Example 1: Find the first four terms of the following sequences whose n^{th} terms are given:

(i) $a_n = 3n+1$ (ii) $a_n = 3n^2-3$

Solution:

(i) $a_n = 3n+1$

Substituting $n = 1, 2, 3$ and 4 we have

$a_1 = 3(1)+1 = 3+\alpha 1 = 3$

Similarly, $a_2 = 3(2)+1 = 6+1 = 7$

$a_3 = 3(3)+1 = 9+1 = 10$

$a_4 = 3(4)+1 = 12+1 = 13$

The first four terms of the sequence are 4, 7, 10, 13.

(ii) $a_n = 3n^2-3$

Substituting $n = 1,2,3$ and 4 we have

$a_1 = 3(1)^2 - 3 = 0$

Similarly, $a_2 = 3(2)^2 - 3 = 3(4) - 3 = 12 - 3 = 9$

$a_3 = 3(3)^2 - 3 = 3(9) - 3 = 27 - 3 = 24$

$a_4 = 3(4)^2 - 3 = 3(16) - 3 = 48 - 3 = 45$

The first four terms of the sequence are 0, 9, 24, 45.

Progressions: Sequences of numbers are also called progressions. Depending on the pattern, the progressions are classified as follows:

(i) Arithmetic progression (ii) Geometric progression (iii) Harmonic progression

Exercise 6.1

1. Find the next four terms of each sequence.

(i) 12, 16, 20, ...

Solution:

$$12, 16, 20, ...$$

From given sequence, it is clear that

$a_1 = 12$

$a_2 = a_1 + 4 = 12 + 4 = 16$

$a_3 = a_2 + 4 = 16 + 4 = 20$

$a_4 = a_3 + 4 = 20 + 4 = 24$

$a_5 = a_4 + 4 = 24 + 4 = 28$

$a_6 = a_5 + 4 = 28 + 4 = 32$

$a_7 = a_6 + 4 = 32 + 4 = 36$

Hence the next four terms of the sequence are

$$24, 28, 32, 36$$

(ii) 3, 1, −1, ...

Solution:

$$3, 1, -1, ...$$

From give sequence, it is clear that

$a_1 = 3$

$a_2 = a_1 + (-2) = 3 - 2 = 1$

$a_3 = a_2 + (-2) = 1 - 2 = -1$

$a_4 = a_3 + (-2) = -1 - 2 = -3$

$a_5 = a_4 + (-2) = -3 - 2 = -5$

$a_6 = a_5 + (-2) = -5 - 2 = -7$

$a_7 = a_6 + (-2) = -7 - 2 = -9$

Hence the next four terms of the sequence are

$$-3, -5, -7, -9.$$

2. Write down the first three terms of each of the following sequences:

(i) $a_n = 3n+5$

Solution:

$$a_n = 3n + 5$$

Put $n = 1, 2, 3$, we have

$a_1 = 3(1) + 5 = 3 + 5 = 8$

$a_2 = 3(2) + 5 = 6 + 5 = 11$

$a_3 = 3(3) + 5 = 9 + 5 = 14$

Hence the first three terms of given sequence are 8, 11, 14

(ii) $a_{n+1} = 4a_n - 7$ and $a_1 = 3$

Solution:

$$a_{n+1} = 4a_n - 7 \text{ and } a_1 = 3$$

Put $n = 1, 2$, we have

$a_{1+1} = 4a_1 - 7 = 4(3) - 7 = 12 - 7$

$\Rightarrow a_2 = 5$

$a_{2+1} = 4a_2 - 7 = 4(5) - 7 = 20 - 7$

$\Rightarrow a_3 = 13$

Hence the first three terms of give sequence are 3, 5, 13.

(iii) $a_n = (n-3)(n+1)$

Solution:

$$a_n = (n-3)(n+1)$$

Put $n = 1,2,3$, we wave

$a_1 = (1-3)(1+1) = (-2)(2) = -4$

$a_2 = (2-3)(2+1) = (-1)(3) = -3$

$a_3 = (3-3)(3+1) = (0)(4) = 0$

Hence the first three terms of given sequence are −4, −3, 0

(iv) $a_1 = -1, a_{n+1} = \dfrac{3}{a_n+2}$

Solution:

$$a_{n+1} = \frac{3}{a_n+2}, a_1 = -1$$

Put $n = 1, 2$, we have

$a_{1+1} = \dfrac{3}{a_1+2} = \dfrac{3}{-1+2} = \dfrac{3}{1}$

$\Rightarrow a_2 = 3$

$a_{2+1} = \dfrac{3}{a_2+2} = \dfrac{3}{3+2}$

$\Rightarrow a_3 = \dfrac{3}{5}$

Hence the first three terms of given sequence are

$$-1, 3, \frac{3}{5}$$

(v) $a_n = 8 - \dfrac{20}{3+n}$

Solution:

$$a_n = 8 - \frac{20}{3+n}$$

Put $n = 1,2,3$, we have

$a_1 = 8 - \dfrac{20}{3+1} = 8 - \dfrac{20}{4} = 8 - 5 = 3$

$a_2 = 8 - \dfrac{20}{3+2} = 8 - \dfrac{20}{5} = 8 - 4 = 4$

$a_3 = 8 - \dfrac{20}{3+3} = 8 - \dfrac{20}{6} = 8 - \dfrac{10}{3} = \dfrac{14}{3}$

Hence the first three terms of given sequence are

$$3, 4, \frac{14}{3}$$

(vi) $a_1 = 1, a_{n+1} = (3a_n + 2)^2$

Solution:

$$a_{n+1} = (3a_n + 2)^2, a_1 = 1$$

Put $n = 1,2$, we have

$a_{1+1} = (3a_1 + 2)^2 = (3(1)+2)^2 = (5)^2$

$\Rightarrow a_2 = 25$

$a_{2+1} = (3a_2 + 2)^2 = (3(25)+2)^2 = (77)^2$

$\Rightarrow a_3 = 5959$

Hence the first three terms of given sequence are

$$1, 25, 5959$$

(vii) $a_n = (-2n)^2$

Solution:

$$a_n = (-2n)^2$$

Put $n = 1, 2, 3$, we have

$a_1 = (-2(1))^2 = (-2)^2 = 4$

$a_2 = (-2(2))^2 = (-4)^2 = 16$

$a_3 = (-2(3))^2 = (-6)^2 = 36$

Hence the first three terms of given sequence are

$$4, 16, 36$$

(viii) $a_n = (-1)^n 7n^2$

Solution:

$$a_n = (-1)^n \cdot 7n^2$$

Put $n = 1, 2, 3$, we have

$a_1 = (-1)^1 \cdot 7(1)^2 = (-1)(7) = -7$

$a_2 = (-1)^2 \cdot 7(2)^2 = (1)(7)(4) = 28$

$a_3 = (-1)^3 \cdot 7(3)^2 = (-1)(7)(9) = -63$

Hence the first three terms of given sequence are −7, 28, −63

3. An expression for the n^{th} triangular number is $\dfrac{n(n+1)}{2}$. Write down the 15th triangular number. Make a triangle of dots by taking $n = 5$.

Solution:

Given that: $a_n = \dfrac{n(n+1)}{2}$

Put $n = 15$, we have

$a_{15} = \dfrac{15(15+1)}{2} = \dfrac{15(16)}{2} = 15(8)$

$a_{15} = 120$

Hence the 15th triangular number is 120. Now, we draw an equilateral triangle with 5 rows.

$$\text{Total no. of dots} = \frac{5(5+1)}{2} = \frac{5(6)}{2} = 15$$

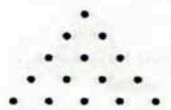
No. of dots in 1st row = 1

No. of dots in and row = 2

No. of dots in 2nd row = 3

No. of dots in 3rd row = 4

No. of dots in 4th row = 5



Triangle of dots for $n = 5$

4. Write down the n^{th} term of each of the following sequences:

(i) 1, 4, 9, ...

Solution:

Given sequence: $1, 4, 9, ..., a_n$

$a_1 = 1 = 1^2$

$a_2 = 4 = 2^2$

$a_3 = 9 = 3^2$

$\vdots$

$a_n = n^2$ (Required nth term)

(ii) 1, 1 + 2, 1 + 2 + 3, ...

Solution:

Given Sequence: $1, 1+2, 1+2+3, ..., a_n$

$a_1 = 1$

$a_2 = 1 + 2$

$a_3 = 1 + 2 + 3$

$a_4 = 1 + 2 + 3 + 4$

$\vdots$

$a_n = 1 + 2 + 3 + \cdots + n$ (Required nth term)

(iii) $a_1b_1, a_2b_2, a_3b_3, \ldots$

Solution:

Given sequence: $a_1b_1, a_2b_2, a_3b_3, \ldots, T_n$

$$T_1 = a_1b_1$$
$$T_2 = a_2b_2$$
$$T_3 = a_3b_3$$
$$\vdots$$
$$T_n = a_nb_n \text{ (Required nth term)}$$

(iv) $x, 2x^2, 3x^3, \ldots$

Solution:

Given Sequence: $x, 2x^2, 3x^3, \ldots, a_n$

$$a_1 = 1x^1$$
$$a_2 = 2x^2$$
$$a_3 = 3x^3$$
$$\vdots$$
$$a_n = nx^n \text{ (Required nth term)}$$

(v) $a_1, a_1 + d, a_1 + 2d, \ldots$

Solution:

Given sequence: $a_1, a_1 + d, a_1 + 2d, \ldots, a_n$

$$a_1 = a_1 = a_1 + 0d = a_1 + (1-1)d$$
$$a_2 = a_1 + 1d = a_1 + (2-1)d$$
$$a_3 = a_1 + 2d = a_1 + (3-1)d$$
$$\vdots$$
$$a_n = a_1 + (n-1)d \text{ (Required nth term)}$$

(vi) $a_1, a_1r, a_1r^2, \ldots$

Solution:

Given sequence: $a_1, a_1r, a_1r^2, \ldots, a_n$

$$a_1 = a_1 = a_1r^0 = a_1r^{1-1}$$
$$a_2 = a_1r^1 = a_1r^{2-1}$$
$$a_3 = a_1r^2 = a_1r^{3-1}$$
$$\vdots$$
$$a_n = a_1r^{n-1} \text{ (Required nth term)}$$

(vii) $\dfrac{a_1}{b_1+c_1}, \dfrac{2a_2}{b_2+c_2}, \dfrac{3a_3}{b_3+c_3}, \ldots$

Solution:

Given sequence: $\dfrac{a_1}{b_1+c_1}, \dfrac{2a_2}{b_2+c_2}, \dfrac{3a_3}{b_3+c_3}, \ldots, T_n$

$$T_1 = \frac{1a_1}{b_1+c_1}$$
$$T_2 = \frac{2a_2}{b_2+c_2}$$
$$T_3 = \frac{3a_3}{b_3+c_3}$$
$$\vdots$$
$$T_n = \frac{na_n}{b_n+c_n} \text{ (Required nth term)}$$

(viii) $\dfrac{1}{a_1}, \dfrac{1}{a_1+d}, \dfrac{1}{a_1+2d}, \ldots$

Solution:

Given Sequence: $\dfrac{1}{a_1}, \dfrac{1}{a_1+d}, \dfrac{1}{a_1+2d}, \ldots, a_n$

$$a_1 = = \frac{1}{a_1} = \frac{1}{a_1+d} = \frac{1}{a_1+(1-1)d}$$
$$a_2 = \frac{1}{a_1+d} = \frac{1}{a_1+(2-1)d}$$
$$a_3 = \frac{1}{a_1+2d} = \frac{1}{a_1+(3-1)d}$$
$$\vdots$$
$$a_n = \frac{1}{a_1+(n-1)d} \text{ (Required nth term)}$$

Arithmetic Progression or Arithmetic Sequence (A.P.)

A sequence $\{a_n\}$ is called an arithmetic sequence or arithmetic progression (A.P.), if $a_n - a_{n-1}$ is the same number for all $n \in N$ and $n > 1$. The difference $a_n - a_{n-1} (n > 1)$ i.e., the difference of two consecutive terms of an A.P. is called the common difference and is usually denoted by d.

Or

An arithmetic progression is a sequence in which each term after the first is obtained by adding fixed constant to the previous term. This fixed constant is called common difference of the arithmetic progression.

For example: Following sequences are in A.P.

(i) 1, 3, 5, 7, … (common difference is $d = 3 - 1 = 5 - 3 = 2$)

(ii) 54, 51, 48, … (common difference is $d = 51 - 54 = 48 - 51 = -3$)

> **Note:**
> If $a_1, a_2, a_3, \ldots a_n, \ldots$ are in A.P. then $d = a_2 - a_1 = a_3 - a_2 = \ldots$ where a_n is n^{th} term of the A.P.

An arithmetic progression with n terms can be written as:

$$a_1, a_1 + d, a_1 + 2d, \ldots, a_1 + (n-1)d$$

The n^{th} term of an arithmetic progression can be written as:

$$a_n = a_1 + (n-1)d$$

> **Note:**
> (i) 1^{st}, 2^{nd}, 3^{rd} and n^{th} terms of an A.P. are denoted by a_1, a_2, a_3 and a_n respectively.
> (ii) n^{th} term from the end of an A.P. is $(m - n + 1)^{th}$ term where 'm' denotes the total number of terms of an A.P.
> (iii) Three numbers a, b, c are in A.P. if and only if $2b = a + c$.
> (iv) Any term (except first and last) in an A.P. is equal to half of the sum of two terms equidistant from it.
> (v) If the term a_1 is unknown or not given, the n^{th} term can be written as $a_n = a_m + (n - m)d, n \geq m$.
> Note that the subscript of the given term and coefficient of d sum to n. i.e., $m + (n - m) = n$

Middle Term of an A.P.: The middle term of an A.P. depends upon the number of terms. That is

- If the total number of terms of an A.P. is even, then there are two middle terms i.e., $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ where n represent the number of terms.
- If the total number of terms of an A.P. is odd, then there is only one middle term i.e., $\left(\frac{n+1}{2}\right)^{th}$ term.

For Examples,

(i) 1,3,[5,7],9,11 is an A.P.

Here $n = 6$ (even number), then

Middle Terms $= \left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ terms

$= \left(\frac{6}{2}\right)^{th}$ and $\left(\frac{6}{2}+1\right)^{th}$ terms

$= 3^{rd}$ and 4^{th} terms

$= a_3$ and a_4 terms

(i) 1,3,5,[7],9,11,13 is an A.P.

Here $n = 7$ (odd number), then

Middle Terms $= \left(\frac{n+1}{2}\right)^{th}$ term

$= \left(\frac{7+1}{2}\right)^{th}$ term

$= 4^{th}$ term

$= a_4$ term

Selection of terms in A.P.:

(i) Three consecutive terms of an A.P. can be chosen as $a-d, a, a+d$ or $a, a+d$, or $a+2d$

(ii) Four consecutive term of an A.P. may be written like $a-3d, a-d, a+d, a+3d$ or $a, a+d, a+2d, a+3d$.

(iii) Last four consecutive terms if ℓ is the last term can be written as below:

$$\ell-3d, \ell-2d, \ell-d, \ell$$

If each term of an A.P. is increased or decreased, multiplied or divided by same non-zero number, then the resulting sequence is also an A.P. that is, if $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in A.P. with common difference d then

(i) $a_1 \pm k, a_2 \pm k, \ldots, a_n \pm k, \ldots$ are also in A.P. with common difference 'd'.

(ii) $ka_1, ka_2, \ldots, ka_n, \ldots$ are also in A.P. with common difference 'kd'.

(iii) $\dfrac{a_1}{k}, \dfrac{a_2}{k}, \ldots, \dfrac{a_n}{k}, \ldots$ are also in A.P. with common difference $\dfrac{d}{k}$.

(iv) Term by term addition or subtraction of two A.Ps. is also an A.P. i.e., If $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$ are in A.P., then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are also n A.P.

Example 2: Find the general term and the eleventh term of the A.P. whose first term and the common difference are 2 and –3 respectively. Also write its first four terms.

Solution:

Here, $a_1 = 2, d = -3$

We know that $a_n = a_1 + (n-1)d$

So, $\quad a_n = 2 + (n-1)(-3) = 2 - 3n + 3$

$a_n = 5 - 3n$ ...(i)

Thus, the general term of the A.P. is $5 - 3n$

Putting $n = 11$ in eq. (i), we have

$a_{11} = 5 - 3(11)$

$= 5 - 33 = -28$

We can find a_2, a_3, a_4 by putting $n = 2, 3, 4$ in eq. (i), that is,

$a_2 = 5 - 3(2) = -1$

$a_3 = 5 - 3(3) = -4$

$a_4 = 5 - 3(4) = -7$

Hence, the first four terms of the sequence are: $2, -1, -4, -7$.

Example 3: If the 5^{th} term of an A.P. is 13 and 17^{th} term is 49, find a_n and a_{13}.

Solution:

Given that, $a_5 = 13$ and $a_{17} = 49$

Putting $n = 5, 17$ in $a_n = a_1 + (n-1)d$, we have

$a_5 = a_1 + (5-1)d$

$a_5 = a_1 + 4d$

$13 = a_1 + 4d$ ...(i) $(\because a_5 = 13)$

$a_{17} = a_1 + (17-1)d$

$a_{17} = a_1 + 16d$

$49 = a_1 + 16d$ ...(ii) $(\because a_{17} = 49)$

Equation(ii) – Equation(i)

$a_1 + 16d = 49$

$\pm a_1 \pm 4d = \pm 13$

$12d = 36 \Rightarrow d = 3$ Put $d = 3$ in eq.(i)

$a_1 = 13 - 4d = 13 - 4(3) = 13 - 12 = 1$

Thus $a_n = a_1 + (n-1)d = 1 + (n-1)(3) = 3n - 2$

and $a_{13} = 3(13) - 2 = 39 - 2 = 37$

Example 4: Find the number of terms in the A.P. ; if $a_1 = 3, d = 7$ and $a_n = 59$

Solution:

Using $a_n = a_1 + (n-1)d$, we have

$59 = 3 + (n-1)\times 7$ $(\because a_n = 59, a_1 = 3 \text{ and } d = 7)$

$56 = (n-1)\times 7 \Rightarrow n - 1 = 8 \Rightarrow n = 9$

Thus, the terms in the A.P. are 9.

Example 5: If $a_{n-2} = 3n - 11$ find the n^{th} term of the sequence.

Solution:

Replacing n by $n+2$, we have

$a_{n+2-2} = 3(n+2) - 11$

$a_n = 3n + 6 - 11$

$a_n = 3n - 5$

Exercise 6.2

1. Find the common difference and write the next two terms of each arithmetic sequence.

(i) 9, 16, 23, ...

Solution:

9, 16, 23, ... (A.P)

$\Rightarrow d = 16 - 9 = 7$ and $a_1 = 9$

Using: $a_n = a_1 + (n-1)d$

$a_4 = a_1 + 3d = 9 + 3(7) = 9 + 21 = 30$

$a_5 = a_1 + 4d = 9 + 4(7) = 9 + 28 = 37$

Hence the common difference is 7 and next two terms of A.P. are 30, 37.

(ii) $5, 5+\sqrt{2}, 5+2\sqrt{2}, \ldots$

Solution:

$5, 5+\sqrt{2}, 5+2\sqrt{2}, \ldots$ (A.P)

$\Rightarrow d = 5 + \sqrt{2} - 5 = \sqrt{2}$ and $a_1 = 5$

Using: $a_n = a_1 + (n-1)d$

$a_4 = a_1 + 3d = 5 + 3\sqrt{2}$

$a_5 = a_1 + 4d = 5 + 4\sqrt{2}$

Hence the common difference is $\sqrt{5}$ and next two terms of A.P. are $5 + 3\sqrt{2}, 5 + 4\sqrt{2}$.

2. Write the first three terms of each arithmetic sequence, with given information.

(i) $a_1 = 2, d = 13$

Solution:

$a_1 = 2, d = 13$ (Aritametic Sequence)

Using: $a_n = a_1 + (n-1)d$

$a_2 = a_1 + d = 2 + 13 = 15$

$a_3 = a_1 + 2d = 2 + 2(13) = 2 + 26 = 28$

Hence the first three terms of A.P are 2, 15, 28.

(ii) $a_1 = 12, d = -13$

Solution:

$a_1 = 12, d = -13$ (Aritametic sequence)

Using: $a_n = a_1 + (n-1)d$

$a_2 = a_1 + d = 12 + (-13) = -1$

$a_3 = a_1 + 2d = 12 + 2(-13) = -14$

Hence the first three terms of A.P. are $12, -1, -14$

3. Find a_{n+1} and a_{2n} if $a_n = 4 + 3n$

Solution:

$a_n = 4 + 3n, a_{n+1} = ?, a_{2n} = ?$

Given that $a_n = 4 + 3n$

Replace n by $n+1$ in (1)

$a_{n+1} = 4 + 3(n+1)$

$= 4 + 3n + 3$

$a_{n+1} = 7 + 3n$

Replace n by $2n$ in (1)

$a_{2n} = 4 + 3(2n)$

$a_{2n} = 4 + 6n$

4. Find the indicated term of each of the following arithmetic sequence.

(i) $a_1 = 3, d = 7, a_{14} = ?$

Solution:

$a_1 = 3, d = 7, a_{14} = ?$

Using: $a_n = a_1 + (n-1)d$

$a_{14} = a_1 + 13d$

$a_{14} = 3 + 13(7) = 3 + 91$

$a_{14} = 94$

(ii) $8, 3, -2, \ldots, a_{12}$

Solution:

$8, 3, -2, \ldots, a_{12}$ (A.P)

Here $a_1 = 8, d = 3 - 8 = -5, a_{12} = ?$

Using: $a_n = a_1 + (n-1)d$

$a_{12} = a_1 + 11d$

$a_{12} = 8 + 11(-5)$

$a_{12} = -47$

5. The 18^{th} and 30^{th} terms of an arithmetic sequence are 367 and 499 respectively. How many term of this sequence are less than 1000?

Solution:

Given that: $a_{18} = 367; a_{30} = 499$

Using: $a_n = a_1 + (n-1)d$

$a_1 + 17d = 367$ ...(1)

$a_1 + 29d = 499$ ...(2)

Equation (2) – Equation (1)

$a_1 + 29d = 499$

$\pm a_1 \pm 17d = \pm 367$

$12d = 132 \Rightarrow d = 11$

Put $d = 11$ in eq. (1), we have

$a_1 + 17(11) = 367$

$a_1 = 367 - 187$

$a_1 = 180$

To find the terms less than 1000, we have

$a_n < 1000$

$a_1 + (n-1)d < 1000$

$180 + (n-1)(11) < 1000$

$11(n - 1) < 1000 - 180$

$$11n - 11 < 820$$
$$11n < 820 + 11$$
$$n < \frac{831}{11}$$
$$n < 75.5454 \text{ approximately}$$
Take $n = 75 \in N$

Hence there are 75 terms of the sequence less than 1000.

6. Is 301 a term of the A.P. of the 5, 11, 17, ...?

Solution:

5, 11, 17, ... (A.P)

Here $a_1 = 5, d = 11 - 5 = 6$

Let $a_n = 301, n = ?$

Using $a_n = a_1 + (n-1)d$
$$301 = 5 + (n-1)(6)$$
$$301 - 5 = 6n - 6$$
$$296 + 6 = 6n$$
$$6n = 302$$
$$n = \frac{302}{6}$$
$$n = \frac{151}{3} \notin N$$

Hence 301 is not a term of the given A.P.

7. If $2x, x+8, 3x+1$ are in A.P., then find the value of x.

Solution:

As $2x, x+8, 3x+1$ are in A.P, so there exists a common difference.
$$(x+8) - 2x = (3x+1) - (x+8)$$
$$x + 8 - 2x = 3x + 1 - x - 8$$
$$-x + 8 = 2x - 7$$
$$8 + 7 = 2x + x$$
$$3x = 15$$
$$x = \frac{15}{3}$$
$$\boxed{x = 5}$$

8. Which term of the A.P., 3, 8, 13, ... is 123?

Solution:

3, 8, 13, ... (A.P)

Here $a_1 = 3, d = 8 - 3 = 5, a_n = 123, n = ?$

Using $a_n = a_1 + (n-1)d$
$$123 = 3 + (n-1)(5)$$
$$123 - 3 = 5n - 5$$
$$120 + 5 = 5n$$
$$5n = 125$$
$$n = \frac{125}{5}$$
$$n = 25$$

Hence 123 is the 25th term of the A.P.

9. Which term of the A.P., 30, 29.5, 29, ... is the first negative term.

Solution:

30, 29.5, 29, ... (A.P)

Here $a_1 = 30, d = 29.5 - 30 = -0.5$

We want to find smallest natural number n, such that
$$a_n < 0$$
$$a_1 + (n-1)d < 0$$
$$30 + (n-1)(-0.5) < 0$$
$$-\frac{1}{2}(n-1) < -30$$

Multiply by -2, we have
$$n - 1 > 60$$
$$n > 61$$

Take $n = 62 \in N$

Now, $a_{62} = a_1 + 61d$
$$= 30 + 61(-0.5)$$
$$= 30 - 30.5$$
$$= -0.5 \text{ (Negative)}$$

Hence 62nd is the first negative term of the A.P.

10. The 7th and 21st terms of an A.P., are 37 and 107 respectively. Find the A.P. and its 100th term.

Solution:

Given that: $a_7 = 37; a_{21} = 107$

Using: $a_n = a_1 + (n-1)d$
$$a_1 + 6d = 37$$
$$a_1 + 20d = 107$$

Equation (2) – Equation (1)
$$a_1 + 20d = 107$$
$$\pm a_1 \pm 6d = \pm 37$$
$$14d = 70 \quad \Rightarrow d = 5$$

Put $d = 5$ in eq. (1), we have
$$a_1 + 6(5) = 37$$
$$a_1 = 37 - 30$$
$$a_1 = 7$$

Now, $a_{100} = a_1 + 99d$
$$= 7 + 99(5)$$
$$= 7 + 495$$
$$a_{100} = 502$$

Hence the required A.P. is

$a_1, (a_1 + d), (a_1 + 2d), ...$

$7, (7+5), (7+2(5)), ...$

$7, 12, 17, ...$

11. If $\frac{1}{a-c}, \frac{1}{b-c}, \frac{1}{b-a}$ are in A.P., the show that $\frac{a-b}{a-c} = \frac{a-c}{b-a}$.

Solution:

As $\frac{1}{a-c}, \frac{1}{b-c}, \frac{1}{b-a}$ are in A.P., so

There exists a common difference
$$\frac{1}{b-c} - \frac{1}{a-c} = \frac{1}{b-a} - \frac{1}{b-c}$$
$$\frac{(a-c)-(b-c)}{(b-c)(a-c)} = \frac{(b-c)-(b-a)}{(b-a)(b-c)}$$
$$\frac{a-c-b+c}{(b-c)(a-c)} = \frac{b-c-b+a}{(b-a)(b-c)}$$
$$\frac{a-b}{(b-c)(a-c)} = \frac{a-c}{(b-a)(b-c)}$$

Multiply both sides by $(b-c)$
$$\frac{a-b}{a-c} = \frac{a-c}{b-a} \text{ (Proved)}$$

12. How many numbers of three digits are divisible by 7?

Solution:

Three digits numbers divisible by 7 are

105, 112, 119, ..., 994 (A.P)

Here $a_1 = 105, d = 112 - 105 = 7, a_n = 994, n = ?$

As we know
$$a_n = a_1 + (n-1)d$$
$$994 = 105 + (n-1)(7)$$
$$994 - 105 = 7n - 7$$
$$889 + 7 = 7n$$
$$7n = 896$$
$$n = \frac{896}{7}$$
$$n = 128$$

Hence there are 128 three digits numbers which are divisible by 7.

13. Find the 8th term from the end of the A.P., 8, 11, 14, ..., 185.

Solution:

8, 11, 14, ..., 185 (A.P)

The reversed A.P is

185, ..., 14, 11, 8 (A.P)

Here $a_1 = 185, d = 11 - 14 = -3, n = 8, a_8 = ?$

As we know
$$a_n = a_1 + (n-1)d$$
$$a_8 = 185 + (8-1)(-3)$$
$$= 185 + (7)(-3)$$
$$= 185 - 21$$
$$= 164 \text{ (Required 8th term from the end)}$$

14. Find the n^{th} term of the progression $\left(\frac{3}{7}\right)^{10}, \left(\frac{10}{7}\right)^{10}, \left(\frac{17}{7}\right)^{10}, ...$ Is the progression an A.P.?

Solution:

Given sequence: $\left(\frac{3}{7}\right)^{10}, \left(\frac{10}{7}\right)^{10}, \left(\frac{17}{7}\right)^{10}, ...$

First, we find n^{th} term of the sequence

3, 10, 17, ...(A.P)

Here $a_1 = 3, d = 10 - 3 = 7, a_n = ?$

As we know
$$a_n = a_1 + (n-1)d$$
$$= 3 + (n-1)(7)$$
$$= 3 + 7n - 7$$
$$a_n = 7n - 4$$

nth term of given sequence $= \left(\frac{a_n}{7}\right)^{10} = \left(\frac{7n-4}{7}\right)^{10}$

As $\left(\frac{10}{7}\right)^{10} - \left(\frac{3}{7}\right)^{10} \neq \left(\frac{17}{7}\right)^{10} - \left(\frac{10}{7}\right)^{10}$

So the given sequence is not an A.P.

15. If the arithmetic progression 3, 10, 17, ... and 63, 65, 67, ... are such that their n^{th} terms are equal, then find the value of n.

Solution:

1st sequence: 3, 10, 17, ... (A.P)

Here $a_1 = 3, d = 10 - 3 = 7$
$$a_n = a_1 + (n-1)d$$
$$= 3 + (n-1)(7)$$
$$= 3 + 7n - 7$$
$$a_n = 7n - 4$$

2nd sequence: 63, 65, 67, ... (A.P)

Here $a'_1 = 63, d' = 65 - 63 = 2$
$$a'_n = a'_1 + (n-1)d'$$
$$= 63 + (n-1)(2)$$
$$= 63 + 2n - 2$$
$$a'_n = 61 + 2n$$

According to the given condition.
$$a_n = a'_n$$
$$7n - 4 = 2n + 61$$
$$7n - 2n = 61 + 4$$
$$5n = 65$$
$$n = \frac{65}{5}$$
$$n = 13$$

16. If the p^{th} term of an A.P. is q and the q^{th} term is p, prove that its n^{th} term is $(p + q - n)$.

Solution:

Let 1st term $= a$, and common difference $= d$

Given that: $a_p = q$; $a_q = p$

Using: $a_n = a_1 + (n-1)d$

$a_1 + (p-1)d = q$ ...(1)

$a_1 + (q-1)d = p$ ...(2)

Equation (2) – Equation (1)

$a_1 + (q-1)d = p$

$\pm a_1 \pm (p-1)d = \pm q$

$(q-1)d - (p-1)d = p - q$

$(q-1-p+1) = p-q$

$(q-p)d = p-q = -(q-p)$

$d = \frac{-(q-p)}{q-p}$

$d = -1$

Put $d = -1$ in eq (1), we have

$a_1 + (p-1)(-1) = q$

$a_1 - p + 1 = q$

$a_1 = p + q - 1$

As we know

$a_n = a_1 + (n-1)d$

$= (p+q-1) + (n-1)(-1)$

$= p+q-1-n+1$

$= p+q-n$ (Required nth term)

17. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P., show that $b = \frac{2ac}{a+c}$.

Solution:

As $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., so there exists a common difference

$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

⇒ $\frac{1}{b} + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$

$\frac{1+1}{b} = \frac{a+c}{ac}$ By taking L.C.M

$\frac{2}{b} = \frac{a+c}{ac}$ ⇒ $b = \frac{2ac}{a+c}$

18. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P., show that the common difference is $\frac{a-c}{2ac}$.

Solution:

As $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., so there exists a common difference

$d = \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

⇒ $d = \frac{1}{b} - \frac{1}{a}$ ...(1) | $d = \frac{1}{c} - \frac{1}{b}$ ...(2)

Equation (1) + Equation (2)

$d = \frac{1}{b} - \frac{1}{a}$

$d = \frac{1}{c} - \frac{1}{b}$

$2d = \frac{1}{c} - \frac{1}{a}$ ⇒ $2d = \frac{a-c}{ac}$ By taking L.C.M

$d = \frac{a-c}{2ac}$

19. If a_k and a_m denotes two different terms of an A.P. show that its n^{th} term is $a_k + (n-k)\left(\frac{a_k - a_m}{k-m}\right)$.

Solution:

Let 1st term = a_1, and common difference = d

Using $a_n = a_1 + (n-1)d$

$a_k = a_1 + (k-1)d$ ...(1)

$a_m = a_1 + (m-1)d$ ...(2)

Equation (1) – Equation (2)

$a_k = a_1 + (k-1)d$

$a_m = \pm a_1 \pm (m-1)d$

$a_k - a_m = (k-1-m+1)d$

$d = \frac{a_k - a_m}{k-m}$ Put this in eq. (1)

$a_k = a_1 + (k-1)\left(\frac{a_k - a_m}{k-m}\right)$

$a_1 = a_k - (k-1)\left(\frac{a_k - a_m}{k-m}\right)$

As we know

$a_n = a_1 + (n-1)d$

$= a_k - (k-1)\left(\frac{a_k - a_m}{k-m}\right) + (n-1)\left(\frac{a_k - a_m}{k-m}\right)$

$= a_k + \{-(k-1)+(n-1)\}\left(\frac{a_k - a_m}{k-m}\right)$

$= a_k + (-k+1+n-1)\left(\frac{a_k - a_m}{k-m}\right)$

$a_n = a_k + (n-k)\left(\frac{a_k - a_m}{k-m}\right)$ (Proved)

20. If $a_1, a_2, a_3, \dots, a_n$ are positive and in A.P., prove that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution:

Given that: $a_1, a_2, a_3, \dots, a_n$ (A.P)

⇒ $a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = d$ ...(1)

As we know

$a_n = a_1 + (n-1)d$

$a_n - a_1 = n - 1$

$d = \frac{a_n - a_1}{n-1}$ ...(2)

L.H.S = $\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$

Rationalizing the denominator by formula

$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$

L.H.S = $\frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1} + \frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2} + \dots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$

$= \frac{\sqrt{a_2}-\sqrt{a_1}}{d} + \frac{\sqrt{a_3}-\sqrt{a_2}}{d} + \dots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d}$ Using (1)

$= \frac{1}{d}\{\sqrt{a_2} - \sqrt{a_1} + \sqrt{a_3} - \sqrt{a_2} + \dots + \sqrt{a_n} - \sqrt{a_{n-1}}\}$

$= \frac{1}{d}(\sqrt{a_n} - \sqrt{a_1})$

$= \frac{n-1}{a_n - a_1}(\sqrt{a_n} - \sqrt{a_1})$ using (2)

$= \frac{(n-1)(\sqrt{a_n} - \sqrt{a_1})}{(\sqrt{a_n})^2 - (\sqrt{a_1})^2}$

$= \frac{(n-1)(\sqrt{a_n} - \sqrt{a_1})}{(\sqrt{a_n} - \sqrt{a_1})(\sqrt{a_n} + \sqrt{a_1})}$

$= \frac{n-1}{\sqrt{a_n} + \sqrt{a_1}}$ = R.H.S (Proved)

21. If the roots of the equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal. Show that a, b, c are in A.P.

Solution:

Given quadratic equation

$(b-c)x^2 + (c-a)x + (a-b) = 0$

Compare it with

$Ax^2 + Bx + C = 0$

⇒ $A = b-c, B = c-a, C = a-b$

For equal roots, disc = 0

$B^2 - 4AC = 0$

$(c-a)^2 - 4(b-c)(a-b) = 0$

$c^2 + a^2 - 2ca - 4(ab - b^2 - ca + bc) = 0$

$a^2 + c^2 - 2ca - 4ab + 4b^2 + 4ca - 4bc = 0$

$a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ca = 0$

$(a)^2 + (-2b)^2 + c^2 + 2(a)(-2b) + 2(-2b)(c) + 2(c)(a) = 0$

$(a - 2b + c)^2 = 0$

Taking the square root of both sides

$a - 2b + c = 0$

⇒ $2b = a + c$

⇒ $b + b = a + c$

⇒ $b - a = c - b$ (common difference exists)

So a, b and c are in A.P.

22. If the sides of a right-angled triangle are in A.P., find the ratio of its sides.

Solution:

Let $a-d, a, a+d$ be the sides of a right triangle, then by Pythagoras theorem.

$(a+d)^2 = (a-d)^2 + a^2$

$a^2 + d^2 + 2ad = a^2 + d^2 - 2ad + a^2$

$2ad + 2ad = a^2$

$4ad = a^2$

Dividing both sides by a ($\because a \neq 0$)

$4d = a$

⇒ Sides of triangle are

$a-d, a, a+d$

$4d-d, 4d, 4d+d$ $\because a = 4d$

$3d, 4d, 5d$

Ratio of sides = $3d : 4d : 5d$

= $3 : 4 : 5$ (Divide by d)

23. If the n^{th} term of a progression is a linear expression in n, then prove that this progression is an A.P.

Solution:

As the nth term of a progression is a linear expression in 'n', so

$a_n = na + b$ where a and b are constants.

Put $n = 1, 2, 3, \dots$

$a_1 = a + b$

$a_2 = 2a + b$

$a_3 = 3a + b$

⋮

Progression: $a+b, 2a+b, 3a+b, \dots$

This progression will be an A.P. if

$(2a+b) - (a+b) = (3a+b) - (2a+b)$

$2a + b - a - b = 3a + b - 2a - b$

$a = a$ (common difference exists)

Hence $a+b, 2a+b, 3a+b, \dots$ is an A.P.

Arithmetic Mean (A.M.)

A number A is said to be the A.M. between the two numbers a and b if a, A, b are in A.P. If d is the common difference of this A.P., then $A-a=d$ and $b-A=d$.

Thus $A-a=b-A$

or $2A=a+b$

$\Rightarrow$ $A=\dfrac{a+b}{2}$

Note: If $A_1, A_2, A_3, \ldots, A_n$ are said to be n A.Ms between two numbers a and b, then $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P.

Example 6: Find three A.Ms. between $\sqrt{2}$ and $3\sqrt{2}$

Solution:

Let A_1, A_2, A_3 be three A.Ms. between $\sqrt{2}$ and $3\sqrt{2}$. Then,

$\sqrt{2}, A_1, A_2, A_3, 3\sqrt{2}$ are in A.P

Here $a_1=\sqrt{2}, n=5, a_5=3\sqrt{2}$

As $a_5=3\sqrt{2} \Rightarrow a_1+4d=3\sqrt{2} \Rightarrow 4d=3\sqrt{2}-a_1 \Rightarrow 4d=3\sqrt{2}-\sqrt{2} \Rightarrow d=\dfrac{2\sqrt{2}}{4}=\dfrac{\sqrt{2}}{2} \Rightarrow d=\dfrac{1}{\sqrt{2}}$

Now,

$$A_1=a_1+d=\sqrt{2}+\frac{1}{\sqrt{2}}=\frac{2+1}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$

$$A_2=A_1+d=\frac{3}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{4}{\sqrt{2}}=2\sqrt{2}$$

$$A_3=A_2+d=2\sqrt{2}+\frac{1}{\sqrt{2}}=\frac{4+1}{\sqrt{2}}=\frac{5}{\sqrt{2}}$$

Therefore, $\dfrac{3}{\sqrt{2}}, 2\sqrt{2}, \dfrac{5}{\sqrt{2}}$ are the three A.Ms between $\sqrt{2}$ and $3\sqrt{2}$.

Exercise 6.3

1. Find A.M. between the given numbers

(i) $2+\sqrt{3i}, 2-\sqrt{3i}$

Solution:

$2+\sqrt{3i}, 2-\sqrt{3i}$

Here $a=2+\sqrt{3i}$ and $b=2\sqrt{3i}$

$$\text{A.M}=\frac{a+b}{2}$$

$$=\frac{2+\sqrt{3i}+2-\sqrt{3i}}{2}=\frac{4}{2}$$

$$\text{A.M}=2$$

(ii) $(a+b)^2, (a-b)^2$

Solution:

$(a+b)^2, (a-b)^2$

Here $a'=(a+b)^2$ and $b'=(a-b)^2$

$$\text{A.M}=\frac{a'+b'}{2}$$

$$=\frac{(a+b)^2+(a-b)^2}{2}$$

$$=\frac{a^2+b^2+2ab+a^2+b^2-2ab}{2}$$

$$=\frac{2a^2+2b^2}{2}=\frac{2(a^2+b^2)}{2}$$

$$\text{A.M}=a^2+b^2$$

2. If 6, 11, 16 are three A.Ms. between a and b, find a and b.

Solution:

Since 6, 11, 16 are three A.Ms between a and b, therefore $a, 6, 11, 16, b$ are in A.P

$\Rightarrow$ $6-a=11-6=b-16$

$6-a=5=b-16$

$6-a=5$ and $b-16=5$

$6-5=a$ | $b=5+16$

$a=1$ | $b=21$

Hence $a=1$ and $b=21$

3. Insert five A.Ms. between $\sqrt{2}$ and $\dfrac{15}{\sqrt{2}}$.

Solution:

Let A_1, A_2, A_3, A_4, A_5 be the five A.Ms between $\sqrt{2}$ and $\dfrac{15}{\sqrt{2}}$, then

$\sqrt{2}, A_1, A_2, A_3, A_4, A_5, \dfrac{15}{\sqrt{2}}$ are in A.P

Here $a_1=\sqrt{2}$ | $a_7=\dfrac{15}{\sqrt{2}}$

$$a_1+6d=\frac{15}{\sqrt{2}}$$

$$\sqrt{2}+6d=\frac{15}{\sqrt{2}} \qquad \because a_1=\sqrt{2}$$

$$6d=\frac{15}{\sqrt{2}}-\sqrt{2}$$

$$6d=\frac{15-2}{\sqrt{2}}$$

$$d=\frac{13}{6\sqrt{2}}$$

Now, $A_1=a_1+d=\sqrt{2}+\dfrac{13}{6\sqrt{2}}=\dfrac{6(\sqrt{2})^2+13}{6\sqrt{2}}$

$$=\frac{6(2)+13}{6\sqrt{2}}=\frac{25}{6\sqrt{2}}$$

$$A_2=a_1+2d=\sqrt{2}+2\left(\frac{13}{6\sqrt{2}}\right)=\sqrt{2}+\frac{13}{3\sqrt{2}}$$

$$=\frac{3(\sqrt{2})^2+13}{3\sqrt{2}}=\frac{6+13}{3\sqrt{2}}=\frac{19}{3\sqrt{2}}$$

$$A_3=a_1+3d=\sqrt{2}+3\left(\frac{13}{6\sqrt{2}}\right)=\sqrt{2}+\frac{13}{2\sqrt{2}}$$

$$=\frac{2(\sqrt{2})^2+13}{2\sqrt{2}}=\frac{4+13}{2\sqrt{2}}=\frac{17}{2\sqrt{2}}$$

$$A_4=a_1+4d=\sqrt{2}+4\left(\frac{13}{6\sqrt{2}}\right)=\sqrt{2}+\frac{26}{3\sqrt{2}}$$

$$=\frac{3(\sqrt{2})^2+26}{3\sqrt{2}}=\frac{6+26}{3\sqrt{2}}=\frac{32}{3\sqrt{2}}$$

$$A_5=a_1+5d=\sqrt{2}+5\left(\frac{13}{6\sqrt{2}}\right)=\sqrt{2}+\frac{65}{6\sqrt{2}}$$

$$=\frac{6(\sqrt{2})^2+65}{6\sqrt{2}}=\frac{12+65}{6\sqrt{2}}=\frac{77}{6\sqrt{2}}$$

Hence the five A.Ms between $\sqrt{2}$ and $\dfrac{15}{\sqrt{2}}$ are $\dfrac{25}{6\sqrt{2}}, \dfrac{19}{3\sqrt{2}}, \dfrac{17}{2\sqrt{2}}, \dfrac{32}{3\sqrt{2}}, \dfrac{77}{6\sqrt{2}}$.

4. The A.M. of two numbers is 7 and their product is 45. Find the numbers.

Solution:

Let a and b be the two required numbers, then by given conditions

A.M $=7$ | Product $=45$

$\dfrac{a+b}{2}=7$ | $ab=45$ ...(2)

$a+b=14$ ...(1)

From eq. (1), we have

$b=14-a$ Put this in eq. (2)

$a(14-a)=45$

$14a-a^2=45$

$0=a^2-14a+45$

$a^2-9a-5a+45=0$

$a(a-9)-5(a-9)=0$

$(a-5)(a-9)=0$

Either $a-5=0$ or $a-9=0$

$a=5$; $a=9$

When $a=5$, then from eq. (1)

$5+b=14 \Rightarrow b=9$

When $a=9$, then from eq. (1)

$9+b=14 \Rightarrow b=5$

Hence required numbers are 5, 9 or 9, 5

5. If n arithmetic means are inserted between a and b, prove that $d=\dfrac{b-a}{n+1}$, where d is the common difference.

Solution:

Let $A_1, A_2, A_3, \ldots, A_n$ be the n A.Ms between a and b, then

$a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P

Here $a_1=a$ and $a_{n+2}=b$

Using: $a_n=a_1+(n-1)d$

$\Rightarrow a_{n+2}=a+(n+2-1)d \qquad \because a_1=a$

$b=a+(n+1)d \qquad \because a_{n+2}=b$

$b-a=(n+1)d$

$d=\dfrac{b-a}{n+1}$ (proved)

6. If A is the A.M. between a and b, prove that $(a-A)^2+(A-b)^2=\dfrac{1}{2}(a-b)^2$.

Solution:

Since 'A' is the A.M between a and b, therefore a, A, b are in A.P.

$\Rightarrow$ $A=\dfrac{a+b}{2}$

L.H.S $=(a-A)^2+(A-b)^2$

$$=\left(a-\frac{a+b}{2}\right)^2+\left(\frac{a+b}{2}-b\right)^2$$

$$= \left(\frac{2a-a-b}{2}\right)^2 + \left(\frac{a+b-2b}{2}\right)^2$$
$$= \left(\frac{a-b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2$$
$$= \frac{1}{4}(a-b)^2 + \frac{1}{4}(a-b)^2$$
$$= \frac{1}{4}\{(a-b)^2 + (a-b)^2\}$$
$$= \frac{1}{4}\cdot 2(a-b)^2$$
$$= \frac{1}{2}(a-b)^2$$
= R.H.S (Proved)

7. For what value of n, $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the A.M. between a and b, where $a \neq b$.

Solution:

Given that: $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ = A.M between a and b

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$
$$2(a^{n+1}+b^{n+1}) = (a+b)(a^n+b^n)$$
$$2a^{n+1}+2b^{n+1} = a^{n+1}+ab^n+a^nb+b^{n+1}$$
$$2a^{n+1}+2b^{n+1}-a^{n+1}-b^{n+1} = ab^n+a^nb$$
$$a^{n+1}+b^{n+1} = ab^n+a^nb$$
$$a^n\cdot a+b^n\cdot b = ab^n+a^nb$$
$$a^na-a^nb = ab^n-b^nb$$
$$a^n(a-b) = b^n(a-b)$$

Since $a \neq b \Rightarrow a-b \neq 0$, so dividing both sides by $a-b$.

$$a^n = b^n$$
$$\frac{a^n}{b^n} = 1$$
$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0 \qquad \because \left(\frac{a}{b}\right)^0 = 1$$

Comparing both sides, we have

$\Rightarrow$ $\boxed{n=0}$

Series:

The sum of the terms of a sequence is called the series of the corresponding sequence.

For example, $1+2+3+\cdots+n$ is a finite series of first n natural numbers.

The sum of first n terms of series is denoted by S_n.

We write, $S_n = a_1+a_2+\ldots+a_n$.

Here,
$$S_1 = a_1$$
$$S_2 = a_1+a_2$$
$$S_3 = a_1+a_2+a_3$$
$S_n = a_1+a_2+a_3+\ldots+a_n$ is known as n^{th} partial sum.

Arithmetic Series: The sum of the terms of an arithmetic sequence is called an arithmetic series.

Formula for the Sum of n Terms of Arithmetic Series:

To develop a formula for the sum of any arithmetic series,

consider $S_n = a_1+a_2+a_3+\ldots+a_n$ (Arithmetic series)

$$S_n = a_1+(a_1+d)+(a_1+2d)+\ldots+(a_n-2d)+(a_n-d)+a_n \qquad \ldots(1)$$

Reverse the order of terms of the series

$$S_n = a_n+(a_n-d)+(a_n-2d)+\ldots+(a_1+2d)+(a_1+d)+a_1 \qquad \ldots(2)$$

Adding eq.(1) and eq.(2), we get

$$2S_n = (a_1+a_n)+(a_1+a_n)+(a_1+a_n)+\cdots+(a_1+a_n)+(a_1+a_n)+(a_1+a_n) \quad \text{[upto n-terms]}$$
$$2S_n = n(a_1+a_n)$$
$$S_n = \frac{n}{2}(a_1+a_n)$$
$$S_n = \frac{n}{2}[a_1+a_1+(n-1)d] \qquad \because a_n = a_1+(n-1)d$$

Thus, $S_n = = \frac{n}{2}[2a_1+(n-1)d]$

Example 7: Find the sum of the first 100 positive integers.

Solution:

First 100 positive integers are 1, 2, 3, …, 100

Series = $1+2+3+\cdots+100$ (A.P.)

Here, $a_1 = 1, a_n = 100, n = 100$ and $d = 1$.

Key concept:
The sum S_n of the first n terms of an arithmetic series is given by
$$S_n = \frac{n}{2}[2a_1+(n-1)d]$$
or $$S_n = \frac{n}{2}(a_1+a_n)$$

Method-1:
$$S_n = \frac{n}{2}(a_1+a_n)$$
$$S_{100} = \frac{100}{2}(1+100)$$
$$S_{100} = 50(101)$$
$$S_{100} = 5050$$

Method-2:
$$S_n = \frac{n}{2}[2a_1+(n+1)d]$$
$$S_{100} = \frac{100}{2}[2(1)+(100-1)1]$$
$$S_{100} = 50(101)$$
$$S_{100} = 5050$$

Example 8: Find the 19^{th} term and the partial sum of 19 terms of the arithmetic series: $2+\frac{7}{2}+5+\frac{13}{2}+\ldots$

Solution:

Here, $a_1 = 2, a_2 = \frac{7}{2}$ and $d = a_2-a_1 = \frac{7}{2}-2 = \frac{3}{2}, n = 19, a_{19} = ?$

Using
$$a_n = a_1+(n-1)d$$
$$a_{19} = 2+(19-1)\frac{3}{2}$$
$$= 2+18\left(\frac{3}{2}\right) = 2+27 = 29$$

Using
$$S_n = \frac{n}{2}(a_1+a_n)$$
$$S_{19} = \frac{19}{2}(a_1+a_{19})$$
$$S_{19} = \frac{19}{2}(2+29) = \frac{19}{2}(31) = \frac{589}{2}$$

Example 9: Find the arithmetic series if its fifth term is 19 and $S_4 = a_9+1$.

Solution:

Given that, $a_5 = 19$, that is,
$$a_1+4d = 19 \qquad \ldots(i)$$

Using the other given condition, we have
$$S_4 = a_9+1$$
$$\frac{4}{2}[2a_1+(4-1)d] = a_1+8d+1 \qquad \because a_9 = a_1+8d \text{ and } S_4 = \frac{4}{2}[2a_1+(4-1)d]$$
$$4a_1+6d = a_1+8d+1$$
$$3a_1-2d = 1 \qquad \ldots(ii)$$

Equation(ii) $- 3 \times$ Equation(i)
$$3a_1-2d = 1$$
$$\pm 3a_1 \pm 12d = \pm 57$$
$$-14d = -56 \Rightarrow d = 4 \text{ Put } d = 4 \text{ in eq.(i)}$$
$$a_1+4(4) = 19 \Rightarrow a_1 = 19-16 \Rightarrow a_1 = 3$$

Required arithmetic series is:

$$a_1 + (a_1 + d) + (a_1 + 2d) + \ldots$$
$$3 + (3+4) + (3+2(4)) + \ldots$$
$$3 + 7 + 11 + \ldots$$

Example 10: How many terms of the series $-9-6-3+0+\ldots$ amount to 66?

Solution:

Here, $a_1 = -9$, $d = -6-(-9) = 3$, $S_n = 66$ and $n = ?$

Given that: $S_n = 66$

Using: $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$$66 = \frac{n}{2}[2(-9) + (n-1)3]$$

$$132 = n[3n-21] \quad \Rightarrow 132 = 3n(n-7) \quad \Rightarrow 44 = n(n-7)$$

$$n^2 - 7n - 44 = 0$$

$$n^2 - 11n + 4n - 44 = 0 \quad \Rightarrow n(n-11) + 4(n-11) = 0 \quad \Rightarrow (n-11)(n+4) = 0$$

Either $n - 11 = 0$ or $n + 4 = 0$

$\Rightarrow$ $n = 11$ or $n = -4$ (But n cannot be negative, because $n \notin N$)

Hence the sum of eleven terms is 66. i.e., $S_{11} = 66$

Example 11: Find the first three terms of an arithmetic series in which $a_1 = 9, a_n = 105$ and $S_n = 741$.

Solution:

Given that: $a_1 = 9, a_n = 105$ and $S_n = 741$.

We want to find: n, d, a_2 and a_3.

Step-I: Find n.

Using: $S_n = \frac{n}{2}(a_1 + a_n)$

$$741 = \frac{n}{2}(9 + 105)$$
$$741 = 57n$$
$$13 = n$$

Step-II: Find d.

Using: $a_n = a_1 + (n-1)d$

$$105 = 9 + (13-1)d$$
$$105 - 9 = 12d$$
$$96 = 12d$$
$$8 = d$$

Step-III: Find a_2 and a_3.

As we know

$$a_2 = a_1 + d$$
$$= 9 + 8 = 17$$
$$a_3 = a_1 + 2d$$
$$= 17 + 8 = 25$$

The first three terms are 9, 17 and 25

Exercise 6.4

1. Sum the series:

(i) $3 + 6 + 9 + \cdots + a_{20}$

Solution:

$3 + 6 + 9 + \cdots + a_{20}$ (Arithmetic Series)

Here $a_1 = 3, d = 6 - 3 = 3, n = 20, S_{20} = ?$

Using the sum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$
$$= \frac{20}{2}\{2(3) + (20-1)(3)\}$$
$$= 10(6 + 57)$$
$$= 10(63)$$
$$S_{20} = 630$$

(ii) $\frac{4}{\sqrt{5}} + \sqrt{5} + \frac{6}{\sqrt{5}} + \ldots + a_n$

Solution:

$\frac{4}{\sqrt{5}} + \sqrt{5} + \frac{6}{\sqrt{5}} + \ldots + a_n$ (Arithmetic Series)

Here $a_1 = \frac{4}{\sqrt{5}}, d = \sqrt{5} - \frac{4}{\sqrt{5}} = \frac{1}{\sqrt{5}}, n = n, S_n = ?$

Using the sum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$
$$= \frac{n}{2}\left\{2\left(\frac{4}{\sqrt{5}}\right) + (n-1)\left(\frac{1}{\sqrt{5}}\right)\right\}$$

$$= \frac{n}{2}\left\{\frac{8}{\sqrt{5}} + (n-1)\cdot\frac{1}{\sqrt{5}}\right\}$$
$$= \frac{n}{2}\left\{\frac{8 + n - 1}{\sqrt{5}}\right\}$$
$$S_n = \frac{n(n+7)}{2\sqrt{5}}$$

2. Find S_n for each arithmetic series:

(i) $a_1 = 4, n = 25, a_n = 100$

Solution:

$$a_1 = 4, n = 25, a_n = 100$$

Using the sum formula

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{25} = \frac{25}{2}(4 + 100)$$
$$= \frac{25}{2}(104)$$
$$= (25)(52)$$
$$S_{25} = 1300$$

(ii) $a_1 = 40, n = 20, d = -3$

Solution:

$$a_1 = 40, n = 20, d = -3$$

Using the bum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$
$$S_{20} = \frac{20}{2}\{2(40) + (20-1)(-3)\}$$
$$= 10(80 - 57)$$
$$= 10(23)$$
$$S_{20} = 230$$

(iii) $a_n = 52, n = 21, d = -4$

Solution:

$$a_n = 52, n = 21, d = -4$$

First, we find a_1, by using the formula

$$a_n = a_1 + (n-1)d$$
$$52 = a_1 + (21-1)(-4)$$
$$52 = a_1 + (20)(-4)$$
$$52 + 80 = a_1$$
$$a_1 = 132$$

Now, by using the sum formula

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{21} = \frac{21}{2}(132 + 52)$$
$$= \frac{21}{2}(184)$$
$$= 21(92)$$
$$S_{21} = 1932$$

3. Find a_1 for arithmetic series: $d = 8, n = 19, S_n = 1786$

Solution:

$a = 8, n = 19, S_n = 1786, a_1 = ?$ (A. Series)

Using the sum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$
$$1786 = \frac{19}{2}\{2a_1 + (19-1)(8)\}$$
$$1786 = \frac{19}{2}\cdot 2\{a_1 + (18)(4)\}$$
$$1786 = 19(a_1 + 72)$$
$$\frac{1786}{19} = a_1 + 72$$
$$94 - 72 = a_1$$
$$\boxed{a_1 = 22}$$

4. How many terms of the series: $96 + 93 + 90 + \ldots$ amount to 1071.

Solution:

$96 + 93 + 90 + \cdots$ amount to 1071

Here $a_1 = 96, d = 93 - 96 = -3$

Let $S_n = 1071, n = ?$

using the bum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$
$$1071 = \frac{n}{2}\{2(96) + (n-1)(-3)\}$$
$$1071 \times 2 = n(192 - 3n + 3)$$
$$2142 = n(-3n + 195)$$
$$3n^2 - 195n + 2142 = 0$$

Dividing both sides by '3'

$$n^2 - 65n + 714 = 0$$
$$n^2 - 51n - 14n + 714 = 0$$
$$n(n-51) - 14(n-51) = 0$$
$$(n-51)(n-14) = 0$$

Either $n - 51 = 0$ or $n - 14 = 0$

$n = 51$; $n = 14$

Hence $S_{51} = 1071$ and $S_{14} = 1071$.

5. If the three sides of a right-angled triangle having perimeter 36 cm are in A.P., find them.

Solution:

Let $a - d, a, a + d$ be the sides of a right triangle, then

Perimeter $= a - d + a + a + d$

$36 = 3a$ $\quad\because$ Perimeter $= 36$ cm

$a = 12$

By Pythagoras theorem

$$(a+d)^2 = (a-d)^2 + a^2$$

Put $a = 12$

$$(12+d)^2 = (12-d)^2 + 12^2$$

$$144 + d^2 + 24d = 144 + d^2 - 24d + 144$$

$$24d + 24d = 144$$

$$48d = 144$$

$$d = \frac{144}{48} \Rightarrow d = 3$$

$\Rightarrow$ Sides of triangle are

$a-d,\quad a,\quad a+d$

$12-3,\quad 12,\quad 12+3$

$9,\quad 12,\quad 15$

$\Rightarrow$ 9 cm, 12 cm, 15 cm

6. Sum the series

(i) $3+5-7+9+11-13+15+17-19+\ldots$ to $3n$ terms.

Solution:

$3+5-7+9+11-13+15+17-19+\ldots$ to $3n$ terms

$=(3+5-7)+(9+11-13)+(15+17-19)+\ldots$ to n terms

$=1+7+13+\ldots$to n terms

Here $a_1 = 1$, $d = 7-1 = 6$, $n = n$, $S_n = ?$

Using the sum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$

$$= \frac{n}{2}\{2(1)+(n-1)6\}$$

$$= \frac{n}{2}(2+6n-6) = \frac{n}{2}(6n-4)$$

$$= n(3n-2)$$

(ii) $1+4-7+10+13-16+19+22-25+\ldots$ to $3n$ terms.

Solution:

$1+4-7+10+13-16+19+22-25+\ldots$to $3n$ terms

$=(1+4-7)+(10+13-16)+(19+22-25)+\ldots$ to n terms

$=-2+7+16+\ldots$to n terms

Here $a_1 = -2$, $d = 7-(-2) = 9$, $n = n$, $S_n = ?$

Using the sum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$

$$= \frac{n}{2}\{2(-2)+(n-1)9\} = \frac{n}{2}\{-4+9n-9\}$$

$$= \frac{n}{2}(9n-13)$$

7. Find the sum of 20 terms of the series whose r^{th} term is $3r+1$.

Solution:

Given that: $a_r = 3r+1$ ...(1)

Putting $r = 1, 2, 3, 4$ and so on in eq. (1)

$a_1 = 3(1)+1 = 3+1 = 4$

$a_2 = 3(2)+1 = 6+1 = 7$

$a_3 = 3(3)+1 = 9+1 = 10$

$a_4 = 3(4)+1 = 12+1 = 13$

Series: $4+7+10+13+\ldots$to 20 terms

Here $a_1 = 4$, $d = 7-4 = 3$, $n = 20$, $S_{20} = ?$

Using the sum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$

$$S_{20} = \frac{20}{2}\{2(4)+(20-1)3\}$$

$$= 10\{8+(19)3\}$$

$$= 10(8+57) = 650$$

8. The 5th and 9th term of an A.P. are 11 and 17 respectively. Find the sum of 20 terms.

Solution:

Given that:

$a_5 = 11$	$a_9 = 17$
$a_1 + 4d = 11$ ...(1)	$a_1 + 8d = 17$ ...(2)

Equation (2) – Equation (1)

$a_1 + 8d = 17$

$\pm a_1 \pm 4d = \pm 11$

$4d = 6 \Rightarrow d = \frac{6}{4} \Rightarrow d = \frac{3}{2}$

Put $d = \frac{3}{2}$ in eq (1), we have

$$a_1 + 4\left(\frac{3}{2}\right) = 11$$

$$a_1 + 6 = 11$$

$$a_1 = 5$$

Here $a_1 = 5, d = \frac{3}{2}, n = 20, S_{20} = ?$

Using the sum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$

$$S_{20} = \frac{20}{2}\left\{2(5)+(20-1)\left(\frac{3}{2}\right)\right\}$$

$$= 10\left\{10+19\left(\frac{3}{2}\right)\right\}$$

$$= 10\left\{\frac{20+57}{2}\right\}$$

$$= 5(77)$$

$$S_{20} = 385$$

9. Obtain the sum of all integers in the first 1000 positive integers which are neither divisible by 5 nor by 2.

Solution:

The first 1000 integers which are neither divisible by 5 nor by 2 are

1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, ..., 991, 993, 997, 999.

Sum $= 1+3+7+9+11+13+17+19+21+23+27+29+\ldots+991+993+997+999$

$=(1+3+7+9)+(11+13+17+19)+(21+23+27+29)+\ldots+(991+993+997+999)$

Sum $= 20+60+100+\ldots+3980$

Here $a_1 = 20$, $d = 60-20 = 40$, $a_n = 3980$, $n = ?$

As we know

$$a_n = a_1 + (n-1)d$$

$$3980 = 20 + (n-1)(40)$$

$$3980 - 20 = 40n - 40$$

$$3960 + 40 = 40n$$

$$4000 = 4n \Rightarrow n = 100$$

Using the sum formula

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{100} = \frac{100}{2}(20+3980) = \frac{100}{2}(4000)$$

$$S_{100} = 50(4000) = 200{,}000$$

10. The sum of 9 terms of an A.P. is 171 and its eighth term is 31. Find the series.

Solution:

Given that:

$a_8 = 31$

$a_1 + 7d = 31$ ...(1)

$S_9 = 171$

$$\frac{9}{2}\{2a_1 + (9-1)d\} = 171$$

$$\because S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$

$$\frac{9}{2}(2a_1 + 8d) = 171$$

$$9(a_1 + 4d) = 171$$

$$a_1 + 4d = 19 \quad \text{...(2)}$$

Equation (1) – Equation (2)

$a_1 + 7d = 31$

$\pm a_1 \pm 4d = \pm 19$

$3d = 12 \Rightarrow d = 4$

Put $d = 4$ in eq (1), we have

$a_1 + 7(4) = 31 \Rightarrow a_1 = 31 - 28 \Rightarrow a_1 = 3$

Hence the required series is

$a_1 + (a_1+d) + (a_1+2d) + (a_1+3d) + \ldots$

$3 + (3+4) + (3+8) + (3+12) + \ldots$

$3+7+11+15+\ldots$

11. The 5th term of an arithmetic progression is 21 and the sum of first six terms is 90. Find the 18th term.

Solution:

Given that:

Here $a_5 = 21$

$a_1 + 4d = 21$ ...(1)

$S_6 = 90$

$$\frac{6}{2}\{2a_1 + (6-1)d\} = 90$$

$$3(2a_1 + 5d) = 90$$

$$2a_1 + 5d = 30 \quad \text{...(2)}$$

2 × Equation (1) – Equation (2)

$2a_1 + 8d = 42$

$\pm 2a_1 \pm 5d = \pm 30$

$3d = 12 \Rightarrow d = 4$

Put $d = 4$ in eq. (1), we have

$$a_1 + 4(4) = 21$$

$$a_1 = 21 - 16 \Rightarrow a_1 = 5$$

Here $a_1 = 5, d = 4, n = 18, a_{18} = ?$

$$a_{18} = a_1 + 17d$$

$$= 5 + 17(4)$$

$$= 5 + 68$$

$$a_{18} = 73$$

12. The sum of three numbers in an A.P. is 24 and their product is 440. Find the numbers.

Solution:

Let the three numbers in A.P. be $a_1 - d, a_1, a_1 + d$

By given conditions

Sum = 24

$a_1 - d + a_1 + a_1 + d = 24$

$3a_1 = 24$

$a_1 = 8$

Product = 440

$(a_1 - d).(a_1).(a_1 + d) = 440$

$a_1(a_1^2 - d^2) = 440$

$8(64 - d^2) = 440 \quad \because a_1 = 8$

$64 - d^2 = 55 \Rightarrow 64 - 55 = d^2$

$\Rightarrow d^2 = 9 \Rightarrow d = \pm 3$

When $d = 3$, $a_1 = 8$ then required numbers are

$a_1 - d, a_1, a_1 + d$

$8-3, 8, 8+3$

$5, 8, 11$

When $d = -3$, $a_1 = 8$ then required numbers are

$a_1 - d, a_1, a_1 + d$

$8+3, 8, 8-3$

$11, 8, 5$

13. The first four terms of an A.P. are 2, 6, 10 and 14. Find the least number of terms needed so that the sum of the terms is greater than 2000.

Solution:

First four terms of A.P. are

$2, 6, 10, 14$

$\Rightarrow a_1 = 2, d = 6-2 = 4$

By using the sum formula

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$

$$= \frac{n}{2}\{2(2)+(n-1)(4)\}$$

$$= \frac{n}{2}\{4+4n-4\}$$

$$= \frac{n}{2}(4n)$$

$$S_n = 2n^2$$

We want to find the least number n, such that

$$S_n > 2000$$

$$\Rightarrow 2n^2 > 2000$$

$$n^2 > 1000$$

$\sqrt{n^2} > \sqrt{1000}$

$n > 31.62$ approximately.

Take $n = 32 \in N$

Hence $n = 32$ is the least number of terms for which $S_{32} > 2000$.

14. Find four numbers in A.P. whose sum is 32 and the sum of whose squares is 276.

Solution:

Let the four numbers in A.P. be $a_1-3d, a_1-d, a_1+d, a_1+3d$

By given conditions

Sum = 32

$a_1-3d+a_1-d+a_1+d+a_1+3d=32$

$4a_1=32$

$a_1=8$

Sum of squares = 276

$(a_1-3d)^2+(a_1-d)^2+(a_1+d)^2+(a_1+3d)^2=276$

$a_1^2+9d^2-6a_1d+a_1^2+d^2-2a_1d+a_1^2+d^2+2a_1d+a_1^2+9d^2+6a_1d=276$

$4a_1^2+20d^2=276$

$4(8)^2+20d^2=276 \qquad \because a_1=8$

$20d^2=276-256 \Rightarrow 20d^2=20 \Rightarrow d^2=1 \Rightarrow d=\pm1$

When $d=1$, $a_1=8$ then required numbers are

$a_1-3d, a_1-d, a_1+d, a_1+3d$

$8-3, 8-1, 8+1, 8+3$

$5, 7, 9, 11$

When $d=-1$, $a_1=8$ then required numbers are

$a_1-3d, a_1-d, a_1+d, a_1+3d$

$8+3, 8+1, 8-1, 8-3$

$11, 9, 7, 5$

15. Find the five numbers in A.P. whose sum is 25 and the sum of whose squares is 135.

Solution:

Let the five numbers in A.P. be $a_1-2d, a_1-d, a_1, a_1+d, a_1+2d$

By given conditions

Sum = 25

$a_1-2d+a_1-d+a_1+a_1+d+a_1+2d=25$

$5a_1=25$

$a_1=5$

Sum of squares = 135

$(a_1-2d)^2+(a_1-d)^2+a_1^2+(a_1+d)^2+(a_1+2d)^2=135$

$a_1^2+4d^2-4a_1d+a_1^2+d^2-2a_1d+a_1^2+a_1^2+d^2+2a_1d+a_1^2+4d^2+4a_1d=135$

$5a_1^2+10d^2=135$

$5(5)^2+10d^2=135 \qquad a_1=5$

$10d^2=135-125 \Rightarrow 10d^2=10 \Rightarrow d^2=1 \Rightarrow d=\pm1$

When $d=1$, $a_1=5$ then required numbers are

$a_1-2d, a_1-d, a_1, a_1+d, a_1+2d$

$5-2, 5-1, 5, 5+1, 5+2$

$3, 4, 5, 6, 7$

When $d=-1$, $a_1=5$ then required numbers are

$a_1-2d, a_1-d, a_1, a_1+d, a_1+2d$

$5+2, 5+1, 5, 5-1, 5-2$

$7, 6, 5, 4, 3$

16. If $\frac{1}{a+b}, \frac{1}{c+a}, \frac{1}{b+c}$ are in A.P. then show that a^2, b^2, c^2 are in A.P.

Solution:

Since $\frac{1}{a+b}, \frac{1}{c+a}, \frac{1}{b+c}$ are in A.P., so we have

$2\left(\frac{1}{c+a}\right)=\frac{1}{a+b}+\frac{1}{b+c}$ (As $2y=x+z$ if x,y,z are in A.P.)

Multiply both sides by LCM $(a+b)(b+c)(c+a)$

$2(a+b)(b+c) = 1(b+c)(c+a)+1(a+b)(c+a)$

$2(a+b)(b+c) = (c+a)(b+c+a+b)$

$2(ab+ac+b^2+bc) = (c+a)(a+2b+c)$

$2ab+2ac+2b^2+2bc = ac+2bc+c^2+a^2+2ab+ac$

$2b^2=a^2+c^2$

$\Rightarrow$ a^2, b^2 and c^2 are in A.P.

17. The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find number of terms.

Solution:

Given that: $a_1 = 11$ and $S_4 = 56$

As we know

$S_n = \frac{n}{2}\{2a_1+(n-1)d\}$

$S_4 = \frac{4}{2}\{2(11)+(4-1)d\}$

$56 = 2(22+3d) \qquad \because S_4=56$

$28 = 22+3d$

$3d = 6 \Rightarrow d=2$

Here $a_1 = 11, d = 2, a_n = ?$

$a_n = a_1+(n-1)d$

$= 11+(n-1)(2)$

$= 11+2n-2$

$a_n = 2n+9 \qquad ...(1)$

Last four terms of A.P are $a_n, a_{n-1}, a_{n-2}, a_{n-3}$

By using eq. (1), we have

$a_{n-1} = 2(n-1)+9 = 2n+7$

$a_{n-2} = 2(n-2)+9 = 2n+5$

$a_{n-3} = 2(n-3)+9 = 2n+3$

Given that: Sum of last four terms = 112

$a_n+a_{n-1}+a_{n-2}+a_{n-3}=112$

$(2n+9)+(2n+7)+(2n+5)(2n+3)=112$

$8n+24=112$

$8n=112-24$

$8n=88 \Rightarrow n=11$

(Required number of terms)

18. The first term of an A.P. is a, the second term is b and the last term is c. show that the sum of A.P. is $\frac{(b+c-2a)(c+a)}{2(b-a)}$.

Solution:

Given that: $a_1=a, a_2=b, a_n=c$

$\Rightarrow d=b-a$

As $a_n=c$

$\Rightarrow$ $a_1+(n-1)d=c$

$a+(n-1)(b-a)=c$

$(n-1)(b-a)=c-a$

$n-1=\frac{c-a}{b-a}$

$n=\frac{c-a}{b-a}+1=\frac{c-a+b-a}{b-a}$

$n=\frac{b+c-2a}{b-a}$

By using the sum formula

$S_n=\frac{n}{2}(a_1+a_n)$

$=\frac{\frac{b+c-2a}{b-a}}{2}(a+c)$

$S_n=\frac{(b+c-2a)(c+a)}{2(b-a)}$ (Proved)

19. Show that the sum of n A.Ms. between a and b is n times the single A.M. between them.

Solution:

Let $A_1, A_2, A_3, \ldots, A_n$ be n A.Ms between a and b, then $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P.

Here $a_1=a$

$a_{n+2}=b$

$a_1+(n+2-1)d=b \qquad$ using $a_n=a_1+(n-1)d$

$a+(n+1)d=b \qquad \because a_1=a$

$(n+1)d=b-a$

$d=\frac{b-a}{n+1}$

Now, $A_1=a_1+1d=a+1\left(\frac{b-a}{n+1}\right)$

$A_2=a_1+2d=a+2\left(\frac{b-a}{n+1}\right)$

$A_3=a_1+3d=a+3\left(\frac{b-a}{n+1}\right)$

$\vdots$

$A_n=a_1+nd=a+n\left(\frac{b-a}{n+1}\right)$

By adding $A_1, A_2, A_3, \ldots, A_n$, we have

$A_1+A_2+A_3+\ldots+A_n=na+\left(\frac{b-a}{n+1}\right)\{1+2+3+\ldots+n\}$

$=na+\left(\frac{b-a}{n+1}\right)\times\frac{n(n+1)}{2} \qquad \because 1+2+3+\ldots+n=\frac{n(n+1)}{2}$

$=na+\frac{(b-a).n}{2}=\frac{2na+nb-na}{2}$

$=\frac{na+nb}{2}=\frac{n(a+b)}{2}$

$=n$(A.M) Proved

Geometric Progression (G.P.)

A geometric progression or geometric sequence in which each term after the first is found by multiplying the previous term by a non-zero constant r called common ratio.

Like arithmetic progression, we can label the terms of a geometric sequence as a_1, a_2, a_3 and so on, $a_1 \neq 0$. The n^{th} term is a_n and the previous term is a_{n-1}. So, $a_n=r(a_{n-1})$. Thus, $r=\frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

Rule for nth term of a G.P.:

Each term after the first term is an r multiple of its preceding term. Thus, we have,

$a_2 = a_1r = a_1r^{2-1}$

$a_3 = a_2r = (a_1r)r = a_1r^2 = a_1r^{3-1}$

$a_4 = a_3r = (a_1r^2)r = a_1r^3 = a_1r^{4-1}$

$\vdots$

$\boxed{a_n = a_1r^{n-1}}$ which is the general term of a G.P

Properties of G.P.

(i) If each term of a G.P. is multiplied or divided by the same non-zero number, then the resulting sequence also a G.P. that is if $g_1, g_2, g_3, ..., g_n, ...$ are in G.P. and k is a non-zero number, then

(a) $kg_1, kg_2, kg_3, ..., kg_n, ...$ are also in G.P.

(b) $\frac{g_1}{k}, \frac{g_2}{k}, \frac{g_3}{k}, ..., \frac{g_n}{k}, ...$ are also in G.P.

(ii) The reciprocals of the term of a G.P. also form a G.P. that is if a, b, c are in G.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in G.P.

(iii) If each term of a G.P. is raised to the same power, the resulting numbers also form a G.P. that is, if a, b, c ... in G.P., then a^n, b^n, c^n are also in G.P.

(iv) Three numbers a, b, c are in G.P. if and only if $b^2 = ac$.

(v) If the set of positive numbers $a_1, a_2, a_3, ..., a_n, ...$ are in G.P., then $\log a_1, \log a_2, \log a_3, ..., \log a_n, ...$ are A.P. and vice-versa.

(vi) Term by term multiplication or division of two G.Ps. are again in G.P. i.e., if $a_1, a_2, a_3, ..., a_n, ...$ & $b_1, b_2, b_3, ..., b_n$, are in G.P. then $a_1b_1, a_2b_2, a_3b_3, ..., a_nb_n$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, ..., \frac{a_n}{b_n}$ are also in G.P.

Example 12: Find the eighth term of a geometric sequence for which $a_1 = -3$ and $r = -2$.

Solution:

Here, $a_1 = -3, r = -2, n = 8, a_8 = ?$

As we know: $a_n = a_1 r^{n-1}$

$a_8 = (-3).(-2)^{8-1} = (-3).(-2)^7 = (-3).(-128)$

$a_8 = 384$

Example 13: Find the n^{th} term of the G.P., 3, 12, 48, ...

Solution:

Here $a_1 = 3, r = 4$

As we know: $a_n = a_1 r^{n-1}$

$a_n = 3.4^{n-1}$

Example 14: Find the tenth term of the G.P., for which $a_4 = 108$ and $r = 3$.

Solution:

Here, $a_4 = 108, r = 3, a_{10} = ?$

Step-1: Find a_1

As we know: $a_n = a_1 r^{n-1}$

$a_4 = a_1 . 3^{4-1}$

$108 = 27a_1$

$a_1 = 4$

Step-2: Find a_{10}

As we know: $a_n = a_1 r^{n-1}$

$a_{10} = 4.(3)^{10-1}$

$a_{10} = 4.(3)^9$

$a_{10} = 78,732$

Example 15: Find the 5^{th} term of the G.P., 3, 6, 12, ...

Solution:

Given G.P. is 3, 6, 12, ...

Here $a_1 = 3, a_2 = 6$, therefore, $r = \frac{a_2}{a_1} = \frac{6}{3} = 2$.

Using $a_n = a_1 r^{n-1}$ for $n = 5$, we have

$a_5 = a_1 r^{5-1}$

$= 3 \cdot 2^{5-1} = 3 \cdot 2^4 = 48$

Example 16: Find a_n if $a_4 = \frac{8}{27}$ and $a_7 = \frac{-64}{729}$ of a G.P.

Solution:

Given that: $a_4 = \frac{8}{27}$ and $a_7 = \frac{-64}{729}$

First, we find a_1 and r.

To find a_n we have to find a_1 and r.

$a_4 = \frac{8}{27}$

$a_1 r^{4-1} = \frac{8}{27} \quad \because a_n = a_1 r^{n-1}$

$a_1 r^3 = \frac{8}{27} \quad ... (i)$

$a_7 = \frac{-64}{729}$

$a_1 r^{7-1} = \frac{-64}{729} \quad \because a_n = a_1 r^{n-1}$

$a_1 r^6 = \frac{-64}{729} \quad ... (ii)$

Equation (ii) ÷ Equation(i)

$$\frac{a_1 r^6}{a_1 r^3} = \frac{\frac{-64}{729}}{\frac{8}{27}} \Rightarrow r^3 = \frac{-8}{27} \Rightarrow r^3 = \left(\frac{-2}{3}\right)^3$$

$\Rightarrow \quad r = -\frac{2}{3}$(taking only real value of r)

Put $r^3 = -\frac{8}{27}$ in (i), we have

$$a_1\left(-\frac{8}{27}\right) = \frac{8}{27} \Rightarrow a_1 = -1$$

Now, $$a_n = a_1 r^{n-1} = (-1)\left(-\frac{2}{3}\right)^{n-1} = (-1)(-1)^{n-1} \cdot \left(\frac{2}{3}\right)^{n-1} = (-1)^n \left(\frac{2}{3}\right)^{n-1}$$

Exercise 6.5

1. Find the 6^{th} term of the G.P.: $-6, -3, \frac{-3}{2}, ...$

Solution:

$-6, -3, -\frac{3}{2}, ...$ (G.P)

Here $a_1 = -6, r = \frac{-3}{-6} = \frac{1}{2}, n = 6, a_6 = ?$

Using $a_n = a_1 r^{n-1}$

$a_6 = (-6)\left(\frac{1}{2}\right)^{6-1}$

$= (-6)\left(\frac{1}{2}\right)^5$

$= (-6)\left(\frac{1}{32}\right)$

$a_6 = \frac{-3}{16}$

2. Find the 8^{th} term of the sequence, $3, 3^2, 3^3, ...$

Solution:

$3, 3^2, 3^3, ...$ (G.P)

Here $a_1 = 3, r = \frac{3^2}{3} = 3, n = 8, a_8 = ?$

Using $a_n = a_1 r^{n-1}$

$a_8 = (3)(3)^{8-1}$

$= (3)(3)^7 = (3)(2189)$

$a_8 = 6561$

3. The n^{th} terms of the sequences $1, 2, 4, 8, ...$ and $256, 128, 64, ...$ are equal. Find the value of n.

Solution:

1st sequence: $1, 2, 4, 8, ...$ (G.P)

Here $a_1 = 1, r = \frac{2}{1} = 2$

Using $a_n = a_1 r^{n-1}$

$= (1)(2)^{n-1} = 2^{n-1}$

2nd Sequence: $256, 128, 64, ...$(G.P)

Here $a_1' = 256, r' = \frac{128}{256} = \frac{1}{2}$

Using $a_n' = a_1'r'^{n-1}$

$= (256)\left(\frac{1}{2}\right)^{n-1}$

$= 2^8 \cdot \left(\frac{1}{2^{n-1}}\right)$

According to the given condition

$a_n = a_n'$

$2^{n-1} = \frac{2^8}{2^{n-1}}$

$(2^{n-1})^2 = 2^8$

$2^{2n-2} = 2^8$

By comparing, we have

$2n - 2 = 8$

$2n = 10$

$n = 5$ (As required)

4. Find the first five terms of each sequence described:

(i) $a_1 = 243, r = \frac{1}{3}$

Solution:

Given that: $a_1 = 243, r = \frac{1}{3}$

Using: $a_1 = a_1 r^{n-1}$

$a_2 = a_1 r$

$= (243)\left(\frac{1}{3}\right) = 81$

$a_3 = a_1 r^2$

$= (243)\left(\frac{1}{3}\right)^2 = (243)\left(\frac{1}{9}\right) = 27$

$a_4 = a_1 r^3$

$= (243)\left(\frac{1}{3}\right)^3 = (243)\left(\frac{1}{27}\right) = 9$

$a_5 = a_1 r^4$

$= 243\left(\frac{1}{3}\right)^4 = 243\left(\frac{1}{81}\right) = 3$

Hence the first five terms of G.P. are

$243, 81, 27, 9, 3$

(ii) $a_1 = 579, r = -\frac{1}{2}$

Solution:

Given that: $a_1 = 579, r = -\frac{1}{2}$

Using: $a_n = a_1 r^{n-1}$

$a_2 = a_1 r$

$= (579)\left(\frac{-1}{2}\right) = \frac{-579}{2}$

$a_3 = a_1 r^2$

$= 579\left(\frac{-1}{2}\right)^2 = \frac{579}{4}$

$a_4 = a_1 r^3$

$= 579\left(\frac{-1}{2}\right)^3 = -\frac{579}{8}$

$a_5 = a_1 r^4$

$= 579\left(\frac{-1}{2}\right)^4 = \frac{579}{16}$

Hence the first five terms of G.P. are

$579, \frac{-579}{2}, \frac{579}{4}, \frac{-579}{8}, \frac{579}{16}$

5. Find the 12^{th} term of $1 + i, 2i, -2 + 2i, \ldots$

Solution:

$1 + i, 2i, -2 + 2i, \ldots$ (G.P)

Here

$a_1 = 1 + i, r = \frac{2i}{1+i} = \frac{2i}{1+i} \times \frac{1-i}{1-i} = \frac{2(i - i^2)}{1 - i^2}$

$= \frac{2(i+1)}{1+1} \Rightarrow r = 1 + i, n = 12, a_{12} = ?$

Using $a_n = a_1 r^{n-1}$

$a_{12} = (1+i)(1+i)^{11} = (1+i)^{12}$

$= [(1+i)^2]^6 = [1 + i^2 + 2i]^6$

$= [1 - 1 + 2i]^6$ $\quad \because i^2 = -1$

$= (2i)^6 = (2)^6 \cdot i^6 = 64 \cdot (i^2)^3$

$= 64 \cdot (1)^3$ $\quad \because i^2 = -1$

$= -64$

6. If the 4^{th} and 9^{th} term of a G.P. are 54 and 13122 respectively. Find the G.P. Also find its general term.

Solution:

Given that:

$a_4 = 54$ | $a_9 = 13122$

$a_1 r^3 = 54 \quad \ldots(1)$ | $a_1 r^8 = 13122 \quad \ldots(2)$

Equation (2) ÷ Equation (1)

$\frac{a_1 r^8}{a_1 r^3} = \frac{13122}{54}$

$r^5 = 243$

$r^5 = 3^5$

$\Rightarrow$ $\boxed{r = 3}$

Put $r = 3$ in eq. (1), we have

$a_1(3)^3 = 54$

$a_1 = \frac{54}{27} \Rightarrow \boxed{a_1 = 2}$

As we know

$a_n = a_1 r^{n-1}$

$= 2(3)^{n-1}$

Required G.P is

$a_1, a_1 r, a_1 r^2, \ldots$

$2, 2(3), 2(3)^2, \ldots$

$2, 6, 18, \ldots$

7. If a, b, c, d are in G.P., prove that:

(i) $a - b, b - c, c - d$ are in G.P.

(ii) $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P.

(iii) $a^2 + b^2, b^2 + c^2, c^2 + d^2$ are in G.P.

Solution:

Since a, b, c, d are in G.P., so there exists common ratio

i.e., $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r$

$\frac{b}{a} = r$	$\frac{c}{b} = r$	$\frac{d}{c} = r$
$b = ar$	$c = br$	$d = cr$
	$c = (ar)r = ar^2$	$d = (ar^2)r = ar^3$

(i) $a - b, b - c, c - d$ are in G.P.

Solution:

$a - b, b - c, c - d$ will be in G.P., if

$\frac{b-c}{a-b} = \frac{c-d}{b-c}$

$\frac{ar - ar^2}{a - ar} = \frac{ar^2 - ar^3}{ar - ar^2}$ Putting values of b, c, d

$\frac{ar(1-r)}{a(1-r)} = \frac{ar^2(1-r)}{ar(1-r)}$

$r = r$ (Common ratio exists)

which shows that $a - b, b - c, c - d$ are in G.P.

(ii) $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P

Solution:

$a^2 - b^2, b^2 - c^2, c^2 - d^2$ will be in G.P., if

$\frac{b^2 - c^2}{a^2 - b^2} = \frac{c^2 - d^2}{b^2 - c^2}$

$\frac{(ar)^2 - (ar^2)^2}{a^2 - (ar)^2} = \frac{(ar^2)^2 - (ar^3)^2}{(ar)^2 - (ar^2)^2}$ Putting values of b, c, d

$\frac{a^2r^2 - a^2r^4}{a^2 - a^2r^2} = \frac{a^2r^4 - a^2r^6}{a^2r^2 - a^2r^4}$

$\frac{a^2r^2(1-r^2)}{a^2(1-r^2)} = \frac{a^2r^4(1-r^2)}{a^2r^2(1-r^2)}$

$r^2 = r^2$ (Common ratio exists)

which shows that $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P.

(iii) $a^2 + b^2, b^2 + c^2, c^2 + d^2$ are in G.P.

Solution:

$a^2 + b^2, b^2 + c^2, c^2 + d^2$ will be in G.P., if

$\frac{b^2 + c^2}{a^2 + b^2} = \frac{c^2 + d^2}{b^2 + c^2}$

$\frac{(ar)^2 + (ar^2)^2}{a^2 + (ar)^2} = \frac{(ar^2)^2 + (ar^3)^2}{(ar)^2 + (ar^2)^2}$ Putting values of b, c, d

$\frac{a^2r^2 + a^2r^4}{a^2 + a^2r^2} = \frac{a^2r^4 + a^2r^6}{a^2r^2 + a^2r^4}$

$\frac{a^2r^2(1+r^2)}{a^2(1+r^2)} = \frac{a^2r^4(1+r^2)}{a^2r^2(1+r^2)}$

$r^2 = r^2$ (Common ratio exists)

which shows that $a^2 + b^2, b^2 + c^2, c^2 + d^2$ are in G.P.

8. If $(p+q)^{th}$ term of a G.P. be m and $(p-q)^{th}$ term be n, then find the p^{th} term.

Solution:

Given that: $a_{p+q} = m$ and $a_{p-q} = n$

As we know

$a_n = a_1 r^{n-1}$, so

$a_1 r^{p+q-1} = m \quad \ldots(1)$

and $a_1 r^{p-q-1} = n \quad \ldots(2)$

Equation (1) × Equation (2)

$(a_1 r^{p+q-1})(a_1 r^{p-q-1}) = mn$

$(a_1)^2 \cdot r^{p+q-1+p-q-1} = mn$

$(a_1)^2 \cdot r^{2p-2} = mn$

$(a_1)^2 \cdot (r^{p-1})^2 = mn$

By taking square root of both sides

$\sqrt{(a_1 r^{p-1})^2} = \sqrt{mn}$

$a_1 r^{p-1} = \sqrt{mn}$

$a_p = \sqrt{mn}$ $\quad \because a_p = a_1 r^{p-1}$

$\Rightarrow$ pth term of G.P = $\sqrt{mn}$.

9. Find three consecutive numbers in G.P. whose sum is 26 and their product is 216.

Solution:

Let the three consecutive numbers in G.P., be $\frac{a_1}{r}, a_1, a_1 r$

By given conditions

Sum = 26 | Product = 216

$\frac{a_1}{r} + a_1 + a_1 r = 26$ | $\left(\frac{a_1}{r}\right)(a_1)(a_1 r) = 216$

$\frac{a_1 + a_1 r + a_1 r^2}{r} = 26$ | $(a_1)^3 = 216$

| $a_1 = 6$

$\Rightarrow a_1 + a_1 r + a_1 r^2 = 26$

Putting $a_1 = 6$

$$6+6r+6r^2=26r$$

$$\Rightarrow \quad 6r^2+6r+6-26r=0$$

$$6r^2-20r+6=0$$

$$3r^2-10r+3=0 \qquad \text{Dividing by '2'}$$

$$3r^2-9r-r+3=0$$

$$\Rightarrow \quad 3r(r-3)-1(r-3)=0$$

$$(r-3)(3r-1)=0$$

Either $r-3=0$; $3r-1=0$ $\quad \Rightarrow r=3, r=\frac{1}{3}$

When $a_1 = 6$ and $r = 3$, then required numbers are

$$\frac{a_1}{r}, a_1, a_1r$$

$$\frac{6}{3}, 6, 6(3)$$

$$2, 6, 18$$

When $a_1 = 6$ and $r=\frac{1}{3}$, then required numbers are

$$\frac{a_1}{r}, a_1, a_1r$$

$$\frac{6}{\frac{1}{3}}, 6, 6\left(\frac{1}{3}\right)$$

$$18, 6, 2$$

10. The 3rd term of a G.P. is the square of 1st term. If the 2nd term is 9 then find the 6th term.

Solution:

Given that: $a_3=(a_1)^2$ and $a_2=9$

Using: $a_n=a_1r^{n-1}$

$$a_1r^2=(a_1)^2$$

Dividing by a_1

$$r^2=a_1 \qquad ...(1)$$

$$a_1r=9 \qquad ...(2)$$

Put $a_1=r^2$ in eq. (2)

$$r^2 . r=9$$

$$r^3=9$$

$$\Rightarrow \quad r=(9)^{\frac{1}{3}} \quad \Rightarrow \quad \boxed{r=9^{\frac{1}{3}}}$$

Put $r=9^{\frac{1}{3}}$ in eq. (2), we have

$$a_1 . 9^{\frac{1}{3}}=9$$

$$a_1=\frac{9}{9^{\frac{1}{3}}}$$

$$a_1=9^{1-\frac{1}{3}}$$

$$\boxed{a_1=9^{\frac{2}{3}}}$$

As we know

$$a_6=a_1r^5$$

$$=9^{\frac{2}{3}}.(9^{\frac{1}{3}})^5=9^{\frac{2}{3}+\frac{5}{3}}=9^{\frac{7}{3}}=\sqrt[3]{9^7}$$

$$=\sqrt[3]{(3^2)^7}=\sqrt[3]{3^{14}}=\sqrt[3]{3^{12}.3^2}$$

$$=(3^{12})^{\frac{1}{3}}.\sqrt[3]{9}$$

$$=3^4.\sqrt[3]{9}=81.\sqrt[3]{9}$$

Hence $a_6=81.\sqrt[3]{9}$

11. If $\frac{1}{a},\frac{1}{b}$ and $\frac{1}{c}$ are in G.P. Show that the common ratio is $\pm\sqrt{\frac{a}{c}}$.

Solution:

Since $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in G.P., so there exists common ratio.

i.e., $$r=\frac{\frac{1}{b}}{\frac{1}{a}}=\frac{\frac{1}{c}}{\frac{1}{b}} \quad \Rightarrow \quad r=\frac{a}{b}=\frac{b}{c}$$

$$\Rightarrow \quad r=\frac{a}{b} \quad ...(1) \qquad\qquad r=\frac{b}{c} \quad ...(2)$$

Equation (1) × Equation (2)

$$r\times r=\frac{a}{b}\times\frac{b}{c} \quad \Rightarrow r^2=\frac{a}{c} \quad \Rightarrow \sqrt{r^2}=\sqrt{\frac{a}{c}}$$

$$\Rightarrow r=\pm\sqrt{\frac{a}{c}}$$

12. If the numbers 1, 4 and 3 are subtracted from three consecutive terms of an A.P., the resulting numbers are in G.P. Find the original numbers if their sum is 21.

Solution:

Let the three consecutive terms in A.P., be a_1-d, a_1, a_1+d

By given conditions

$$a_1-d+a_1+a_1+d=21$$

$$3a_1=21 \Rightarrow a_1=7$$

a_1-d-1, a_1-4, a_1+d-3 are in G.P.

Put $a_1=7$

$7-d-1, 7-4, 7+d-3$ are in G.P.

$6-d, 3, 4+d$ are in G.P.

$$\Rightarrow \quad \frac{3}{6-d}=\frac{4+d}{3} \qquad \text{(common ratio exists)}$$

$$3\times3=(4+d).(6-d)$$

$$\Rightarrow \quad 9=24-4d+6d-d^2$$

$$0=24+2d-d^2-9$$

$$\Rightarrow \quad d^2-2d-15=0$$

$$d^2-5d+3d-15=0$$

$$\Rightarrow \quad d(d-5)+3(d-5)=0$$

$$(d-5)(d+3)=0$$

Either $d-5=0$ or $d+3=0$

$$d=5 \quad ; \quad d=-3$$

when $a_1=7$ and $d=5$, then required numbers are

$$a_1-d, a_1, a_1+d$$

$$7-5, 7, 7+5$$

$$2, 7, 12$$

when $a_1=7$ and $d=-3$, then required numbers are

$$a_1-d, a_1, a_1+d$$

$$7+3, 7, 7-3$$

$$10, 7, 4$$

13. If three consecutive numbers in A.P. are increased by 1, 4, 15 respectively, the resulting numbers are in G.P. Find the original numbers if their sum is 6.

Solution:

Let the three consecutive terms in A.P., be a_1-d, a_1, a_1+d

By given conditions

$$a_1-d+a_1+a_1+d=6$$

$$3a_1=6$$

$$a_1=2$$

a_1-d+1, a_1+4, a_1+d+15 are in G.P.

Put $a_1=2$

$2-d+1, 2+4, 2+d+15$ are in G.P.

$3-d, 6, 17+d$ are in G.P.

$$\Rightarrow \quad \frac{6}{3-6}=\frac{17+d}{6} \qquad \text{(Common ratio exists)}$$

$$6\times6=(17+d).(3-d)$$

$$36=51-17d+3d-d^2$$

$$0=51-14d-d^2-36$$

$$d^2+14d-15=0$$

$$d^2+15d-1d-15=0 \qquad \text{(By factorization)}$$

$$d(d+15)-1(d+15)=0 \quad \Rightarrow (d+15)(d-1)=0$$

Either $d+15=0$ or $d-1=0$

$$d=-15 \quad ; \quad d=1$$

when $a_1=2$ and $d=-15$, then required numbers are

$$a_1-d, a_1, a_1+d$$

$$2+15, 2, 2-15$$

$$17, 2, -13$$

when $a_1=2$ and $d=1$, then required numbers are

$$a_1-d, a_1, a_1+d$$

$$2-1, 2, 2+1$$

$$1, 2, 3$$

14. If p^{th}, q^{th} terms of a G.P. are q and p respectively, show that $(p+q)^{th}$ term is $(q^p+p^q)^{\frac{1}{p-q}}$.

Solution:

Given that: $a_p=q$ and $a_q=p$

Using: $a_n=a_1r^{n-1}$

$$a_1r^{p-1}=q \qquad ...(1)$$

$$a_1r^{q-1}=p \qquad ...(2)$$

Equation (1) ÷ Equation (2)

$$\frac{a_1r^{p-1}}{a_1r^{q-1}}=\frac{q}{p}$$

$$r^{p-1-q+1}=\frac{q}{p}$$

$$r^{p-q}=\frac{q}{p}$$

$$\Rightarrow \quad r=\left(\frac{q}{p}\right)^{\frac{1}{p-q}} \qquad \text{Put this in eq. (1)}$$

$$a_1\left\{\left(\frac{q}{p}\right)^{\frac{1}{p-q}}\right\}^{p-1}=q$$

$$a_1\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}}=q$$

$$a_1=q.\left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$$

$$=q^1.\frac{p^{\frac{p-1}{p-q}}}{q^{\frac{p-1}{p-q}}}$$

$$=q^{1-\frac{p-1}{p-q}}.p^{\frac{p-1}{p-q}}$$

$$=q^{\frac{p-q-p+1}{p-q}}.p^{\frac{p-1}{p-q}}$$

$$a_1=q^{\frac{1-q}{p-q}}.p^{\frac{p-1}{p-q}}$$

As we know

$$a_n=a_1r^{n-1}$$

$$a_{p+q}=q^{\frac{1-q}{p-q}}.p^{\frac{p-1}{p-q}}.\left\{\left(\frac{q}{p}\right)^{\frac{1}{p-q}}\right\}^{p+q-1}$$

$$=q^{\frac{1-q}{p-q}}.p^{\frac{p-1}{p-q}}.\frac{q^{\frac{p+q-1}{p-q}}}{p^{\frac{p+q-1}{p-q}}}$$

$$=q^{\frac{1-q}{p-q}+\frac{p+q-1}{p-q}}.p^{\frac{p-1}{p-q}-\frac{p+q-1}{p-q}}$$

$$= q^{\frac{1-q+p+q-1}{p-q}} \cdot p^{\frac{p-1-p-q+1}{p-q}}$$

$$= q^{\frac{p}{p-q}} \cdot p^{\frac{-q}{p-q}}$$

$$= \frac{q^{\frac{p}{p-q}}}{p^{\frac{q}{p-q}}}$$

$$= \left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$$

$$a_{p+q} = (q^p \div p^q)^{\frac{1}{p-q}} \text{ (As required)}$$

15. If $a, 2a+2, 3a+3, \ldots$ are in G.P., then find the fifth term.

Solution:

$a, 2a+2, 3a+3, \ldots$ (G.P)

$\Rightarrow \quad (2a+2)^2 = a(3a+3)$ (As $y^2 = x \cdot y$ if x, y, z are in G.P)

$4a^2 + 4 + 8a = 3a^2 + 3a$

$4a^2 + 4 + 8a - 3a^2 - 3a = 0$

$a^2 + 5a + 4 = 0$

$a^2 + 4a + a + 4 = 0$

$a(a+4) + 1(a+4) = 0$

$(a+1)(a+4) = 0$

Either $a + 1 = 0$ or $a + 4 = 0$

$a = -1$ or $a = -4$

If $a = -1$, then the terms are

$-1, (-2+2), (-3+3), \ldots$

$-1, 0, 0, \ldots$ (Not a valid G.P.)

If $a = -4$, then the terms are

$-4, (2(-4)+2), (3(-4)+3), \ldots$

$-4, -6, -9, \ldots$

$\Rightarrow \quad r = \frac{-6}{-4} = \frac{3}{2}$

Hence $a_5 = ar^4$

$= (-4)\left(\frac{3}{2}\right)^4$

$= -4\left(\frac{81}{16}\right)$

$$\boxed{a_5 = \frac{-81}{4}}$$

Geometric Mean (G.M.)

A number G is said to be a geometric mean (G.M.) between two numbers a and b if a, G, b are in G.P. Therefore

$$\frac{G}{a} = \frac{b}{G}$$

$\Rightarrow \quad G^2 = ab$

$\Rightarrow \quad G = \pm\sqrt{ab}$

> **Note:** $G_1, G_2, G_3, \ldots, G_n$ are said to be n G.Ms. between two numbers a and b if $a, G_1, G_2, G_3, \ldots, G_n, b$ are in G.P.

Example 17: Insert three G.Ms. between 2 and $\frac{1}{2}$.

Solution:

Let G_1, G_2, G_3 be three G.Ms. between 2 and $\frac{1}{2}$. Therefore

$2, G_1, G_2, G_3, \frac{1}{2}$ are in G.P.

Here $a_1 = 2, a_5 = \frac{1}{2}$ and $n = 5$.

As we know: $a_n = a_1 r^{n-1}$

$a_5 = a_1 r^4$

$\frac{1}{2} = 2r^4 \quad \Rightarrow 4r^4 - 1 = 0 \quad \Rightarrow (2r^2)^2 - 1^2 = 0 \quad \Rightarrow (2r^2-1)(2r^2+1) = 0$

$\Rightarrow \quad 2r^2 - 1 = 0 \quad ; \quad 2r^2 + 1 = 0$

$r^2 = \frac{1}{2} \quad ; \quad r^2 = -\frac{1}{2}$

Taking square root of both sides, we have

$r = \pm\frac{1}{\sqrt{2}} \quad ; \quad r = \pm\frac{i}{\sqrt{2}}$

When $r = \frac{1}{\sqrt{2}}$, we have	When $r = \frac{-1}{\sqrt{2}}$, we have	When $r = \frac{i}{\sqrt{2}}$, we have	When $r = \frac{-i}{\sqrt{2}}$, we have
$G_1 = a_1 r = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$	$G_1 = a_1 r = 2\left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{2}$	$G_1 = a_1 r = 2\left(\frac{i}{\sqrt{2}}\right) = \sqrt{2}i$	$G_1 = a_1 r = 2\left(\frac{-i}{\sqrt{2}}\right) = -\sqrt{2}i$
$G_2 = a_1 r^2 = 2\left(\frac{1}{\sqrt{2}}\right)^2 = 1$	$G_2 = a_1 r^2 = 2\left(\frac{-1}{\sqrt{2}}\right)^2 = 1$	$G_2 = a_1 r^2 = 2\left(\frac{i}{\sqrt{2}}\right)^2 = -1$	$G_2 = a_1 r^2 = 2\left(\frac{-i}{\sqrt{2}}\right)^2 = -1$
$G_3 = a_1 r^3 = 2\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{\sqrt{2}}$	$G_3 = a_1 r^3 = 2\left(\frac{-1}{\sqrt{2}}\right)^3 = -\frac{1}{\sqrt{2}}$	$G_3 = a_1 r^3 = 2\left(\frac{i}{\sqrt{2}}\right)^3 = -\frac{i}{\sqrt{2}}$	$G_3 = a_1 r^3 = 2\left(\frac{-i}{\sqrt{2}}\right)^3 = \frac{i}{\sqrt{2}}$

> **Note:** The real values of r are usually taken but here other cases are considered to widen the outlook of the students.

Exercise 6.6

1. Find G.M. between:

(i) -2 and 8 (ii) $-2i$ and $8i$ (iii) 6 and 9

(i) -2 and 8

Solution:

Here $a = -2$ and $b = 8$, so

G.M $= \pm\sqrt{ab}$

$= \pm\sqrt{(-2)(8)}$

$= \pm\sqrt{-16} = \pm 4\sqrt{-1}$

G.M $= \pm 4i$ $\quad \because \sqrt{-1} = i$

Hence the G.M between -2 and 8 is $4i$ or $-4i$.

(ii) $-2i$ and $8i$

Solution:

Here $a = -2i$ and $b = 8i$, so

G.M $= \pm\sqrt{ab}$

$= \pm\sqrt{(-2i)(8i)}$

$= \pm\sqrt{-16i^2} = \pm\sqrt{-16(-1)}$ $\quad \because i^2 = -1$

$= \pm\sqrt{16} = \pm 4$

Hence the G.M between -2 and 8 is 4 or -4.

(iii) 6 and 9

Solution:

Here $a = 6, b = 9$

G.M $= \pm\sqrt{ab}$

$= \pm\sqrt{(6)(9)}$

$= \pm 3\sqrt{6}$

Hence the G.M between 6 and 9 is $3\sqrt{6}$ or $-3\sqrt{6}$.

2. Insert four real geometric means between 3 and 96.

Solution:

Let G_1, G_2, G_3, G_4 be four G.Ms. between 3 and 96, then 3, G_1, G_2, G_3, G_4, 96 are in G.P.

Here $a_1 = 3$ | $a_6 = 96$

$ar^5 = 96$ using $a_n = a_1 r^{n-1}$

$3r^5 = 96$ $\quad \because a_1 = 3$

$r^5 = 32 \Rightarrow r^5 = 2^5 \Rightarrow r = 2$

(Taking only the real value of r)

Now, $G_1 = a_1 r = 3 \cdot (2) = 3 \cdot 2 = 6$

$G_2 = a_1 r^2 = 3 \cdot (2)^2 = 3 \cdot 4 = 12$

$G_3 = a_1 r^3 = 3 \cdot (2)^3 = 3 \cdot 8 = 24$

$G_4 = a_1 r^4 = 3 \cdot (2)^4 = 3 \cdot 16 = 48$

3. If both x and y are positive distinct real numbers, show that the geometric mean between x and y is less than their arithmetic mean.

Solution:

Given that x and y are two distinct positive real numbers, then

A.M $= \frac{x+y}{2}$ and G.M $= \sqrt{xy}$

We want to prove that G.M < A.M i.e., A.M − G.M > 0

A.M − G.M $= \frac{x+y}{2} - \sqrt{xy}$

$= \frac{x+y-2\sqrt{xy}}{2}$

$= \frac{(\sqrt{x})^2 + (\sqrt{y})^2 - 2(\sqrt{x})(\sqrt{y})}{2}$

$= \frac{(\sqrt{x}-\sqrt{y})^2}{2} > 0$

$\Rightarrow$ A.M − G.M > 0 which is required result.

4. For what value of n, $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between a and b?

Solution:

Given that:

$\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ = +ive G.M between a and b.

$$\frac{a^n+b^n}{a^{n-1}+b^{n-1}}=\sqrt{ab}$$

$$a^n+b^n=a^{\frac{1}{2}}b^{\frac{1}{2}}.(a^{n-1}+b^{n-1})$$

$$a^n+b^n=a^{n-1+\frac{1}{2}}.b^{\frac{1}{2}}+a^{\frac{1}{2}}.b^{n-1+\frac{1}{2}}$$

$$a^{n-\frac{1}{2}}.a^{\frac{1}{2}}+b^{n-\frac{1}{2}}.b^{\frac{1}{2}}=a^{n-\frac{1}{2}}.b^{\frac{1}{2}}+a^{\frac{1}{2}}.b^{n-\frac{1}{2}}$$

$$a^{n-\frac{1}{2}}.a^{\frac{1}{2}}-a^{n-\frac{1}{2}}.b^{\frac{1}{2}}=a^{\frac{1}{2}}.b^{n-\frac{1}{2}}-b^{n-\frac{1}{2}}.b^{\frac{1}{2}}$$

$$a^{n-\frac{1}{2}}\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)=b^{n-\frac{1}{2}}\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)$$

$$\frac{a^{n-\frac{1}{2}}}{b^{n-\frac{1}{2}}}=\frac{\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)}{\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)} \qquad \Rightarrow\left(\frac{a}{b}\right)^{n-\frac{1}{2}}=1$$

$$\left(\frac{a}{b}\right)^{n-\frac{1}{2}}=\left(\frac{a}{b}\right)^{0} \qquad \because\left(\frac{a}{b}\right)^{0}=1$$

Comparing both sides, we have

$$\Rightarrow \quad n-\frac{1}{2}=0 \qquad \Rightarrow n=\frac{1}{2}$$

5. The A.M. of two positive integral numbers exceeds their (positive) G.M. by 2 and their sum is 20, find the numbers.

Solution:

Let a and b be the two positive integers, then by given conditions

Sum = 20	A.M = G.M + 2
$a+b=20$ ...(1)	A.M − G.M = 2

$$\frac{a+b}{2}-\sqrt{ab}=2$$

$$\frac{20}{2}-\sqrt{ab}=2 \qquad \text{using eq. (1)}$$

$$10-2=\sqrt{ab} \qquad \Rightarrow \sqrt{ab}=8$$

$$ab=64 \qquad \text{Squaring both sides}$$

$$b=\frac{64}{a} \qquad \text{...(2)}$$

Putting $b=\frac{64}{a}$ in eq (1), we have

$$a+\frac{64}{a}=20$$

$$a^2+64=20a \qquad \text{Multiplying by 'a'}$$

$$a^2-20a+64=0 \qquad \Rightarrow a^2-16a-4a+64=0$$

$$a(a-16)-4(a-16)=0 \qquad \Rightarrow (a-16)(a-4)=0$$

Either $a-16=0$ or $a-4=0$

$a=16$; $a=4$

When $a=16$, then from eq (2)	When $a=4$, then from eq (2)
$b=\frac{64}{16}=4$	$b=\frac{64}{4}=16$

Hence the required numbers are 16, 4 or 4, 16.

6. The A.M. between two numbers is 5 and their (positive) G.M. is 4. Find the numbers.

Solution:

Let a and b be the two required numbers, then by given conditions

A.M = 5	Positive G.M. = 4
$\frac{a+b}{2}=5$	$\sqrt{ab}=4$
$a+b=10$ ...(1)	$ab=16$ Squaring both sides
	$b=\frac{16}{a}$ ...(2)

Putting $b=\frac{16}{a}$ in eq (1), we have

$$a+\frac{16}{a}=10$$

$$a^2+16=10a \qquad \text{Multiplying by 'a'}$$

$$a^2-10a+16=0$$

$$a^2-8a-2a+16=0$$

$$a(a-8)-2(a-8)=0$$

$$(a-8)(a-2)=0$$

Either $a-8=0$ or $a-2=0$

$a=8$; $a=2$

When $a=8$, then from eq (2)	When $a=2$, then from eq (2)
$b=\frac{16}{8}=2$	$b=\frac{16}{2}=8$

Hence the required numbers are 8, 2 or 2, 8.

7. The arithmetic mean between two positive numbers a and b is double their geometric mean. Prove that $a:b=2+\sqrt{3}:2-\sqrt{3}$

Solution:

Since a and b are the two positive numbers, so

$$A=\frac{a+b}{2}, G=\sqrt{ab}$$

According to the given condition

$$A=2G$$

Putting values of A and G

$$\frac{a+b}{2}=2\sqrt{ab}$$

$$\frac{a+b}{2\sqrt{ab}}=\frac{2}{1}$$

By componendo and dividendo theorem

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\frac{2+1}{2-1}$$

$$\frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2}=\frac{3}{1}$$

By taking square root of both sides

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{3}}{1}$$

Again, by componendo and dividendo theorem

$$\frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}-\sqrt{a}+\sqrt{b}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\frac{2\sqrt{a}}{2\sqrt{b}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Squaring both sides, we have

$$\left(\sqrt{\frac{a}{b}}\right)^2=\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2$$

$$\frac{a}{b}=\frac{3+1+2\sqrt{3}}{3+1-2\sqrt{3}}$$

$$\frac{a}{b}=\frac{4+2\sqrt{3}}{4-2\sqrt{3}}$$

$$\frac{a}{b}=\frac{2(2+\sqrt{3})}{2(2-\sqrt{3})}$$

$$\frac{a}{b}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$$a:b=2+\sqrt{3}:2-\sqrt{3} \text{ (Proved)}$$

8. If one geometric mean G and two arithmetic means p and q be inserted between two positive numbers, show that $G^2=(2p-q)(2q-p)$

Solution:

Let a and b be the two positive numbers.

Since G is a G.M between a and b, so

a, G, b are in G.P

$$\Rightarrow \qquad G^2=ab \qquad \text{...(1)}$$

Since p and q are two A.Ms between a and b, so

a, p, q, b are in A.P.

$\Rightarrow p-a=q-p$	$q-p=b-q$
$p-q+p=a$	$q-p+q=b$
$a=2p-q$	$b=2q+p$

Put $a=2p-a$ and $b=2q+p$ in eq (1), we have

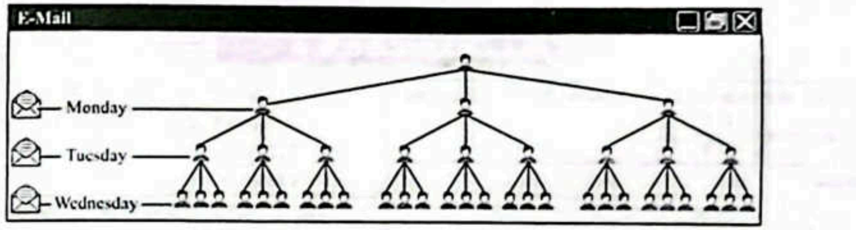
$$G^2=ab$$

$$G^2=(2p-a)(2q-p) \text{ (Proved)}$$

Geometric Series:

Geometric Series: The sum of the terms of a geometric sequence is called a geometric series.

For example, suppose you e-mail an Islamic quote to three friends on Monday. Each of those friends send it to three of their friends on Tuesday. Each person who receives the quote on Tuesday sends it to three more people on Wednesday and so on.



Notice that every day, the number of people who read your Islamic quote is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the quote is $1+3+9+27+81+243+729+2187$ or 3280. The numbers 1, 3, 9, 27, 81, 243, 729 and 2187 form a geometric sequence in which $a_1=1$ and $r=3$. The indicated sum of the numbers in the sequence, $1+3+9+27+81+243+729+2187$ is called a geometric series.

Formula for the Sum of n Terms of Geometric Series:

The sum of a geometric progression can be written as: $S_n=\frac{a_1(1-r^n)}{1-r}, r\neq 1$

To develop a formula for the sum of a geometric series,

Consider $\quad S_n=a_1+a_2+a_3+\ldots+a_n$ (Geometric series)

$$S_n=a_1+a_1r+a_1r^2+\ldots+a_1r^{n-3}+a_1r^{n-2}+a_1r^{n-1} \qquad \text{... (i)}$$

Multiplying both sides by r, we have

$$rS_n = a_1r + a_1r^2 + \dots + a_1r^{n-3} + a_1r^{n-2} + a_1r^{n-1} + a_1r^n \quad \dots (ii)$$

Subtracting (ii) from (i), we get

$$S_n - rS_n = a_1 - a_1r^n$$

$$S_n(1-r) = a_1(1-r^n)$$

$$S_n = \frac{a_1(1-r^n)}{1-r}, r \neq 1$$

Note: If $r = 1$, then form eq. (i) $S_n = na_1$

Example 18: Find the sum of n terms of the geometric series if $a_n = (-3)\left(\frac{2}{5}\right)^n$.

Solution:

Given That: $a_n = (-3)\left(\frac{2}{5}\right)^n$

$$a_1r^{n-1} = (-3)\left(\frac{2}{5}\right)^{1+(n-1)} \quad \because a_n = a_1r^{n-1}$$

$$a_1r^{n-1} = (-3)\left(\frac{2}{5}\right)^1\left(\frac{2}{5}\right)^{n-1}$$

$$a_1r^{n-1} = \left(\frac{-6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$$

Comparing both sides, we have $a_1 = -\frac{6}{5}$ and $r = \frac{2}{5}$

As we know: $$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{-\frac{6}{5}\left[1-\left(\frac{2}{5}\right)^n\right]}{1-\frac{2}{5}} = \frac{-\frac{6}{5}\left[1-\left(\frac{2}{5}\right)^n\right]}{\frac{3}{5}} = \left(-\frac{6}{5}\right)\left(\frac{5}{3}\right)\left[1-\left(\frac{2}{5}\right)^n\right] = (-2)\left[1-\left(\frac{2}{5}\right)^n\right]$$

Exercise 6.7

1. Find the sum of first 15 terms of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \dots$

Solution:

$1, \frac{1}{3}, \frac{1}{9}, \dots$ (G.P)

Here $a_1 = 1, r = \frac{\frac{1}{3}}{1} = \frac{1}{3} < 1, n = 15, S_{15} = ?$

Using $S_n = \frac{a_1(1-r^n)}{1-r} \quad \because |r| < 1$

$$S_{15} = \frac{1\left\{1-\left(\frac{1}{3}\right)^{15}\right\}}{1-\frac{1}{3}} = \frac{1-\frac{1}{3^{15}}}{\frac{2}{3}}$$

$$= \left(\frac{3^{15}-1}{3^{15}}\right) \times \frac{3}{2} = \frac{3^{15}-1}{3^{14} \times 2}$$

$$= \frac{14348907-1}{4782969 \times 2} = \frac{14348906}{4782969 \times 2}$$

$$S_{15} = \frac{7174453}{4782969}$$

---

2. The 3rd term of a G.P. is 16 and the 6th term is -128. Find the first term and the sum of the first seven terms.

Solution:

Given that:

$a_3 = 16$ and $a_6 = -128$

$a_1r^2 = 16 \dots (1)$ and $a_1r^5 = -128 \dots (2)$

Equation (2) ÷ Equation (1)

$$\frac{a_1r^5}{a_1r^2} = \frac{-128}{16}$$

$$r^3 = -8$$

$$r^3 = (-2)^3$$

$\Rightarrow$ $\boxed{r = -2}$ Put this in eq. (1), we have

$$a_1(-2)^2 = 16$$

$$a_1 = \frac{16}{4} \Rightarrow \boxed{a_1 = 4}$$

Using the sum formula

$$S_n = \frac{a_1(r^n-1)}{r-1} \quad \because |r| > 1$$

Put $n = 7, a_1 = 4, r = -2$

$$S_7 = \frac{4((-2)^7-1)}{-2-1}$$

$$= \frac{4(-128-1)}{-3} = \frac{4(-129)}{-3} = \frac{-516}{-3}$$

$$S_7 = 172$$

Hence $a_1 = 4$ and $S_7 = 172$

---

3. Sum to n terms the series:

(i) $0.2 + 0.22 + 0.222 + \cdots$

(ii) $3 + 33 + 333 + \cdots$

(i) $0.2 + 0.22 + 0.222 + \cdots$

Solution:

$$S_n = .2 + .22 + .222 + \dots \text{ to } n \text{ terms}$$

$$S_n = 2(.1 + .11 + .111 + \dots \text{ to } n \text{ terms})$$

Multiplying and dividing by '9'

$$S_n = \frac{2}{9}(.9 + .99 + .999 + \dots \text{ to } n \text{ terms})$$

$$S_n = \frac{2}{9}[(1-0.1) + (1-0.01) + (1-0.001) + \dots \text{ to } n \text{ terms}]$$

$$S_n = \frac{2}{9}\{(1+1+1+\dots \text{ to } n \text{ terms}) - (0.1 + 0.01 + 0.001 + \dots \text{ to } n \text{ terms})\}$$

$$S_n = \frac{2}{9}\left\{n - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots \text{ to } n \text{ terms}\right)\right\}$$

↓ (Geometric Series)

Here $a_1 = \frac{1}{10}, r = \frac{\frac{1}{100}}{\frac{1}{10}} = \frac{1}{10} < 1$, using $S_n = \frac{a_1(1-r^n)}{1-r} \quad \because |r| < 1$

$$S_n = \frac{2}{9}\left[n - \frac{\frac{1}{10}\left(1-\left(\frac{1}{10}\right)^n\right)}{1-\frac{1}{10}}\right]$$

$$S_n = \frac{2}{9}\left[n - \frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{\frac{9}{10}}\right]$$

$$S_n = \frac{2}{9}\left[n - \frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]$$

(ii) $3 + 33 + 333 + \cdots$

Solution:

$$S_n = 3 + 33 + 333 + \dots \text{ to } n \text{ terms}$$

Multiplying and dividing by '9'

$$S_n = \frac{1}{3}(9 + 99 + 999 + \dots \text{ to } n \text{ terms})$$

$$S_n = \frac{1}{3}\{(10-1) + (100-1) + (1000-1) + \dots \text{ to } n \text{ terms}\}$$

$$S_n = \frac{1}{3}\{(10 + 100 + 1000 + \dots \text{ to } n \text{ terms}) - (1+1+1+\dots \text{ to } n \text{ terms})\}$$

$$S_n = \frac{1}{3}\{(10 + 100 + 1000 + \dots \text{ to } n \text{ terms}) - n\}$$

↓ (Geometric Series)

Here $a_1 = 10, r = \frac{100}{10} = 10 > 1$, using $S_n = \frac{a_1(r^n-1)}{r-1} \quad \because |r| > 1$

$$S_n = \frac{1}{3}\left\{\frac{10(10^n-1)}{10-1} - n\right\}$$

$$S_n = \frac{1}{3}\left\{\frac{10}{9}(10^n-1) - n\right\}$$

---

4. Sum to n terms the series

(i) $1 + (a+b) + (a^2+ab+b^2) + (a^3+a^2b+ab^2+b^3) + \cdots$

(ii) $r + (1+k)r^2 + (1+k+k^2)r^3 + \cdots$

(i) $1 + (a+b) + (a^2+ab+b^2) + \cdots$

Solution:

$$S_n = 1 + (a+b) + (a^2+ab+b^2) + n \text{ terms}$$

Multiplying and dividing by $(a-b)$

$$S_n = \frac{1}{a-b}\{(a-b) + (a-b)(a+b) + (a-b)(a^2+ab+b^2) + \dots n \text{ terms}\}$$

$$S_n = \frac{1}{a-b}\{(a-b) + (a^2-b^2) + (a^3-b^3) + \dots n \text{ terms}\}$$

$$S_n = \frac{1}{a-b}\{(a+a^2+a^3+\dots n \text{ terms}) - (b+b^2+b^3+\dots n \text{ terms})\}$$

↓ (Geometric Series) ↓

Here $a_1 = a, r = \frac{a^2}{a} = a$; Here $a_1 = b, r = \frac{b^2}{b} = b$

$$S_n = \frac{1}{a-b}\left[\frac{a(a^n-1)}{a-1} - \frac{b(b^n-1)}{b-1}\right]$$

$$= \frac{1}{a-b}\left[\frac{a(b-1)(a^n-1) - b(a-1)(b^n-1)}{(a-1)(b-1)}\right]$$

$$S_n = \frac{a(b-1)(a^n-1) - b(a-1)(b^n-1)}{(a-b)(a-1)(b-1)}$$

(ii) $r + (1+k)r^2 + (1+k+k^2)r^3 + \cdots$

Solution:

$S_n = r + (1+k)r^2 + (1+k+k^2)r^3 + \ldots n$ terms

Multiplying and dividing by $(1-k)$

$$S_n = \frac{1}{1-k}\{(1-k)r + (1-k)(1+k)r^2 + (1-k)(1+k+k^2)r^3 + \ldots n \text{ terms}\}$$

$$S_n = \frac{1}{1-k}\{(1-k)r + (1-k^2)r^2 + (1-k^3)r^3 + \ldots n \text{ terms}\}$$

$$S_n = \frac{1}{1-k}\{(r + r^2 + r^3 + r^4 + \ldots n \text{ term}) - (kr + k^2r^2 + k^3r^3 + k^4r^4 + n \text{ terms})\}$$

↓ (Geometric series) ↓

Here $a_1 = r, r = \frac{r^2}{r} = r$; Here $a_1 = kr$, $r = \frac{k^2r^2}{kr} = kr$

$$S_n = \frac{1}{1-k}\left[\frac{r(r^n-1)}{r-1} - \frac{rk(k^nr^n-1)}{rk-1}\right]$$

$$S_n = \frac{r}{1-k}\left[\frac{r^n-1}{r-1} - \frac{k(k^nr^n-1)}{rk-1}\right]$$

5. Sum the series $2 + (1-i) + \left(\frac{1}{i}\right) + \ldots$ to 8 terms.

Solution:

$$2 + (1-i) + \left(\frac{1}{i}\right) + \ldots \text{ to 8 terms}$$

Here $a_1 = 2, r = \frac{1-i}{2}, n = 8, S_8 = ?$ using $S_n = \frac{a_1(1-r^n)}{1-r}$

$$S_8 = \frac{2\left(1-\left(\frac{1-i}{2}\right)^8\right)}{1-\frac{1-i}{2}} = \frac{2\left(1-\frac{(1-i)^8}{2^8}\right)}{\frac{2-1+i}{2}}$$

$$= \frac{2\left(\frac{2^8-((1-i)^2)^4}{2^8}\right)}{\frac{1+i}{2}} = \frac{2^8-(1+i^2-2i)^4}{2^7} \times \frac{2}{1+i}$$

$$= \frac{2^8-(1-1-2i)^4}{2^6.(1+i)} = \frac{256-(-2i)^4}{64.(1+i)} = \frac{256-2^4i^4}{64.(1+i)}$$

$$= \frac{256-16}{64.(1+i)} \quad \because i^4 = (i^2)^2 = (-1)^2 = 1$$

$$= \frac{240}{64.(1+i)} = \frac{15}{4.(1+i)} \times \frac{1-i}{1-i}$$

$$= \frac{15(1-i)}{4(1^2-i^2)} = \frac{15(1-i)}{4(1+1)}$$

$$S_8 = \frac{15}{8}(1-i)$$

6. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$, where r is the common ratio of G.P.

Solution:

Let $a_1 = 1^{st}$ term and r = common ratio.

The sum of first n terms of a G.P. is

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Replace n by $2n$, we have

$$S_{2n} = \frac{a_1(1-r^{2n})}{1-r}$$

The sum of terms from $(n+1)$ to $2n$ is

$$S' = S_{2n} - S_n$$

$$= \frac{a_1(1-r^{2n})}{1-r} - \frac{a_1(1-r^n)}{1-r}$$

$$= \frac{a_1(1-r^{2n}) - a_1(1-r^n)}{1-r}$$

$$= \frac{a_1(1-r^{2n}-1+r^n)}{1-r}$$

$$= \frac{a_1(r^n-r^{2n})}{1-r}$$

$$S' = \frac{a_1.r^n(1-r^n)}{1-r}$$

Required ratio $= S_n : S'$

$$= \frac{S_n}{S'}$$

$$= \frac{\frac{a_1(1-r^n)}{1-r}}{\frac{a_1 \cdot r^n(1-r^n)}{1-r}}$$

$$S_n : S' = \frac{1}{r^n} \text{ (Proved)}$$

Arithmetico-Geometric Progression (A.G.P.):

Suppose $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an A.P., and $b_1, b_2, b_3, \ldots, b_n, \ldots$ is a G.P. then the sequence formed by multiplying the corresponding terms of A.P. and G.P., that is, $a_1b_1, a_2b_2, a_3b_3, \ldots, a_nb_n, \ldots$ is said to be an arithmetico-geometric sequence.

Consider an A.P., $a, a+d, a+2d, \ldots, \{a+(n-1)d\}$ and a G.P., $b, br, br^2, \ldots, br^{n-1}$ where $r \neq 1$.

Multiplying the corresponding terms of A.P. and G.P., we get an arithmetico-geometric sequence

$$ab, (a+d)br, (a+2d)br^2, \ldots, \{a+(n-1)d\}br^{n-1}$$

Note:
Note that the n^{th} term of arithmetico-geometric sequence is product of n^{th} term of A.P. and n^{th} term of G.P.

Arithmetico-Geometric Series:

Sum of the terms of arithmetico-geometric sequence is called arithmetico-geometric series. Thus, arithmetico-geometric series has the form

$$ab + (a+d)br + (a+2d)br^2 + \cdots + \{a+(n-1)d\}br^{n-1}$$

Sum of first n Terms of Arithmetico-Geometric Series:

Let $S_n = ab + (a+d)br + (a+2d)br^2 + \cdots + [a+(n-1)d]br^{n-1}$ ... (i)

Multiplying both sides by r, we have

Then $rS_n = abr + (a+d)br^2 + \cdots + [a+(n-2)d]br^{n-1} + [a+(n-1)d]br^n$ ... (ii)

Subtracting (ii) from (i), we get

$$S_n - rS_n = ab + (a+d-a)br + (a+2d-a-d)br^2 + \cdots + [a+nd-d-a-nd+2d]br^{n-1} - [a+(n-1)d]br^n$$

$$(1-r)S_n = ab + \underbrace{\left[dbr + dbr^2 + \cdots + dbr^{n-1}\right]}_{\text{G. Series}} - [a+(n-1)d]br^n$$

Here $a_1 = dbr, r = r, S_{n-1} = \frac{a_1(1-r^{n-1})}{1-r}$

$$(1-r)S_n = ab + \frac{dbr(1-r^{n-1})}{1-r} - [a+(n-1)d]br^n$$

$$(1-r)S_n = ab + \frac{dbr}{1-r} - \frac{dbr^n}{1-r} - [a+(n-1)d]br^n$$

$$\boxed{S_n = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2} - \frac{dbr^n}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r}} \quad \ldots \text{(iii)}$$

which is the sum of the n terms of arithmetico-geometric series.

Sum to Infinity of Arithmetico-Geometric Series:

If $|r| < 1$, then $r^n \to 0$ and $nr^n \to 0$ as $n \to \infty$

Therefore, eq. (iii) reduces to $\boxed{S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}}$

which is the sum to infinity of arithmetico-geometric series.

Example 19: Sum the series upto n terms: $2 \cdot 1 + 3 \cdot 2 + 4 \cdot 4 + 5 \cdot 8 + \cdots$

Solution:

Let $S_n = 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 2^2 + 5 \cdot 2^3 + \cdots$ to n terms

n^{th} term of the A.P., 2, 3, 4, 5, $\cdots$ is $a_1 + (n-1)d = 2 + (n-1)(1)$
$= 2 + n - 1 = n + 1$

n^{th} term of the G.P., 1, 2, 2^2, 2^3, … is $a_1r^{n-1} = 1.2^{n-1} = 2^{n-1}$

So, $S_n = 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 2^2 + 5 \cdot 2^3 + \cdots + (n+1)2^{n-1}$ ...(i)

Multiplying both sides by common ratio of G.P., we get

$$2S_n = 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + 5 \cdot 2^4 + \cdots + (n)2^{n-1} + (n+1)2^n \quad \ldots \text{(ii)}$$

Subtracting (ii) from (i), we get

$$S_n - 2S_n = 2\cdot1 + (3-2)\cdot2 + (4-3)\cdot2^2 + (5-4)\cdot2^3 + \cdots + (n+1-n)2^{n-1} - (n+1)2^n$$

$$-S_n = 2\cdot1 + 1\cdot2 + 1\cdot2^2 + 1\cdot2^3 + \cdots + 1\cdot2^{n-1} - (n+1)2^n$$

$$-S_n = 2 + \{2 + 2^2 + 2^3 + \cdots + 2^{n-1}\} - (n+1)2^n$$

$$-S_n = 2 + \frac{2(2^{n-1}-1)}{2-1} - (n+1)\,2^n \qquad \because\ S_{n-1} = \frac{a_1(r^{n-1}-1)}{r-1}$$

$$-S_n = 2 + 2^n - 2 - n\cdot2^n - 2^n$$

$$-S_n = -n\cdot2^n$$

$$S_n = n\cdot2^n$$

Example 20: Sum the series upto n terms: $2 + \frac{4}{3} + \frac{6}{9} + \frac{8}{27} + \cdots$

Solution:

Let $S_n = 2 + \frac{4}{3} + \frac{6}{9} + \frac{8}{27} + \cdots$ to n terms

n^{th} term of the A.P., 2,4,6,8,... is $a_1 + (n-1)d = 2 + (n-1)(2) = 2 + 2n - 2 = 2n$

n^{th} term of the G.P., $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $a_1 r^{n-1} = (1)\left(\frac{1}{3}\right)^{n-1} = \frac{1}{3^{n-1}}$

So, $$S_n = 2 + \frac{4}{3} + \frac{6}{9} + \frac{8}{27} + \cdots + \frac{2n}{3^{n-1}} \quad \ldots \text{(i)}$$

$$\frac{1}{3}S_n = \frac{2}{3} + \frac{4}{9} + \frac{6}{27} + \cdots + \frac{2n-2}{3^{n-1}} + \frac{2n}{3^n} \quad \ldots \text{(ii)}$$

Subtracting (ii) from (i), we get

$$\left(1 - \frac{1}{3}\right)S_n = 2 + \frac{4-2}{3} + \frac{6-4}{9} + \frac{8-6}{27} + \cdots + \frac{2n-2n+2}{3^{n-1}} - \frac{2n}{3^n}$$

$$\frac{2}{3}S_n = 2 + \left[\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}}\right] - \frac{2n}{3^n}$$

$$\frac{2}{3}S_n = 2 + \left[\frac{\frac{2}{3}\left\{1 - \left(\frac{1}{3}\right)^{n-1}\right\}}{1 - \frac{1}{3}}\right] - \frac{2n}{3^n} \qquad \because\ S_{n-1} = \frac{a_1\left(1 - r^{n-1}\right)}{1-r}$$

$$\frac{2}{3}S_n = 2 + \frac{\frac{2}{3}\left\{1 - \left(\frac{1}{3}\right)^{n-1}\right\}}{\frac{2}{3}} - \frac{2n}{3^n}$$

$$= 2 + 1 - \left(\frac{1}{3}\right)^{n-1} - 2n\left(\frac{1}{3}\right)^n$$

$$S_n = \frac{3}{2}\left[3 - \left(\frac{1}{3}\right)^{n-1} - 2n\left(\frac{1}{3}\right)^n\right]$$

$$S_n = \frac{9}{2} - \frac{3}{2}\left(\frac{1}{3}\right)^{n-1} - 3n\left(\frac{1}{3}\right)^n$$

Example 21: Find the sum to n terms of the series: $1 + 2x + 3x^2 + 4x^3 + \cdots$ where $x \neq 1$. If $|x| < 1$, sum the series to infinity.

Solution:

Let $$S_\infty = 1x^0 + 2x + 3x^2 + 4x^3 + \cdots \infty$$

Arithmetic Part

1, 2, 3, 4, ⋯ (A.P.)

Here $a = 1$

$d = 2 - 1$

$= 1$

Geometric Part

$x^0, x, x^2, x^3 + \cdots$ (G.P.)

Here $b = 1$

$r = \frac{x}{x^0} = x$

As we know: $$S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

$$S_\infty = \frac{1.1}{1-x} + \frac{1.(1)x}{(1-x)^2} = \frac{1(1-x)+x}{(1-x)^2} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$$

Exercise 6.8

1. Find the 8^{th} term of the arithmetico-geometric sequence, where the arithmetic part is 1, 4, 7, ... and the geometric part is 5, 10, 20, ...

Solution:

Arithmetic Part

1, 4, 7, ... (A.P)

Here $a_1 = 1$, $d = 4 - 1 = 3$

$a_8 = a_1 + 7d$

$= 1 + 7(3)$

$= 22$

Geometric Part

5, 10, 20, ... (G.P)

Here $b_1 = 5$, $r = \frac{10}{5} = 2$

$b_8 = b_1 r^7$

$= 5(2)^7$

$= 5(128) = 640$

8^{th} term of A.G.P. $= a_8 \cdot b_8$

$= (22)(640)$

$= 14080$

2. Find the n^{th} term of the arithmetic-geometric sequence, where the arithmetic part is 3, 7, 11, ... and the geometric part is 2, 6, 18, ...

Solution:

Arithmetic Part

3, 7, 11, ... (A.P)

Here $a_1 = 3$, $d = 7 - 3 = 4$

$a_n = a_1 + (n-1)d$

$= 3 + (n-1)(4)$

$= 3 + 4n - 4$

$= 4n - 1$

Geometric Part

2, 6, 18, ... (G.P)

Here $b_1 = 2$, $r = \frac{6}{2} = 3$

$b_n = b_1 r^{n-1}$

$= 2(3)^{n-1}$

$= 2 \cdot 3^{n-1}$

n^{th} term of A.G.P. $= a_n \cdot b_n$

$= (4n-1)(2 \cdot 3^{n-1})$

$= 2(4n-1) \cdot 3^{n-1}$

3. Consider the arithmetico-geometric sequence defined by arithmetic part: $a_{n+1} = 2n + 5$ and geometric part: $b_{n-2} = \frac{1}{9}(-3)^n$. Find the n^{th} term and the sum of first three terms of the arithmetico-geometric sequence.

Solution:

Arithmetic Part

$a_{n+1} = 2n + 5$

Replace n by $n - 1$

$a_{n-1+1} = 2(n-1) + 5$

$a_n = 2n - 2 + 5$

$a_n = 2n + 3$

Geometric Part

$b_{n-2} = \frac{1}{9}(-3)^n$

Replace n by $n + 2$

$a_{n+2-2} = \frac{1}{9}(-3)^{n+2}$

$b_n = \frac{1}{9}(-3)^n.(-3)^2$

$b_n = \frac{1}{9}(-3)^n.9$

$= (-3)^n$

nth term of A.G.P. $= a_n \cdot b_n$

$= (2n+3)\cdot(-3)^n$

First three terms of A.G.P. are

$= (2(1)+3)(-3)^1, (2(2)+3)(-3)^2, (2(3)+3)(-3)^3$

$= (5)(-3), (7)(9), (9)(-27)$

$= -15, 63, -243$

Sum of first three terms of A.G.P $= -15 + 63 - 243$

$= -195$

4. Sum to n terms the following series:

(i) $1\cdot2 + 3\cdot4 + 5\cdot8 + 7\cdot16 + \cdots$

Solution:

Let $S_n = 1\cdot2 + 3\cdot4 + 5\cdot8 + 7\cdot16 + \cdots$ to n terms

Arithmetic Part

$1, 3, 5, 7, \ldots$ (A.P)

Here $a_1 = 1,\ d = 3 - 1 = 2$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 1 + (n-1)(2) \\ &= 1 + 2n - 2 \\ &= 2n - 1 \end{aligned}$$

Geometric Part

$2, 4, 8, 16, \ldots$ (G.P)

Here $b_1 = 2,\ r = \frac{4}{2} = 2$

$$\begin{aligned} b_n &= b_1 r^{n-1} \\ &= 2\,.\,(2)^{n-1} = 2^{n-1+1} = 2^n \end{aligned}$$

$$\Rightarrow \quad S_n = 1\cdot 2 + 3\cdot 4 + 5\cdot 8 + 7\cdot 16 + \ldots + (2n-1)\cdot 2^n \quad \ldots(1)$$

Multiply both sides by 2

$$2S_n = 1\cdot 4 + 3\cdot 8 + 5\cdot 16 + \ldots + (2n-1)\cdot 2^{n+1}$$

$$2S_n = 1\cdot 4 + 3\cdot 8 + 5\cdot 16 + \ldots + (2n-3)\cdot 2^n + (2n-1)\cdot 2^{n+1} \quad \ldots(2)$$

Equation (1) – Equation (2)

$$S_n - 2S_n = 1\cdot 2 + (3-1)4 + (5-3)8 + (7-5)16 + \ldots + ((2n-1)-(2n-3))2^n - (2n-1)\cdot 2^{n+1}$$

$$-S_n = 2 + 2\cdot 4 + 2\cdot 8 + 2\cdot 16 + \ldots + (2n-1-2n+3)\cdot 2^n - (2n-1)\cdot 2^{n+1}$$

$$= 2 + 2\cdot 2^2 + 2\cdot 2^3 + 2\cdot 2^4 + \ldots + 2\cdot 2^n - (2n-1)\cdot 2^{n+1}$$

$$= 2 + 2\left(2^2 + 2^3 + 2^4 + \ldots + 2^n\right) - (2n-1)\cdot 2^{n+1}$$

$$= 2 + 2\left\{\frac{2^2\left(2^{n-1}-1\right)}{2-1}\right\} - (2n-1)\cdot 2^{n+1} = 2 + 2^3\left(2^{n-1}-1\right) - (2n-1)\cdot 2^{n+1}$$

$$= 2 + 2^{n+2} - 8 - (2n-1)\cdot 2^{n+1} = -6 + \left(2^2 - (2n-1)\cdot 2^1\right)\cdot 2^n = -6 + (4 - 4n + 2)\cdot 2^n$$

$$-S_n = -6 + (-4n+6)\cdot 2^n = -[6 + (4n-6)2^n]$$

$$\Rightarrow \quad \boxed{S_n = 6 + (4n-6)\cdot 2^n}$$

(ii) $2 \cdot 3 + 4 \cdot 3^2 + 6 \cdot 3^3 + 8 \cdot 3^4 + \cdots$

Solution:

Let $S_n = 2 \cdot 3 + 4 \cdot 3^2 + 6 \cdot 3^3 + 8 \cdot 3^4 + \cdots$ to n terms

Arithmetic Part

$2, 4, 6, 8, \ldots$ (A.P)

Here $a_1 = 2,\ d = 4 - 2 = 2$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 2 + (n-1)(2) \\ &= 2 + 2n - 2 \\ a_n &= 2n \end{aligned}$$

Geometric Part

$3, 3^2, 3^3, 3^4, \ldots$ (G.P)

Here $b_1 = 3,\ r = \frac{3^2}{3} = 3$

$$\begin{aligned} b_n &= b_1 r^{n-1} \\ &= 3^1\,.\,3^{n-1} = 3^n \end{aligned}$$

$$\Rightarrow \quad S_n = 2\cdot 3 + 4\cdot 3^2 + 6\cdot 3^3 + 8\cdot 3^4 + \ldots + 2n\cdot 3^n \quad \ldots(1)$$

Multiply both sides by 3

$$3S_n = 2\cdot 3^2 + 4\cdot 3^3 + 6\cdot 3^4 + \ldots + 2n\cdot 3^{n+1}$$

$$3S_n = 2\cdot 3^2 + 4\cdot 3^3 + 6\cdot 3^4 + \ldots + 2(n-1)\cdot 3^n + 2n\cdot 3^{n+1} \quad \ldots(2)$$

Equation (1) – Equation (2)

$$S_n - 3S_n = 2\cdot 3 + (4-2)\cdot 3^2 + (6-4)\cdot 3^3 + (8-6)\cdot 3^4 + \ldots + (2n-2n+2)3^n - 2n\,.\,3^{n+1}$$

$$-2S_n = 2\cdot 3 + 2\cdot 3^2 + 2\cdot 3^3 + 2\cdot 3^4 + \ldots + 2\cdot 3^n - 2n\cdot 3^{n+1}$$

$$-2S_n = 2\underbrace{(3 + 3^2 + 3^3 + 3^4 + \ldots + 3^n)}_{\uparrow\ \text{(Geometric Series)}} - 2n\cdot 3^{n+1}$$

Using $S_n = \dfrac{a_1(r^n - 1)}{r-1} \quad \because |r| > 1$

$$-2S_n = 2\left\{\frac{3(3^n-1)}{3-1}\right\} - 2n\cdot 3^{n+1} = 2\frac{3(3^n-1)}{2} - 2n\cdot 3^{n+1} = 3(3^n-1) - 2n\cdot 3^{n+1}$$

$$-2S_n = 3^{n+1} - 3 - 2n\cdot 3^{n+1}$$

$$2S_n = 3 + 2n\cdot 3^{n+1} - 3^{n+1} \qquad \text{Multiplying both sides by } -1$$

$$2S_n = 3[1 + 2n\,.\,3^n - 3^n]$$

$$S_n = \frac{3}{2}[1 + (2n-1)\cdot 3^n]$$

(iii) $2 + \frac{5}{4} + \frac{8}{4^2} + \frac{11}{4^3} + \ldots$

Solution:

Let $S_n = 2 + \frac{5}{4} + \frac{8}{4^2} + \frac{11}{4^3} + \ldots$ to n terms

Arithmetic Part

$2, 5, 8, 11, \ldots$ (A.P)

Here $a_1 = 2,\ d = 5 - 2 = 3$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 2 + (n-1)(3) \\ &= 2 + 3n - 3 \\ &= 3n - 1 \end{aligned}$$

Geometric Part

$1, \frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots$ (G.P)

Here $b_1 = 1,\ r = \frac{1}{4}$

$$b_n = b_1 r^{n-1} = 1\cdot\left(\frac{1}{4}\right)^{n-1} = \frac{1}{4^{n-1}}$$

$$\Rightarrow \quad S_n = 2 + \frac{5}{4} + \frac{8}{4^2} + \frac{11}{4^3} + \ldots + \frac{3n-1}{4^{n-1}} \quad \ldots(1)$$

Multiply both sides by $\frac{1}{4}$

$$\frac{1}{4}S_n = \frac{2}{4} + \frac{5}{4^2} + \frac{8}{4^3} + \frac{11}{4^4} + \ldots + \frac{3n-1}{4^n}$$

$$\frac{1}{4}S_n = \frac{2}{4} + \frac{5}{4^2} + \frac{8}{4^3} + \ldots + \frac{3n-4}{4^{n-1}} + \frac{3n-1}{4^n} \quad \ldots(2)$$

Equation (1) – Equation (2)

$$\left(1 - \frac{1}{4}\right)S_n = 2 + (5-2)\cdot\frac{1}{4} + (8-5)\cdot\frac{1}{4^2} + (11-8)\cdot\frac{1}{4^3} + \ldots + (3n-1-3n+4)\cdot\frac{1}{4^{n-1}} - \frac{3n-1}{4^n}$$

$$\frac{3}{4}S_n = 2 + \left\{\frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \ldots + \frac{3}{4^{n-1}}\right\} - \frac{3n-1}{4^n}$$

$$= 2 + \left\{\frac{\frac{3}{4}\left(1 - \left(\frac{1}{4}\right)^{n-1}\right)}{1 - \frac{1}{4}}\right\} - \frac{3n-1}{4^n} = 2 + \left\{\frac{\frac{3}{4}\left(1 - \frac{1}{4^{n-1}}\right)}{\frac{3}{4}}\right\} - \frac{3n-1}{4^n}$$

$$\frac{3}{4}S_n = 2 + 1 - \frac{1}{4^{n-1}} - \frac{3n-1}{4^n}$$

$$S_n = \frac{4}{3}\left\{3 - \frac{1}{4^{n-1}} - \frac{3n-1}{4^n}\right\}$$

$$S_n = 4 - \frac{4}{3}\left(\frac{1}{4}\right)^{n-1} - \frac{4}{3}(3n-1)\left(\frac{1}{4}\right)^n$$

(iv) $1 + \frac{3}{5} + \frac{5}{5^2} + \frac{7}{5^3} + \ldots$

Solution:

Let $S_n = 1 + \frac{3}{5} + \frac{5}{5^2} + \frac{7}{5^3} + \ldots$ to n terms

Arithmetic Part

1, 3, 5, 7, ... (A.P)

Here $a_1 = 1$, $d = 3 - 1 = 2$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 1 + (n-1)(2) \\ &= 1 + 2n - 2 \\ &= 2n - 1 \end{aligned}$$

Geometric Part

$1, \frac{1}{5}, \frac{1}{5^2}, \frac{1}{5^3}, \ldots$ (G.P)

Here $b_1 = 1$, $r = \frac{1}{5}$

$$\begin{aligned} b_n &= b_1 r^{n-1} \\ &= 1 \cdot \left(\frac{1}{5}\right)^{n-1} = \frac{1}{5^{n-1}} \end{aligned}$$

$$\Rightarrow \quad S_n = 1 + \frac{3}{5} + \frac{5}{5^2} + \frac{7}{5^3} + \ldots + \frac{2n-1}{5^{n-1}} \quad \ldots(1)$$

Multiply both sides by $\frac{1}{5}$

$$\frac{1}{5}S_n = \frac{1}{5} + \frac{3}{5^2} + \frac{5}{5^3} + \frac{7}{5^4} + \ldots + \frac{2n-1}{5^n}$$

$$\frac{1}{5}S_n = \frac{1}{5} + \frac{3}{5^2} + \frac{5}{5^3} + \frac{7}{5^4} + \ldots + \frac{2n-3}{5^{n-1}} + \frac{2n-1}{5^n} \quad \ldots(2)$$

Equation (1) – Equation (2)

$$\left(1 - \frac{1}{5}\right)S_n = 1 + (3-1)\cdot\frac{1}{5} + (5-3)\cdot\frac{1}{5^2} + (7-5)\cdot\frac{1}{5^3} + \ldots + (2n-1-2n+3)\frac{1}{5^{n-1}} - \frac{2n-1}{5^n}$$

$$\frac{4}{5}S_n = 1 + \left(\frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \ldots + \frac{2}{5^{n-1}}\right) - \frac{2n-1}{5^n}$$

$$= 1 + \frac{\frac{2}{5}\left(1 - \left(\frac{1}{5}\right)^{n-1}\right)}{1 - \frac{1}{5}} - \frac{2n-1}{5^n} = 1 + \frac{\frac{2}{5}\left(1 - \frac{1}{5^{n-1}}\right)}{\frac{4}{5}} - \frac{2n-1}{5^n}$$

$$\frac{4}{5}S_n = 1 + \frac{1}{2}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{2n-1}{5^n}$$

$$S_n = \frac{5}{4}\left\{1 + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{5^{n-1}} - \frac{2n-1}{5^n}\right\} = \frac{5}{4}\left\{\frac{3}{2} - \frac{1}{2}\cdot\frac{1}{5^{n-1}} - \frac{2n-1}{5^n}\right\}$$

$$S_n = \frac{15}{8} - \frac{5}{8}\left(\frac{1}{5}\right)^{n-1} - \frac{5}{4}(2n-1)\cdot\left(\frac{1}{5}\right)^n$$

(v) $1 + \frac{4}{3} + \frac{7}{9} + \frac{10}{27} + \ldots$

Solution:

Let $S_n = 1 + \frac{4}{3} + \frac{7}{9} + \frac{10}{27} + \ldots$ to n terms

$S_n = 1 + 4\cdot\frac{1}{3} + 7\cdot\frac{1}{9} + 10\cdot\frac{1}{27} + \ldots$ to n terms

Arithmetic Part

1, 4, 7, 10, ... (A.P)

Here $a_1 = 1$, $d = 4 - 1 = 3$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 1 + (n-1)(3) \\ &= 1 + 3n - 3 \\ &= 3n - 2 \end{aligned}$$

Geometric Part

$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ (G.P)

Here $b_1 = 1$, $r = \frac{1}{3}$

$$b_n = b_1 r^{n-1} = 1\cdot\left(\frac{1}{3}\right)^{n-1} = \frac{1}{3^{n-1}}$$

$$\Rightarrow \quad S_n = 1 + \frac{4}{3} + \frac{7}{9} + \frac{10}{27} + \ldots + \frac{3n-2}{3^{n-1}} \quad \ldots(1)$$

Multiply both sides by $\frac{1}{3}$

$$\frac{1}{3}S_n = \frac{1}{3} + \frac{4}{3^2} + \frac{7}{3^3} + \frac{10}{3^4} + \ldots + \frac{3n-2}{3^n}$$

$$\frac{1}{3}S_n = \frac{1}{3} + \frac{4}{3^2} + \frac{7}{3^3} + \frac{10}{3^4} + \ldots + \frac{3n-5}{3^{n-1}} + \frac{3n-2}{3^n} \quad \ldots(2)$$

Equation (1) – Equation (2)

$$S_n - \frac{1}{3}S_n = 1 + (4-1)\cdot\frac{1}{3} + (7-4)\cdot\frac{1}{3^2} + (10-7)\cdot\frac{1}{3^3} + \ldots + (3n-2-3n+5)\cdot\frac{1}{3^{n-1}} - \frac{3n-2}{3^n}$$

$$\left(1 - \frac{1}{3}\right)S_n = 1 + \left(\frac{3}{3} + \frac{3}{3^2} + \frac{3}{3^3} + \ldots + \frac{3}{3^{n-1}}\right) - \frac{3n-2}{3^n}$$

$$\frac{2}{3}S_n = 1 + \frac{\frac{3}{3}\left(1 - \left(\frac{1}{3}\right)^{n-1}\right)}{1 - \frac{1}{3}} - \frac{3n-2}{3^n} = 1 + \frac{\left(1 - \frac{1}{3^{n-1}}\right)}{\frac{2}{3}} - \frac{3n-2}{3^n} = 1 + \frac{3}{2}\left(1 - \frac{1}{3^{n-1}}\right) - \frac{3n-2}{3^n}$$

$$\frac{2}{3}S_n = 1 + \frac{3}{2} - \frac{3}{2}\left(\frac{1}{3}\right)^{n-1} - (3n-2)\cdot\left(\frac{1}{3}\right)^n$$

$$S_n = \frac{3}{2}\left\{\frac{5}{2} - \frac{3}{2}\left(\frac{1}{3}\right)^{n-1} - (3n-2)\left(\frac{1}{3}\right)^n\right\}$$

$$S_n = \frac{15}{4} - \frac{9}{4}\left(\frac{1}{3}\right)^{n-1} - \frac{3}{2}(3n-2)\left(\frac{1}{3}\right)^n$$

5. Sum the following infinite series:

(i) $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots$

Solution:

Let $S_\infty = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots\infty$

$$= 1\cdot 1 + 3\cdot\frac{1}{2} + 5\cdot\frac{1}{2^2} + 7\cdot\frac{1}{2^3} + \ldots\infty$$

Arithmetic Part

1, 3, 5, 7, ... (A.P)

Here $a = 1$, $d = 3 - 1 = 2$

Geometric Part

$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$ (G.P)

Here $b = 1$, $r = \frac{1}{2}$, $|r| < 1$

As we know $S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$

$$= \frac{(1)(1)}{1 - \frac{1}{2}} + \frac{(2)(1)\left(\frac{1}{2}\right)}{\left(1 - \frac{1}{2}\right)^2}$$

$$= \frac{1}{\frac{1}{2}} + \frac{1}{\left(\frac{1}{2}\right)^2} = 2 + \frac{1}{\frac{1}{4}}$$

$S_\infty = 2 + 4$

$\boxed{S_\infty = 6}$

(ii) $2 + \frac{5}{3} + \frac{8}{9} + \frac{11}{27} + \ldots$

Solution:

Let $S_\infty = 2 + \frac{5}{3} + \frac{8}{9} + \frac{11}{27} + \ldots\infty$

$$= 2\cdot 1 + 5\cdot\frac{1}{3} + 8\cdot\frac{1}{3^2} + 11\cdot\frac{1}{3^3} + \ldots\infty$$

Arithmetic Part

2, 5, 8, 11, ... (A.P)

Here $a = 2$

$d = 5 - 2 = 3$

Geometric Part

$1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots$ (G.P)

Here $b = 1$, $r = \frac{1}{3}$, $|r| < 1$

As we know $S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$

$$= \frac{(2)(1)}{1 - \frac{1}{3}} + \frac{(3)(1)\left(\frac{1}{3}\right)}{\left(1 - \frac{1}{3}\right)^2}$$

$$= \frac{\frac{2}{3}}{\frac{2}{3}} + \frac{1}{\left(\frac{2}{3}\right)^2}$$

$$= 2\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2$$

$$= 3 + \frac{9}{4}$$

$$S_\infty = \frac{12+9}{4}$$

$$\boxed{S_\infty = \frac{21}{4}}$$

6. Show that $2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdot 16^{\frac{1}{16}} \ldots \infty = 4$

Solution:

$$2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdot 16^{\frac{1}{16}} \ldots \infty = 4$$

$$\text{L.H.S} = 2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdot 16^{\frac{1}{16}} \ldots \infty$$

$$= 2^{\frac{1}{2}} \cdot (2^2)^{\frac{1}{4}} \cdot (2^3)^{\frac{1}{8}} \cdot (2^4)^{\frac{1}{16}} \ldots \infty$$

$$= 2^{\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots} = 2^{S_\infty} \quad \ldots(i)$$

Let $S_\infty = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots \infty$

Arithmetic Part	Geometric Part
1, 2, 3, 4, ... (A.P)	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ (G.P)
Here $a = 1$,	Here $b = \frac{1}{2}$
$d = 2 - 1 = 1$	$r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}, \lvert r \rvert < 1$

As we know $S_\infty = \frac{ab}{1-r} + \frac{abr}{(1-r)^2}$

$$= \frac{(1)\left(\frac{1}{2}\right)}{1-\frac{1}{2}} + \frac{(1)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^2}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{1}{4}}{\frac{1}{4}}$$

$$= 1 + 1$$

$$\boxed{S_\infty = 2}$$

L.H.S $= 2^2$ using (i)

$= 4 =$ R.H.S (Proved)

7. Show that $\sqrt{4} \cdot \sqrt[4]{16} \cdot \sqrt[8]{64} \cdot \sqrt[16]{256} \ldots \infty = 16$

Solution:

$$\sqrt{4} \cdot \sqrt[4]{16} \cdot \sqrt[8]{64} \cdot \sqrt[16]{256} \ldots \infty = 16$$

$$\text{L.H.S} = 4^{\frac{1}{2}} \cdot 16^{\frac{1}{4}} \cdot 64^{\frac{1}{8}} \cdot 256^{\frac{1}{16}} \ldots \infty$$

$$= 4^{\frac{1}{2}} \cdot (4^2)^{\frac{1}{4}} \cdot (4^3)^{\frac{1}{8}} \cdot (4^4)^{\frac{1}{16}} \ldots \infty$$

$$= 4^{\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots} = 4^{S_\infty}$$

Let $S_\infty = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots \infty$

$$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots \infty$$

Arithmetic Part	Geometric Part
1, 2, 3, 4, ... (A.P)	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ (G.P)
Here $a = 1$,	Here $b = \frac{1}{2}$
$d = 2 - 1 = 1$	$r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}, \lvert r \rvert < 1$

As we know $S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$

$$= \frac{(1)\left(\frac{1}{2}\right)}{1-\frac{1}{2}} + \frac{(1)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^2}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{1}{4}}{\frac{1}{4}}$$

$$S_\infty = 1 + 1 = 2$$

L.H.S $= 4^2$ using (i)

$= 16 =$ R.H.S (Proved)

8. Sum to n terms the series $2 + 4x + 6x^2 + 8x^3 + \cdots$ where $x \neq 1$

Solution:

Let $S_n = 2 + 4x + 6x^2 + 8x^3 + \ldots$ to n terms

Arithmetic Part	Geometric Part
2, 4, 6, 8, ... (A.P)	$1, x, x^2, x^3, \ldots$ (G.P)
Here $a_1 = 2$, $d = 4 - 2 = 2$	Here $b_1 = 1$, $r = \frac{x}{1} = x$
$a_n = a_1 + (n-1)d$	$b_n = b_1 r^{n-1}$
$= 2 + (n-1)(2)$	$= 1 \cdot x^{n-1}$
$= 2 + 2n - 2$	$= x^{n-1}$
$= 2n$	

$$\Rightarrow \quad S_n = 2 + 4x + 6x^2 + 8x^3 + \ldots + (2n) \cdot x^{n-1} \quad \ldots(1)$$

Multiply both sides by 'x'

$$xS_n = 2x + 4x^2 + 6x^3 + 8x^4 + \ldots + (2n) \cdot x^n$$

$$xS_n = 2x + 4x^2 + 6x^3 + 8x^4 + \ldots + 2(n-1)x^{n-1} + 2n \cdot x^n \ldots(2)$$

Equation (1) – Equation (2)

$$S_n - xS_n = 2 + (4-2)x + (6-4)x^2 + (8-6)x^3 + \ldots + (2n - 2n + 2)x^{n-1} - 2n \cdot x^n$$

$$(1-x)S_n = 2 + 2 \cdot x + 2x^2 + 2x^3 + \ldots + 2 \cdot x^{n-1} - 2nx^n$$

$$= 2 + (2x + 2x^2 + 2x^3 + \ldots + 2x^{n-1}) - 2n \cdot x^n$$

$$= 2 + \frac{2x(1-x^{n-1})}{1-x} - 2n \cdot x^n$$

$$(1-x)S_n = \frac{2 - 2x + 2x - 2x^n - 2n(1-x) \cdot x^n}{1-x}$$

$$S_n = \frac{2 - 2x^n - 2nx^n + 2nx^{n+1}}{(1-x)(1-x)}$$

$$S_n = \frac{2 - 2(n+1)x^n + 2nx^{n+1}}{(1-x)^2}$$

9. Find the sum to n terms of the series: $\frac{2n+1}{2n-1} + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \cdots$

Solution:

Let $S_n = 1\left(\frac{2n+1}{2n-1}\right) + 3\left(\frac{3n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + 7\left(\frac{2n+1}{2n-1}\right)^4 + \ldots$ to n terms

Let $\frac{2n+1}{2n-1} = x$

$\Rightarrow \quad S_n = 1 \cdot x + 3x^2 + 5x^3 + 7x^4 + \ldots$ to n terms

Arithmetic Part	Geometric Part
1, 3, 5, 7, ... (A.P)	$x, x^2, x^3, x^4, \ldots$ (G.P)
Here $a_1 = 1$, $d = 3 - 1 = 2$	Here $b_1 = x$, $r = \frac{x^2}{x} = x$
$a_n = a_1 + (n-1)d$	$b_n = b_1 r^{n-1}$
$= 1 + (n-1)(2)$	$= x \cdot x^{n-1}$
$= 1 + 2n - 2$	$= x^n$
$= 2n - 1$	

$$\Rightarrow \quad S_n = 1 \cdot x + 3x^2 + 5x^3 + 7x^4 + \ldots + (2n-1) \cdot x^n \quad \ldots(1)$$

Multiply both sidles by x

$$xS_n = 1 \cdot x^2 + 3x^3 + 5x^4 + \ldots + (2n-1)x^{n+1}$$

$$xS_n = 1 \cdot x^2 + 3x^3 + 5x^4 + \ldots + (2n-3)x^n + (2n-1)x^{n+1} \ldots(2)$$

Equation (1) – Equation (2)

$$(1-x)S_n = 1 \cdot x + (3-1)x^2 + (5-3)x^3 + (7-5)x^4 + \ldots + (2n - 1 - 2n + 3)x^n - (2n-1)x^{n+1}$$

$$= 1 \cdot x + (2 \cdot x^2 + 2 \cdot x^3 + 2 \cdot x^4 + \ldots + 2 \cdot x^n) - (2n-1)x^{n+1}$$

$$(1-x)S_n = x + \frac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1}$$

Dividing both sides by $(1-x)$

$$S_n = \frac{x}{1-x} + \frac{2(x^2 - x^{n+1})}{(1-x)^2} - \frac{(2n-1) \cdot x^{n+1}}{1-x}$$

Put $x = \frac{2n+1}{2n-1}$ and $1 - x = 1 - \frac{2n+1}{2n-1} = \frac{-2}{2n-1}$

$$S_n = \frac{\frac{2n+1}{2n-1}}{\frac{-2}{2n-1}} + \frac{2\left\{\frac{(2n+1)^2}{(2n-1)^2} - \frac{(2n+1)^{n+1}}{(2n-1)^{n+1}}\right\}}{\left(\frac{-2}{2n-1}\right)^2} - \frac{(2n-1) \cdot \frac{(2n+1)^{n+1}}{(2n-1)^{n+1}}}{\frac{-2}{2n-1}}$$

$$= -\frac{1}{2}(2n+1) + \left(\frac{2n-1}{-2}\right)^2 \cdot 2\left\{\frac{(2n+1)^2}{(2n-1)^2} - \frac{(2n+1)^{n+1}}{(2n-1)^{n-1+2}}\right\} - \frac{(2n-1)^2}{-2} \cdot \frac{(2n+1)^{n+1}}{(2n-1)^{n-1+2}}$$

$$= -\frac{1}{2}(2n+1) + \frac{(2n-1)^2}{4} \cdot 2 \cdot \frac{1}{(2n-1)^2}\left\{(2n+1)^2 - \frac{(2n+1)^{n+1}}{(2n-1)^{n-1}}\right\} + \frac{1}{2}(2n-1)^2 \cdot \frac{1}{(2n-1)^2} \cdot \frac{(2n+1)^{n+1}}{(2n-1)^{n-1}}$$

$$= -\frac{1}{2}(2n+1) + \frac{1}{2}\left\{(2n+1)^2 - \frac{(2n+1)^{n+1}}{(2n-1)^{n-1}}\right\} + \frac{1(2n+1)^{n+1}}{2(2n-1)^{n-1}}$$

$$= -\frac{1}{2}(2n+1) + \frac{1}{2}(2n+1)^2 - \frac{1(2n+1)^{n+1}}{2(2n-1)^{n-1}} + \frac{1(2n+1)^{n+1}}{2(2n-1)^{n-1}} = \frac{1}{2}(2n+1)(-1+2n+1) = \frac{1}{2}(2n+1)(2n)$$

$$S_n = n(2n+1)$$

10. Prove that:

$$1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + \ldots \text{ to } n \text{ terms} = n^2$$

Solution:

Let $S_n = 1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + \ldots$ to n terms

Put $1 + \frac{1}{n} = x$

$S_n = 1.1 + 2x + 3x^2 + \cdots$ to n rems

Arithmetic Part	Geometric Part
1, 2, 3, ... (A.P)	$1, x, x^2, \ldots$ (G.P)
Here $a_1 = 1$, $d = 2 - 1 = 1$	Here $b_1 = 1$, $r = \frac{x}{1} = x$
$a_n = a_1 + (n-1)d$	$b_n = b_1 r^{n-1}$
$= 1 + (n-1)(1)$	$= 1 \cdot x^{n-1}$
$= 1 + n - 1$	$= x^{n-1}$
$= n$	

$\Rightarrow \quad S_n = 1.1 + 2x + 3x^2 + 4x^3 + \ldots + n \cdot x^{n-1} \quad \ldots(1)$

Multiply both sides by x

$xS_n = 1 \cdot x + 2x^2 + 3x^3 + 4x^4 + \ldots + nx^n$

$xS_n = 1 \cdot x + 2x^2 + 3x^3 + \ldots + (n-1)x^{n-1} + nx^n \quad \ldots(2)$

Equation (1) - Equation (2)

$S_n - xS_n = 1 + (2-1)x + (3-2)x^2 + (4-3)x^3 + \ldots + (n-n+1)x^{n-1} - 1$

$(1-x)S_n = 1 + (1.x + 1.x^2 + 1.x^3 + \ldots + 1.x^{n-1}) - nx^n$

$(1-x)S_n = 1 + \frac{x(1-x^{n-1})}{1-x} - nx^n$

Dicing both sides by $(1-x)$

$$S_n = \frac{1}{1-x} + \frac{x - x^n}{(1-x)^2} - \frac{nx^n}{1-x}$$

$$= \frac{1 - x + x - x^n}{(1-x)^2} - \frac{nx^n}{1-x}$$

$$= \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}$$

Put $x = 1 + \frac{1}{n}$ and $1 - x = -\frac{1}{n}$

$$S_n = \frac{1 - \left(1+\frac{1}{n}\right)^n}{\left(-\frac{1}{n}\right)^2} - \frac{n\left(1+\frac{1}{n}\right)^n}{-\frac{1}{n}}$$

$$= n^2\left\{1 - \left(1+\frac{1}{n}\right)^n\right\} + n^2\left(1+\frac{1}{n}\right)^n$$

$$= n^2 - n^2\left(1+\frac{1}{n}\right)^n + n^2\left(1+\frac{1}{n}\right)^n$$

$$S_n = n^2$$

Hence, $1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + \ldots$ to terms $= n^2$

11. Sum the series to n terms $2 + 5x + 8x^2 + 11x^3 + \ldots$ and deduce the sum to infinity if $|x| < 1$.

Solution:

Let $S_n = 2 \cdot 1 + 5x + 8x^2 + 11x^3 + \cdots$ to n terms

Arithmetic Part	Geometric Part
2, 5, 8, 11, ... (A.P)	$1, x, x^2, x^3 \ldots$ (G.P)
Here $a_1 = 2$, $d = 5 - 2 = 3$	Here $b_1 = 1$, $r = x$
$a_n = a_1 + (n-1)d$	$b_n = b_1 r^{n-1}$
$= 2 + (n-1)(3)$	$= 1 \cdot x^{n-1}$
$= 2 + 3n - 3$	$= x^{n-1}$
$= 3n - 1$	

$\Rightarrow \quad S_n = 2 \cdot 1 + 5x + 8x^2 + 11x^3 + \ldots + (3n-1)x^{n-1} \quad \ldots(1)$

Multiply both sides by 'x'

$xS_n = 2 \cdot x + 5x^2 + 8x^3 + 11x^4 + \ldots (3n-1)x^n$

$xS_n = 2 \cdot x + 5x^2 + 8x^3 + \ldots + (3n-4)x^{n-1} + (3n-1)x^n \quad \ldots(2)$

Equation (1) – Equation (2)

$S_n - xS_n = 2 + (5-2)x + (8-5)x^2 + (11-8)x^3 + \ldots + (3n-1-3n+4)x^{n-1} - (3n-1)x^n$

$(1-x)S_n = 2 + (3x + 3x^2 + 3x^3 + \ldots + 3x^{n-1}) - (3n-1)x^n$

$(1-x)S_n = 2 + \frac{3x(1-x^{n-1})}{1-x} - (3n-1)x^n$

Dividing both sides by $(1-x)$

$$S_n = \frac{2}{1-x} + \frac{3x(1-x^{n-1})}{(1-x)^2} - \frac{(3n-1)x^n}{1-x}$$

If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$

$$S_\infty = \frac{2}{1-x} + \frac{3x(1-0)}{(1-x)^2} - \frac{(3n-1)(0)}{1-x}$$

$$= \frac{2}{1-x} + \frac{3x}{(1-x)^2} - 0$$

$$= \frac{2(1-x)+3x}{(1-x)^2} = \frac{2-2x+3x}{(1-x)^2}$$

$$S_\infty = \frac{2+x}{(1-x)^2}$$

Harmonic Progression (H.P.)

A sequence of numbers is called a Harmonic Sequence or Harmonic Progression if the reciprocals of its terms are in arithmetic progression.

For Example, the sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ is a harmonic sequence since their reciprocals 1, 3, 5, 7 are in A.P.

- Remember that the reciprocal of zero is not defined, so zero cannot be the term of a harmonic sequence.

The general form of the harmonic sequence is $\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots, \frac{1}{a_1+(n-1)d}$

Example 22: Find the n^{th} and 8^{th} terms of H.P. : $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$

Solution:

Given that $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ is a H.P.

2, 5, 8, ... is an A.P. (Reciprocal of terms of H.P.)

Here $a_1 = 2$, $d = 5 - 2 = 3$

As we know: $a_n = a_1 + (n-1)d = 2 + (n-1)(3) = 2 + 3n - 3$

$\boxed{a_n = 3n - 1}$

$a_8 = 3(8) - 1 \quad$ Put $n = 8$ in a_n

$\boxed{a_8 = 23}$

Thus, the n^{th} and 8^{th} term of the given H.P. are $\frac{1}{a_n} = \frac{1}{3n-1}$ and $\frac{1}{a_8} = \frac{1}{23}$ respectively.

Example 23: If the 4^{th} and 7^{th} terms of the H.P. are $\frac{2}{13}$ and $\frac{2}{25}$ respectively, find the sequence.

Solution:

In H.P. $a_4 = \frac{2}{13}$ and $a_7 = \frac{2}{25}$

In A.P. $a_4 = \frac{13}{2}$ | $a_7 = \frac{25}{2}$

$a_1 + 3d = \frac{13}{2} \quad \ldots$ (i) | $a_1 + 6d = \frac{25}{2} \quad \ldots$ (ii) $\quad$ Using $a_n = a_1 + (n-1)d$

Equation(ii) – Equation(i)

$a_1 + 6d = \frac{25}{2}$

$\pm a_1 \pm 3d = \pm\frac{13}{2}$

$3d = \frac{25}{2} - \frac{13}{2} = \frac{12}{2} = 6 \quad \Rightarrow d = 2 \quad$ Put $d = 2$ in eq.(i)

$$a_1 = \frac{13}{2} - 3d = \frac{13}{2} - 3(2) = \frac{13}{2} - 6 = \frac{13-12}{2} \quad \Rightarrow a_1 = \frac{1}{2}$$

Hence, A.P is $\quad a_1 + (a_1 + d) + (a_1 + 2d) + \ldots$

$$\frac{1}{2} + \left(\frac{1}{2} + 2\right) + \left(\frac{1}{2} + 2(2)\right) + \ldots$$

$$\frac{1}{2} + \frac{5}{2} + \frac{9}{2} + \ldots$$

Hence the required H.P. is $\frac{2}{1}, \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \ldots$

Harmonic Mean (H.M.)

A number H is said to be the harmonic mean (H.M.) between two numbers a and b if a, H, b are in H.P.

Let a, b be the two numbers and H be their H.M. Then $\frac{1}{a}, A, \frac{1}{b}$ are in A.P.

Therefore, $\quad A = \dfrac{\frac{1}{a} + \frac{1}{b}}{2} = \dfrac{\frac{b+a}{ab}}{2} = \dfrac{a+b}{2ab}$

and $\quad H = \dfrac{1}{\frac{a+b}{2ab}} = \dfrac{2ab}{a+b}$ (Required H.M.)

For example, H.M. between 3 and 7 is $\dfrac{2ab}{a+b} = \dfrac{2\times3\times7}{3+7} = \dfrac{2\times21}{10} = \dfrac{21}{5}$

***n* Harmonic Means between two Numbers:**

$H_1, H_2, H_3, \cdots, H_n$ are called n harmonic means (H.Ms.) between a and b if $a, H_1, H_2, H_3, \ldots, H_n, b$ are in H.P. If we want to insert n H.Ms., between a and b, we first find n A.Ms $A_1, A_2, \ldots, A_n$ between $\frac{1}{a}$ and $\frac{1}{b}$, then take their reciprocals to get n H.Ms. between a and b, that is, $\frac{1}{A_1}, \frac{1}{A_2}, \ldots, \frac{1}{A_n}$ will be the required n H.Ms. between a and b.

Example 24: Find three harmonic means between $\frac{1}{5}$ and $\frac{1}{17}$.

Solution:

Let $\quad H_1, H_2, H_3$ be three H.Ms. between $\frac{1}{5}$ and $\frac{1}{17}$, then

A_1, A_2, A_3 be three A.Ms. between 5 and 17, that is,

$5, A_1, A_2, A_3, 17$ are in A.P.

Here, $\quad a_1 = 5$ and $a_5 = 17 \quad \Rightarrow a_1 + 4d = 17 \quad \Rightarrow 5 + 4d = 17 \quad \Rightarrow 4d = 12 \quad \Rightarrow d = 3$

Thus, $\quad A_1 = 5 + 3 = 8 \quad \Rightarrow H_1 = \frac{1}{8}$

$A_2 = 5 + 2(3) = 11 \quad \Rightarrow H_2 = \frac{1}{11}$

$A_3 = 5 + 3(3) = 14 \quad \Rightarrow H_3 = \frac{1}{14}$

Hence $\frac{1}{8}, \frac{1}{11}, \frac{1}{14}$ are the required harmonic means between $\frac{1}{5}$ and $\frac{1}{17}$.

Exercise 6.9

1. Find the 9th term of the following harmonic sequences:

(i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

Solution:

$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ is a H.P.

3, 5, 7, ... is an A.P. (Reciprocals of terms of H.P)

Here $\quad a_1 = 3, d = 5 - 3 = 2, n = 9, a_9 = ?$

Using $\quad a_n = a_1 + (n-1)d$

$a_9 = a_1 + 8d$

$= 3 + 8(2) = 3 + 16 = 19$

Hence the 9th term of given H.P. $= \frac{1}{19}$.

(ii) $\frac{-1}{5}, \frac{-1}{3}, -1, \ldots$

Solution:

$-\frac{1}{5}, -\frac{1}{3}, -1, \ldots$ is a H.P.

$-5, -3, -1, \ldots$ is an A.P. (Reciprocals of terms of H.P)

Here $\quad a_1 = -5, d = -3 + 5 = 2, n = 9, a_9 = ?$

Using $\quad a_n = a_1 + (n-1)d$

$a_9 = a_1 + 8d$

$= -5 + 8(2) = -5 + 16 = 11$

Hence the 9th term of given H.P. $= \frac{1}{11}$.

2. Insert five harmonic means between the following given numbers:

(i) $\frac{-2}{5}$ and $\frac{2}{13}$

Solution:

Let H_1, H_2, H_3, H_4, H_5 be five H.Ms between $-\frac{2}{5}$ and $\frac{2}{13}$, then

$-\frac{2}{5}, H_1, H_2, H_3, H_4, H_5, \frac{2}{13}$ is a H.P.

Reciprocals of terms of H.P

$\Rightarrow \quad -\frac{5}{2}, A_1, A_2, A_3, A_4, A_5, \frac{13}{2}$ is an A.P.

Here $a_1 = -\frac{5}{2} \quad | \quad a_7 = \frac{13}{2}$

$a_1 + 6d = \frac{13}{2}$ using $a_n = a_1 + (n-1)d$

$-\frac{5}{2} + 6d = \frac{13}{2} \qquad \because a_1 = -\frac{5}{2}$

$6d = \frac{13}{2} + \frac{5}{2} = \frac{13+5}{2} = \frac{18}{2}$

$6d = 9$

$d = \frac{9}{6} = \frac{3}{2}$

Now $A_1 = a_1 + d = -\frac{5}{2} + \frac{3}{2} = \frac{-5+3}{2} = \frac{-2}{2} = -1 \Rightarrow H_1 = -1$

$A_2 = a_1 + 2d = -\frac{5}{2} + 2\left(\frac{3}{2}\right) = -\frac{5}{2} + \frac{6}{2} = \frac{-5+6}{2} = \frac{1}{2} \Rightarrow H_2 = 2$

$A_3 = a_1 + 3d = -\frac{5}{2} + 3\left(\frac{3}{2}\right) = -\frac{5}{2} + \frac{9}{2} = \frac{-5+9}{2} = \frac{4}{2} = 2 \Rightarrow H_3 = \frac{1}{2}$

$A_4 = a_1 + 4d = -\frac{5}{2} + 4\left(\frac{3}{2}\right) = -\frac{5}{2} + \frac{12}{2} = \frac{-5+12}{2} = \frac{7}{2} \Rightarrow H_4 = \frac{2}{7}$

$A_5 = a_1 + 5d = -\frac{5}{2} + 5\left(\frac{3}{2}\right) = -\frac{5}{2} + \frac{15}{2} = \frac{-5+15}{2} = \frac{10}{2} = 5 \Rightarrow H_5 = \frac{1}{5}$

Hence the five H.Ms between $-\frac{2}{5}$ and $\frac{2}{13}$ are

$$-1, 2, \frac{1}{2}, \frac{2}{7}, \frac{1}{5}$$

(ii) $\frac{1}{4}$ and $\frac{1}{24}$

Solution:

Let H_1, H_2, H_3, H_4, H_5 be five H.Ms between $\frac{1}{4}$ and $\frac{1}{24}$, then

$\frac{1}{4}, H_1, H_2, H_3, H_4, H_5, \frac{1}{24}$ is a H.P.

Reciprocals of terms of H.P

$\Rightarrow \quad 4, A_1, A_2, A_3, A_4, A_5, 24$ is an A.P.

Here $a_1 = 4 \quad | \quad a_7 = 24$

$a_1 + 6d = 24$ using $a_n = a_1 + (n-1)d$

$4 + 6d = 24 \qquad \because a_1 = 4$

$6d = 24 - 4$

$6d = 20$

$d = \frac{20}{6} = \frac{10}{3}$

Now, $A_1 = a_1 + 1d = 4 + \frac{10}{3} = \frac{12+10}{3} = \frac{22}{3} \Rightarrow H_1 = \frac{3}{22}$

$A_2 = a_1 + 2d = 4 + 2\left(\frac{10}{3}\right) = 4 + \frac{20}{3} = \frac{12+20}{3} = \frac{32}{3} \Rightarrow H_2 = \frac{3}{32}$

$A_3 = a_1 + 3d = 4 + 3\left(\frac{10}{3}\right) = 4 + \frac{30}{3} = \frac{12+30}{3} = \frac{42}{3} = 14 \Rightarrow H_3 = \frac{1}{14}$

$A_4 = a_1 + 4d = 4 + 4\left(\frac{10}{3}\right) = 4 + \frac{40}{3} = \frac{12+40}{3} = \frac{52}{3} \Rightarrow H_4 = \frac{3}{52}$

$A_5 = a_1 + 5d = 4 + 5\left(\frac{10}{3}\right) = 4 + \frac{50}{3} = \frac{12+50}{3} = \frac{62}{3} \Rightarrow H_5 = \frac{3}{62}$

Hence the five H.Ms between $\frac{1}{4}$ and $\frac{1}{24}$ are

$\frac{3}{22}, \frac{3}{32}, \frac{1}{14}, \frac{3}{52}, \frac{3}{62}$.

3. The first term of an H.P. is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.

Solution:

In H.P. $a_1 = -\frac{1}{3}$, $a_5 = \frac{1}{5}$, $a_9 = ?$

In A.P.

$a_1 = -3$ | $a_5 = 5$

$a_1 + 4d = 5$ using $a_n = a_1 + (n-1)d$

$-3 + 4d = 5$ $\because a_1 = -3$

$4d = 5 + 3$

$4d = 8$

$d = 2$

Now, $a_9 = a_1 + 8d$

$a_9 = -3 + 8(2) = -3 + 16 = 13$ in A.P.

Hence the 9th term of H.P. $= \frac{1}{13}$.

4. If 5 is the harmonic mean between 2 and b, find b.

Solution:

Here $a = 2$, $b = b$, H.M $= 5$

We know that

$$\text{H.M} = \frac{2ab}{a+b}$$

$$5 = \frac{2(2)b}{2+b}$$

$$10 + 5b = 4b$$

$$5b - 4b = -10$$

$$b = -10$$

5. If the numbers $\frac{1}{k}$, $\frac{1}{2k+1}$ and $\frac{1}{4k-1}$ are in harmonic sequence, find k.

Solution:

Since $\frac{1}{k}$, $\frac{1}{2k+1}$, $\frac{1}{4k-1}$ are in H.P., therefore

$k, 2k+1, 4k-1$ are in A.P.

$(2k+1) - k = (4k-1) - (2k+1)$ (Common difference exists)

$2k + 1 - k = 4k - 1 - 2k - 1$

$k + 1 = 2k - 2$

$2k - k = 1 + 2$

$k = 3$

6. Find n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be H.M between a and b.

Solution:

Given that:

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \text{H.M between } a \text{ and } b$$

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{2ab}{a+b}$$

$$(a^{n+1}+b^{n+1})(a+b) = 2ab(a^n+b^n)$$

$$a^{n+2}+a^{n+1}b+ab^{n+1}+b^{n+2} = 2a^{n+1}b+2ab^{n+1}$$

$$a^{n+2}+a^{n+1}b-2a^{n+1}b = 2ab^{n+1}-ab^{n+1}-b^{n+2}$$

$$a^{n+2}-a^{n+1}b = ab^{n+1}-b^{n+2}$$

$$a^1.a^{n+1}-a^{n+1}b = ab^{n+1}-b^1.b^{n+1}$$

$$a^{n+1}(a-b) = b^{n+1}(a-b)$$

$$\frac{a^{n+1}}{b^{n+1}} = \frac{(a-b)}{(a-b)}$$

$$\left(\frac{a}{b}\right)^{n+1} = 1$$

$$\left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^0 \quad \because \left(\frac{a}{b}\right)^0 = 1$$

Comparing both sides, we have

$\Rightarrow$ $n + 1 = 0$

$\Rightarrow$ $n = -1$

7. If a^2, b^2 and c^2 are in A.P. show that $a+b, c+a$ and $b+c$ are in H.P.

Solution:

Since a^2, b^2, c^2 are in A.P. so

There exists common difference exists

$\Rightarrow$ $b^2 - a^2 = c^2 - b^2$

$(b+a)(b-a) = (c-b)(c+b)$

$$b - a = \frac{(c-b)(c+b)}{a+b} \quad \text{...(i)}$$

Now, $a+b, c+a, b+c$ will be in H.P.

$\Rightarrow$ $\frac{1}{a+b}, \frac{1}{c+a}, \frac{1}{b+c}$ will be in A.P. if

$$\frac{1}{c+a} - \frac{1}{a+b} = \frac{1}{b+c} - \frac{1}{c+a} \quad \text{(Common difference)}$$

$$\frac{a+b-c-a}{(c+a)(a+b)} = \frac{c+a-b-c}{(b+c)(c+a)}$$

$$\frac{b-c}{(c+a)(a+b)} = \frac{a-b}{(b+c)(c+a)}$$

$$\frac{b-c}{(a+b)} = \frac{(b-a)}{(b+c)}$$

$$\frac{b-c}{(a+b)} = \frac{\frac{(c-b)(c+b)}{a+b}}{(b+c)} \quad \text{using eq. (1)}$$

$$\frac{b-c}{(a+b)} = \frac{(c-b)(c+b)}{(a+b)(b+c)}$$

$$\frac{b-c}{a+b} = \frac{(c-b)}{a+b}$$

$$\frac{b-c}{a+b} = \frac{b-c}{a+b} \quad \text{(Common difference exists)}$$

Hence $a+b, c+a, b+c$ are in H.P.

8. If the H.M. and A.M. between two numbers are 4 and $\frac{9}{2}$ respectively, find the numbers.

Solution:

Let a and b be the required numbers

By given conditions

H.M = 4 | A.M = $\frac{9}{2}$

$\frac{2ab}{a+b} = 4$ | $\frac{a+b}{2} = \frac{9}{2}$

$2ab = 4(a+b)$ | $a + b = 9$ ...(1)

$2ab = 4(9)$ using eq. (1)

$ab = 18$

$a = \frac{18}{b}$ ...(2)

Putting $a = \frac{18}{b}$ in eq (1), we have

$\frac{18}{b} + b = 9$

$18 + b^2 = 9b$ Multiply by 'b'

$b^2 - 9b + 18 = 0$ $\Rightarrow$ $b^2 - 6b - 3b + 18 = 0$

$b(b-6) - 3(b-6) = 0$ $\Rightarrow$ $(b-6)(b-3) = 0$

Either $b - 6 = 0$ or $b - 3 = 0$

$b = 6$; $b = 3$

When $b = 6$, then from eq (2)

$a = \frac{18}{6} = 3$

When $b = 3$, then from eq (2)

$a = \frac{18}{3} = 6$

Hence the required numbers are 3, 6 or 6, 3.

9. If the (positive) G.M. and H.M. between two numbers are 4 and $\frac{16}{5}$, find the numbers.

Solution:

Let a and b be the required numbers

By given conditions

+ve G.M = 4 | H.M = $\frac{16}{5}$

$\sqrt{ab} = 4$ | $\frac{2ab}{a+b} = \frac{16}{5}$

$ab = 16$ | $10ab = 16(a+b)$

$a = \frac{16}{b}$ ...(1) | $10(16) = 16(a+b)$ $\because ab = 16$

$a + b = 10$ ...(2)

Putting $a = \frac{16}{b}$ in eq (2), we have

$\frac{16}{b} + b = 10$

$16 + b^2 = 10b$ Multiply by 'b'

$b^2 - 10b + 16 = 0$

$b^2 - 8b - 2b + 16 = 0$

$b(b-8) - 2(b-8) = 0$

$(b-8)(b-2) = 0$

Either $b - 8 = 0$ or $b - 2 = 0$

$b = 8$; $b = 2$

When $b = 8$, then from eq (1)

$a = \frac{16}{8} = 2$

When $b = 2$, then from eq (1)

$a = \frac{16}{2} = 8$

Hence the required numbers are 2, 8 or 8, 2.

10. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., show that a, b, c are in H.P.

Solution:

Given that: $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

Adding '2' in each term

$\frac{b+c-a}{a}+2, \frac{c+a-b}{b}+2, \frac{a+b-c}{c}+2$ are in A.P.

$\frac{b+c-a+2a}{a}, \frac{c+a-b+2b}{b}, \frac{a+b-c+2c}{c}$ are in A.P.

$\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P.

Dividing each term by $(a+b+c)$

$\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

By taking reciprocal of each term

$\Rightarrow$ a, b, c are in H.P. (Proved)

11. If a, b, c, d are in H.P., show that $3(a-b)(c-d) = (b-c)(a-d)$.

Solution:

Since a, b, c, d are in H.P, therefore

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in A.P.

Let $\frac{1}{a}=x-3d, \frac{1}{b}=x-d, \frac{1}{c}=x+d, \frac{1}{d}=x+3d$

Taking reciprocal

$$a=\frac{1}{x-3d}, b=\frac{1}{x-d}, c=\frac{1}{x+d}, d=\frac{1}{x+3d}$$

Now, $a-b=\frac{1}{x-3d}-\frac{1}{x-d}$

$$=\frac{x-d-x+3d}{(x-3d)(x-d)}=\frac{2d}{(x-3d)(x-d)}$$

$$b-c=\frac{1}{x-d}-\frac{1}{x+d}$$

$$=\frac{x+d-x+d}{(x-d)(x+d)}=\frac{2d}{(x-d)(x+d)}$$

$$c-d=\frac{1}{x+d}-\frac{1}{x+3d}$$

$$=\frac{x+3d-x-d}{(x+d)(x+3d)}=\frac{2d}{(x+d)(x+3d)}$$

$$a-d=\frac{1}{x-3d}-\frac{1}{x+3d}$$

$$=\frac{x+3d-x+3d}{(x-3d)(x+3d)}=\frac{6d}{(x-3d)(x+3d)}$$

L.H.S $=3(a-b)(c-d)$

$$=3\cdot\frac{2d}{(x-3d)(x-d)}\cdot\frac{2d}{(x+d)(x+3d)}$$

$$=\frac{12d^2}{(x^2-d^2)(x^2-9d^2)}$$

R.H.S $=(b-c)(a-d)$

$$=\frac{2d}{(x-d)(x+d)}\cdot\frac{6d}{(x-3d)(x+3d)}$$

$$=\frac{12d^2}{(x^2-d^2)(x^2-9d^2)}$$

Hence proved, L.H.S = R.H.S

12. If between any two numbers there are inserted two A.Ms. A_1, A_2, two G.Ms. G_1, G_2 and two H.Ms. H_1, H_2; show that $\frac{A_1+A_2}{G_1G_2}=\frac{H_1+H_2}{H_1H_2}$.

Solution:

Let a and b be any two numbers.

As A_1, A_2 are two A.Ms between a and b, so a, A_1, A_2, b are in A.P.

$\Rightarrow \quad A_1-a=b-A_2$

$A_1+A_2=a+b \quad \ldots(1)$

As G_1, G_2 are two G.Ms between a and b, so a, G_1, G_2, b are in G.P.

$\Rightarrow \quad \frac{G_1}{a}=\frac{b}{G_2}$

$G_1 . G_2=ab \quad \ldots(2)$

As H_1, H_2 are two H. Ms between a and b, so $a, H_1, H_2 b$ are in H.P.

$\Rightarrow \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

$$\frac{1}{H_1}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H_2}$$

$$\frac{1}{H_1}+\frac{1}{H_2}=\frac{1}{a}+\frac{1}{b}$$

$$\frac{H_1+H_2}{H_1\cdot H_2}=\frac{a+b}{ab}$$

By using eq (1) and eq (2), we have

$$\frac{H_1+H_2}{H_1\cdot H_2}=\frac{A_1+A_2}{G_1\cdot G_2}$$

$\Rightarrow \quad \frac{A_1+A_2}{G_1\cdot G_2}=\frac{H_1+H_2}{H_1\cdot H_2}$ (Proved)

13. The H.M. of two numbers is 4. The A.M., A and G.M., G satisfy the relation $2A+G^2=27$. Find the numbers.

Solution:

Let two required numbers be a and b. Then

$A=\frac{a+b}{2}$

$a+b=2A \quad \ldots(1)$

$G=\sqrt{ab}$

$G^2=ab \quad \ldots(2)$

Given that:

$H=4$

$\frac{2ab}{a+b}=4$

$\frac{2G^2}{2A}=4$

$G^2=4A \quad \ldots(3)$

$2A+G^2=27$

$\Rightarrow 2A+4A=27 \quad \because G^2=4A$

$6A=27$

$A=\frac{27}{6} \quad \Rightarrow A=\frac{9}{2}$

Put $A=\frac{9}{2}$ in eq (1), we have

$a+b=2\left(\frac{9}{2}\right)$

$a+b=9 \quad \ldots(4)$

Put $A=\frac{9}{2}$ in eq (3), we have

$G^2=4\left(\frac{9}{2}\right)$

$G^2=18$

$ab=18 \ldots(5)$ using eq (1)

From eq. (4), we have

$b=9-a$ Put this in eq (5)

$a(9-a)=18$

$9a-a^2=18$

$0=a^2-9a+18$

$a^2-6a-3a+18=0$

$a(a-6)-3(a-6)=0$

$(a-3)(a-6)=0$

Either $a-3=0$ or $a-6=0$

$a=3$ or $a=6$

When $a=3$, then from eq. (5)

$3b=18$

$b=6$

When $a=6$, then from eq. (5)

$6b=18$

$b=3$

Hence the required numbers are 3, 6 or 6, 3.

14. First three of the four numbers a, b, c, d are in A.P., and the next three are in H.P., show that $ad=bc$.

Solution:

Given four numbers a, b, c, d

As a, b, c are in A.P., so

$2b=a+c$

$b=\frac{a+c}{2} \quad \ldots(1)$

As b, c, d are in H.P., so

$\frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in A.P.

$\Rightarrow \quad 2\left(\frac{1}{c}\right)=\frac{1}{b}+\frac{1}{d}$

$\frac{2}{c}=\frac{d+b}{bd}$

$2bd=c(b+d)$

$2d\left(\frac{a+c}{2}\right)=c(b+d)$ using eq. (1)

$d(a+c)=c(b+d)$

$ad+dc=bc+cd$

$ad=bc$ (Proved)

15. If a, b, c are in G.P., show that $\log_a x, \log_b x, \log_c x$ are in H.P.

Solution:

Since a, b, c are in G.P., so

$b^2=ac$

Taking $\log_x$ on both sides

$\log_x(b^2)=\log_x(a.c)$

$2\log_x b=\log_x a+\log_x c$

$\Rightarrow \log_x a, \log_x b, \log_x c$ are in A.P.

By taking reciprocal of each term $\frac{1}{\log_x a}, \frac{1}{\log_x b}, \frac{1}{\log_x c}$ are in H.P.

$\Rightarrow \log_a x, \log_b x, \log_c x$ are in H.P.

Hence, if a, b, c are in G.P, then $\log_a x, \log_b x, \log_c x$ are in H.P.

16. If a, b, c are in H.P., show that

(i) $\frac{a-b}{b-c}=\frac{a}{c}$

Solution:

Since a, b, c are in H.P., therefore

$b=$ H.M between a and c

$\Rightarrow \quad b=\frac{2ac}{a+c}$

$b(a+c)=2ac$

$ab+bc=ac+ac$

$ab-ac=ac-bc$

$a(b-c)=c(a-b)$

$\frac{a}{c}=\frac{a-b}{b-c}$

$\Rightarrow \quad \frac{a-b}{b-c}=\frac{a}{c}$ (Proved)

(ii) $(a-c)^2=(a+c)(a-2b+c)$

Solution:

Since a, b, c are in H.P., therefore

$b=$ H.M between a and c

$b=\frac{2ac}{a+c}$

R.H.S $=(a+c)(a-2b+c)$

$=(a+c)\left(a-\frac{4ac}{a+c}+c\right)$ Put $b=\frac{2ac}{a+c}$

$=(a+c)\left(\frac{a(a+c)-4ac+c(a+c)}{a+c}\right)$

$=a^2+ac-4ac+ac+c^2$

$=a^2-2ac+c^2$

$=(a-c)^2$

= L.H.S (Proved)

17. If $2+x, 5+x$ and $9+x$ are in H.P., find the value of x.

Solution:

Given that $2+x, 5+x, 9+x$ are in H.P.

Reciprocals of terms

$\frac{1}{2+x}, \frac{1}{5+x}, \frac{1}{9+x}$ are in A.P.

$\Rightarrow \quad 2\left(\frac{1}{5+x}\right)=\frac{1}{2+x}+\frac{1}{9+x}$

$\frac{2}{5+x}=\frac{9+x+2+x}{(2+x)(9+x)}$

$\frac{2}{5+x}=\frac{2x+11}{18+11x+x^2}$

$2(18+11x+x^2)=(2x+11)(5+x)$

$36+22x+2x^2=10x+2x^2+55+11x$

$36+22x=21x+55$

$22x-21x=55-36$

$\boxed{x=19}$

18. If the roots of the equation
$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$
are equal, prove that a, b, c are in H.P.

Solution:

$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$

Compare it with

$Ax^2 + Bx + C = 0$

Here $A = a(b-c), B = b(c-a), C = c(a-b)$

For equal roots, disc $= 0$

$B^2 - 4AC = 0$

$(b(c-a))^2 - 4(a(b-c))(c(a-b)) = 0$

$b^2(c^2 + a^2 - 2ca) - 4ac(ab - b^2 - ca + bc) = 0$

$b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc + 4ab^2c + 4a^2c^2 - 4abc^2 = 0$

$b^2c^2 + a^2b^2 + 2ab^2c - 4a^2bc + 4a^2c^2 - 4abc^2 = 0$

$a^2b^2 + b^2c^2 + 4a^2c^2 + 2ab^2c - 4abc^2 - 4a^2bc = 0$

$(ab)^2 + (bc)^2 + (-2ac)^2 + 2(ab)(bc) + 2(bc)(-2ac) + 2(-2ac)(ab) = 0$

$(ab + bc - 2ac)^2 = 0$

Taking square root on both sides

$ab + bc - 2ac = 0$

$ab + bc = 2ac$

Divide both sides by "abc"

$$\frac{ab}{abc} + \frac{bc}{abc} = \frac{2ac}{abc}$$

$$\frac{1}{c} + \frac{1}{a} = \frac{2}{b}$$

$$2\left(\frac{1}{b}\right) = \frac{1}{a} + \frac{1}{c}$$

$\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Reciprocals of terms

$\Rightarrow$ a, b, c are in H.P.

Miscellaneous Series:

The Greek letter Σ(sigma) is used to denote sums of different types.

For examples:

- The notation $\sum_{i=m}^{n} a_i$ is used to express the sum $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$
- The sum expression $1 + 3 + 5 + \cdots$ to n terms is written as $\sum_{k=1}^{n}(2k-1)$, where $2k-1$ is the k^{th} term of the sum and k is called the index of summation. 1 and n are called the lower limit and upper limit of summation respectively.

The sum of the first n natural numbers $= 1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k$

The sum of squares of the first n natural numbers $= 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2$

The sum of the cubes of the first n natural numbers $= 1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{k=1}^{n} k^3$

We evaluate $\sum_{k=1}^{n}\left[k^m - (k-1)^m\right]$ for any positive integer m and we shall use this result to find out formulae for the expressions stated above.

$$\sum_{k=1}^{n}\left[k^m - (k-1)^m\right] = (1^m - 0^m) + (2^m - 1^m) + (3^m - 2^m) + \cdots + \left[(n-1)^m - (n-2)^m\right] + \left[n^m - (n-1)^m\right] = n^m$$

i.e., $\boxed{\sum_{k=1}^{n}\left[k^m - (k-1)^m\right] = n^m}$

- If $m = 1$, then $\sum_{k=1}^{n}\left[(k^1 - (k-1)^1\right] = n^1$ i.e., $\sum_{k=1}^{n}(1) = n$
- If $m = 2$, then $\sum_{k=1}^{n}\left[k^2 - (k-1)^2\right] = n^2$

Properties of Summation:

(i) $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(ii) $\sum_{k=1}^{n} \alpha a_k = \alpha \sum_{k=1}^{n} a_k$

To Find the Formulae for the Sums:

(i) $\sum_{k=1}^{n} k$ (ii) $\sum_{k=1}^{n} k^2$ (iii) $\sum_{k=1}^{n} k^3$

Proof: (i)

To Prove: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

We know that $(k-1)^2 = k^2 - 2k + 1$

$k^2 - (k-1)^2 = 2k - 1$

Taking $\sum_{k=1}^{n}$ on both sides, we have

$$\sum_{k=1}^{n}\left[k^2 - (k-1)^2\right] = \sum_{k=1}^{n}(2k-1)$$

$$n^2 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n}(1) \qquad \because \sum_{k=1}^{n}\left[k^2 - (k-1)^2\right] = n^2$$

$$n^2 = 2\sum_{k=1}^{n} k - n \qquad \because \sum_{k=1}^{n} 1 = n$$

$$2\sum_{k=1}^{n} k = n^2 + n$$

Thus $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Similarly, we can prove easily

(ii) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (iii) $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example 25: Find the sum of the series $1^3 + 3^3 + 5^3 + \cdots$ to n terms.

Solution:

nth term of A.P., 1, 3, 4, ...

$\Rightarrow$ $a_n = a_1 + (n-1)d$

$= 1 + (n-1)(2) = 1 + 2n - 2 = 2n - 1$

Let T_n be the nth term of given series, then

$T_n = (2n-1)^3$

$\Rightarrow$ $T_k = (2k-1)^3$

$= 8k^3 - 12k^2 + 6k - 1$

Taking $\sum_{k=1}^{n}$ both sides, we have

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(8k^3 - 12k^2 + 6k - 1\right)$$

or

$$S_n = \sum_{k=1}^{n}\left(8k^3 - 12k^2 + 6k - 1\right)$$

$$= 8\sum_{k=1}^{n} k^3 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= 8\left[\frac{n(n+1)}{2}\right]^2 - 12\left[\frac{n(n+1)(2n+1)}{6}\right] + 6\left[\frac{n(n+1)}{2}\right] - n$$
$$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$$
$$= 2n^2(n^2+2n+1) - 2n(2n^2+3n+1) + n(3n+3) - n$$
$$= 2n[(n^3+2n^2+n) - (2n^2+3n+1)] + n(3n+3-1)$$
$$= 2n(n^3-2n-1) + n(3n+2)$$
$$= 2n(n^3-2n-1) + n(3n+2)$$
$$= n[2n^3-4n-2+3n+2]$$
$$= n[2n^3-n] = n[n(2n^2-1)]$$
$$= n^2[2n^2-1]$$

Example 26: Find the sum of n terms of series whose n^{th} terms is $n^3 + \frac{3}{2}n^2 + \frac{1}{2}n + 1$.

Solution:

Given that: $T_n = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n + 1$

$\Rightarrow$ $T_k = k^3 + \frac{3}{2}k^2 + \frac{1}{2}k + 1$

Taking $\sum_{k=1}^{n}$ both sides, we have

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(k^3 + \frac{3}{2}k^2 + \frac{1}{2}k + 1\right)$$
$$S_n = \sum_{k=1}^{n}k^3 + \frac{3}{2}\sum_{k=1}^{n}k^2 + \frac{1}{2}\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$
$$= \frac{n^2(n+1)^2}{4} + \frac{3}{2}\times\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\times\left[\frac{n(n+1)}{2}\right] + n$$
$$= \frac{n}{4}\left[n(n^2+2n+1) + (2n^2+3n+1) + (n+1) + 4\right] = \frac{n}{4}(n^3+2n^2+n+2n^2+3n+1+n+1+$$
$$S_n = \frac{n}{4}(n^3+4n^2+5n+6)$$

Exercise 6.10

1. Sum the following series upto n terms:

(i) $1\times3+2\times5+3\times7+\ldots$

Solution:

$1\times3+2\times5+3\times7+\ldots$ to n terms

nth term of A.P., 1, 2, 3, ...

$a_n = a_1 + (n-1)d$

$= 1 + (n-1)(1) = 1 + n - 1 = n$

nth term of A.P., 3, 5, 7, ...

$b_n = b_1 + (n-1)d'$

$= 3 + (n-1)(2) = 3 + 2n - 2 = 2n + 1$

Let T_n be the nth term of given series, then

$T_n = a_n \times b_n$

$T_n = n(2n+1) = 2n^2 + n$

$\Rightarrow$ $T_k = 2k^2 + k$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(2k^2+k)$$
$$S_n = 2\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k$$

$$S_n = 2\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left\{\frac{2(2n+1)}{3} + 1\right\}$$
$$= \frac{n(n+1)}{2}\left(\frac{4n+2+3}{3}\right)$$
$$S_n = \frac{n(n+1)(4n+5)}{6}$$

(ii) $1\times5+2\times8+3\times11+\ldots$

Solution:

$1\times5+2\times8+3\times11+\cdots$ to n terms

nth term of A.P., 1, 2, 3, ...

$a_n = a_1 + (n-1)d$

$= 1 + (n-1)(1) = 1 + n - 1 = n$

nth term of A.P., 5, 8, 11, ...

$b_n = b_1 + (n-1)d$

$= 5 + (n-1)(3) = 5 + 3n - 3 = 3n + 2$

Let T_n be the nth term of given series, then

$T_n = a_n \times b_n$

$T_n = n(3n+2) = 3n^2 + 2n$

$\Rightarrow$ $T_k = 3k^2 + 2k$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = 3\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k$$
$$S_n = \frac{3n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\{(2n+1)+2\}$$
$$S_n = \frac{n(n+1)(2n+3)}{2}$$

(iii) $1\times2+2\times5+3\times8+\ldots$

Solution:

$1\times2+2\times5+3\times8+\ldots$ to n terms

nth term of A.P., 1, 2, 3, ...

$a_n = a_1 + (n-1)d$

$= 1 + (n-1)(1) = 1 + n - 1 = n$

nth term of A.P, 2, 5, 8, ...

$b_n = b_1 + (n-1)d$

$= 2 + (n-1)(3) = 2 + 3n - 3 = 3n - 1$

Let T_n be the nth term of given series

$T_n = a_n \times b_n$

$T_n = n(3n-1) = 3n^2 - n$

$\Rightarrow$ $T_k = 3k^2 - k$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = 3\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k$$
$$S_n = \frac{3n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\{(2n+1)-1\}$$
$$= \frac{n(n+1)(2n)}{2}$$
$$S_n = n^2(n+1)$$

(iv) $1\times3\times5+2\times4\times6+3\times5\times7+\ldots$

Solution:

$1\times3\times5+2\times4\times6+3\times5\times7+\cdots$ to n terms

nth term of A.P., 1, 2, 3, ...

$a_n = a_1 + (n-1)d$

$= 1 + (n-1)(1) = 1 + n - 1 = n$

nth term of A.P., 3, 4, 5, ...

$b_n = b_1 + (n-1)d$

$= 3 + (n-1)(1) = 3 + n - 1 = n + 2$

nth term of A.P., 5, 6, 7, ...

$c_n = c_1 + (n-1)d$

$= 5 + (n-1)(1) = 5 + n - 1 = n + 4$

Let T_n be the nth term of given series, then

$T_n = a_n \times b_n \times c_n$

$T_n = n(n+2)(n+4)$

$T_n = n(n^2+6n+8) = n^3 + 6n^2 + 8n$

$T_k = k^3 + 6k^2 + 8k$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^3+6k^2+8k)$$
$$S_n = \sum_{k=1}^{n}k^3 + 6\sum_{k=1}^{n}k^2 + 8\sum_{k=1}^{n}k$$
$$= \frac{n^2(n+1)^2}{4} + \frac{6n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2}$$
$$= \frac{n^2(n+1)^2}{4} + n(n+1)(2n+1) + 4n(n+1)$$
$$= n(n+1)\left\{\frac{n(n+1)}{4} + (2n+1) + 4\right\}$$
$$= n(n+1)\left\{\frac{n^2+n+8n+4+16}{4}\right\}$$
$$= \frac{1}{4}n(n+1)(n^2+9n+20)$$
$$= \frac{1}{4}n(n+1)(n^2+5n+4n+20)$$

$$= \frac{1}{4}n(n+1)\big(n(n+5)+4(n+5)\big)$$

$$S_n = \frac{n(n+1)(n+4)(n+5)}{4}$$

(v) $1\times2\times4+2\times3\times7+3\times4\times10+\dots$

Solution:

$1\times2\times4+2\times3\times7+3\times4\times10+\cdots$ to n terms

nth term of A.P., 1, 2, 3, ...

$$a_n = a_1 + (n-1)d$$
$$= 1+(n-1)(1) = 1+n-1 = n$$

nth term of A.P., 2, 3, 4, ...

$$b_n = b_1 + (n-1)d$$
$$= 2+(n-1)(1) = 2+n-1 = n+1$$

nth term of A.P., 4, 7, 10, ...

$$c_n = c_1 + (n-1)d$$
$$= 4+(n-1)(3) = 4+3n-3 = 3n+1$$

Let T_n be the nth term of given series, then

$$T_n = a_n \times b_n \times c_n$$
$$T_n = n(n+1)(3n+1) = n(3n^2+4n+1)$$
$$T_n = 3n^3+4n^2+n$$
$$\Rightarrow \quad T_k = 3k^3+4k^2+k$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = 3\sum_{k=1}^{n}k^3 + 4\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k$$

$$S_n = \frac{3n^2(n+1)^2}{4} + \frac{4n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{3n(n+1)}{2} + \frac{4(2n+1)}{3} + 1\right\}$$

$$= \frac{n(n+1)}{2}\left\{\frac{9n(n+1)+8(2n+1)+6}{6}\right\}$$

$$= n(n+1)\left(\frac{9n^2+9n+16n+8+6}{6}\right)$$

$$= \frac{n(n+1)(9n^2+25n+14)}{6}$$

$$= \frac{n(n+1)(9n^2+18n+7n+14)}{12}$$

$$= \frac{n(n+1)[9n(n+2)+7(n+2)]}{12}$$

$$S_n = \frac{n(n+1)(n+2)(9n+7)}{12}$$

(vi) $2^2+4^2+6^2+\dots$

Solution:

$2^2+4^2+6^2+\cdots$ to n terms

nth term of A.P., 2, 4, 6, ...

$$a_n = a_1 + (n-1)d$$
$$= 2+(n-1)(2) = 2+2n-2 = 2n$$

Let T_n be the nth term of given series, Then

$$T_n = (a_n)^2$$
$$T_n = (2n)^2 = 4n^2$$
$$\Rightarrow \quad T_k = 4k^2$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = 4\sum_{k=1}^{n}k^2$$

$$S_n = \frac{4n(n+1)(2n+1)}{6}$$

$$S_n = \frac{2n(n+1)(2n+1)}{3}$$

(vii) $3^2+6^2+9^2+\dots$

Solution:

$3^2+6^2+9^2+\cdots$ to n terms

nth term of A.P., 3, 6, 9, ...

$$a_n = a_1 + (n-1)d$$
$$= 3+(n-1)(3) = 3+3n-3 = 3n$$

Let T_n be the nth term of given series, then

$$T_n = (3n)^2$$
$$T_n = 9n^2$$
$$\Rightarrow \quad T_k = 9k^2$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = 9\sum_{k=1}^{n}k^2$$

$$S_n = \frac{9n(n+1)(2n+1)}{6}$$

$$S_n = \frac{3n(n+1)(2n+1)}{2}$$

(viii) $4\times1^2+7\times2^2+10\times3^2+\dots$

Solution:

$4\times1^2+7\times2^2+10\times3^2+\cdots$ to n terms

nth term of A.P., 4, 7, 10, ...

$$a_n = a_1 + (n-1)d$$
$$= 4+(n-1)(3) = 4+3n-3 = 3n+1$$

nth term of A.P., 1, 2, 3, ...

$$b_n = b_1 + (n-1)d$$
$$= 1+(n-1)(1) = 1+n-1 = n$$

Let T_n be the nth term of given series, Then

$$T_n = a_n \times (b_n)^2$$
$$T_n = (3n+1)(n^2) = 3n^3+n^2$$
$$\Rightarrow \quad T_k = 3k^3+k^2$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(3k^3+k^2)$$

$$S_n = 3\sum_{k=1}^{n}k^3 + \sum_{k=1}^{n}k^2$$

$$= \frac{3n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left\{\frac{3n(n+1)}{2} + \frac{2n+1}{3}\right\}$$

$$= \frac{n(n+1)}{2}\left\{\frac{9n(n+1)+4n+2}{6}\right\}$$

$$S_n = \frac{n(n+1)(9n^2+13n+2)}{12}$$

(ix) $3+(3+7)+(3+7+11)+\dots$

Solution:

$3+(3+7)+(3+7+11)+\dots$ to n terms

Let T_n be n^{th} term of given series, then

$$T_n = 3+7+11+\dots+a_n$$

Here $a_1 = 3,\ d = 7-3 = 4$ using $S_n = \frac{n}{2}[2a_1+(n-1)d]$

$$T_n = \frac{n}{2}[2(3)+(n-1)(4)]$$

$$= \frac{n}{2}[6+4n-4] = \frac{n}{2}[2+4n]$$

$$= \frac{n}{2}\cdot 2(1+2n)$$

$$T_n = 2n^2+n$$
$$\Rightarrow \quad T_k = 2k^2+k$$

Taking $\sum_{k=1}^{n}$ on both sides

$$\sum_{k=1}^{n} T_k = 2\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k$$

$$S_n = \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{2(2n+1)}{3}+1\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{4n+2+3}{3}\right]$$

$$= \frac{n(n+1)}{2}\left(\frac{4n+5}{3}\right)$$

$$S_n = \frac{n(n+1)(4n+5)}{6}$$

(x) $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\dots$

Solution:

Let T_n be the nth term of the given series, then

$$T_n = 1^2+2^2+3^2+\dots+n^2$$

$$T_n = \frac{n(n+1)(2n+1)}{6} = \frac{n(2n^2+3n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$\Rightarrow \quad T_k = \frac{1}{6}(2k^3+3k^2+k)$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \frac{1}{6}\left\{2\sum_{k=1}^{n}k^3 + 3\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k\right\}$$

$$S_n = \frac{1}{6}\left\{2\left\{\frac{n(n+1)}{2}\right\}^2 + 3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{6}\left\{\frac{2n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{6}\left\{\frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{6}\times\frac{n(n+1)}{2}(n(n+1)+2n+1+1)$$

$$= \frac{n(n+1)}{12}(n^2+n+2n+2)$$

$$= \frac{n(n+1)(n^2+3n+2)}{12}$$

$$= \frac{n(n+1)(n^2+2n+n+2)}{12}$$

$$= \frac{n(n+1)[n(n+2)+1(n+2)]}{12}$$

$$= \frac{n(n+1)[(n+1)(n+2)]}{12}$$

$$S_n = \frac{n(n+1)^2(n+2)}{12}$$

2. Sum the series:

(i) $1^2-2^2+3^2-4^2+\dots+(2n-1)^2-(2n)^2$

Solution:

Let T_n be the nth ... of the given series, then

$$T_n = (2n-1)^2-(2n)^2$$
$$T_n = 4\ \ +1-4n-4n^2$$
$$T_n\ \ .\ -4n$$
$$\Rightarrow \quad T_k = 1-4k$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}1 - 4\sum_{k=1}^{n}k$$

$$S_n = n - 4\frac{n(n+1)}{2}$$
$$= n - 2n(n+1)$$
$$= n - 2n^2 - 2n$$
$$= -2n^2 - n$$
$$S_n = -n(2n+1)$$

(ii) $\frac{1^2}{1}+\frac{1^2+2^2}{2}+\frac{1^2+2^2+3^2}{3}+\dots$ **to n terms:**

Solution:

Let T_n be the nth term of the given series, then

$$T_n = \frac{1^2+2^2+3^2+\dots+n^2}{n}$$

As we know $1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$$T_n = \frac{\frac{n(n+1)(2n+1)}{6}}{n}$$
$$= \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}$$
$$= \frac{(2n^2+3n+1)}{6}$$
$$\Rightarrow T_k = \frac{1}{6}(2k^2+3k+1)$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \frac{1}{6}\sum_{k=1}^{n}(2k^2+3k+1)$$
$$S_n = \frac{1}{6}\left\{2\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\right\}$$
$$S_n = \frac{1}{6}\left\{2\frac{n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{2}+n\right\}$$
$$= \frac{n}{6}\left\{\frac{(n+1)(2n+1)}{3}+\frac{3(n+1)}{2}+1\right\}$$
$$= \frac{n}{6}\left\{\frac{(2n^2+n+2n+1)}{3}+\frac{(3n+3)}{2}+1\right\}$$
$$= \frac{n}{6}\left\{\frac{2(2n^2+3n+1)+3(3n+3)+6}{6}\right\}$$
$$= \frac{n}{36}(4n^2+6n+2+9n+9+16)$$
$$S_n = \frac{n}{36}(4n^2+15n+17)$$

3. Find the sum to n terms of the series whose nth terms are given:

(i) $5n^2+2n+3$

Solution:

Let $T_n = 5n^2+2n+3$

$$\Rightarrow T_k = 5k^2+2k+3$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(5k^2+2k+3)$$
$$S_n = 5\sum_{k=1}^{k}k^2+2\sum_{k=1}^{n}k+3\sum_{k=1}^{n}(1)$$
$$= \frac{5n(n+1)(2n+1)}{6}+\frac{2n(n+1)}{2}+3n$$
$$= n\left\{\frac{5(n+1)(2n+1)}{6}+(n+1)+3\right\}$$
$$= n\left\{\frac{5(2n^2+3n+1)+6n+6+18}{6}\right\}$$
$$= \frac{n}{6}(10n^2+15n+5+6n+24)$$
$$S_n = \frac{n}{6}(10n^2+21n+29)$$

(ii) n^2+2n-3

Solution:

Let $T_n = n^2+2n-3$

$$\Rightarrow T_k = k^2+2k-3$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+2k-3)$$
$$S_n = \sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k-3\sum_{k=1}^{n}(1)$$
$$= \frac{n(n+1)(2n+1)}{6}+\frac{2n(n+1)}{2}-3n$$
$$= n\left\{\frac{(n+1)(2n+1)}{6}+(n+1)-3\right\}$$
$$= n\left\{\frac{2n^2+3n+1+6n+6-18}{6}\right\}$$
$$S_n = \frac{n(2n^2+9n-11)}{6}$$

4. Given n^{th} terms of the series, find the sum to $2n$ terms:

(i) $3n^2+5n+2$

Solution:

Let $T_n = 3n^2+5n+2$

$$\Rightarrow T_k = 3k^2+5k+2$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(3k^2+5k+2)$$
$$S_n = 3\sum_{k=1}^{n}k^2+5\sum_{k=1}^{n}k+2\sum_{k=1}^{n}(1)$$
$$= \frac{3n(n+1)(2n+1)}{6}+\frac{5n(n+1)}{2}+2n$$
$$= n\left\{\frac{(n+1)(2n+1)}{2}+\frac{5(n+1)}{2}+2\right\}$$
$$= n\left\{\frac{2n^2+3n+1+5n+5+4}{2}\right\}$$
$$= \frac{n}{2}(2n^2+8n+10)$$
$$= \frac{2n}{2}(n^2+4n+5)$$
$$S_n = n(n^2+4n+5)$$

Replace n by $2n$, we have

$$S_{2n} = 2n((2n)^2+4(2n)+5)$$
$$= 2n(4n^2+8n+5)$$

(ii) n^2+n-2

Solution:

Let $T_n = n^2+n-2$

$$\Rightarrow T_k = k^2+k-2$$

Taking $\sum_{k=1}^{n}$ both sides

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+k-2)$$
$$S_n = \sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k-2\sum_{k=1}^{n}(1)$$
$$= \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}-2n$$
$$= n\left\{\frac{(n+1)(2n+1)}{6}+\frac{n+1}{2}-2\right\}$$
$$= n\left\{\frac{2n^2+3n+1+3n+3-12}{6}\right\}$$
$$= \frac{n}{6}(2n^2+6n-8)$$
$$= \frac{2n}{6}(n^2+3n-4)$$
$$S_n = \frac{n}{3}(n^2+3n-4)$$

Replace n by $2n$, we have

$$S_{2n} = \frac{2n}{3}((2n)^2+3(2n)-4)$$
$$S_{2n} = \frac{2n}{3}(4n^2+6n-4)$$
$$S_{2n} = \frac{4n}{3}(2n^2+3n-2)$$

Real Life Problems involving Sequences and Series:

Example 27: Vehicle Arrival Sequence

Vehicles arrive at a toll booth at a rate of 4 cars every 5 minutes. Represent the number of cars arriving over time as a sequence and predict the total number of cars after an hour.

Solution:

The sequence of car arrivals is: 4, 8, 12, 16,… (Arithmetic Progression)

Here, $a_1 = 4, d = 4, n = \frac{60}{5} = 12, a_{12} = ?$

Using the formula for arithmetic sequence

$$a_n = a_1 + (n-1)d$$
$$a_{12} = 4 + (12-1)(4) = 4 + 11(4)$$
$$= 4 + 44 = 48$$

Thus, after one hour there will be 48 cars.

Simple Interest on Loan (Arithmetic Sequence with Particular Term)

Example 28: To buy furniture for a new apartment Tayyab borrowed Rs. 50,000 at 8% simple interest for 11 years. How much interest will he pay?

Solution:

Since 8% is the yearly interest rate, we have

Interest after one year $= \text{Rs.}50{,}000 \times \frac{8}{100} \times 1 = \text{Rs.}4000$

$$\text{Interest after two years} = Rs.50{,}000\times\frac{8}{100}\times 2 = Rs.8000$$

Therefore, we have the A.P.

$$4000, 8000, 12000, \ldots$$

Here, $a_1 = 4000, a_2 = 8000, d = a_2 - a_1 = 4000, n = 11$

Using the formula for arithmetic sequence

$$a_n = a_1 + (n-1)d$$
$$a_{11} = 4000 + (11-1)(4000)$$
$$= 4000 + 10(4000) = 4000 + 40000$$
$$= Rs.44000$$

Thus, Tayyab will pay a total interest of Rs. 44000 on borrowed amount of Rs 50,000 after 11 years.

Compound Interest on Loan (Geometric Sequence with Particular Term):

Example 29: Amna invests Rs. 200000 at 5% interest compounded annually. What total amount will she get after 10 years?

Solution:

Let the principal amount be P. Then,

$$\text{The interest for the first year} = 5\%\text{ of } P = P\times\frac{5}{100} = P(0.05)$$
$$\text{The total amount after first year} = P + P(0.05) = P(1+0.05)$$
$$\text{The interest for the second year} = P(1+0.05)\times 0.05$$
$$\text{The total amount after second year} = P(1+0.05) + P(1+0.05)\times 0.05$$
$$= P(1+0.05)(1+0.05)$$
$$= P(1+0.05)^2$$
$$\text{Similarly, the total amount after third year} = P(1+0.05)^3$$

Thus, we have sequence of amounts

$$P(1.05), P(1.05)^2, P(1.05)^3, \ldots \text{ (G.P.)}$$

$$\text{Here, } a_1 = P(1.05), r = 1.05, n = 10, a_{10} = ?$$

Using the geometric sequence formula

$$a_n = a_1 r^{n-1}$$
$$a_{10} = a_1 r^{10-1}$$
$$= P(1.05)\times(1.05)^9$$
$$= (200000)(1.05)^{10} \qquad \because P = 200000$$
$$= (200000)(1.62889)$$
$$= 325778.92$$

Thus, the total amount Amna will get after 10 years will be Rs. 325778.92

Grid Column Distribution (Arithmetic Series Sum of Terms):

Example 30: A web designer is using a 12-column grid system where each column increases in width by 10px from the previous one. The first column width is 50px wide. Find the total width occupied by all 12 columns.

Solution:

This follows an arithmetic series with:

First term = $a_1 = 50$, Common difference = $d = 10$

Number of terms = $n = 12$

Using the formula for the sum of an arithmetic series:

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$S_{12} = \frac{12}{2}[2(50) + (12-1)(10)] = 6[100 + 110] = 6[210] = 1260p$$

Thus, the total width of all 12 columns is 1260px.

Example 31: Motor Vehicle Leasing Using Arithmetic Sequence

A company leases a motor vehicle with the following terms:

- The first monthly payment is Rs. 15,000.
- Each subsequent payment increases by Rs. 500 due to inflation adjustments.
- The lease term is 24 months.

Find:

(i) What is the payment in the 24 th month? (ii) What is the total amount paid over 24 months?

(iii) If the company can only afford to pay a total of Rs. 400,000, can they complete the 24 -months lease?

(iv) Find maximum months n such that total, payment $S_n \leq 400{,}000$.

Solution:

Given that: First term = $a_1 = 15000$

Common difference = $d = 500$

Number of terms = $n = 24$

(i) Payment in 24th month:

Using the formula

$$a_n = a_1 + (n-1)d$$
$$a_{24} = 15000 + (24-1)(500)$$
$$= 15000 + 23\times 500 = 15000 + 11500 = Rs.26500$$

(ii) Total payment over 24 months using the formula

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$= \frac{24}{2}(15000 + 26500) = 12(41500) = Rs.498000$$

(iii) Can the company afford the lease? No, total payments (Rs. 498000) exceed the budget of Rs. 400,000 by Rs. 98,000.

(iv) Using: $S_n = \frac{n}{2}[2a_1 + (n-1)d] \leq 400{,}000$

Substituting the values:

$$\frac{n}{2}[2(15000) + (n-1)(500)] \leq 400{,}000$$
$$n[15000 + 250n - 250] \leq 400{,}000$$
$$n(250n + 14750) \leq 400{,}000$$
$$250n^2 + 14750n - 400000 \leq 0$$
$$n^2 + 59n - 1600 \leq 0$$

Associated equation is $n^2 + 59n - 1600 = 0$

$$n = \frac{-59 \pm \sqrt{(59)^2 - 4(1)(-1600)}}{2(1)}$$
$$n = \frac{-59 \pm 99.4}{2}$$
$$n = \frac{-59 - 99.4}{2}, \quad n = \frac{-59 + 99.4}{2}$$
$$n = -79.2, \quad n = 20.2$$

Clearly $n = 20$ satisfy the inequality.

So, $n = 20$ is the maximum months such that payment $S_n \leq 400{,}000$.

Exercise 6.11

1. A sum of Rs. 10400 is paid off in 40 instalment such that each instalment is Rs. 10 more than the preceding instalment. Calculate the value of the first instalment.

Solution:

Total sum = 10400

Let first installment = a

The sequence of installments is

$a, a+10, a+20, \ldots, a_{40}$ (A.P)

Here $a_1 = a$, $d = 10$, $n = 40$, $S_{40} = 10400$

As we know $S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$$S_{40} = \frac{40}{2}(2a + (40-1)(10))$$

$$10400 = 20(2a + 390)$$

$$\frac{10400}{20} = 2a + 390$$

$$520 - 390 = 2a$$

$$2a = 130$$

$$a = 65$$

Hence, the value of first installment is Rs. 65.

2. An investor invests Rs. 150000 at an annual compound interest rate of 6% for 8 years. Find the total amount will be get after 8 years.

Solution:

Principal amount = 150000

Rate of interest = 6%

$\Rightarrow$ $r = 1 + 6\% = 1 + 0.06 = 1.06$

Here $a_1 = 150000$, $n = 9$, $a_9 = ?$

As we know $a_n = a_1 r^{n-1}$

$$a_9 = 150000(1.06)^{9-1}$$

$$= 150000(1.59385)$$

$$= 239077.50$$

Thus, the total amount the investor will get after 8 years will be Rs. 239077.50.

3. The population of a town is 4084101 at present and five years ago it was 3200000. Find its rate of increase if it increased geometrically.

Solution:

Population five years ago = a_1 = 3200000

Current population = a_6 = 4084101

As we know $a_n = a_1 r^{n-1}$

$a_6 = a_1 r^{6-1}$ Put $n = 6$

$$4084101 = (3200000)r^5$$

$$\frac{4084101}{3200000} = r^5$$

$$1.2762815625 = r^5$$

By taking fifth root of both sides

$$r = (1.2762815625)^{\frac{1}{5}}$$

$$r = 1.05$$

$$r = 1 + 0.05$$

$$r = 1 + \frac{5}{100}$$

$$r = 1 + 5\%$$

$\Rightarrow$ Rate of increase = 5%

4. Determine the total worth of a yearly Rs. 5000 investment after 20 years if the interest rate is 5% compounded annually.

Solution:

Current amount = 5000

Rate of interest = 5%

$\Rightarrow$ $r = 1 + 5\% = 1 + 0.05 = 1.05$

Here $a_1 = 5000$, $n = 21$, $S_{21} = ?$

By using sum formula

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

$$S_{21} = \frac{5000((1.05)^{21} - 1)}{1.05 - 1}$$

$$= \frac{5000(1.78596)}{0.05}$$

$$= 178596$$

Hence, the total worth of the investment after 20 years is Rs. 178596.

5. A water tank develops a leak. Each week, the tank loses 5 gallons of water due to the leak. Initially, the tank is full and container 2000 gallons.

(i) How many gallons are in the tank 20 weeks later?

(ii) How many weeks until the tank is half-full?

(iii) How many weeks until the tank is empty?

Solution:

If the water tank starts with 2000 gallons and loses 5 gallons each week due to leakage, then

Sequence: 1995, 1900, 1985, ..., a_n (A.P.)

(i) Here $a_1 = 1995$, $d = 1900 - 1995 = -5$

$n = 20$, $a_{20} = ?$

As we know $a_n = a_1 + (n-1)d$

$$a_{20} = 1995 + 19(-5)$$

$$= 1995 - 95$$

$$= 1900 \text{ gallons}$$

Thus, the tank contains 1900 gallons after 20 weeks.

(ii) If the tank is half full, then

$a_n = 1000$, $n = ?$, $a_1 = 1995$, $d = -5$

As we know $a_n = a_1 + (n-1)d$

$$1000 = 1995 + (n-1)(-5)$$

$$1000 - 1995 = -5n + 5$$

$$-995 - 5 = -5n$$

$$-5n = -1000$$

$$n = \frac{1000}{5}$$

$$n = 200 \text{ weeks}$$

Thus, the tank is half-full after 200 weeks.

(iii) If the tank is empty, then

$a_n = 0$, $n = ?$, $a_1 = 1995$, $d = -5$

As we know $a_n = a_1 + (n-1)d$

$$0 = 1995 + (n-1)(-5)$$

$$1995 - 5n + 5 = 0$$

$$5n = 2000$$

$$n = 400 \text{ weeks}$$

Thus, the tank is empty after 400 weeks.

6. A drug company has manufactured 7 million doses of a vaccine to date. They promise additional production at a rate of 1.4 million doses/month over the next year.

(i) How many doses of the vaccine, in total, will have been produced after a year?

(ii) The general term a_n describes the total number of doses of the vaccine produced. Describe the meaning of the variable n in the context of this problem. Find the general term a_n

(iii) Find the value of a_{10} and interpret its meaning in words.

Solution:

According to the statement, we have the sequence

7, 8.4, 9.8, ..., a_n (A.P)

(i) Here $a_1 = 7$, $d = 1.4$, $n = 12$, $a_{12} = ?$

As we know $a_n = a_1 + nd$

$$a_{12} = 7 + 12(1.4)$$

$$a_{12} = 7 + 16.8$$

$$a_{12} = 23.8 \text{ million doses}$$

Hence, the company will produce 23.8 million doses after a year.

(ii) The general term a_n is

$$a_n = a_1 + nd$$

$$a_n = 7 + n(1.4)$$

where a_n is the total no. of doses in millions and n is the number of months passed since now.

(iii) Here $a_1 = 7$, $a_{10} = ?$, $n = 10$, $d = 1.4$

As we know $a_n = a_1 + nd$

$$a_{10} = 7 + 10(1.4)$$

$$= 7 + 14$$

$$= 21 \text{ million doses}$$

Interpretation: After 10 months, the total number of vaccine doses produced is 21 million.

7. At a toll booth, the number of vehicles passing through during the first minute is 100 . Due to road congestion, each minute only 80% of the vehicles from the previous minute manage to pass.

(i) Represent the number of vehicles passing each minute as a sequence.

(ii) Find the total number of vehicles that pass through in 15 minutes.

(iii) What is the maximum number of vehicles that can pass in the long run (as time $t \to \infty$)

Solution:

(i) Number of vehicles during 1st minute = 100

Number of vehicles during 2nd minute = 80%(100)

$$= \frac{80}{100}(100) = 80$$

Number of vehicles during 3rd minute = 80%(80)

$$= \frac{80}{100}(80) = 64$$

Sequence: 100, 80, 64, ... (G.P)

(ii) Number of vehicles during 1st minute = a_1 = 100

Rate of decrease = 20%

$\Rightarrow$ $r = 1 - 20\% = 1 - 0.20 = 0.80$

Here $n = 15$ and $S_{15} = ?$

By using the sum formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_{15} = \frac{100(1 - (0.80)^{15})}{1 - 0.80}$$

$$= \frac{100(1 - 0.03518)}{0.20}$$

$$= \frac{100(0.96482)}{0.20}$$

S_{15} = 482.41 = 482 approximately

(iii) Here $a_1 = 100$, $r = 0.80$, $S_\infty = ?$

As we know $S_\infty = \frac{a_1}{1-r}$ $\because |r| = 0.80 < 1$

$$= \frac{100}{1 - 0.80}$$

$$= \frac{100}{0.20}$$

$$S_\infty = 500$$

8. A sum of Rs. 5000 is inverted at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an A.P.? If so find the interest at the end of 20 years making use of this fact.

Solution:

Total sum = 5000

Interest at the end of 1st year = 8%(5000)

$= \frac{8}{100}(5000) = 400$

Interest at the end of 2^{nd} year $= 2 \times 8\%(5000)$

$= 2\left(\frac{8}{100}\times 5000\right)$

$= 2(400) = 800$

Interest at the end of 3^{rd} year $= 3 \times 8\%(5000)$

$= 3\left(\frac{8}{100}\times 5000\right)$

$= 3(400) = 1200$

The sequence of interest is

400, 800, 1200, ...

As 800 - 400 = 1200 - 800

⇒ 400 = 400 (Common difference exists)

So, the sequence form an A.P.

Here $a_1 = 400, d = 400, n = 20, a_{20} = ?$

Using: $a_n = a_1 + (n-1)d$

$a_{20} = 400 + (20-1)(400)$

$= 400 + (19)(400)$

$= 8000$

Thus, the simple interest at the end of 20 years is Rs. 8000.

9. A machine is purchased for Rs.20,000. Depreciates at 6% per annum for the first four years and after that 8% per annum for the next six years. Depreciation being calculated on diminishing value. Find the value of the machine after a period of 10 years.

Solution:

Initial value of machine $= a_1 = 20000$

Rate of depreciation for first four years = 6%

⇒ $r_1 = 1 - 6\% = 1 - 0.06 = 0.94$

For value of machine after four years

$n_1 = 5, a_5 = ?$

Using: $a_n = a_1 r_1^{n_1-1}$

$a_5 = 20000(0.94)^{5-1}$

$= 20000(0.94)^4$

$= 15614.9792$

Let $a = a_5 = 15614.9792$

Rate of depreciation for next six years = 8%

⇒ $r_1 = 1 - 8\% = 1 - 0.08 = 0.92$

For value of machine after next six years $n_2 = 7$

Value after ten years $= ar_2^{n_2-1}$

$= (15614.9792)(0.92)^{7-1}$

$= (15614.9792)(0.92)^6$

$= (15614.9792)(0.60636)$

$= 9468.30$ approximately

10. Two cars start together in the same direction from the same place. The first goes with uniform speed of 20 km/h. The second goes at a speed of 12 km/h in the first hour and increase the speed by 1 km/h each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop?

Solution:

Suppose the 2^{nd} car overtakes the first car after t hours. Then, the two cars travel the same distance in t hours.

Distance travelled by 1^{st} car is

$D_1 = 20 + 20 + 20 + \dots + 20$ (Arithmetic series)

Here $a_1 = 20, a_t = 20$

$D_1 = \frac{t}{2}(a_1 + a_t)$

$= \frac{t}{2}(20+20) = \frac{t}{2}(40) = 20t$

Distance travelled by 2^{nd} car is

$D_2 = 12 + 13 + 14 + \dots$ to t terms (Arithmetic series)

Here $a_1 = 12, d = 13 - 12 = 1$

$D_2 = \frac{t}{2}(2a_1 + (t-1)d)$

$= \frac{t}{2}(2(12) + (t-1)(1)) = \frac{t}{2}(24 + t - 1)$

$= \frac{t}{2}(t+23)$

As $D_1 = D_2$

⇒ $20t = \frac{t}{2}(t+23)$

$40t = t^2 + 23t$

$0 = t^2 - 17t$

$t(t-17) = 0$

⇒ $t - 17 = 0 \quad (\because t \neq 0)$

$t = 17$

Thus, the second car will overtakes the first after 17 hours.

11. 150 workers were engaged to finish a piece of work in a certain number of days. Five workers dropped the second day, five more workers dropped the third day and so on. It takes 10 more days to finish the work now. Find the number of days in which the work was completed.

Solution:

Let n be the number of days in which 150 workers complete the work.

According to the given condition

$150(n) = 150 + 145 + 140, \dots$ to $(n+10)$ days

↓ Arithmetic Series

Here $a_1 = 150, d = 145 - 150 = -5$

Using $S_n = \frac{n}{2}(2a_1 + (n-1)d)$

⇒ $150n = \frac{n+10}{2}(2(150) + (n+10-1)(-5))$

$300n = (n+10)(300 + (n+9)(-5))$

$300n = (n+10)(300 - 5n - 45)$

$300n = (n+10)(255 - 5n)$

$300n = 255 - 5n^2 + 2550 - 50n$

$5n^2 + 50n + 300n - 255n - 2550 = 0$

$5n^2 + 95n - 2550 = 0$

$n^2 + 19n - 510 = 0$ (Divide by 5)

$n^2 + 34n - 15n - 510 = 0$

$n(n+34) - 15(n+34) = 0$

$(n-15)(n+34) = 0$

$n - 15 = 0$ or $n + 34 = 0$

$n = 15$ or $n = -34$ (Not possible)

Number of days $= n + 10$

$= 15 + 10 = 25$

Thus, the work was completed in 25 days.

12. A radioactive product has a half life of 5 years. If the radioactivity level is 68 microcuries after 20 years. Determine the original level of radioactivity.

Solution:

Let original level of radio activity $= a_1$

After 5-years life of radioactivity $= \frac{1}{2}a_1$

After 10-years life of radioactivity $= \frac{1}{2}\left(\frac{1}{2}a_1\right) = \frac{1}{4}a_1$

After 15-years life of radioactivity $= \frac{1}{2}\left(\frac{1}{4}a_1\right) = \frac{1}{8}a_1$

After 20-years life of radioactivity $= \frac{1}{2}\left(\frac{1}{8}a_1\right) = \frac{1}{16}a_1$

Then the sequence of radioactivity life is

$a_1, \frac{1}{2}a_1, \frac{1}{4}a_1, \frac{1}{8}a_1, \frac{1}{16}a_1$

Here $a_1 = a_1, r = \frac{1}{2}, n = 5$

As radioactivity level after 20-years is 68 microcuries, so we have

$a_5 = 68$

$a_1 r^{5-1} = 68 \qquad \because a_n = a_1 r^{n-1}$

$a_1 r^4 = 68$

$a_1\left(\frac{1}{2}\right)^4 = 68$

$a_1 = 68 \times 2^4 = 1088$

Thus, the original level of radioactivity was 1088 microcuries.

13. An object moving in a line is given an initial velocity of 4.5 m/s and a constant acceleration of 2.5 m/s². How long will it take the object to reach a velocity of 20 m/s?

Solution:

Let Initial velocity $= a_1 = 4.5$ m/s

Constant acceleration $= d = 2.5$ m/s²

Final velocity $= a_n = 20$ m/s

Sequence: 4.5, 7, 9.5, ..., a_n (A.P)

Using $a_n = a_1 + nd$

$20 = 4.5 + n(2.5)$

$20 - 4.5 = 2.5n$

$15.5 = 2.5n$

$\frac{15.5}{2.5} = n$

$6.2 = n$

Thus, it will take the object 6.2 seconds to reach a velocity of 20 m/s.

14. In an integrated circuit with an initial current of 1080mA, the temperature in the components decreases from 20% to 17% to 14%. Assuming that each temperature decrease is caused by a decrease in the initial current, what is the value of the current at fourth measurement?

Solution:

Initial current = 1080 mA

The sequence of temperature decreases is 20%, 17%, 14%, 11% (A.P)

Current of 1^{st} measurement $= 1080 - 20\%(1080)$

$= 1080 - \frac{20}{100}(1080)$

$= 1080 - 216$

$= 864$ mA

Current at 2^{nd} measurement $= 864 - 17\%$

$= 864 - \frac{17}{100}(1080)$

$= 864 - 183.6$

$= 680.4$ mA

Current at 3^{rd} measurement $= 680.4 - 14\%(1080)$

$= 680.4 - \frac{14}{100}(1080)$

$= 680.4 - 151.2$

$= 529.2$ mA

Current at 4^{th} measurement $= 529.2 - 11\%(1080)$

$= 529.2 - \frac{11}{100}(1080)$

$= 529.2 - 118.8$

$= 410.4$ mA

15. Show that the amount of a certain sum of money at compound interest of $r\%$ per year for n year from a G.P.

Solution:

Let the principal sum of money = P

Annual rate of compound interest = $r\%$

The amount for one year = $P + r\%$ of P

$$= P + \frac{r}{100}P$$

$$= P\left(1+\frac{r}{100}\right)$$

The amount for two years $= P\left(1+\frac{r}{100}\right) + r\%$ of $P\left(1+\frac{r}{100}\right)$

$$= P\left(1+\frac{r}{100}\right)\left(1+\frac{r}{100}\right)$$

$$= P\left(1+\frac{r}{100}\right)^2$$

The amount for three years $= P\left(1+\frac{r}{100}\right)^2 + r\%$ of $P\left(1+\frac{r}{100}\right)^2$

$$= P\left(1+\frac{r}{100}\right)^2\left(1+\frac{r}{100}\right) = P\left(1+\frac{r}{100}\right)^3$$

⋮ ⋮ ⋮

The amount for n years $= P\left(1+\frac{r}{100}\right)^n$

Sequence:

$$P\left(1+\frac{r}{100}\right), P\left(1+\frac{r}{100}\right)^2, P\left(1+\frac{r}{100}\right)^3, \ldots, P\left(1+\frac{r}{100}\right)^n \quad \ldots(I)$$

As $\dfrac{P\left(1+\frac{r}{100}\right)^2}{P\left(1+\frac{r}{100}\right)} = \dfrac{P\left(1+\frac{r}{100}\right)^3}{P\left(1+\frac{r}{100}\right)^2}$

$\Rightarrow \left(1+\frac{r}{100}\right) = \left(1+\frac{r}{100}\right)$ (Common ratio exists)

So, sequence (I) forms a G.P.

Formula Sheet

Arithmetic	1. $a_n = a_1 + (n-1)d$	2. $A = \frac{a+b}{2}$
	3. $S_n = \frac{n}{2}[2a_1 + (n-1)d]$	4. $S_n = \frac{n}{2}(a_1 + a_n)$
Geometric	5. $a_n = a_1 r^{n-1}$	6. $G = \pm\sqrt{ab}$
	7. $S_n = \frac{a_1(1-r^n)}{1-r}, \lvert r\rvert<1$	8. $S_n = \frac{a_1(r^n-1)}{r-1}, \lvert r\rvert>1$
Arithmetico-Geometric	9. $S_n = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2} - \frac{dbr^n}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r}$	
	10. $S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$	
Harmonic	11. $a_n = \frac{1}{a_1 + (n-1)d}$	12. $H = \frac{2ab}{a+b}$
Miscellaneous Series	13. $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k$	14. $\sum_{k=1}^{n}\alpha a_k = \alpha\sum_{k=1}^{n}a_k$
	15. $\sum_{k=1}^{n}\left[(k^m - (k-1)^m\right] = n^m$	16. $\sum_{k=1}^{n}k = \frac{n(n+1)}{2}$
	17. $\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$	18. $\sum_{k=1}^{n}k^3 = \left[\frac{n(n+1)}{2}\right]^2$

Multiple Choice Questions (MCQs)

Exercise 6.1

1. A sequence is a function whose domain is a set of - - - - -
(A) real numbers (B) natural numbers (C) integers (D) none of these

2. Sequence is denoted by - - - - -
(A) a_n (B) Σa_n (C) $\{a_n\}$ (D) $\{S_n\}$

3. An infinite sequence has no - - - - -
(A) 1^{st} term (B) middle term (C) tenth term (D) last term

4. If $a_n - a_{n-1} = n + 2$, $a_1 = 2$ then a_3 = - - - - -
(A) 6 (B) 4 (C) 11 (D) 17

5. The next term of the sequence 7, 9, 12, ... is:
(A) 16 (B) 15 (C) 14 (D) 18

Exercise 6.2

6. A sequence $\{a_n\}$ is an arithmetic sequence if $\forall$ n $\in$ N and n > 1:
(A) $\frac{a_n}{a_{n-1}}$ is same (B) $a_n - a_{n-1}$ is same (C) $a_{n+1} - a_{n-1}$ is different (D) $\frac{a_{n-1}}{a_n}$ is same

7. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P then the common difference is - - - - -
(A) $\frac{a-c}{2ac}$ (B) $\frac{2ac}{a-c}$ (C) $\frac{a+c}{2ac}$ (D) $\frac{2ac}{a+c}$

8. If $a_{n-3} = 2n - 5$, its nth term is - - - - -
(A) 2n + 1 (B) 2n + 3 (C) 2n − 2 (D) 2n − 8

9. The 7th term of an A.P., whose first term is 2 and common difference is zero is - - -
(A) 14 (B) 7 (C) 2 (D) 0

10. $a_1, a_1 + d, a_1 + 2d, \ldots$ is the general form of:
(A) G.P. (B) A.P. (C) H.P. (D) none of these

Exercise 6.3

11. Middle term of three consecutive terms in A.P is the - - - - -
(A) A.M (B) G.M (C) H.M (D) none of these

12. Arithmetic mean between $\frac{1}{a}$ and $\frac{1}{b}$ is - - - - -
(A) $\frac{a+b}{ab}$ (B) $\frac{a+b}{2ab}$ (C) $\frac{2ab}{a+b}$ (D) $\frac{ab}{a+b}$

13. The arithmetic mean between $2+\sqrt{2}$ and $2-\sqrt{2}$ is - - - - -
(A) 2 (B) $\sqrt{2}$ (C) 0 (D) 4

14. If 5 is A.M. between −5 and b, then b is equal to - - - - -
(A) 0 (B) 15 (C) −5 (D) 10

Exercise 6.4

15. Sum of terms of an A.P. is called - - - - -
(A) geometric series (B) arithmetic series (C) arithmetic progression (D) harmonic progression

16. Formula for the sum of n terms of A.P. (Arithmetic progression) is - - - - -
(A) $a_n = a_1 + (n-1)d$ (B) $S_n = \frac{n}{2}(a_1 + a_n)$ (C) $S_n = \frac{a_1(1-r^n)}{1-r}$ (D) $S = \frac{a}{1-r}$

17. The sum $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ is called - - - - - of the sequence $\{a_n\}$.
(A) nth partial sum (B) sequence (C) 3rd partial sum (D) none of these

Exercise 6.5

18. No term of a geometric sequence can be -----
(A) 0 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) 1

19. If $\frac{1}{a}, \frac{1}{b}$, and $\frac{1}{c}$ are in G.P then common ratio is equal to -----
(A) $\pm\sqrt{\frac{c}{a}}$ (B) $\pm\sqrt{\frac{a}{c}}$ (C) $\pm\sqrt{a+c}$ (D) $\pm\sqrt{a-c}$

20. If a, b, c are in G.P and a > 0, b > 0, c > 0, then the reciprocals of a, b, c form -----
(A) A.P (B) G.P (C) H.P (D) none of these

21. The sequence $5, \frac{5}{2}, \frac{5}{4}, \ldots$ is -----
(A) series (B) A.P. (C) G.P. (D) H.P

Exercise 6.6

22. If a, G, b be the G.P. where a, b are numbers then G.M. is equal to -----
(A) $\sqrt{ab}$ (B) $-\sqrt{ab}$ (C) $\pm\sqrt{ab}$ (D) ab

23. G.M between 1 and –16 is:
(A) –4 (B) 4 (C) ±4i (D) $\frac{1}{4}$

24. For what value of n, $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between a and b?
(A) n = –1 (B) 1 (C) $\frac{1}{2}$ (D) 0

Exercise 6.7

25. Sum of n terms of G.P. with first term 'a' common ratio 'r' is -----
(A) $\frac{a+b}{2}$ (B) $\frac{a(1-r^n)}{1-r}$ (C) $\frac{2ab}{a+b}$ (D) $\frac{n}{2}\{2a+(n-1)d\}$

26. If S_n tends to a definite number limit as $n \to \infty$, then the series is -----
(A) convergent (B) divergent (C) oscillatory (D) none of these

27. An infinite geometric series is divergent if -----
(A) $|r| = 1$ (B) $|r| > 1$ (C) both (A) & (B) (D) $|r| \le 1$

28. If $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$ an infinite geometric series, then sum is:
(A) $\frac{1}{4}$ (B) $\frac{5}{4}$ (C) $\frac{4}{5}$ (D) $\frac{3}{4}$

Exercise 6.8

29. The sequence formed by multiplying the corresponding terms of A.P. and G.P. is called -----
(A) A.P. (B) G.P. (C) H.P. (D) A.G.P.

30. Sum of the terms of ----- is called arithmetico-geometric series.
(A) A.P. (B) G.P. (C) A.P and G.P (D) A.G.P.

31. The series of the form $ab+(a+d)br+(a+2d)br^2+\cdots+\{a+(n-1)d\}br^{n-1}$ is called -----
(A) arithmetic series (B) geometric series (C) harmonic series (D) arithmetico-geometric series

32. Sum to infinity of arithmetico-geometric series is -----
(A) $S_\infty = \frac{a}{1-r}$ (B) $S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$ (C) $S_\infty = \frac{dbr}{1-r} + \frac{ab}{(1-r)^2}$ (D) $S_\infty = \frac{ab}{1+r} + \frac{dbr}{(1+r)^2}$

Exercise 6.9

33. The reciprocal of the terms of H.P is -----
(A) A.P (B) G.P (C) H.P (D) none of these

34. No term of a harmonic sequence can be -----
(A) 3 (B) 2 (C) 1 (D) 0

35. The harmonic mean between two numbers a and b is -----
(A) $\frac{2}{a+b}$ (B) $\pm\sqrt{ab}$ (C) $\frac{a+b}{2}$ (D) $\frac{2ab}{a+b}$

36. If A, G, H are arithmetic, geometric, harmonic means between a and b respectively then A, G, H are -----
(A) in G.P. (B) in A.P. (C) in H.P. (D) all of these

Exercise 6.10

37. $\sum_{k=1}^{n} 4$ equals to:
(A) n (B) 4n (C) 8n (D) 2n

38. If n is any positive integer then $1^3 + 2^3 + 3^3 + \ldots + n^3$ equals -----
(A) $\frac{n(n+1)}{2}$ (B) $\frac{n(n+1)^2}{2}$ (C) $\frac{n^2(n+1)^2}{4}$ (D) $\frac{n^2(n+1)}{4}$

39. $\sum_{k=1}^{n} k$ is equal to -----
(A) $\frac{n}{2}$ (B) $\frac{n+1}{2}$ (C) $\frac{n(n-1)}{2}$ (D) $\frac{n(n+1)}{2}$

Exercise 6.11

40. Vehicles arrive at a toll booth at a rate of 4 cars every 5 minutes, then the sequence of car arrivals is:
(A) 4, 10, 15, 20,... (B) 4, 4, 4, 4,... (C) 4, 8, 12, 16,... (D) 5, 5, 5, 5,...

41. Vehicles arrive at a toll booth at a rate of 4 cars every 5 minutes. Total number of cars after an hour will be:
(A) 48 cars (B) 50 cars (C) 20 cars (D) 28 cars

42. A web designer is using a 12-column grid system where each column increases in width by $10px$ from the previous one. If first column width is $50px$ wide, then the width of 5^{th} column will be:
(A) 150 (B) 200 (C) 350 (D) 300

ANSWER KEY

1.	B	2.	C	3.	D	4.	C	5.	A	6.	B	7.	A	8.	A	9.	C	10.	B
11.	A	12.	B	13.	A	14.	B	15.	B	16.	B	17.	A	18.	A	19.	B	20.	B
21.	C	22.	C	23.	C	24.	C	25.	B	26.	A	27.	C	28.	A	29.	D	30.	D
31.	D	32.	B	33.	A	34.	D	35.	D	36.	A	37.	B	38.	C	39.	D	40.	C
41.	A	42.	C																

Unit 7 Permutations and Combinations

Introduction

In our daily life, permutations and combinations play a vital role in counting total number of possibilities, as well as in arrangements and selections of objects. They are used in many fields of science. For example,

- In probability theory, permutations and combinations are used to compute how many times an event can occur in various scenarios and to estimate the odds of winning a lottery.
- In biology, these are used to find out the total numbers of possible DNA sequences.
- In computer science, these are used to count the possible number of passwords of a given length by using some specific characters.
- Moreover, these are the important parts of many encryption algorithms to ensure the privacy and integrity of a data set.

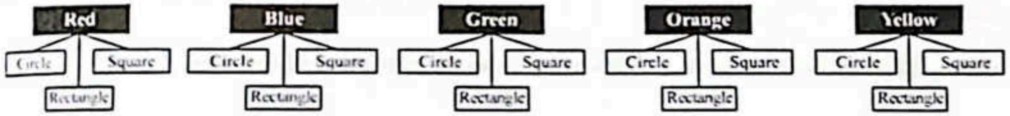
Fundamental Principle of Counting:

Challenge:

Danish wants to prepare invitation cards of 5 different colours (red, blue, green, orange and yellow) by changing any of 3 shapes (circle, square and rectangle). How many cards can Danish make?

> **Challenge:**
> Make a tree diagram and find how many cards can Danish make?

1^{st} **Way:** By making tree diagram.



From tree diagram, it is clearer there are 15 choices for Danish to make cards of 5 different colours (red, blue, green, orange and yellow) by changing any of 3 shapes (circle, square and rectangle).

2^{nd} **Way:** To find the total number of different cards Danish can make, multiply the number of colors by the number of shapes:

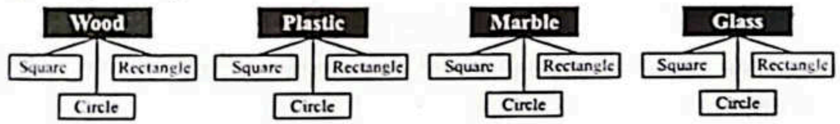
$$\text{Total number of cards made} = \text{Total no. of colours} \times \text{Total no. of shape}$$
$$= 5 \times 3 = 15 \text{ cards}$$

Problem:

Danish's father wants to buy a table and has asked his son to help him decide. He narrowed down his options for manufacturer, types of material (wood, plastic, glass and marble) and types of shape (circle, square and rectangle). Find the total number of table choices from the above options.

Again the problem is to count the total number of ways in which Danish's father can choose a table.

1^{st} **Way:** By making tree diagram.



From tree diagram, it is clearer there are 12 choices for Danish's father to buy a table with one type of material and one type of shape.