Improper Rational Fraction:

A rational fraction $\frac{P(x)}{Q(x)}$ is called an **Improper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator.

For example, $\frac{x}{2x-3}$, $\frac{(x-2)(x+1)}{(x-1)(x+4)}$ and $\frac{x^3-x^2+x+1}{x^2+5}$ are improper rational fractions or improper fractions.

Mixed Form: Any improper rational fraction can be reduced by division to a mixed form, consisting of the sum of a polynomial and a proper rational fraction.

For example, $\frac{3x^2+1}{x-2}$ is an improper rational fraction.

$$\begin{array}{r|l} & 3x+6 \\ \hline x-2 & 3x^2+1 \\ & \pm 3x^2 \mp 6x \\ \hline & 6x+1 \\ & \pm 6x \mp 12 \\ \hline & 13 \end{array}$$

By long division, we obtain

$$\frac{3x^2+1}{x-2} = Q+\frac{R}{D}$$

$$= (3x+6)+\frac{13}{x-2}$$

i.e., an improper rational fraction has $\frac{3x^2-1}{x-2}$ been reduced to the sum of a polynomial $3x+6$ and a proper rational fraction $\frac{13}{x-2}$.

- When a rational fraction is separated into partial fractions, the result is an identity; i.e., it is true for all values of the variable in the domain of identity.

The evaluation of the coefficients of the partial fractions is based on the following theorem:

> *"If two polynomials are equal for all values of the variable, then the polynomials have same degree and the coefficients of like powers of the variable in both the polynomials must be equal".*
>
> For example, if $px^3+qx^2-ax+b=2x^3-3x^2-4x+5, \forall x$ then $p=2, q=-3, a=4$ and $b=5$.

Resolution of a Rational Fraction $\frac{P(x)}{Q(x)}$ into Partial Fractions:

Following are the main points of resolving a rational fraction $\frac{P(x)}{Q(x)}$ into partial fractions:

(i) The degree of $P(x)$ must be less than that of $Q(x)$. If not, divide and work with the remainder theorem.
(ii) Factor the denominator $Q(x)$ into its irreducible factor, write the rational fraction into partial fractions.
(iii) Multiply the identity with the denominator of left hand side.
(iv) Equate the coefficients of like terms (powers of x).
(v) Solve the resulting equations for the coefficients.

We now discuss the following cases of partial fractions resolution.

Case I: Resolution of $\frac{P(x)}{Q(x)}$ into partial fractions when $Q(x)$ has only non-repeated linear factors:

The polynomial $Q(x)$ may be written as:

$$Q(x)=(x-a_1)(x-a_2)\ldots(x-a_n), \text{ where } a_1 \neq a_2 \neq \ldots \neq a_n$$

$$\therefore \quad \frac{P(x)}{Q(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n} \text{ is an identity.}$$

Where $A_1, A_2, \ldots, A_n$ are numbers to be found.

The method is explained by the following examples:

Example 1: Resolve $\frac{7x+25}{(x+3)(x+4)}$ into partial fractions.

Solution:

Suppose $$\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3}+\frac{B}{x+4} \qquad \ldots(1)$$

Multiplying both sides by $(x+3)(x+4)$, we get

$$\Rightarrow \quad 7x+25 = A(x+4)+B(x+3) \qquad \ldots(2)$$

Put $x+3=0 \Rightarrow x=-3$ in eq. (2), we have

$$7(-3)+25 = A(-3+4)$$
$$-21+25 = A(1)$$
$$\boxed{A=4}$$

Put $x+4=0 \Rightarrow x=-4$ in eq. (2), we have

$$7(-4)+25 = B(-4+3)$$
$$-28+25 = B(-1)$$
$$\boxed{B=3}$$

Putting values of A and B in eq. (1), we have

Hence, $$\frac{7x+25}{(x+3)(x+4)}=\frac{4}{x+3}+\frac{3}{x+4}.$$

Example 2: Resolve $\frac{x^2-10x+13}{(x-1)(x^2-5x+6)}$ into partial fractions.

Solution:

The polynomial x^2-5x+6 in the denominator can be factorized and its factors are $x-3$ and $x-2$.

$$\frac{x^2-10x+13}{(x-1)(x^2-5x+6)} = \frac{x^2-10x+13}{(x-1)(x^2-3x-2x+6)}$$

$$= \frac{x^2-10x+13}{(x-1)(x(x-3)-2(x-3))}$$

$$= \frac{x^2-10x+13}{(x-1)(x-2)(x-3)}$$

Suppose $$\frac{x^2-10x+13}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3} \qquad \ldots(1)$$

Multiplying both sides by $(x-1)(x-2)(x-3)$, we get

$$\Rightarrow \quad x^2-10x+13 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \qquad \ldots(2)$$

Put $x-1=0 \Rightarrow x=1$ in eq. (2), we have

$$(1)^2-10(1)+13 = A(1-2)(1-3)$$
$$1-10+13 = A(-1)(-2)$$
$$4 = 2A$$
$$\boxed{A=2}$$

Put $x-2=0 \Rightarrow x=2$ in eq. (2), we have

$$(2)^2-10(2)+13 = B(2-1)(2-3)$$
$$4-20+13 = B(1)(-1)$$
$$-3 = -B$$
$$\boxed{B=3}$$

Put $x-3=0 \Rightarrow x=3$ in eq. (2), we have

$$(3)^2-10(3)+13 = C(3-1)(3-2) \Rightarrow 9-30+13 = C(2)(1) \Rightarrow -8 = 2C \Rightarrow \boxed{C=-4}$$

Putting values of A, B and C in eq.(1), we have

Hence, $$\frac{x^2-10x+13}{(x-1)(x^2-5x+6)}=\frac{2}{x-1}+\frac{3}{x-2}-\frac{4}{x-3}$$

Example 3: Resolve $\frac{2x^3+x^2-x-3}{x(2x+3)(x-1)}$ into Partial Fractions.

Solution:

$$\frac{2x^3+x^2-x-3}{x(2x+3)(x-1)}=\frac{2x^3+x^2-x-3}{x(2x^2+x-3)}$$

$$=\frac{2x^3+x^2-x-3}{2x^3+x^2-3x} \text{ (Improper Fraction)}$$

$$=Q+\frac{R}{D}$$

$$\begin{array}{r|l} & \quad 1 \\ 2x^3+x^2-3x & \overline{)\,2x^3+x^2-x-} \\ & \underline{\pm 2x^3 \pm x^2 \mp 3x} \\ & \quad 2x- \end{array}$$

$$\therefore \quad \frac{2x^3+x^2-x-3}{x(2x+3)(x-1)}=1+\frac{2x-3}{x(2x+3)(x-1)}$$

Suppose $\quad \frac{2x-3}{x(2x+3)(x-1)}=\frac{A}{x}+\frac{B}{2x+3}+\frac{C}{x-1}$

Multiplying both sides by $(x-1)(x-2)(x-3)$, we get

$$\Rightarrow \quad 2x-3=A(2x+3)(x-1)+B(x)(x-1)+C(x)(2x+3)$$

Put $x=0$ in eq. (2), we have

$$2(0)-3=A(2(0)+3)(0-1)$$
$$0-3=A(3)(-1)$$
$$-3=-3A$$
$$\boxed{A=1}$$

Put $2x+3=0 \Rightarrow x=-\frac{3}{2}$ in eq. (2), we have

$$2\left(-\frac{3}{2}\right)-3=B\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)$$

$$-6=B\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) \Rightarrow -6=B\left(\frac{15}{4}\right) \Rightarrow -6\times\frac{4}{15}=B \Rightarrow \boxed{B=-}$$

Put $x-1=0 \Rightarrow x=1$ in eq. (2), we have

$$2(1)-3=C(1)(2(1)+3) \Rightarrow -1=C(5) \Rightarrow \boxed{C=-\frac{1}{5}}$$

Putting values of A, B and C in eq.(1), we have

$$\frac{2x-3}{x(2x+3)(x-1)}=\frac{1}{x}-\frac{8}{5(2x+3)}-\frac{1}{5(x-1)}$$

From eq. (a), we have

Hence, $\frac{2x^3+x^2-x-3}{x(2x+3)(x-1)}=1+\frac{1}{x}-\frac{8}{5(2x+3)}-\frac{1}{5(x-1)}$

Case II: When $Q(x)$ has repeated linear factors:

If the polynomial $Q(x)$ has a repeated linear factors $(x-a)^n$, $n \geq 2$ and n is a positive integer, then $\frac{P(x)}{Q(x)}$ ma written as the following identity:

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_n}{(x-a)^n}$$

where $A_1, A_2, \ldots, A_n$ are numbers to be found.

The method is explained by the following examples:

Example 4: Resolve $\frac{x^2+x-1}{(x+2)^3}$ into partial fractions.

Solution:

Suppose $\quad \frac{x^2+x-1}{(x+2)^3}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3} \quad \ldots (1)$

Multiplying both sides by $(x+2)^3$, we get

$$x^2+x-1=A(x+2)^2+B(x+2)+C \quad \ldots (2)$$

Put $x+2=0 \Rightarrow x=-2$ in eq. (2), we have

$$(-2)^2+(-2)-1=A(0)+B(0)+C$$

$$\Rightarrow \quad \boxed{1=C}$$

Re-arrange eq (2)

$$x^2+x-1=A(x^2+4x+4)+B(x+2)+C$$
$$x^2+x-1=Ax^2+4Ax+4A+Bx+2B+C$$
$$x^2+x-1=Ax^2+(4A+B)x+(4A+2B+C)$$

Equating the coefficients of x^2 and x, we get $\boxed{A=1}$

and $\quad 1=4A+B$

$\Rightarrow \quad 1=4+B \Rightarrow \boxed{B=-3}$

Putting values of A, B and C in eq.(1), we have

Hence, $\frac{x^2+x-1}{(x+2)^3}=\frac{1}{x+2}-\frac{3}{(x+2)^2}+\frac{1}{(x+2)^3}$

Example 5: Resolve $\frac{1}{(x+1)^2(x^2-1)}$ into partial fractions.

Solution:

$$\because \quad \frac{1}{(x+1)^2(x^2-1)}=\frac{1}{(x+1)^2(x-1)(x+1)}=\frac{1}{(x-1)(x+1)^3}$$

Suppose $\quad \frac{1}{(x-1)(x+1)^3}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3} \quad \ldots (1)$

Multiplying both sides by $(x-1)(x+1)^3$, we get

$$1=A(x+1)^3+B(x+1)^2(x-1)+C(x-1)(x+1)+D(x-1) \quad \ldots (2)$$

Put $x-1=0 \Rightarrow x=1$ in eq. (2), we have

$$1=A(1+1)^3$$
$$1=A(8) \Rightarrow \boxed{A=\frac{1}{8}}$$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2), we have

$$1=D(-1-1)$$
$$1=D(-2) \Rightarrow \boxed{D=-\frac{1}{2}}$$

Re-arrange eq.(2)

$$1=A(x^3+3x^2+3x+1)+B(x^3+x^2-x-1)+C(x^2-1)+D(x-1)$$
$$1=(A+B)x^3+(3A+B+C)x^2+(3A-B+D)x+(A-B-C-D)$$

Equating the coefficients of x^3 and x^2, we get

$$0=A+B \Rightarrow B=-A \Rightarrow \boxed{B=-\frac{1}{8}}$$

and $\quad 0=3A+B+C \Rightarrow 0=\frac{3}{8}-\frac{1}{8}+C \Rightarrow 0=\frac{2}{8}+C \Rightarrow \boxed{C=-\frac{1}{4}}$

Putting values of A, B, C and D in eq (1), we have

$$\frac{1}{(x-1)(x+1)^3}=\frac{\frac{1}{8}}{x-1}+\frac{-\frac{1}{8}}{x+1}+\frac{-\frac{1}{4}}{(x+1)^2}+\frac{-\frac{1}{2}}{(x+1)^3}$$

Hence, $\dfrac{1}{(x+1)^2(x^2-1)}=\dfrac{1}{8(x-1)}-\dfrac{1}{8(x+1)}-\dfrac{1}{4(x+1)^2}-\dfrac{1}{2(x+1)^3}$

Exercise 5.1

Resolve the following into partial fractions:

1. $\dfrac{2}{x^2-1}$

Solution:

$$\frac{2}{x^2-1}=\frac{2}{(x-1)(x+1)}$$

Let $\dfrac{2}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$ ...(1)

Multiply each term by LCM $(x-1)(x+1)$

$2=A(x+1)+B(x-1)$ ...(2)

Put $x-1=0\Rightarrow x=1$ in eq (2)

$2=A(1+1)\Rightarrow 2=2A \quad \Rightarrow \boxed{A=1}$

Put $x+1=0\Rightarrow x=-1$ in eq (2)

$Z=B(-1-1)\Rightarrow Z=-2B \Rightarrow \boxed{B=-1}$

Putting values of A and B in eq. (1), we have

$$\frac{2}{(x-1)(x+1)}=\frac{1}{x-1}-\frac{1}{x+1}$$

Hence, $\dfrac{2}{x^2-1}=\dfrac{1}{x-1}-\dfrac{1}{x+1}$

These are required partial fractions.

2. $\dfrac{a-b}{(x-a)(x-b)}$

Solution:

Let $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{A}{x-a}+\dfrac{B}{x-b}$ ...(1)

Multiply each term by LCM $(x-a)(x-b)$

$a-b=A(x-b)+B(x-a)$ ...(2)

Put $x-a=0\Rightarrow x=a$ in eq. (2)

$a-b=A(a-b)\Rightarrow A=\dfrac{a-b}{a-b} \quad \Rightarrow \boxed{A=1}$

Put $x-b=0\Rightarrow x=b$ in eq (2)

$a-b=B(b-a)$

$a-b=-B(a-b)$

$B=\dfrac{a-b}{-(a-b)} \quad \Rightarrow \boxed{B=-1}$

Putting the values of A and B in eq (1), we have

Hence, $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{1}{x-a}-\dfrac{1}{x-b}$

These are required partial fractions.

3. $\dfrac{x^2+1}{(x+1)(x-1)}$

Solution:

$$\frac{x^2+1}{(x+1)(x-1)}=\frac{x^2+1}{x^2-1} \text{ (Improper Fraction)}$$

$$=Q+\frac{R}{D}$$

$$=1+\frac{2}{x^2-1}$$

$$\begin{array}{r} 1 \\ x^2-1\overline{)\;x^2+1} \\ \underline{\pm x^2\mp 1} \\ 2 \end{array}$$

$\dfrac{x^2+1}{(x+1)(x-1)}=1+\dfrac{2}{(x+1)(x-1)}$ ...(a)

Let $\dfrac{2}{(x+1)(x-1)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}$ ...(i)

Multiplying each term by L.C.M. $(x+1)(x-1)$

$2=A(x-1)+B(x+1)$ ...(ii)

Put $x+1=0 \quad \Rightarrow x=-1$ in eq (ii)

$2=A(-1-1)$

$2=-2A$

$\Rightarrow \boxed{A=-1}$

Put $x-1=0 \quad \Rightarrow x=1$ in eq (ii)

$2=B(1+1)$

$2=2B$

$\Rightarrow \boxed{B=1}$

Putting values of A and B in eq (i), we have

$$\frac{2}{(x+1)(x-1)}=\frac{-1}{x+1}+\frac{1}{x-1}$$

From eq. (a), we have

$$\frac{x^2+1}{(x+1)(x-1)}=1-\frac{1}{x+1}+\frac{1}{x-1}$$

These are the required partial fractions.

4. $\dfrac{2x+3}{(x+1)(x+2)(x+3)}$

Solution:

Let $\dfrac{2x+3}{(x+1)(x+2)(x+3)}=\dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{x+3}$ ...(1)

Multiply each term by LCM $(x+1)(x+2)(x+3)$

$2x+3=A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$ ...(2)

Put $x+1=0\Rightarrow x=-1$ in eq (1)

$2(-1)+3=A(-1+2)(-1+3)$

$-2+3=A(1)(2)\Rightarrow 1=2A \quad \Rightarrow \boxed{A=\frac{1}{2}}$

Put $x+2=0\Rightarrow x=-2$ in eq. (1)

$2(-2)+3=B(-2+1)(-2+3)$

$-4+3=B(-1)(1)\Rightarrow -1=-B \quad \Rightarrow \boxed{B=1}$

Put $x+3=0\Rightarrow x=-3$ in eq (1)

$2(-3)+3=C(-3+1)(-3+2)$

$-6+3=C(-2)(-1)\Rightarrow -3=2C \quad \Rightarrow \boxed{C=\frac{-3}{2}}$

Putting values of A, B and C in eq (1), we have

$$\frac{2x+3}{(x+1)(x+2)(x+3)}=\frac{\frac{1}{2}}{x+1}+\frac{1}{x+2}+\frac{\frac{-3}{2}}{x+3}$$

Hence, $\dfrac{2x+3}{(x+1)(x+2)(x+3)}=\dfrac{1}{2(x+1)}+\dfrac{1}{x+2}-\dfrac{3}{2(x+3)}$

These are required partial fractions.

5. $\dfrac{x^2+4x+5}{(x+1)(x^2+5x+6)}$

Solution:

$$\frac{x^2+4x+5}{(x+1)(x^2+5x+6)}=\frac{x^2+4x+5}{(x+1)(x^2+3x+2x+6)}$$

$$=\frac{x^2+4x+5}{(x+1)(x(x+3)+2(x+3))}$$

$$=\frac{x^2+4x+5}{(x+1)(x+2)(x+3)}$$

Let $\dfrac{x^2+4x+5}{(x+1)(x+2)(x+3)}=\dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{x+3}$ ...(1)

Multiply each term by LCM $(x+1)(x+2)(x+3)$

$x^2+4x+5=A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$ ...(2)

Put $x+1=0\Rightarrow x=-1$ in eq. (2)

$(-1)^2+4(-1)+5=A(-1+2)(-1+3)$

$1-4+5=A(1)(2)$

$2=2A \quad \Rightarrow \boxed{A=1}$

Put $x+2=0\Rightarrow x=-2$ in eq. (2)

$(-2)^2+4(-2)+5=B(-2+1)(-2+3)$

$4-8+5=B(-1)(1)$

$1=-B \quad \Rightarrow \boxed{B=-1}$

Put $x+3=0\Rightarrow x=-3$ in eq (2)

$(-3)^2+4(-3)+5=C(-3+1)(-3+2)$

$9-12+5=C(-2)(-1)$

$2=2C \quad \Rightarrow \boxed{C=1}$

Putting values of A, B and C in eq (1)

$$\frac{x^2+4x+5}{(x+1)(x+2)(x+3)}=\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}$$

Hence, $\dfrac{x^2+4x+5}{(x+1)(x^2+5x+6)}=\dfrac{1}{x+1}-\dfrac{1}{x+2}+\dfrac{1}{x+3}$

These are required partial fractions.

6. $\dfrac{4x^3+5x^2-3x-2}{x^2-1}$

Solution:

$$\frac{4x^3+5x^2-3x-2}{x^2-1} \text{ (Improper fraction)}$$

$$\frac{4x^3+5x^2-3x-2}{x^2-1}=Q\dotplus\frac{R}{D}$$

$$\begin{array}{r} 4x+5 \\ x^2-1\overline{)\;4x^3+5x^2-3x-2} \\ \underline{\pm 4x^3 \qquad \mp 4x} \\ 5x^2+x-2 \\ \underline{\pm 5x^2 \quad \mp 5} \\ x+3 \end{array}$$

$$=(4x+5)+\frac{x+3}{x^2-1}$$

$\dfrac{4x^3+5x^2-3x-2}{x^2-1}=(4x+5)+\dfrac{x+3}{(x-1)(x+1)}$ ...(a)

Let $\dfrac{x+3}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$ ...(1)

Multiply each term by LCM $(x-1)(x+1)$

$x+3=A(x+1)+B(x-1)$ ...(2)

Put $x-1=0\Rightarrow x=1$ in eq (2)

$1+3=A(1+1)\Rightarrow 4=2A \quad \Rightarrow \boxed{A=2}$

Put $x+1=0\Rightarrow x=-1$ in eq (2)

$-1+3=B(-1-1)\Rightarrow 2=-2B \Rightarrow \boxed{B=-1}$

Putting values of A and B in eq. (1), we have

$$\frac{x+3}{(x-1)(x+1)}=\frac{2}{x-1}-\frac{1}{x+1}$$

From eq. (a), we have

Hence, $\dfrac{4x^3+5x^2-3x-2}{x^2-1}=(4x+5)+\dfrac{2}{x-1}-\dfrac{1}{x+1}$

These are required partial fractions.

7. $\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$

Solution:

Let $\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$ ...(1)

Multiply each Term by LCM $(x-1)(x-2)(x-3)$

$$3x^2-12x+11=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \quad \ldots(2)$$

Put $x-1=0 \Rightarrow x=1$ in eq (2)

$$3(1)^2-12(1)+11=A(1-2)(1-3)$$
$$3-12+11=A(-1)(-2) \Rightarrow 2=2A \Rightarrow \boxed{A=1}$$

Put $x-2=0 \Rightarrow x=2$ in eq (2)

$$3(2)^2-12(2)+11=B(2-1)(2-3)$$
$$12-24+11=B(1)(-1) \Rightarrow -1=-B \Rightarrow \boxed{B=1}$$

Put $x-3=0 \Rightarrow x=3$ in eq (2)

$$3(3)^2-12(3)+11=C(3-1)(3-2)$$
$$27-36+11=C(2)(1) \Rightarrow 2=2C \quad \Rightarrow \boxed{C=1}$$

Putting values of A, B and C in $eq(1)$, we have

Hence, $$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$$

These are required partial fractions.

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8. $$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$$

Solution:

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} \text{ (Improper fraction)}$$

$$\frac{(x-1)(x^2-5x+6)}{(x-4)(x^2-11x+30)}=\frac{x^3-5x^2+6x-x^2+5x-6}{x^3-11x^2+30x-4x^2+44x-120}$$

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}=\frac{x^3-6x^2+11x-6}{x^3-15x^2+74x-120}$$
$$=Q+\frac{R}{D}$$

$$\begin{array}{r|l} & 1 \\ \hline x^3-15x^2+74x-120 & x^3-6x^2+11x-6 \\ & \pm x^3 \mp 15x^2 \pm 74x \mp 120 \\ \hline & 9x^2-63x+114 \end{array}$$

$$=1+\frac{9x^2-63x+114}{x^3-15x^2+74x-120}$$

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}=1+\frac{9x^2-63x+114}{(x-4)(x-5)(x-6)} \quad \ldots(a)$$

Let $$\frac{9x^2-63x+114}{(x-4)(x-5)(x-6)}=\frac{A}{x-4}+\frac{B}{x-5}+\frac{C}{x-6} \quad \ldots(1)$$

Multiply each term by LCM $(x-4)(x-5)(x-6)$

$$9x^2-63x+114=A(x-5)(x-6)+B(x-4)(x-6)+C(x-4)(x-5) \quad \ldots(2)$$

Put $x-4=0 \Rightarrow x=4$ in eq (2)

$$9(4)^2-63(4)+114=A(4-5)(4-6)$$
$$144-252+114=A(-1)(-2)$$
$$6=2A \quad \Rightarrow \boxed{A=3}$$

Put $x-5=0 \Rightarrow x=5$ in eq (2)

$$9(5)^2-63(5)+114=B(5-4)(5-6)$$
$$225-315+114=B(1)(-1)$$
$$24=-B \quad \Rightarrow \boxed{B=-24}$$

Put $x-6=0 \Rightarrow x=6$ in eq (2)

$$9(6)^2-63(6)+114=C(6-4)(6-5)$$
$$324-378+114=C(2)(1)$$
$$60=2C \quad \Rightarrow \boxed{C=30}$$

Putting values of A, B and C in eq. (1), we have

$$\frac{9x^2-63x+114}{(x-4)(x-5)(x-6)}=\frac{3}{x-4}-\frac{24}{x-5}+\frac{30}{x-6}$$

From eq. (a), we have

Hence, $$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}=1+\frac{3}{x-4}-\frac{24}{x-5}+\frac{30}{x-6}$$

These are required partial fractions.

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9. $$\frac{x^2}{(x^2+a)(x^2+b)(x^2+c)}$$

Solution:

$$\frac{x^2}{(x^2+a)(x^2+b)(x^2+c)} \text{ Put } x^2=y$$

Let $$\frac{y}{(y+a)(y+b)(y+c)}=\frac{A}{y+a}+\frac{B}{y+b}+\frac{C}{y+c}$$

Multiply each term by LCM $(y+a)(y+b)(y+c)$

$$y=A(y+b)(y+c)+B(y+a)(y+c)+C(y+a)(y+b) \ldots(2)$$

Put $y+a=0 \Rightarrow y=-a$ in eq (2)

$$-a=A(-a+b)(-a+c)$$
$$-a=-A(a-b)(c-a) \quad \Rightarrow \boxed{A=\frac{a}{(a-b)(c-a)}}$$

Put $y+b=0 \Rightarrow y=-b$ in eq (2)

$$-b=B(-b+a)(-b+c)$$
$$-b=-B(a-b)(b-c) \quad \Rightarrow \boxed{B=\frac{b}{(a-b)(b-c)}}$$

Put $y+c=0 \Rightarrow y=-c$ in eq (2)

$$-c=C(-c+a)(-c+b)$$
$$-c=-C(c-a)(b-c) \quad \Rightarrow \boxed{C=\frac{c}{(c-a)(b-c)}}$$

Putting values of A, B and C in eq (1)

$$\frac{y}{(y+a)(y+b)(y+c)}=\frac{\frac{a}{(a-b)(c-a)}}{y+a}+\frac{\frac{b}{(a-b)(b-c)}}{y+b}+\frac{\frac{c}{(c-a)(b-c)}}{y+c}$$

Put $y=x^2$

Hence, $$\frac{x^2}{(x^2+a)(x^2+b)(x^2+c)}=\frac{a}{(a-b)(c-a)(x^2+a)}+\frac{b}{(a-b)(b-c)(x^2+b)}+\frac{c}{(c-a)(b-c)(x^2+c)}$$

These are required partial fractions.

---

10. $$\frac{x+1}{(x-1)^2}$$

Solution:

Let $$\frac{x+1}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2} \quad \ldots(1)$$

Multiply each term by LCM $(x-1)^2$

$$x+1=A(x-1)+B \quad \ldots(2)$$

Put $x-1=0 \Rightarrow x=1$ in eq (2)

$$1+1=B$$
$$\Rightarrow \boxed{B=2}$$

From eq. (2), we have

$$1x+1=Ax-A+B$$

Equating the coefficients of x, we get

$$\boxed{A=1}$$

Putting the values of A and B in eq (1), we have

Hence, $$\frac{x+1}{(x-1)^2}=\frac{1}{x-1}+\frac{2}{(x-1)^2}$$

These are required partial fractions.

---

11. $$\frac{x^2+x}{(x^2-1)^2}$$

Solution:

$$\frac{x^2+x}{(x^2-1)^2}=\frac{x(x+1)}{(x+1)^2(x-1)^2}=\frac{x}{(x+1)(x-1)^2}$$

Let $$\frac{x}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2} \quad \ldots(1)$$

Multiply each Term by LCM $(x+1)(x-1)^2$

$$x=A(x-1)^2+B(x+1)(x-1)+C(x+1) \ldots(2)$$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2)

$$-1=A(-1-1)^2 \Rightarrow -1=4A \quad \Rightarrow \boxed{A=-\frac{1}{4}}$$

Put $x-1=0 \Rightarrow x=1$ in eq (2)

$$1=C(1+1) \Rightarrow 1=2C \quad \Rightarrow \boxed{C=\frac{1}{2}}$$

From eq. (2), we have

$$x=A(x^2+1-2x)+B(x^2-1)+Cx+C$$
$$x=Ax^2+A-2Ax+Bx^2-B+Cx+C$$
$$x=(A+B)x^2+(-2A+C)x+(A-B+C)$$

Equating the coefficients of x^2, we get

$$A+B=0 \Rightarrow -\frac{1}{4}+B=0 \quad \Rightarrow \boxed{B=\frac{1}{4}}$$

Putting the values of A, B and C in eq (1), we have

$$\frac{x}{(x+1)(x-1)^2}=\frac{-\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1}+\frac{\frac{1}{2}}{(x-1)^2}$$

$$\frac{x^2+x}{(x^2-1)^2}=-\frac{1}{4(x+1)}+\frac{1}{4(x-1)}+\frac{1}{2(x-1)^2}$$

These are required partial fractions.

---

12. $$\frac{3x^2+4x-5}{(x-1)^3}$$

Solution:

Let $$\frac{3x^2+4x-5}{(x-1)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3} \quad \ldots(1)$$

Multiply each term by LCM $(x-1)^3$

$$3x^2+4x-5=A(x-1)^2+B(x-1)+C \quad \ldots(2)$$

Put $x-1=0 \Rightarrow x=1$ in eq (2)

$$3(1)^2+4(1)-5=C$$
$$\Rightarrow \boxed{C=2}$$

From eq (2), we have

$$3x^2+4x-5=A(x^2+1-2x)+Bx-B+C$$
$$3x^2+4x-5=Ax^2+A-2Ax+Bx-B+C$$
$$3x^2+4x-5=Ax^2+(-2A+B)x+(A-B+C)$$

Equating coefficients of x^2 and x, we get

$$\boxed{A=3}$$

and $-2A+B=4 \Rightarrow -2(3)+B=4$

$$B=4+6$$
$$\boxed{B=10}$$

Putting the values of A, B and C in eq (1), we have

Hence, $$\frac{3x^2+4x-5}{(x-1)^3}=\frac{3}{x-1}+\frac{10}{(x-1)^2}+\frac{2}{(x-1)^3}$$

These are required partial fractions.

---

13. $$\frac{1}{x(x+1)^3}$$

Solution:

c

Let $$\frac{1}{x(x+1)^3}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3} \quad \ldots(1)$$

Multiply each term by LCM $x(x+1)^3$

$$1=A(x+1)^3+B(x)(x+1)^2+C(x)(x+1)+D(x) \quad \ldots(2)$$

Put $x=0$ in eq. (2)

$$1=A(0+1)^3$$
$$1=A(1)^3 \quad \Rightarrow \boxed{A=1}$$

Put $x+1=0 \Rightarrow x=-1$ in eq (2)

$$1=D(-1) \quad \Rightarrow \boxed{D=-1}$$

From eq (2), we have

$$1 = A(x^3+1^3+3x^2+3x)+B(x)(x^2+1+2x)+C(x^2+x)+Dx$$

$$1 = Ax^3+A+3Ax^2+3Ax+Bx^3+Bx+2Bx^2+Cx^2+Cx+Dx$$

$$1 = (A+B)x^3+(3A+2B+C)x^2+(3A+B+C+D)x+A$$

Equating the coefficients of x^3 and x^2, we get

$$A+B=0$$
$$1+B=0$$
$$\boxed{B=-1}$$

$$3A+2B+C=0$$
$$3(1)+2(-1)+C=0$$
$$3-2+C=0$$
$$\boxed{C=-1}$$

Putting values of A, B, C and D in eq (1), we have

Hence, $\frac{1}{x(x+1)^3} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}$

These are required partial fractions.

14. $\mathbf{\frac{4x^2-3x+1}{(x+1)(x-1)^2}}$

Solution:

Let $\frac{4x^2-3x+1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ ...(1)

Multiply each term by LCM$(x+1)(x-1)^2$

$$4x^2-3x+1 = A(x-1)^2+B(x+1)(x-1)+C(x+1) \quad ...(2)$$

Put $x+1=0 \Rightarrow x=-1$ in eq. (2)

$$4(-1)^2-3(-1)+1 = A(-1-1)^2$$

$$4+3+1=4A \Rightarrow A=\frac{8}{4} \Rightarrow \boxed{A=2}$$

Put $x-1=0 \Rightarrow x=1$ in eq (2)

$$4(1)^2-3(1)+1 = C(1+1)$$

$$4-3+1=2C \Rightarrow C=\frac{2}{2} \Rightarrow \boxed{C=1}$$

From eq (2), we have

$$4x^2-3x+1 = A(x^2+1-2x)+B(x^2-1)+Cx+C$$

$$4x^2-3x+1 = Ax^2+A-2Ax+Bx^2-B+Cx+C$$

$$4x^2-3x+1 = (A+B)x^2+(-2A+C)x+(A-B+C)$$

Equating the coefficients of x^2, we get

$$A+B=4 \Rightarrow 2+B=4 \Rightarrow \boxed{B=2}$$

Putting the values of A, B and C in eq (1), we have

Hence, $\frac{4x^2-3x+1}{(x+1)(x-1)^2} = \frac{2}{x+1} + \frac{2}{x-1} + \frac{1}{(x-1)^2}$

These are required partial fractions.

15. $\mathbf{\frac{12x^2-48}{(x-2)^2(x+2)^2}}$

Solution:

$$\frac{12x^2-48}{(x-2)^2(x+2)^2} = \frac{12(x^2-4)}{(x-2)^2(x+2)^2} = \frac{12(x+2)(x-2)}{(x-2)^2(x+2)^2}$$

$$= \frac{12}{(x-2)(x+2)}$$

Let $\frac{12}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$ ...(1)

Multiply each term by LCM $(x-2)(x+2)$

$$12 = A(x+2)+B(x-2)$$

Put $x-2=0 \Rightarrow x=2$ in eq. (2)

$$12 = A(2+2) \Rightarrow A=\frac{12}{4} \Rightarrow \boxed{A=3}$$

Put $x+2=0 \Rightarrow x=-2$ in eq (2)

$$12 = B(-2-2) \Rightarrow B=\frac{12}{-4} \Rightarrow \boxed{B=-3}$$

Putting values of A and B in eq (1), we have

$$\frac{12}{(x-2)(x+2)} = \frac{3}{x-2} - \frac{3}{x+2}$$

Hence, $\frac{12x^2-48}{(x-2)^2(x+2)^2} = \frac{3}{x-2} - \frac{3}{x+2}$

These are required partial fractions.

Definition: A quadratic factor is irreducible if it cannot be written as the product of two linear factors with real coefficients. For example, x^2+x+1 and x^2+3 are irreducible quadratic factors.

Case III: When $Q(x)$ contains non-repeated irreducible quadratic factors

If the polynomial $Q(x)$ contains non-repeated irreducible quadratic factors then $\frac{P(x)}{Q(x)}$ may be written as the identity having partial fractions of the form:

$$\frac{Ax+B}{ax^2+bx+c} \text{ where } A \text{ and } B \text{ are the numbers to be found}$$

The method is explained by the following examples:

Example 6: Resolve $\frac{3x-11}{(x^2+1)(x+3)}$ into partial fractions.

Solution:

Suppose $\frac{3x-11}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3}$ ...(1)

Multiplying both sides by $(x^2+1)(x+3)$, we get

$$3x-11 = (Ax+B)(x+3)+C(x^2+1) \quad ...(2)$$

$$3x-11 = (A+C)x^2+(3A+B)x+(3B+C)$$

Put $x+3=0 \Rightarrow x=-3$ in eq. (2), we get

$$-9-11 = C(9+1) \quad \Rightarrow \quad \boxed{C=-2}$$

Re-arrange eq.(2)

$$3x-11 = Ax^2+3Ax+Bx+3B+Cx^2+C$$

$$3x-11 = (A+C)x^2+(3A+B)x+(3B+C)$$

Equating the coefficients of x^2 and x, we get

$$0=A+C \quad \Rightarrow A=-C \quad \Rightarrow \boxed{A=2}$$

and $\quad 3=3A+B \quad \Rightarrow B=3-3A \quad \Rightarrow B=3-6 \Rightarrow \boxed{B=-3}$

Putting values of A, B and C in eq.(1), we have

Hence, $\frac{3x-11}{(x^2+1)(x+3)} = \frac{2x-3}{x^2+1} - \frac{2}{x+3}$

Example 7: Resolve $\frac{4x^2+8x}{x^4+2x^2+9}$ into partial fractions.

Solution:

$$\frac{4x^2+8x}{x^4+2x^2+9} = \frac{4x^2+8x}{(x^2)^2+3^2+2(3)(x^2)-2(3)(x^2)+2x^2}$$

$$= \frac{4x^2+8x}{(x^2+3)^2-6x^2+2x^2}$$

$$= \frac{4x^2+8x}{(x^2+3)^2-(2x)^2}$$

$$= \frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)}$$

Suppose $\frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{x^2-2x+3}$ ...(1)

Multiplying both sides by $(x^2+2x+3)(x^2-2x+3)$, we get

$$4x^2+8x = (Ax+B)(x^2-2x+3)+(Cx+D)(x^2+2x+3)$$

$$4x^2+8x = Ax^3-2Ax^2+3Ax+Bx^2-2Bx+3B+Cx^3+2Cx^2+3Cx+Dx^2+2Dx+3D$$

$$4x^2+8x = (A+C)x^3+(-2A+B+2C+D)x^2+(3A-2B+3C+2D)x+3B+3D$$

Equating the coefficients of x^3, x^2, x and x^0, we have

$$0=A+C$$
$$A=-C \quad ...(2)$$
$$\boxed{A=-1}$$

$$4=-2A+B+2C+D$$
Put $A=-C$ and $B=-D$
$$4=2C-D+2C+D$$
$$4=4C \quad \Rightarrow \boxed{C=1} \text{ put in eq.(2)}$$

$$8=3A-2B+3C+2D$$
Put $A=-C$ and $B=-D$
$$8=-3C+\text{[illegible]}+3C+2D$$
$$8=4D \quad \Rightarrow \boxed{D=2} \text{ put in eq.(3)}$$

$$0=3B+3D$$
$$B=-D \quad ...(3)$$
$$\boxed{B=-2}$$

Putting values of A, B, C and D in eq.(1), we have

Hence, $\frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)} = \frac{-x-2}{x^2+2x+3} + \frac{x+\text{[illegible]}}{x^2-2x+3}$

Case IV: When $Q(x)$ has repeated irreducible quadratic factors

If the polynomial $Q(x)$ contains a repeated irreducible quadratic factors $(ax^2 + bx + c)^n$, $n \geq 2$ and n is a positive integer, then $\frac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where $A_1, B_1, A_2, B_2, \dots, A_n, B_n$ are numbers to be found. The method is explained through the following example

Example 8: Resolve $\frac{4x^2}{(x^2+1)^2(x-1)}$ into partial fractions.

Solution:

Suppose $$\frac{4x^2}{(x^2+1)^2(x-1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1} \quad \dots(1)$$

Multiplying both sides by $(x^2+1)^2(x-1)$, we get

$$4x^2 = (Ax + B)(x^2 + 1)(x - 1) + (Cx + D)(x - 1) + E(x^2 + 1)^2 \quad \dots(2)$$

Put $x - 1 = 0 \Rightarrow x = 1$ in (2), we get

$$4(1)^2 = E(1^2 + 1)^2 \Rightarrow 4 = 4E \quad \Rightarrow \boxed{E = 1}$$

Re-arrange eq.(2)

$$4x^2 = (Ax + B)(x^3 - x^2 + x - 1) + Cx^2 - Cx + Dx - D + E(x^4 + 1 + 2x^2)$$
$$4x^2 = Ax^4 - Ax^3 + Ax^2 - Ax + Bx^3 - Bx^2 + Bx - B + Cx^2 - Cx + Dx - D + Ex^4 + E + 2Ex^2$$
$$4x^2 = (A + E)x^4 + (-A + B)x^3 + (A - B + C + 2E)x^2 + (-A + B - C + D)x + (-B - D + E)$$

Equating the coefficients of x^4, x^3, x^2 and x, we get

$$0 = A + E \Rightarrow A = -E \Rightarrow \boxed{A = -1}$$
$$0 = -A + B \Rightarrow B = A \Rightarrow \boxed{B = -1}$$
$$4 = A - B + C + 2E \Rightarrow C = 4 - A + B - 2E = 4 + 1 - 1 - 2 \Rightarrow \boxed{C = 2}$$

and $$0 = -A + B - C + D \Rightarrow D = A - B + C = -1 + 1 + 2 = 2 \Rightarrow \boxed{D = 2}$$

Putting values of A, B, C, D and E in eq.(1), we have

Hence, $$\frac{4x^2}{(x^2+1)^2(x-1)} = \frac{-x-1}{x^2+1} + \frac{2x+2}{(x^2+1)^2} + \frac{1}{x-1}$$

Exercise 5.2

Resolve into partial fractions:

1. $\frac{2x^2+3x+3}{(x+1)(x^2+1)}$

Solution:

Let $$\frac{2x^2+3x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \quad \dots(1)$$

Multiply each term by LCM $(x+1)(x^2+1)$

$$2x^2 + 3x + 3 = A(x^2+1) + (Bx+C)(x+1) \quad \dots(2)$$

Put $x + 1 = 0 \Rightarrow x = -1$ in eq (2)

$$2(-1)^2 + 3(-1) + 3 = A((-1)^2 + 1)$$
$$2 - 3 + 3 = 2A \Rightarrow A = \frac{2}{2} \Rightarrow \boxed{A = 1}$$

From eq (2), we have

$$2x^2 + 3x + 3 = Ax^2 + A + Bx^2 + Bx + Cx + C$$
$$2x^2 + 3x + 3 = (A + B)x^2 + (B + C)x + (A + C)$$

Equating coefficients of x^2 and x, we get

$$\begin{array}{r|r} A + B = 2 & B + C = 3 \\ 1 + B = 2 & 1 + C = 3 \\ B = 2 - 1 & C = 3 - 1 \\ \boxed{B = 1} & \boxed{C = 2} \end{array}$$

Putting the values of A, B and C in eq (1), we have

Hence, $$\frac{2x^2+3x+3}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{1x+2}{x^2+1}$$

These are required partial fractions.

2. $\frac{2x+1}{(x-2)(x^2+3x+5)}$

Solution:

Let $$\frac{2x+1}{(x-2)(x^2+3x+5)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+3x+5}$$

Multiply each term by LCM $(x-2)(x^2+3x+5)$

$$2x + 1 = A(x^2+3x+5) + (Bx+C)(x-2) \quad \dots(2)$$

Put $x - 2 = 0 \Rightarrow x = 2$ in eq. (2)

$$2(2) + 1 = A((2)^2 + 3(2) + 5)$$
$$5 = A(15) \Rightarrow A = \frac{5}{15} \quad \Rightarrow \boxed{A = \frac{1}{3}}$$

From eq. (2), we have

$$2x + 1 = Ax^2 + 3Ax + 5A + Bx^2 - 2Bx + Cx - 2C$$
$$2x + 1 = (A + B)x^2 + (3A - 2B + C)x + (5A - 2C)$$

Equating coefficients of x^2 and x, we get

$$\begin{array}{r|r} A + B = 0 & 3A - 2B + C = 2 \\ \frac{1}{3} + B = 0 & 3\left(\frac{1}{3}\right) - 2\left(-\frac{1}{3}\right) + C = 2 \\ \boxed{B = -\frac{1}{3}} & C = 2 - 1 - \frac{2}{3} = 1 - \frac{2}{3} \\ & \boxed{C = \frac{1}{3}} \end{array}$$

Putting the values of A, B and C in eq. (1), we have

$$\frac{2x+1}{(x-2)(x^2+3x+5)} = \frac{\frac{1}{3}}{x-2} + \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+3x+5}$$

Hence, $$\frac{2x+1}{(x-2)(x^2+3x+5)} = \frac{1}{3(x-2)} - \frac{x-1}{3(x^2+3x+5)}$$

These are required partial fractions.

3. $\frac{2x+32}{(x-2)(x^2+2)^2}$

Solution:

Let $$\frac{2x+32}{(x-2)(x^2+2)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \quad \dots(1)$$

Multiply each term by LCM $(x-2)(x^2+2)^2$

$$2x + 32 = A(x^2+2)^2 + (Bx+C)(x-2)(x^2+2) + (Dx+E)(x-2)$$

Put $x - 2 = 0 \Rightarrow x = 2$ in eq. (2)

$$2(2) + 32 = A(2^2 + 2)^2$$
$$36 = 36A \quad \Rightarrow \boxed{A = 1}$$

From eq. (2), we have

$$2x + 32 = A(x^4 + 4 + 4x^2) + (Bx + C)(x^3 + 2x - 2x^2 - 4) + Dx^2 - 2Dx + Ex - 2E$$
$$= Ax^4 + 4A + 4Ax^2 + Bx^4 + 2Bx^2 - 2Bx^3 - 4Bx + Cx^3 + 2Cx - 2Cx^2 - 4C + Dx^2 - 2Dx + Ex - 2E$$

$$2x + 32 = (A+B)x^4 + (-2B+C)x^3 + (4A+2B-2C+D)x^2 + (-4B+2C-2D+E)x + (4A-4C-2E)$$

Equating coefficients of x^4, x^3, x^2 and x, we get

$$A + B = 0 \Rightarrow 1 + B = 0$$
$$\Rightarrow \boxed{B = -1}$$
$$-2B + C = 0 \Rightarrow 2 + C = 0$$
$$\Rightarrow \boxed{C = -2}$$
$$4A + 2B - 2C + D = 0$$
$$4(1) + 2(-1) - 2(-2) + D = 0$$
$$\Rightarrow \boxed{D = -6}$$
$$-4B + 2C - 2D + E = 2$$
$$-4(-1) + 2(-2) - 2(-6) + E = 2$$
$$4 - 4 + 12 + E = 2 \Rightarrow E = 2 - 12$$
$$\Rightarrow \boxed{E = -10}$$

Putting values of A, B, C, D and E in eq (1), we have

$$\frac{2x+32}{(x-2)(x^2+2)^2} = \frac{1}{x-2} + \frac{-1x-2}{x^2+2} + \frac{-6x-10}{(x^2+2)^2}$$

Hence, $$\frac{2x+32}{(x-2)(x^2+2)^2} = \frac{1}{x-2} - \frac{x+2}{x^2+2} - \frac{2(3x+5)}{(x^2+2)^2}$$

These are required partial fractions.

4. $\frac{3x^2+3}{x^3+1}$

Solution:

$$\frac{3x^2+3}{x^3+1} = \frac{3x^2+3}{(x+1)(x^2-x+1)}$$

Let $$\frac{3x^2+3}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \quad \dots(1)$$

Multiply each Term by LCM $(x+1)(x^2-x+1)$

$$3x^2 + 3 = A(x^2 - x + 1) + (Bx + C)(x + 1) \quad \dots(2)$$

Put $x + 1 = 0 \Rightarrow x = -1$ in eq. (2)

$$3(-1)^2 + 3 = A((-1)^2 + 1 + 1) \Rightarrow A = \frac{6}{3} \Rightarrow \boxed{A = 2}$$

From eq. (2), we have

$$3x^2 + 3 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$
$$3x^2 + 3 = (A + B)x^2 + (-A + B + C)x + (A + C)$$

Equating coefficients of x^2 and x, we get

$$\begin{array}{r|r} A + B = 3 & -A + B + C = 0 \\ 2 + B = 3 & -2 + 1 + C = 0 \\ B = 3 - 2 & -1 + C = 0 \\ \boxed{B = 1} & \boxed{C = 1} \end{array}$$

Putting the values of A, B and C, in eq (1), we have

$$\frac{3x^2+3}{(x+1)(x^2-x+1)} = \frac{2}{x+1} + \frac{1x+1}{x^2-x+1}$$

Hence, $$\frac{3x^2+3}{x^3+1} = \frac{2}{x+1} + \frac{x+1}{x^2-x+1}$$

These are required partial fractions.

5. $\dfrac{x^4}{x^4+2x^2+1}$

Solution:

$\dfrac{x^4}{x^4+2x^2+1}$ (Improper fraction)

$$\frac{x^4}{x^4+2x^2+1} = Q + \frac{R}{D} \qquad x^4+2x^2+1\overline{\big)\,x^4\,}\;1$$

$$\begin{array}{r} \pm x^4 \pm 2x^2 \pm 1 \\ \hline -2x^2-1 \end{array}$$

$$= 1 + \frac{-2x^2-1}{x^4+2x^2+1}$$

$$= 1 - \frac{2x^2+1}{(x^2)^2+1^2+2x^2}$$

$$= 1 - \frac{2x^2+1}{(x^2+1)^2} \quad \text{(a)}$$

Let $\dfrac{2x^2+1}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$ (1)

Multiply each term by LCM $(x^2+1)^2$

$$2x^2+1 = (Ax+B)(x^2+1) + (Cx+D)$$
$$2x^2+1 = Ax^3+Ax+Bx^2+B+Cx+D$$
$$2x^2+1 = Ax^3+Bx^2+(A+C)x+D$$

Equating coefficients of x^3, x^2, x and x^0, we have

$A = 0$
$B = 2$
$A + C = 0 \Rightarrow 0 + C = 0 \Rightarrow C = 0$
$D = 1$

Putting the values of A, B, C and D in eq (1), we have

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{0x+2}{x^2+1} + \frac{0x+1}{(x^2+1)^2}$$

$$= \frac{2}{x^2+1} + \frac{1}{(x^2+1)^2}$$

From eq. (a), we have

$$\frac{x^4}{x^4+2x^2+1} = 1 - \frac{2}{x^2+1} - \frac{1}{(x^2+1)^2}$$

These are required partial fractions.

6. $\dfrac{6x^4+40x^2}{(4-x^2)(x^2+4)^2}$

Solution:

$$\frac{6x^4+40x^2}{(4-x^2)(x^2+4)^2} = \frac{6x^4+40x^2}{(2-x)(2+x)(x^2+4)^2}$$

Let $\dfrac{6x^4+40x^2}{(2-x)(2+x)(x^2+4)^2} = \dfrac{A}{2-x} + \dfrac{B}{2+x} + \dfrac{(x+1)}{x^2+4} + \dfrac{Ex+F}{(x^2+4)^2}$ ...(1)

Multiply each term by LCM $(2-x)(2+x)(x^2+4)^2$

$$6x^4+40x^2 = A(2+x)(x^2+4)^2 + B(2-x)(x^2+4)^2 + (Cx+D)(2-x)(2+x)(x^2+4) + (Ex+F)(2-x)(2+x) \quad ...(2)$$

Put $2 - x = 0 \Rightarrow x = 2$ in eq. (2)

$6(2)^4 + 40(2)^2 = A(2+2)(2^2+4)^2$

$96 + 160 = A(4)(64) \Rightarrow A = \dfrac{256}{256} \Rightarrow \boxed{A=1}$

Put $2 + x = 0 \Rightarrow x = -2$ in eq. (2)

$6(-2)^4 + 40(-2)^2 = B(2+2)((-2)^2+4)^2$

$96 + 160 = B(4)(64) \Rightarrow B = \dfrac{256}{256} \Rightarrow \boxed{B=1}$

From eq (2), we have

$$6x^4+40x^2 = A(2+x)(x^4+16+8x^2) + B(2-x)(x^4+16+8x^2) + (Cx+D)(4-x^2)(x^2+4) + (Ex+F)(4-x^2)$$

$$6x^4+40x^2 = A(2x^4+32+16x^2+x^5+16x+8x^3) + B(2x^4+32+16x^2-x^5-16x-8x^3) +$$
$$(Cx+D)(\cancel{4x^2}+16-x^4-\cancel{4x^2}) + 4Ex - Ex^3 + 4F - Fx^2$$

$$6x^4+40x^2 = 2Ax^4+32A+16Ax^2+Ax^5+16Ax+8Ax^3+2Bx^4+32B+16Bx^2-Bx^5-16Bx-8Bx^3+16Cx-Cx^5+$$
$$16D-Dx^4+4Ex-Ex^3+4F-Fx^2$$

$$6x^4+40x^2 = (A-B-C)x^5+(2A+2B-D)x^4+(8A-8B-E)x^3+(16A+16B-F)x^2+(16A-16B+16C+4E)x+$$
$$(32A+32B+16D+4F)$$

Equating coefficients of x^5, x^4, x^3 and x^2, we get

$A - B - C = 0$
$1 - 1 - C = 0$
$\boxed{C=0}$

$2A + 2B - D = 6$
$2(1) + 2(1) - D = 6$
$\boxed{D=-2}$

$8A - 8B - E = 0$
$8(1) - 8(1) - E = 0$
$\boxed{E=0}$

$16A + 16B - F = 40$
$16(1) + 16(1) - F = 40$
$\boxed{F=-8}$

Putting all the values in eq (1), we have

$$\frac{6x^4+40x^2}{(2-x)(2+x)(x^2+4)^2} = \frac{1}{2-x} + \frac{1}{2+x} + \frac{0x-2}{x^2+4} + \frac{0x-8}{(x^2+4)^2}$$

Hence, $\dfrac{6x^4+40x^2}{(4-x^2)(x^2+4)^2} = \dfrac{1}{2-x} + \dfrac{1}{2+x} - \dfrac{2}{x^2+4} - \dfrac{8}{(x^2+4)^2}$

These are required partial fractions.

Formula Sheet

Case I: Resolution of $\dfrac{P(x)}{Q(x)}$ into partial fractions when $Q(x)$ has only non-repeated linear factors:

The polynomial $Q(x)$ may be written as:

$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ where $a_1 \neq a_2 \neq \cdots \neq a_n$

$\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{x-a_1} + \dfrac{A_2}{x-a_2} + \cdots + \dfrac{A_n}{x-a_n}$ is an identity

Case II: When $Q(x)$ has repeated linear factors:

The polynomial $Q(x)$ may be written as:

$Q(x) = (x-a)^n, n \geq 2$ and n is a positive integer, then

$\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{(x-a)} + \dfrac{A_2}{(x-a)^2} + \cdots + \dfrac{A_n}{(x-a)^n}$ is an identity

Case III: When $Q(x)$ contains non-repeated irreducible quadratic factors

The polynomial $Q(x)$ may be written as:

$Q(x) = (a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)$ (Irreducible quadratic factors)

$\dfrac{P(x)}{Q(x)} = \dfrac{Ax+B}{a_1x^2+b_1x+c_1} + \dfrac{Cx+D}{a_2x^2+b_2x+c_2}$ is an identity

Case IV: When $Q(x)$ has repeated irreducible quadratic factors

The polynomial $Q(x)$ may be written as:

$Q(x) = (ax^2+bx+c)^n, n \geq 2$ and n is a positive integer, then

$\dfrac{P(x)}{Q(x)} = \dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$ is an identity

Multiple Choice Questions (MCQs)

Exercise 5.1

1. **A relation in which the equality is true for only finite values of unknowns is called a/an:**
   (A) identity (B) conditional equation (C) trigonometric equation (D) algebraic relation
2. **$x^2 + x - 6 = 0$ is - - - - -**
   (A) conditional equation (B) identity (C) proper fraction (D) improper fraction
3. **$(x+3)(x+4) = x^2 + 7x + 12$ is - - - - -**
   (A) equation (B) function (C) identity (D) conditional equation

4. From the identity $5x + 4 = A(x - 1) + B(x + 2)$, then value of $B =$ - - - - -
(A) -3 (B) 3 (C) -2 (D) 2

5. If degree of $P(x) = 3$ and degree of $Q(x) = 4$, then $\frac{P(x)}{Q(x)}$ will be - - - - -
(A) proper rational fraction (B) improper rational fraction
(C) polynomial (D) none of these

6. $\frac{x^3 + 1}{q(x)}$ will be proper fraction if Degree of $q(x) =$ - - - - -
(A) 4 (B) 3 (C) 2 (D) 1

7. An improper fraction can be reduced to mixed form by the operation - - - - -
(A) addition (B) subtraction (C) multiplication (D) division

8. Resolution of $\frac{x^2 + 6}{(x + 2)(x + 3)}$ into partial fraction is of the form - - - - -
(A) $\frac{A}{x+2} + \frac{B}{x+3}$ (B) $\frac{A}{x+2} + \frac{Bx+C}{x+3}$ (C) $\frac{Ax+B}{x+2} + \frac{C}{x+3}$ (D) $1 + \frac{A}{x+2} + \frac{B}{x+3}$

9. $\frac{9}{(x + 2)^2 (x - 1)} = \frac{A}{x-1} + \frac{B}{x+2} +$ - - - - -
(A) $\frac{C}{x+2}$ (B) $\frac{C}{(x+2)^2}$ (C) $\frac{C}{x-2}$ (D) none of these

Exercise 5.2

10. A quadratic factor is - - - - - if it cannot be written as the product of two linear factors with real coefficients.
(A) reducible (B) irreducible (C) solvable (D) none of these

11. The partial fraction of $\frac{9x - 7}{(x^2 + 1)(x + 3)}$ are of the form - - - - -
(A) $\frac{Ax+B}{x^2+1} + \frac{C}{x+3}$ (B) $\frac{A}{x^2+1} + \frac{B}{x+3}$ (C) $\frac{A}{x^2+1}$ (D) $\frac{B}{x+3}$

12. Partial fractions of $\frac{1}{x^3 + 1}$ will be of the form - - - - -
(A) $\frac{A}{x+1} + \frac{B}{x^2+x+1}$ (B) $\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ (C) $\frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$ (D) $\frac{Ax+B}{x^2+1} + \frac{C}{x^2-x+1}$

13. Partial fraction of $\frac{2x^4 - 3x^3 - 4x^2}{(x^2 + 2)^2 (x + 1)^2}$ is of the form - - - - -
(A) $\frac{A}{(x^2+2)^2} + \frac{B}{(x+1)^2}$ (B) $\frac{A}{x^2+2} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2}$
(C) $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+2)^2}$ (D) $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+d}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$

ANSWER KEY

1	B	2	A	3	C	4	B	5	A	6	A	7	D	8	D	9	B	10	B
11	A	12	B	13	D														

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Unit 6 Sequences and Series

Introduction

In this unit, students will learn to analyze and solve problems involving arithmetic, geometric, and harmonic sequences and series, including their real-world applications. Students will identify various sequence types, compute finite and infinite sums, and utilize sigma notation. Additionally, they will explore practical scenarios such as motor vehicle leasing, investment planning, and financial calculations. This unit also emphasizes applying these concepts to diverse fields, including healthcare, finance, and traffic modeling. Finally, students will be able to solve both theoretical and real-life problems using sequences and series effectively.

Let us observe the following pattern of numbers:

(i) 5, 11, 17, 23, ... (ii) 6, 12, 24, 48, ...

(iii) 4, 2, 0, -2, -4, ... (iv) $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

In example (i), every number (except 5) is formed by adding 6 to the previous numbers. Hence a specific pattern is followed in the arrangement of these numbers. Similarly, in example (ii), every number is obtained by multiplying the previous number by 2. Similar cases are followed in example (iii) and (iv).

Note:
When a set of numbers follows a pattern and there is a clear rule for finding next number in the pattern, then we have sequence as in above examples.

Sequence:

A sequence is a function whose domain is the set N of all natural numbers, whereas the range may be any subset of real numbers or complex numbers. The numbers in a sequence are called its **terms**. We denote the first term of a sequence as a_1, second term as a_2 and so on. The n^{th} term of a sequence is denoted by a_n, which may also be referred to as the general term of the sequence, and the terms immediately preceding it are called the $(n - 1)^{st}$ term, the $(n - 2)^{nd}$ terms and so on.

Finite and Infinite Sequences:

Finite Sequence: A sequence which consists of a finite number of terms is called a finite sequence. For example, 2, 5, 8, 11, 14, 17, 20, 23 is a finite sequence of 8 terms.

Infinite Sequence: A sequence which consists of an infinite number of terms is called an infinite sequence. For example: 3, 10, 17, 24, ... is an infinite sequence, or more generally as $3, 10, 17, 24, \ldots, 7n - 4, \ldots$ to show how each term was generated.

Note:
If a sequence is given, then we can find its n^{th} term and if the n^{th} term of a sequence is given then we can find the terms of the sequence.