{1}{p-q}}$$

Put $1-qx=0 \Rightarrow x=\frac{1}{q}$ in eq (2)

$$\frac{1}{q}=B\left(1-\frac{p}{q}\right)$$

$$\frac{1}{q}=B\left(\frac{q-p}{q}\right)$$

$$\Rightarrow \boxed{B=-\frac{1}{p-q}}$$

Putting the values of A and B in eq (1), we have

$$\frac{x}{(1-px)(1-qx)}=\frac{\frac{1}{p-q}}{1-px}-\frac{\frac{1}{p-q}}{1-qx}$$

$$=\frac{1}{p-q}\left(\frac{1}{1-px}-\frac{1}{1-qx}\right)$$

$$=\frac{1}{p-q}\{(1-px)^{-1}-(1-qx)^{-1}\}$$

$$=\frac{1}{p-q}\left\{\left(1+px+\frac{-1(-1-1)}{2!}(-px)^2+\cdots\right)-\left(1+qx+\frac{-1(-1-1)}{2!}(-qx)^2+\cdots\right)\right\}$$

$$=\frac{1}{p-q}\{(1+px+p^2x^2+\cdots+p^rx^r+\cdots)-(1+qx+q^2x^2+\cdots+q^rx^r+\cdots)\}$$

The terms involving x^r:

$$=\frac{1}{p-q}\{p^rx^r-q^rx^r\}=\left(\frac{p^r-q^r}{p-q}\right)x^r$$

Thus, the coefficient of $x^r=\frac{p^r-q^r}{p-q}$

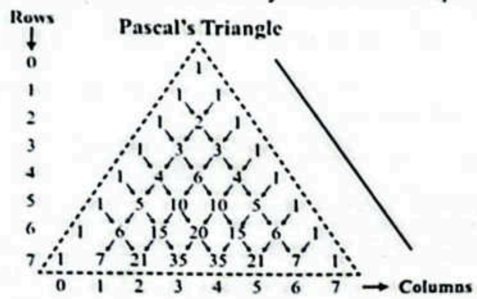
Binomial Coefficients Using Pascal's Triangle:

Binomial coefficients arise in the binomial expansion of powers of a binomial expression, such as $(x+y)^n$. These coefficients are denoted by:

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}, \text{ where } 0\le r\le n.$$

Pascal's Triangle provides a combinatorial method to compute binomial coefficients without directly using factorials. The construction of Pascal's triangle follows these rules:

1. The first row (corresponding to $=0$) consists of a single entry: 1.
2. Each subsequent row begins and ends with 1.
3. Every interior entry is the sum of the two entries directly above it from the previous row.



Mathematically, this is expressed by Pascal's Rule:

$$\text{Pascal's Rule: }\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}, \text{ for } 0<k<n$$

The entries in the n^{th} row of Pascal's Triangle correspond to the binomial coefficients $\binom{n}{0},\binom{n}{1},\ldots,\binom{n}{n}$

For example, the binomial coefficients corresponding to $n=4$ are:

$$\binom{4}{0}=1,\ \binom{4}{1}=4,\ \binom{4}{2}=6,\ \binom{4}{3}=4,\ \binom{4}{4}=1$$

Example 22: Expand $(x+y)^4$ using Pascal's triangle.

Solution:

In Pascal's triangle: For $n=4$, the binomial coefficients are 1 4 6 4 1

Thus, the binomial expansion using Pascal's triangle is

$$(x+y)^4=1x^4+4x^3y+6x^2y^2+4xy^3+1y^4$$

$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

Example 23: Expand $(x-2)^5$ use the binomial theorem and using Pascal's triangle.

Solution: Expand using Binomial Theorem:

$$(x-2)^5={}^5C_0x^5(-2)^0+{}^5C_1x^{5-1}(-2)^1+{}^5C_2x^{5-2}(-2)^2+{}^5C_3x^{5-3}(-2)^3+{}^5C_4x^{5-4}(-2)^4+{}^5C_5x^{5-5}(-2)^5$$

$$=x^5-10x^4y+40x^3y^2-80x^2y^3+80xy^4-32y^5$$

In Pascal's triangle: For $n=5$, the binomial coefficients are 1 5 10 10 5 1

$$(a+b)^5={}^5C_0a^5b^0+{}^5C_1a^4b^1+{}^5C_2a^3b^2+{}^5C_3a^2b^3+{}^5C_4a^1b^4+{}^5C_5a^0b^5$$

$$(a+b)^5=1a^5b^0+5a^4b^1+10\,a^3b^2+10\,a^2b^3+5\,a^1b^4+1\,a^0b^5$$

Put $a = x, b = -2$, we have

$$(x+2)^5 = x^5(-2)^0 + 5x^4(-2)^1 + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5$$
$$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5.$$

Applications of Binomial Theorem:

Finding Approximate Value Using Binomial Theorem:

Approximations: We have seen in the particular cases of the expansion of $(1+x)^n$ that the power of x goes on increasing in each expansion. Since $|x| < 1$, so

$$|x|^r < |x| \text{ for } r = 2, 3, 4, \ldots$$

This fact shows that terms in each expansion go on decreasing numerically if $|x| < 1$. Thus, some initial terms of the binomial series are enough for determining the approximate values of binomial expansions having indices as negative integers or fractions.

Summation of infinite series: The binomial series are conveniently used for summation of infinite series. The series (whose sum is required) is compared with

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

to find out the values of n and x. Then the sum is calculated by putting the values of n and x in $(1+x)^n$.

Example 24: Expand $(1-2x)^{1/3}$ to four terms and apply it to evaluate $(0.8)^{1/3}$ correct to three places of decimal.

Solution: This expansion is valid only if $|2x| < 1$ or $2|x| < 1$ or $|x| < \frac{1}{2}$, that is

$$(1-2x)^{1/3} = 1 + \frac{1}{3}(-2x) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-2x)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-2x)^3 - \ldots$$
$$= 1 - \frac{2}{3}x + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2.1}(4x^2) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3.2.1}(-8x^3) - \ldots$$
$$= 1 - \frac{2}{3}x - \frac{4}{9}x^2 - \frac{1.2.5}{3.3.3}\cdot\frac{1}{3.2.1}(8x^3) - \ldots$$
$$= 1 - \frac{2}{3}x - \frac{4}{9}x^2 - \frac{40}{81}x^3 - \ldots$$

Putting $x = 0.1$ in the above expansion we have

$$(1-2(0.1))^{1/3} = 1 - \frac{2}{3}(0.1) - \frac{4}{9}(0.1)^2 - \frac{40}{81}(0.1)^3 - \cdots$$
$$= = 1 - \frac{0.2}{3} - \frac{0.04}{9} - \frac{0.04}{81} - \cdots \qquad \because 40 \times 0.001 = 0.04$$
$$= 1 - 0.06666 - 0.00444 - 0.00049 = 1 - 0.07159 = 0.92841$$

Thus $(.8)^{1/3} \approx .928$

Alternative method:

$$(0.8)^{1/3} = (1-0.2)^{1/3}$$
$$= 1 + \frac{1}{3}(-0.2) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-0.2)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-0.2)^3 + \cdots$$

$$= 1 - 0.0667 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}(0.04) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}(-0.008) + \cdots$$
$$= 1 - 0.0667 - \frac{1}{9}(0.04) - \frac{5}{81}(0.008) + \cdots$$
$$= 1 - 0.0667 - 0.0044 - 0.0005 - \cdots$$
$$= 0.9284 \approx 0.928$$

Example 25: Evaluate $\sqrt[3]{30}$ correct to three places of decimal.

Solution: $\sqrt[3]{30} = (30)^{1/3} = (27+3)^{\frac{1}{3}}$

$$= \left[27\left(1+\frac{3}{27}\right)\right]^{1/3} = (27)^{1/3}\left(1+\frac{1}{9}\right)^{1/3} = 3\left(1+\frac{1}{9}\right)^{1/3}$$
$$= 3\left[1 + \frac{1}{3}\cdot\frac{1}{9} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{1}{9}\right)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{1}{9}\right)^3 + \ldots\right]$$
$$= 3\left[1 + \frac{1}{3}\cdot\frac{1}{9} - \frac{1}{9}\left(\frac{1}{9}\right)^2 + \frac{5}{81}\left(\frac{1}{9}\right)^3 + \ldots\right] = 3\left[1 + \frac{1}{27} - \left(\frac{1}{27}\right)^2 + \ldots\right]$$
$$\approx 3[1 + 0.03704 - 0.001372] = 3[1.035668] = 3.107004$$

Thus $\sqrt[3]{30} \approx 3.107$

Finding the Remainder Using Binomial Theorem:

Example 26: Using binomial theorem, find the remainder when 5^{99} is divided by 13.

Solution: $5^{99} = 5\cdot 5^{98} = 5\cdot\left(5^2\right)^{49} = 5\cdot(25)^{49}$

$$= 5(26-1)^{49}$$
$$= 5\left[\binom{49}{0}26^{49}1^0 - \binom{49}{1}26^{48}1^1 + \binom{49}{2}26^{47}1^2 + \ldots - \binom{49}{49}26^0 1^{49}\right]$$
$$= 5\left[26^{49} - \binom{49}{1}26^{48} + \binom{49}{2}26^{47} + \ldots - 1\right]$$
$$= 5\cdot 26^{49} - 5\cdot\binom{49}{1}26^{48} + 5\cdot\binom{49}{2}26^{47} + \ldots - 5$$
$$= \left[5\cdot 26^{49} - 5\cdot\binom{49}{1}26^{48} + 5\cdot\binom{49}{2}26^{47} + \ldots - 13\right] + 8$$
$$= 13k + 8, \text{ where } k \text{ is an integer}$$

Hence, 8 is the remainder when 5^{99} is divided by 13.

Example 27: Using the binomial theorem, show that $11^n - 10n$ leaves a remainder 1 when divided by 100 for all positive integers n.

Solution:

$$11^n = (1+10)^n = \binom{n}{0}1^n 10^0 + \binom{n}{1}1^{n-1}10^1 + \binom{n}{2}1^{n-2}10^2 + \binom{n}{3}1^{n-3}10^3 + \cdots + \binom{n}{n}1^0 10^n$$
$$11^n = 1 + 10n + \binom{n}{2}100 + \binom{n}{3}1000 + \cdots + 10^n$$

$$11^n - 10n = 1 + 100\left[\binom{n}{2} + \binom{n}{3}(10) + \cdots + 10^{n-2}\right]$$

$$11^n - 10n = 1 + 100k \text{, where } k \text{ is an integer.}$$

$$11^n - 10n = 100k + 1$$

This show that $11^n - 10n$ leaves a remainder 1 when divided by 100.

Finding Last Digit of a Number:

Example 28: Using binomial theorem, find the last two digits of the number 11^{12}.

Solution: $(11)^{12} = (10+1)^{12}$

$$= \binom{12}{0}10^{12}1^0 + \binom{12}{1}10^{11}1^1 + \binom{12}{2}10^{10}1^2 + \ldots + \binom{12}{1}10^1 1^{11} + \binom{12}{12}10^0 1^{12}$$

The last two digits can be found by the last two terms, because the remaining terms are the multiples of 100 and hence do not affect the last two digits

$$\binom{12}{1}10^1 1^{11} + \binom{12}{12}10^0 1^{12} = 120 + 1 = 121$$

The last two digits of 121 are 2, 1 .

Hence the last two digits of 11^{12} are 2,1 .

Divisibility Test

Example 29: Show that $(15)^{13} + (13)^{15}$ is divisible by 14.

Solution $(15)^{13} + (13)^{15} = (14+1)^{13} + (14-1)^{15}$

$$= \left[{}^{13}C_0(14)^{13} + {}^{13}C_1(14)^{12} + {}^{13}C_2(14)^{11} + \cdots + {}^{13}C_{12}(14) + {}^{13}C_{13}\right]$$
$$+\left[{}^{15}C_0(14)^{15} - {}^{15}C_1(14)^{14} + {}^{15}C_2(14)^{13} - \cdots + {}^{15}C_{14}(14) - {}^{15}C_{15}\right]$$

$$= \left[{}^{13}C_0(14)^{13} + {}^{13}C_1(14)^{12} + {}^{13}C_2(14)^{11} + \cdots {}^{13}C_{12}(14) + 1 + {}^{15}C_0(14)^{15} - {}^{15}C_1(14)^{14} + \cdots + {}^{15}C_{14}(14) - 1\right]$$

$$= 14\left[{}^{13}C_0(14)^{12} + {}^{13}C_1(14)^{11} + {}^{13}C_2(14)^{10} + \cdots + {}^{13}C_{12} + {}^{15}C_0(14)^{14} - {}^{15}C_1(14)^{13} + \cdots + {}^{15}C_{14}\right]$$

$= 14k$, where k is an integer.

Thus, $14k$ is divisible by 14.

Comparing Two Large Numbers

Example 30: Which number is larger 51^{25} or $49^{25} + 50^{25}$?

Solution:

$$51^{25} = (50+1)^{25}$$
$$= \binom{25}{0}(50)^{25}(1)^0 + \binom{25}{1}(50)^{24}(1)^1 + \binom{25}{2}(50)^{23}(1)^2 + \binom{25}{3}(50)^{22}(1)^3 + \cdots$$
$$= (50)^{25} + 25\cdot(50)^{24} + \frac{25\cdot 24}{1\cdot 2}(50)^{23} + \frac{25\cdot 24\cdot 23}{1\cdot 2\cdot 3}(50)^{22} + \cdots$$

Similarly

$$49^{25} = (50-1)^{25}$$
$$= (50)^{25} - 25\cdot(50)^{24} + \frac{25\cdot 24}{1\cdot 2}(50)^{23} - \frac{25\cdot 24\cdot 23}{1\cdot 2\cdot 3}(50)^{22} + \ldots$$

By subtracting, we get

$$51^{25} - 49^{25} = 2(25)\cdot(50)^{24} + 2.\frac{25\cdot 24\cdot 23}{1\cdot 2\cdot 3}\cdot(50)^{22} + \ldots$$

$$51^{25} - 49^{25} = \left[(50)^{25} + 2\cdot\frac{25\cdot 24\cdot 23}{1\cdot 2\cdot 3}\cdot(50)^{22} + \ldots\right] > 50^{25}$$

$\Rightarrow (51)^{25} - (49)^{25} > 50^{25} \Rightarrow (51)^{25} > (49)^{25} + 50^{25}$

Hence, $(51)^{25}$ is greater than $49^{25} + 50^{25}$.

Economic Forecasting with Compound Interest:

Example 31: A bank offers a compound interest rate of 5% per year. Sumaira invests Rs. 100,000 for 3 years. How much will her investment be worth at the end of 3 years?

Solution: Using the compound interest formula, the future value A of the investment is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where, $P = 100{,}000$ (the principal), $r = 0.05$ (the interest rate), $n = 1$ (compounding once per year), $t = 3$ (the time in years).

Substitute the values:

$$A = 100000(1+0.05)^{1\cdot 3} = 1000(1.05)^3 \quad \ldots (1)$$

Using the binomial expansion for $(1.05)^3$:

$$(1+0.05)^3 = {}^3C_0(1)^3(0.05)^0 + {}^3C_1(1)^2(0.05)^1 + {}^3C_2(1)^1(0.05)^2 + {}^3C_3(1)^0(0.05)^3$$
$$= 1 + 3(0.05) + 3(0.05)^2 + (0.05)^3$$
$$= 1 + 0.15 + 0.0075 + 0.000125$$
$$= 1.157625$$

Now calculate the future value: $A = 100000 \times 1.157625 = 115762.5$ Using (1)

So, after 3 years, the investment will be worth Rs. 115762.5

Exercise 8.4

1. Using binomial theorem find the value of the following to three places of decimals:

(i) $\sqrt{99}$

Solution:

$$\sqrt{99} = (99)^{\frac{1}{2}}$$
$$= (100-1)^{\frac{1}{2}} = \left[100\left(1 - \frac{1}{100}\right)\right]^{\frac{1}{2}} = 100^{\frac{1}{2}}\left(1 - \frac{1}{100}\right)^{\frac{1}{2}}$$
$$= 10(1-0.01)^{\frac{1}{2}}$$
$$= 10\left(1 + \left(\frac{1}{2}\right)(-0.01) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-0.01)^2 + \ldots\right) = 10\left[1 - 0.005 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(0.0001)\right]$$
$$= 10\left[1 - 0.005 - \frac{1}{8}(0.0001)\right] = 10(1 - 0.0050 - 0) = 10(0.995)$$
$$\sqrt{99} = 9.950 \text{ approx.}$$

(ii) $(1.03)^{\frac{1}{3}}$

Solution:

$$(1.03)^{\frac{1}{3}} = (1+0.03)^{\frac{1}{3}}$$

$$= 1+\left(\frac{1}{3}\right)(0.03)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}(0.03)^2+\dots = 1+0.01+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}(0.0009)$$

$$= 1+0.01-\frac{2}{18}(0.0009) = 1+0.01-\frac{1}{9}(0.0009)=1+0.01-0.0001=1.0099$$

$(1.03)^{\frac{1}{3}} = 1.010$ approx

(iii) $\frac{1}{\sqrt[5]{252}}$

Solution:

$$\frac{1}{\sqrt[5]{252}} = (252)^{-\frac15} = (243+9)^{-\frac15} = \left[243\left(1+\frac{9}{243}\right)\right]^{-\frac15}$$

$$=243^{-\frac15}\left(1+\frac{1}{27}\right)^{-\frac15}$$

$$=(3^5)^{-\frac15}\left(1+\left(-\frac15\right)\left(\frac{1}{27}\right)+\frac{\left(-\frac15\right)\left(-\frac15-1\right)}{2!}\left(\frac1{27}\right)^2+\dots\right)=3^{-1}\left[1-\frac{1}{135}+\frac{\left(-\frac15\right)\left(-\frac65\right)}{2}\left(\frac{1}{729}\right)\right]$$

$$=\frac13\left[1-\frac1{135}+\left(\frac{6}{50}\right)\left(\frac{1}{729}\right)\right]=\frac13\left[1-\frac{1}{135}+\frac{1}{6075}\right]$$

$$=\frac13(1-0.0074+0.0002)=\frac13(0.9928)=0.3309$$

$\frac{1}{\sqrt[5]{252}}=0.331$ approx.

(iv) $\frac{\sqrt7}{\sqrt8}$

Solution:

$$\frac{\sqrt7}{\sqrt8}=\sqrt{\frac78}=\sqrt{1-\frac18}=\left(1-\frac18\right)^{\frac12}$$

$$=1+\left(\frac12\right)\left(-\frac18\right)+\frac{\left(\frac12\right)\left(\frac12-1\right)}{2!}\left(-\frac18\right)^2+\dots=1-\frac1{16}+\frac{\left(\frac12\right)\left(-\frac12\right)}{2}\left(\frac1{64}\right)+\dots$$

$$=1-\frac1{16}-\left(\frac18\right)\left(\frac1{64}\right)=1-\frac1{16}-\frac1{512}=1-0.0625-0.0020=0.9355$$

$\frac{\sqrt7}{\sqrt8}=0.936$ approx

2. Find the remainder when 8^{100} is divided by 7.

Solution:

$$8^{100}=(1+7)^{100}$$

$$=\binom{100}{0}1^{100}\cdot7^0+\binom{100}{1}1^{99}\cdot7^1+\binom{100}{2}1^{98}\cdot7^2+\dots+\binom{100}{100}1^0\cdot7^{100}$$

$$=(1)(1)(1)+\left\{\binom{100}{1}7+\binom{100}{2}7^2+\dots+(1)(1)7^{100}\right\}$$

$$=1+\left\{\binom{100}{1}\cdot7+\binom{100}{2}\cdot7^2+\dots+7^{100}\right\}$$

$8^{100}=1+7k$, where $k\in Z$

Hence, 1 is the remainder, when 8^{100} is divided by 7.

3. Find the remainder when 2^{100} is divided by 3.

Solution:

$$2^{100}=(2^2)^{50}=(4)^{50}$$

$$=(1+3)^{50}$$

$$=\binom{50}{0}1^{50}\cdot3^0+\binom{50}{1}1^{49}\cdot3^1+\binom{50}{2}1^{48}\cdot3^2+\dots+\binom{50}{50}\cdot3^{50}$$

$$=(1)(1)(1)+\left\{\binom{50}{1}\cdot3+\binom{50}{2}\cdot3^2+\dots+(1)(1)\cdot3^{50}\right\}$$

$$=1+\left\{\binom{50}{1}3^1+\binom{50}{2}3^2+\dots+3^{50}\right\}$$

$$=1+3k, \text{ where } k\in Z$$

Hence, 1 is the remainder, when 2^{100} is divided by 3.

4. Using the binomial theorem, find which number is larger:

(i) $19^{10}+20^{10}$ or 21^{10}

Solution:

$$21^{10}=(20+1)^{10}$$

$$=\binom{10}{0}20^{10}\cdot1^0+\binom{10}{1}20^9\cdot1^1+\binom{10}{2}20^8\cdot1^2+\binom{10}{3}20^7\cdot1^3+\cdots$$

$$=20^{10}+10\cdot20^9+\binom{10}{2}\cdot20^8+\binom{10}{3}20^7+\cdots$$

Similarly, $19^{10}=(20-1)^{10}$

$$=\binom{10}{0}20^{10}\cdot(-1)^0+\binom{10}{1}20^9\cdot(-1)^1+\binom{10}{2}20^8\cdot(-1)^2+\binom{10}{3}20^7\cdot(-1)^3+\cdots$$

$$=20^{10}-10\cdot20^9+\binom{10}{2}20^8-\binom{10}{3}20^7+\cdots$$

By subtracting, we get

$$21^{10}-19^{10}=2(10\cdot20^9)+2\binom{10}{3}20^7+\cdots$$

$=20.20^9+k$, where k is some positive constant.

$$21^{10}-19^{10}=20^{10}+k$$

$$\Rightarrow 21^{10}-19^{10}>20^{10}$$

$$\Rightarrow 21^{10}>19^{10}+20^{10}$$

Hence 21^{10} is greater than $19^{10}+20^{10}$

(ii) $29^{15}+30^{15}$ or 31^{15}

Solution:

$$31^{15}=(30+1)^{15}$$