For example number of the elements of the in matrix $\begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -1 \end{bmatrix}$ is $4 \times 3 = 12$.

General Definition of a Matrix: Generally, a bracketed rectangular array of $m \cdot n$ elements $a_{ij}(1,2,3,\ldots,m; j =$ $1,2,3,\ldots,n)$, arranged in m rows and n columns such as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad \ldots \text{(i)}$$

is called an m by n matrix (written as $m \times n$ matrix), where $m \times n$ is called the **order** of the matrix.

- The matrices are usually represented by the capital letters such as A, B, C, X, Y, etc.,
- Small letters such as a, b, c, l, m, n, or $a_{11}, a_{12}, a_{13}, \ldots$, etc., are used to indicate the entries of the matrices

Let the matrix in (i) be denoted by A. The i th row and the j th column of A are indicated in the following tabul representation of A.

$$\begin{array}{cc} & \begin{array}{cccccccc} & & & & \text{jth column} & & \\ & & & & \downarrow & & \end{array} \\ & \\ A = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \quad \ldots \text{(ii)}$$

$$\begin{array}{c} \\ \\ \\ \\ i\text{th row} \rightarrow \\ \\ \\ \end{array}$$

The elements of the ith row of A are $a_{i1}\, a_{i2}\, a_{i3} \ldots a_{ij} \ldots a_{in}$ while the elements of the jth column of A $a_{1j}\, a_{2j}\, a_{3j} \ldots a_{ij} \ldots a_{mj}$.

- We note that a_{ij} is the element of the ith row and jth column of A.
- The double subscripts are useful to name the elements of the matrices.

For example, the element 7 is at a_{23} position in the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$.

- For convenience, we shall write the matrix A as: $A = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]$, for $i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots,$ where a_{ij} is the element of the ith row and jth column of A.
- a_{ij} is also known as the (i, j)th element or entry of A.
- The elements (entries) of matrices need not always be numbers but in the study of matrices, we shall take the elements of the matrices from R or C.

Note:
The matrix A is called real m if all of its elements are real.

Row Matrix or Row vector: A matrix, which has only one row is called a row matrix. e.g., 1×3 matrix of t form $[1 \quad -1 \quad 3 \quad 4]$ is said to be a row matrix or a row vector.

Column Matrix or Column Vector: A matrix which has only one column is called a column matrix. e.g., an 3×1 matrix of the form $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is said to be a column matrix or a column vector.

Rectangular Matrix: If $m \neq n$, then the matrix is called a rectangular matrix of order $m \times n$.

Or

The matrix in which the number of rows is not equal to the number of columns, is said to be a rectangular mat

For example $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & 4 \\ 3 & -1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ are rectangular matrices of orders 2×3 and 4×3 respectively.

Square Matrix: If $m = n$, then the matrix of order $m \times n$ is said to be a square matrix of order n or m.

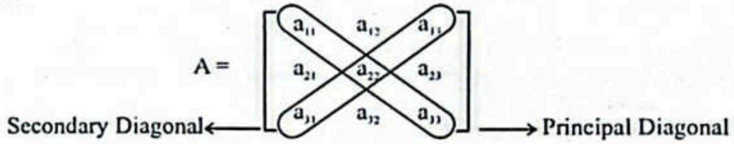
Or

The matrix which has the same number of rows and columns is called a square matrix.

For example: $[0], \begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 8 \\ 3 & 5 & 4 \end{bmatrix}$ are square matrices of orders 1,2 and 3 respectively.

Diagonals of a Matrix: Let $A = \left[a_{ij}\right]$ be a square matrix of order n, then the entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ form the principal diagonal for the matrix A and the entries $a_{1n}, a_{2\,n-1}, a_{3\,n-2}, \ldots, a_{n-1\,2}, a_{n1}$ form the secondary diagonal for the matrix A.

For example,



the entries of the principal diagonal are a_{11}, a_{22}, a_{33} and the entries of the secondary diagonal are a_{13}, a_{22}, a_{31}.

- The principal diagonal of a square matrix is also called the leading diagonal or main diagonal of the matrix.

Diagonal Matrix: A square matrix A is said to be diagonal matrix, if all of its elements are zero except the principal diagonal elements. Although some of the elements of principal diagonal may be zero but not all.

Or

Let $A = \left[a_{ij}\right]$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and at least one $a_{ij} \neq 0$ for $i = j$, then the matrix A is called a diagonal matrix.

For example, the matrices $[7], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ are diagonal matrices.

Scalar Matrix: A square matrix A is said to be scalar matrix, if all of its elements are zero except the principal diagonal elements and also all the principal diagonal elements are same non-zero scalar.

Or

Let $A = \left[a_{ij}\right]$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = k$ (some non-zero scalar) for all $i = j$, then the matrix A is called a scalar matrix of order n.

For example, the matrices $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} (a \neq 0)$ and $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ are scalar matrices of order 2, 3 and 4 respectively.

Unit Matrix or Identity Matrix: A square matrix A is said to be unit or identity matrix of order n, if all of its elements are zero except the principal diagonal elements and also all the principal diagonal elements are '1' we denote such matrix by I_n or simply I.

Or

Let $A = [a_{ij}]$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, then the matrix A is called a unit matrix or identity matrix of order n . We denote such a matrix by I_n or simply I and it is of the form:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

For example, the identity matrix of order 3 is denoted by I_3, that is, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Null Matrix or Zero Matrix: A square or rectangular matrix whose each element is zero, is called a null or zero matri

Or

An $m \times n$ matrix with all its elements equal to zero, is denoted by $O_{m \times n}$. Null matrices may be of any order. He are some examples:

$$[0],\ [0\ \ 0\ \ 0],\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are null matrices of order $1,\ 1 \times 3,\ 2 \times 3,\ 2 \times 2,\ 3 \times 1,\ 3 \times 4$ respectively.

Note:
O may be used to denote null matrix of any order if there is no confusion

Equal Matrices: Two matrices of the same order are said to be equal if their corresponding entries are equal.

Or

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal, i.e., $A = B$ iff $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \ldots, m, j = 1, 2, 3, \ldots, n$. I other words, A and B represent the same matrix.

Transpose of a Matrix: If A is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of A, is called the transpose of A. It is denoted by A^t.

Or

Let $A = [a_{ij}]_{m \times n}$, then the transpose of A is defined as: $A^t = [a'_{ij}]_{n \times m}$ where $a'_{ij} = a_{ji}$ for $i = 1, 2, 3, \ldots, m, j =$ $1, 2, 3, \ldots, n$.

For example, if $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, then $B^t = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \\ b_{14} & b_{24} & b_{34} \end{bmatrix}$

Note that the 2nd row of B has the same entries respectively as the 2nd column of B^t and the 3rd row of B^t has th same entries respectively as the 3rd column of B etc.

Matrix Operations

Matrix operations involve various techniques and procedures applied to matrices. These operations are foundatio in linear algebra and have applications in numerous fields such as computer graphics, physics, statistics, etc. He are some key matrix operations:

Addition of Matrices

Two matrices are conformable for addition if they are of the same order.

The sum $A + B$ of two $m \times n$, matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the $m \times n$ matrix $C = [c_{ij}]$ formed by addi the corresponding entries of A and B together. In symbols, we write as $C = A + B$, that is:

$$[c_{ij}] = [a_{ij} + b_{ij}] \text{ where } c_{ij} = a_{ij} + b_{ij} \text{ for } i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n$$

Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, then we define subtraction of B from A as:

$$\begin{aligned} A - B &= A + (-B) \\ &= [a_{ij}] + [-b_{ij}] = [a_{ij} + (-b_{ij})] = [a_{ij} - b_{ij}] \text{ for } i = 1, 2, 3, \ldots, m\ ;\ j = 1, 2, 3, \ldots, n \end{aligned}$$

Thus, the matrix $A - B$ is formed by subtracting each entry of B from the corresponding entry of A.

Example 1: If $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$, then show that $(A+B)^t = A^t + B^t$

Solution:

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1+2 & 0+(-1) & -1+3 & 2+1 \\ 3+1 & 1+3 & 2+(-1) & 5+4 \\ 0+3 & -2+1 & 1+2 & 6+(-1) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 & 2 & 3 \\ 4 & 4 & 1 & 9 \\ 3 & -1 & 3 & 5 \end{bmatrix} \end{aligned}$$

$$\text{and} \quad (A+B)^t = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \qquad \ldots(i)$$

Taking transpose of A and B, we have

$$A^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} \text{ and } B^t = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^t + B^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \qquad \ldots(ii)$$

From (i) and (ii), we have $(A+B)^t = A^t + B^t$

Scalar Multiplication

If $A = [a_{ij}]$ is $m \times n$ matrix and k is a real or complex number, then the product of k and A, denoted by kA, is the matrix formed by multiplying each entry of A by k, that is $kA = [ka_{ij}]$

Note:
If n is a positive integer, then $A + A + A + \ldots$ to n terms $= nA$.

Obviously, order of kA is $m \times n$.

Multiplication of two Matrices

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B.

Note 1. In general, $AB \neq BA$

Note 2. If the product AB is defined, then the order of the product can be illustrated as given below:

$$\begin{tikzcd}[row sep=small] \text{Order of } A & m \times n \arrow[dd, bend right] \\ \text{Order of } B & n \times p \arrow[d, bend left] \\ \text{Order of } AB & m \times p \end{tikzcd}$$

Example 2: If $A=\begin{bmatrix}2 & -1 & 0\\ 1 & 2 & -3\\ 1 & 2 & -2\end{bmatrix}$ and $B=\begin{bmatrix}2 & -2 & 3\\ -1 & -4 & 6\\ 0 & -5 & 5\end{bmatrix}$, then compute A^2B.

Solution:

$$A^2 = A\cdot A=\begin{bmatrix}2 & -1 & 0\\ 1 & 2 & -3\\ 1 & 2 & -2\end{bmatrix}\begin{bmatrix}2 & -1 & 0\\ 1 & 2 & -3\\ 1 & 2 & -2\end{bmatrix}$$

$$=\begin{bmatrix}4-1+0 & -2-2+0 & 0+3+0\\ 2+2-3 & -1+4-6 & 0-6+6\\ 2+2-2 & -1+4-4 & 0-6+4\end{bmatrix}=\begin{bmatrix}3 & -4 & 3\\ 1 & -3 & 0\\ 2 & -1 & -2\end{bmatrix}$$

$$\therefore \quad A^2B=\begin{bmatrix}3 & -4 & 3\\ 1 & -3 & 0\\ 2 & -1 & -2\end{bmatrix}\begin{bmatrix}2 & -2 & 3\\ -1 & -4 & 6\\ 0 & -5 & 5\end{bmatrix}$$

$$=\begin{bmatrix}6+4+0 & -6+16-15 & 9-24+15\\ 2+3+0 & -2+12+0 & 3-18+0\\ 4+1+0 & -4+4+10 & 6-6-10\end{bmatrix}=\begin{bmatrix}10 & -5 & 0\\ 5 & 10 & -15\\ 5 & 10 & -10\end{bmatrix}$$

Note: Powers of square matrices are defined as:

$$A^2 = A\times A,\ \ A^3=A\times A\times A$$

$$A^n = A\times A\times A\times\cdots \text{ to } n \text{ factors.}$$

Properties of Matrix Addition, Scalar Multiplication and Matrix Multiplication

If A, B and C are conformable for the indicated sum or product of matrices and c and d are scalars, then following properties are true:

(i) Commutative property w.r.t. addition: $A+B=B+A$

(ii) Associative property w.r.t. addition: $(A+B)+C=A+(B+C)$

(iii) Associative property of scalar multiplication: $(cd)A=c(dA)$

(iv) Existence of additive identity: $A+O=O+A=A$ (O is null matrix and A is a square matrix)

(v) Existence of multiplicative identity: $IA=AI=A$ (I is unit/identity matrix)

(vi) Distributive property w.r.t scalar multiplication: (a) $c(A+B)=cA+cB$ (b) $(c+d)A=cA+dA$

(vii) Associative property w.r.t. multiplication: $A(BC)=(AB)C$

(viii) Left distributive property: $A(B+C)=AB+AC$

(ix) Right distributive property: $(A+B)C=AC+BC$

(x) $c(AB)=(cA)B=A(cB)$

Example 3: Find AB and BA if $A=\begin{bmatrix}2 & 0 & 1\\ 1 & 4 & 2\\ 3 & 0 & 6\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & -1\\ 1 & -2 & 3\end{bmatrix}$

Solution:

$$AB=\begin{bmatrix}2 & 0 & 1\\ 1 & 4 & 2\\ 3 & 0 & 6\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & -1\\ 1 & -2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}2\times1+0\times2+1\times1 & 2\times(-1)+0\times3+1\times(-2) & 2\times0+0\times(-1)+1\times3\\ 1\times1+4\times2+2\times1 & 1\times(-1)+4\times3+2\times(-2) & 1\times0+4\times(-1)+2\times3\\ 3\times1+0\times2+6\times1 & 3\times(-1)+0\times3+6\times(-2) & 3\times0+0\times(-1)+6\times3\end{bmatrix}$$

$$=\begin{bmatrix}3 & -4 & 3\\ 11 & 7 & 2\\ 9 & -15 & 18\end{bmatrix} \qquad \ldots(i)$$

$$BA=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & -1\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 1 & 4 & 2\\ 3 & 0 & 6\end{bmatrix}$$

$$=\begin{bmatrix}1\times2+(-1)\times1+0\times3 & 1\times0+(-1)\times4+0\times0 & 1\times1+(-1)\times2+0\times6\\ 2\times2+3\times1+(-1)\times3 & 2\times0+3\times4+(-1)\times0 & 2\times1+3\times2+(-1)\times6\\ 1\times2+(-2)\times1+3\times3 & 1\times0+(-2)\times4+3\times0 & 1\times1+(-2)\times2+3\times6\end{bmatrix}$$

$$=\begin{bmatrix}1 & -4 & -1\\ 4 & 12 & 2\\ 9 & -8 & 15\end{bmatrix} \qquad \ldots(ii)$$

Thus, from (i) and (ii), $AB\neq BA$

Note:
Matrix multiplication is not commutative in general.

Exercise 4.1

1. If $A=[a_{ij}]_{3\times4}$, then show that

(i) $I_3A=A$

Solution:

$I_3A=A$

As $A=[a_{ij}]_{3\times4}$, so $A=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix}$ and

$$I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\text{L.H.S}=I_3A$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix}$$

$$=\begin{bmatrix}a_{11}+0+0 & a_{12}+0+0 & a_{13}+0+0 & a_{14}+0+0\\ 0+a_{21}+0 & 0+a_{22}+0 & 0+a_{23}+0 & 0+a_{24}+0\\ 0+0+a_{31} & 0+0+a_{32} & 0+0+a_{33} & 0+0+a_{34}\end{bmatrix}$$

$$=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix} \text{ (Proved)}$$

(ii) $AI_4=A$

Solution:

$AI_4=A$

As $A = [a_{ij}]_{3\times4}$, so $A=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix}$ and

$$I_4=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$\text{L.H.S}=AI_4$

$$=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}a_{11}+0+0+0 & 0+a_{12}+0+0 & 0+0+a_{13}+0 & 0+0+0+a_{14}\\ a_{21}+0+0+0 & 0+a_{22}+0+0 & 0+0+a_{23}+0 & 0+0+0+a_{24}\\ a_{31}+0+0+0 & 0+a_{32}+0+0 & 0+0+a_{33}+0 & 0+0+0+a_{34}\end{bmatrix}$$

$$=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix}=A=\text{R.H.S (Proved)}$$

2. If $A=\begin{bmatrix}0 & -1 & 2\\ 3 & 2 & 1\\ -1 & 0 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & 1 & -1\\ 1 & 2 & 4\\ -1 & 2 & 1\end{bmatrix}$ **and** $C=\begin{bmatrix}1 & 0 & -2\\ -1 & 5 & 0\\ 3 & 4 & -1\end{bmatrix}$, **then find**

(i) $A-B$

Solution:

$$A-B=\begin{bmatrix}0 & -1 & 2\\ 3 & 2 & 1\\ -1 & 0 & 4\end{bmatrix}-\begin{bmatrix}2 & 1 & -1\\ 1 & 2 & 4\\ -1 & 2 & 1\end{bmatrix}=\begin{bmatrix}-2 & -2 & 3\\ 2 & 0 & -3\\ 0 & -2 & 3\end{bmatrix}$$

(ii) $B-C$

Solution:

$$B-A=\begin{bmatrix}2&1&-1\\1&2&4\\-1&2&1\end{bmatrix}-\begin{bmatrix}1&0&-2\\-1&5&0\\3&4&-1\end{bmatrix}$$

$$=\begin{bmatrix}1&1&1\\2&-3&4\\-4&-2&2\end{bmatrix}$$

(iii) $(A-B)-C$

Solution:

$(A-B)-C$

$$=\left(\begin{bmatrix}0&-1&2\\3&2&1\\-1&0&4\end{bmatrix}-\begin{bmatrix}2&1&-1\\1&2&4\\-1&2&1\end{bmatrix}\right)-\begin{bmatrix}1&0&-2\\-1&5&0\\3&4&-1\end{bmatrix}$$

$$=\begin{bmatrix}-2&-2&3\\2&0&-3\\0&-2&3\end{bmatrix}-\begin{bmatrix}1&0&-2\\-1&5&0\\3&4&-1\end{bmatrix}=\begin{bmatrix}-3&-2&5\\3&-5&-3\\-3&-6&4\end{bmatrix}$$

(iv) $A-(B-C)$

Solution:

$A-(B-C)$

$$=\begin{bmatrix}0&-1&2\\3&2&1\\-1&0&4\end{bmatrix}-\left(\begin{bmatrix}2&1&-1\\1&2&4\\-1&2&1\end{bmatrix}-\begin{bmatrix}1&0&-2\\-1&5&0\\3&4&-1\end{bmatrix}\right)$$

$$=\begin{bmatrix}0&-1&2\\3&2&1\\-1&0&4\end{bmatrix}-\begin{bmatrix}1&1&1\\2&-3&4\\-4&-2&2\end{bmatrix}=\begin{bmatrix}-1&-2&1\\1&5&-3\\3&2&2\end{bmatrix}$$

3. If $A=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}$, $B=\begin{bmatrix}-i&1\\2i&1\end{bmatrix}$ and $C=\begin{bmatrix}2i&1\\-i&i\end{bmatrix}$, then show that:

(i) $(AB)C=A(BC)$

Solution:

L.H.S $=(AB)C$

$$=\left(\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}-i&1\\2i&1\end{bmatrix}\right)\begin{bmatrix}2i&1\\-i&i\end{bmatrix}$$

$$=\begin{bmatrix}-i^2+4i^2&i+2i\\-i-2i^2&1-i\end{bmatrix}\begin{bmatrix}2i&1\\-i&i\end{bmatrix}$$

$$=\begin{bmatrix}3(-1)&3i\\2-i&1-i\end{bmatrix}\begin{bmatrix}2i&1\\-i&i\end{bmatrix}\qquad \because i^2=-1$$

$$=\begin{bmatrix}3(-1)&3i\\2-i&1-i\end{bmatrix}\begin{bmatrix}2i&1\\-i&i\end{bmatrix}$$

$$=\begin{bmatrix}-6i-3i^2&-3+3i^2\\4i-2i^2-i+i^2&2-i+i-i^2\end{bmatrix}$$

$$=\begin{bmatrix}-6i-3(-1)&-3+3(-1)\\3i-(-1)&2-(-1)\end{bmatrix}\qquad \because i^2=-1$$

$$=\begin{bmatrix}3-6i&-6\\1+3i&3\end{bmatrix}$$

R.H.S $=A(BC)$

$$=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\left(\begin{bmatrix}-i&1\\2i&1\end{bmatrix}\begin{bmatrix}2i&1\\-i&i\end{bmatrix}\right)$$

$$=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}-2i^2-i&-i+i\\4i^2-i&2i+i\end{bmatrix}$$

$$=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}-2(-1)-i&0\\4(-1)-i&3i\end{bmatrix}\qquad \because i^2=-1$$

$$=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}2-i&0\\-4-i&3i\end{bmatrix}$$

$$=\begin{bmatrix}2i-i^2-8i-2i^2&0+6i^2\\2-i-4i+i^2&0-3i^2\end{bmatrix}=\begin{bmatrix}-6i-3i^2&6i^2\\2-3i+i^2&-3i^2\end{bmatrix}$$

$$=\begin{bmatrix}-6i-3(-1)&6(-1)\\2+3i+(-1)&-3(-1)\end{bmatrix}\qquad \because i^2=-1$$

$$\text{R.H.S}=\begin{bmatrix}3-6i&-6\\1+3i&3\end{bmatrix}$$

Hence proved L.H.S = R.H.S

(ii) $A(B+C)=AB+AC$

Solution:

L.H.S $=A(B+C)$

$$=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\left(\begin{bmatrix}-i&1\\2i&1\end{bmatrix}+\begin{bmatrix}2i&1\\-i&i\end{bmatrix}\right)$$

$$=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}i&2\\i&1+i\end{bmatrix}$$

$$=\begin{bmatrix}i^2+2i^2&2i+2i+2i^2\\i-i^2&2-i-i^2\end{bmatrix}$$

$$=\begin{bmatrix}3i^2&4i+2i^2\\i-i^2&2-i-i^2\end{bmatrix}$$

$$=\begin{bmatrix}3(-1)&4i+2(-1)\\i-(-1)&2-i-(-1)\end{bmatrix}\qquad \because i^2=-1$$

$$\text{L.H.S}=\begin{bmatrix}-3&-2+4i\\1+i&3-i\end{bmatrix}$$

R.H.S $=AB+AC$

$$=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}-i&1\\2i&1\end{bmatrix}+\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}2i&1\\-i&i\end{bmatrix}$$

$$=\begin{bmatrix}-i^2+4i^2&i+2i\\-i-2i^2&1-i\end{bmatrix}+\begin{bmatrix}2i^2-2i^2&i+2i^2\\2i+i^2&1-i^2\end{bmatrix}$$

$$=\begin{bmatrix}3i^2&3i\\-i-2i^2&1-i\end{bmatrix}+\begin{bmatrix}0&i+2i^2\\2i+i^2&1-i^2\end{bmatrix}$$

$$=\begin{bmatrix}3(-1)&3i\\-i-2(-1)&1-i\end{bmatrix}+\begin{bmatrix}0&i+2(-1)\\2i+(-1)&1-(-1)\end{bmatrix}\qquad \because i^2=-1$$

$$=\begin{bmatrix}-3&3i\\-i+2&1-i\end{bmatrix}+\begin{bmatrix}0&i-2\\-1+2i&2\end{bmatrix}$$

$$\text{R.H.S}=\begin{bmatrix}-3&-2+4i\\1+i&3-i\end{bmatrix}$$

Hence proved L.H.S = R.H.S

4. If A and B are square matrices of the same order, then explain why in general:

(i) $(A+B)^2\neq A^2+2AB+B^2$

Solution:

$(A+B)^2\neq A^2+2AB+B^2$

Consider $(A+B)^2$

$=(A+B).(A+B)$

$=A(A+B)+B(A+B)$

$=A.A+A.B+B.A+B.B$

$=A^2+AB+BA+B^2$

$\neq A^2+2AB+B^2$ $\quad\because$ In general $AB\neq BA$

Hence $(A+B)^2\neq A^2+2AB+B^2$

(ii) $(A-B)^2\neq A^2-2AB+B^2$

Solution:

$(A-B)^2\neq A^2-2AB+B^2$

Consider $(A-B)^2$

$=(A-B)(A-B)$

$=A(A-B)-B(A-B)$

$=A.A-A.B-B.A+B.B$

$=A^2-AB-BA+B^2$

$\neq A^2-2AB+B^2$ $\quad\because$ In general $AB\neq BA$

Hence $(A-B)^2\neq A^2-2AB+B^2$

(iii) $(A+B)(A-B)\neq A^2-B^2$

Solution:

$(A+B)(A-B)\neq A^2-B^2$

Consider $(A+B)(A-B)$

$=(A+B)(A-B)=A(A-B)+B(A-B)$

$=A.A-A.B+B.A-B.B$

$=A^2-AB+BA-B^2$

$\neq A^2-B^2$ $\quad\because$ In general $AB\neq BA$

Hence $(A+B)(A-B)\neq A^2-B^2$

5. If $A=\begin{bmatrix}-1&2&3\\1&0&2\\-3&5&3\end{bmatrix}$, then find $A+A^t, A-A^t, AA^t, A^tA$ and $(A^t)^t$.

Solution:

$$A=\begin{bmatrix}-1&2&3\\1&0&2\\-3&5&3\end{bmatrix}\Rightarrow A^t=\begin{bmatrix}-1&1&-3\\2&0&5\\3&2&3\end{bmatrix}$$

$$A+A^t=\begin{bmatrix}-1&2&3\\1&0&2\\-3&5&3\end{bmatrix}+\begin{bmatrix}-1&1&-3\\2&0&5\\3&2&3\end{bmatrix}$$

$$=\begin{bmatrix}-1-1&2+1&3-3\\1+2&0+0&2+5\\-3+3&5+2&3+3\end{bmatrix}=\begin{bmatrix}-2&3&0\\3&0&7\\0&7&6\end{bmatrix}$$

$$A-A^t=\begin{bmatrix}-1&2&3\\1&0&2\\-3&5&3\end{bmatrix}-\begin{bmatrix}-1&1&-3\\2&0&5\\3&2&3\end{bmatrix}$$

$$=\begin{bmatrix}-1+1&2-1&3+3\\1-2&0-0&2-5\\-3-3&5-2&3-3\end{bmatrix}=\begin{bmatrix}0&1&6\\-1&0&-3\\-6&3&0\end{bmatrix}$$

$$AA^t=\begin{bmatrix}-1&2&3\\1&0&2\\-3&5&3\end{bmatrix}\begin{bmatrix}-1&1&-3\\2&0&5\\3&2&3\end{bmatrix}$$

$$=\begin{bmatrix}1+4+9&-1+0+6&3+10+9\\-1+0+6&1+0+4&-3+0+6\\3+10+9&-3+0+6&9+25+9\end{bmatrix}$$

$$=\begin{bmatrix}14&5&22\\5&5&3\\22&3&43\end{bmatrix}$$

$$A^tA=\begin{bmatrix}-1&1&-3\\2&0&5\\3&2&3\end{bmatrix}\begin{bmatrix}-1&2&3\\1&0&2\\-3&5&3\end{bmatrix}$$

$$=\begin{bmatrix}1+1+9&-2+0-15&-3+2-9\\-2+0-15&4+0+25&6+0+15\\-3+2-9&6+0+15&9+4+9\end{bmatrix}$$

$$=\begin{bmatrix}11&-17&-10\\-17&29&21\\-10&21&22\end{bmatrix}$$

$$(A^t)^t=\begin{bmatrix}-1&1&-3\\2&0&5\\3&2&3\end{bmatrix}^t=\begin{bmatrix}-1&2&3\\1&0&2\\-3&5&3\end{bmatrix}$$

6. Solve the matrix equations $A^2 - 5A + 4I - X = 0$ if

$$A=\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}$$

Solution:

$$A^2 = A\,.\,A$$

$$=\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}$$

$$=\begin{bmatrix}4+0+1&0+0-1&2+0+0\\4+2+3&0+1-3&2+3+0\\2-2+0&0-1+0&1-3+0\end{bmatrix}$$

$$A^2=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}$$

Given Equation: $A^2 - 5A + 4I - X = 0$

$A^2 - 5A + 4I = X$

Putting values of A^2, A and I

$$X=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}-5\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}+4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$X=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}-\begin{bmatrix}10&0&5\\10&5&15\\5&-5&0\end{bmatrix}+\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

$$X=\begin{bmatrix}-5&-1&-3\\-1&-7&-10\\-5&4&-2\end{bmatrix}+\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

$$X=\begin{bmatrix}-5+4&-1+0&-3+0\\-1+0&-7+4&-10+0\\-5+0&4+0&-2+4\end{bmatrix}$$

$$X=\begin{bmatrix}-1&-1&-3\\-1&-3&-10\\-5&4&2\end{bmatrix}$$

7. If A and B are two matrices such that $AB = B$ and $BA = A$, show that $A^2 + B^2 = A + B$.

Solution:

Given that: $AB = B$ and $BA = A$

$$\begin{aligned}\text{L.H.S} &= A^2 + B^2\\ &= A \cdot A + B \cdot B\\ &= A(BA) + B(AB) \quad \because A = BA, AB = B\end{aligned}$$

By associative law

$$\begin{aligned}&= (AB)A + (BA)B\\ &= BA + AB\\ &= A + B\\ &= \text{R.H.S (Proved)}\end{aligned}$$

Determinants

The determinants of square matrices of order $n \geq 3$, can be written by the following pattern. For example, if n =

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}, \text{ then the determinant of } A=|A|=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}$$

Now our aim is to compute the determinants of matrices of various orders.

Minor and Cofactor of an Element of a Matrix or its Determinant

Minor of an Element: Let us consider a square matrix A of order n, then the minor of an element a_{ij}, denoted M_{ij} is the determinant formed by deleting the ith row and the jth column of $A(\text{or}|A|)$.

For example, consider a square matrix A of order 3, $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

To find the minor of the element a_{12}, delete the first row and second column of A

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}, \text{ that is } M_{12}=\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}$$

Cofactor of an Element: The cofactor of an element a_{ij} of a square matrix A denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$

For example, $A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix} = -\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}$

Determinant of a Square Matrix of Order $n = 3$

If A is a matrix of order 3, that is, $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$, then:

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} \quad \text{for } i = 1, 2, 3$$

or $$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} \quad \text{for } j = 1, 2, 3$$

For example, for $i = 1, j = 1$ and $j = 2$, we have

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \quad \text{... (i)}$$

or $$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \quad \text{... (ii)}$$

or $$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \quad \text{... (iii)}$$

(iii) can be written as:

$$|A| = a_{12}(-1)^{1+2}M_{12} + a_{22}(-1)^{2+2}M_{22} + a_{32}(-1)^{3+2}M_{32}$$

$$|A| = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32} \quad \text{... (iv)}$$

Similarly (i) can be written as:

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \quad \text{... (v)}$$

Putting the values of M_{11}, M_{12} and M_{13} in (v), we obtain

$$|A| = a_{11}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix} - a_{12}\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix} + a_{13}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}$$

or $$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \quad \text{... (vi)}$$

or $$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \quad \text{... (vii)}$$

Equation (vii) is the required expansion of determinant of square matrix of order 3.

Example 4: Evaluate the determinant if $A=\begin{bmatrix}1&-2&3\\-2&3&1\\4&-3&2\end{bmatrix}$

Solution: $|A| = \begin{vmatrix}1&-2&3\\-2&3&1\\4&-3&2\end{vmatrix}$

Expanding by R_1, we get

$$|A| = 1\begin{vmatrix}3&1\\-3&2\end{vmatrix} - (-2)\begin{vmatrix}-2&1\\4&2\end{vmatrix} + 3\begin{vmatrix}-2&3\\4&-3\end{vmatrix}$$

$$= 1[6-1(-3)] + 2[(-2)(2)-(1)(4)] + 3[(-2)(-3)-12]$$

$$= (6+3) + 2(-4-4) + 3(6-12) = 9 - 16 - 18 = -25$$

Example 5: Find the cofactors A_{12}, A_{22} and A_{32} of $A=\begin{bmatrix}1&-2&3\\-2&3&1\\4&-3&2\end{bmatrix}$ and find $|A|$.

Solution: We first find A_{12}, A_{22} and A_{32},

$$A_{12} = (-1)^{1+2}\begin{vmatrix}-2&1\\4&2\end{vmatrix} = (-1)^3(-4-4) = -1(-8) = 8$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix} = (-1)^4(2-12) = 1(-10) = -10$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 3\\ -2 & 1\end{vmatrix} = (-1)^5(1-(-6)) = -1(7) = -7$$

Here $a_{12} = -2, a_{22} = 3, a_{32} = -3$

Expanding by R_1, we get

$$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

$$= (-2)8 + 3(-10) + (-3)(-7) = -16 - 30 + 21 = -25$$

Properties of Determinants

i. For a square matrix $A, |A| = |A^t|$

ii. If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.

iii. If a square matrix A has two identical rows or two identical columns, then $|A| = 0$.

iv. If all the entries of a row (or a column) of a square matrix A are zero, then $|A| = 0$.

v. If the entries of a row (or a column) in a square matrix A are multiplied by a number $k \in R$, then the determinant of the resulting matrix is $k|A|$.

vi. If each entry of a row (or a column) of a square matrix consists of two terms, then its determinant can be written as the sum of two determinants, i.e., if

$$B = \begin{bmatrix}a_{11}+b_{11} & a_{12} & a_{13}\\ a_{21}+b_{21} & a_{22} & a_{23}\\ a_{31}+b_{31} & a_{32} & a_{33}\end{bmatrix}, \text{ then } |B| = \begin{vmatrix}a_{11}+b_{11} & a_{12} & a_{13}\\ a_{21}+b_{21} & a_{22} & a_{23}\\ a_{31}+b_{31} & a_{32} & a_{33}\end{vmatrix} = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} + \begin{vmatrix}b_{11} & a_{12} & a_{13}\\ b_{21} & a_{22} & a_{23}\\ b_{31} & a_{32} & a_{33}\end{vmatrix}$$

vii. If any row (column) of a determinant is multiplied by a non-zero number k and the result is added to the corresponding entries of another row (column), the value of the determinant does not change.

viii. If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Note: We shall define triangular matrices on following pages (page no. 113).

Example 6: Without expansion, show that $\begin{vmatrix}x & a+x & b+c\\ x & b+x & c+a\\ x & c+x & a+b\end{vmatrix} = 0$

Solution: Adding the entries of C_3 to the corresponding entries of C_2.

$$\text{L.H.S} = \begin{vmatrix}x & a+x & b+c\\ x & b+x & c+a\\ x & c+x & a+b\end{vmatrix}$$

$$= \begin{vmatrix}x & a+b+c+x & b+c\\ x & a+b+c+x & c+a\\ x & a+b+c+x & a+b\end{vmatrix} \qquad \text{By } C_2 + C_3$$

$$= x(a+b+c+x)\begin{vmatrix}1 & 1 & b+c\\ 1 & 1 & c+a\\ 1 & 1 & a+b\end{vmatrix} \qquad \begin{pmatrix}\text{by taking common } x \text{ from } C_1\\ \text{and } (a+b+c+x) \text{ from } C_2\end{pmatrix}$$

$$= x(a+b+c+x)\cdot 0 \qquad \because C_1 \text{ and } C_2 \text{ are identical.}$$

$$= 0 = \text{R.H.S (Proved)}$$

Adjoint and Inverse of a Square Matrix

Let $A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$, then the matrix of co-factors of $A = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$, then

$$\text{adj } A = (\text{matrix of cofactors of } A)^t = \begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}$$

Inverse of a Square Matrix of Order $n \geq 3$: Let A be a non singular $(|A| \neq 0)$ square matrix of order n. If there exists a matrix B such that $AB = BA = I_n$, then B is called the multiplicative inverse of A and is denoted by A^{-1}. It is obvious that the order of A^{-1} is $n \times n$.
Thus, $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

If A is non-singular matrix then $A^{-1} = \frac{1}{|A|}\text{adj}A$

Example 7: Find A^{-1} if $A = \begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 1 & -1 & 1\end{bmatrix}$.

Solution:

$$A = \begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 1 & -1 & 1\end{bmatrix}$$

$$|A| = 1\begin{vmatrix}2 & 1\\ -1 & 1\end{vmatrix} - 0\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix} + 2\begin{vmatrix}0 & 2\\ 1 & -1\end{vmatrix} \qquad \text{Expanding by } R_1$$

$$= 1(2+1) - 0 + 2(0-2) = 1(3) + 0 + 2(-2) = 3 + 0 - 4 = -1 \neq 0, \text{ so } A^{-1} \text{ exists.}$$

We first find the cofactor of the elements of A.

$$A_{11} = (-1)^{1+1}\begin{vmatrix}2 & 1\\ -1 & 1\end{vmatrix} = 1\cdot(2+1) = 3$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix} = (-1)(0-1) = 1$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix}0 & 2\\ 1 & -1\end{vmatrix} = 1.(0-2) = -2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}0 & 2\\ -1 & 1\end{vmatrix} = (-1)(0+2) = -2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 2\\ 1 & 1\end{vmatrix} = 1.(1-2) = -1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}1 & 0\\ 1 & -1\end{vmatrix} = (-1)(-1-0) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}0 & 2\\ 2 & 1\end{vmatrix} = 1\cdot(0-4) = -4$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix} = (-1)(1-0) = -1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix} = 1\cdot(2-0) = 2$$

$$\text{Matrix of cofactors of } A = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix} = \begin{bmatrix}3 & 1 & -2\\ -2 & -1 & 1\\ -4 & -1 & 2\end{bmatrix}$$

$$\text{adj } A = (\text{matrix of cofactors of } A)^t = \begin{bmatrix}3 & -2 & -4\\ 1 & -1 & -1\\ -2 & 1 & 2\end{bmatrix}$$

$$\text{Using: } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix}3 & -2 & -4\\ 1 & -1 & -1\\ -2 & 1 & 2\end{bmatrix} = \begin{bmatrix}-3 & 2 & 4\\ -1 & 1 & 1\\ 2 & -1 & -2\end{bmatrix}$$

Exercise 4.2

1. Evaluate the following determinants:

(i) $\begin{vmatrix} 1 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 3 & 2 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 3 & 2 \end{vmatrix}$$

Expanding by R_1

$$= 1\begin{vmatrix} -1 & -3 \\ 3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix}$$

$$= 1(-2+9) + 2(6-6) - 4(9-2)$$

$$= 1(7) + 2(0) - 4(7)$$

$$= 7 + 0 - 28 = -21$$

(ii) $\begin{vmatrix} a+b & a-b & a \\ a & a+b & a-b \\ a-b & a & a+b \end{vmatrix}$

Solution:

$$\begin{vmatrix} a+b & a-b & a \\ a & a+b & a-b \\ a-b & a & a+b \end{vmatrix}$$

Expanding by R_1

$$= (a+b)\begin{vmatrix} a+b & a-b \\ a & a+b \end{vmatrix} - (a-b)\begin{vmatrix} a & a-b \\ a-b & a+b \end{vmatrix} + a\begin{vmatrix} a & a+b \\ a-b & a \end{vmatrix}$$

$$= (a+b)\{(a+b)^2 - a(a-b)\} - (a-b)\{a(a+b) - (a-b)^2\} + a\{a^2 - (a^2 - b^2)\}$$

$$= (a+b)(a^2 + b^2 + 2ab - a^2 + ab) - (a-b)(a^2 + ab - a^2 - b^2 + 2ab) + a(a^2 - a^2 + b^2)$$

$$= (a+b)(b^2 + 3ab) - (a-b)(3ab - b^2) + a(b^2)$$

$$= ab^2 + 3a^2b + b^3 + 3ab^2 - 3a^2b + ab^2 + 3ab^2 - b^3 + ab^2$$

$$= 9ab^2$$

(iii) $\begin{vmatrix} 2x & x & x \\ y & 2y & y \\ z & z & 2z \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2x & x & x \\ y & 2y & y \\ z & z & 2z \end{vmatrix}$$

$$= 2x\begin{vmatrix} 2y & y \\ z & 2z \end{vmatrix} - x\begin{vmatrix} y & y \\ z & 2z \end{vmatrix} + x\begin{vmatrix} y & 2y \\ z & z \end{vmatrix}$$

$$= 2x(4yz - yz) - x(2yz - yz) + x(yz - 2yz)$$

$$= 8xyz - 2xyz - 2xyz + xyz + xyz - 2xyz$$

$$= 10xyz - 6xyz = 4xyz$$

2. Without expansion show that:

(i) $\begin{vmatrix} 7 & 8 & 9 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{vmatrix} = 0$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 7 & 8 & 9 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{vmatrix}$$

By $C_1 + C_3$

$$= \begin{vmatrix} 16 & 8 & 9 \\ 12 & 6 & 7 \\ 6 & 3 & 4 \end{vmatrix}$$

By taking 2 common from C_1

$$= 2\begin{vmatrix} 8 & 8 & 9 \\ 6 & 6 & 7 \\ 3 & 3 & 4 \end{vmatrix}$$

$= 2(0)$ $\quad \because C_1$ and C_2 are identical

$= 0 = \text{R.H.S}$ (Proved)

(ii) $\begin{vmatrix} 5 & 6 & -1 \\ 2 & 2 & 0 \\ 2 & -8 & 10 \end{vmatrix} = 0$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 5 & 6 & -1 \\ 2 & 2 & 0 \\ 2 & -8 & 10 \end{vmatrix}$$

By $C_2 + C_3$

$$= \begin{vmatrix} 5 & 6+(-1) & -1 \\ 2 & 2+0 & 0 \\ 2 & -8+10 & 10 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 5 & -1 \\ 2 & 2 & 0 \\ 2 & 2 & 10 \end{vmatrix}$$

$= 0$ $\quad \because C_1$ and C_2 are identical

$= \text{R.H.S}$ (Proved)

(iii) $\begin{vmatrix} -a & 0 & b \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix} = 0$

Solution:

$$\text{L.H.S} = \begin{vmatrix} -a & 0 & b \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix}$$

Multiply and divide C_1 by b, C_2 by c, C_3 by a

$$\text{L.H.S} = \frac{1}{bca}\begin{vmatrix} -ab & 0 & ab \\ 0 & ac & -ac \\ bc & -bc & 0 \end{vmatrix}$$

By $C_1 + (C_2 + C_3)$

$$= \frac{1}{abc}\begin{vmatrix} -ab+0+ab & 0 & ab \\ 0+ac-ac & ac & -ac \\ bc-bc+0 & -bc & 0 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} 0 & 0 & ab \\ 0 & ac & -ac \\ 0 & -bc & 0 \end{vmatrix}$$

$= \frac{1}{abc}(0)$ $\quad \because C_1$ is zero.

$= 0 = \text{R.H.S}$ (Proved)

(iv) $\begin{vmatrix} \ell & m+n & 1 \\ m & n+\ell & 1 \\ n & \ell+m & 1 \end{vmatrix} = 0$

Solution:

$$\text{L.H.S} = \begin{vmatrix} \ell & m+n & 1 \\ m & n+\ell & 1 \\ n & \ell+m & 1 \end{vmatrix}$$

By $C_2 + C_1$

$$= \begin{vmatrix} \ell & \ell+m+n & 1 \\ m & \ell+m+n & 1 \\ n & \ell+m+n & 1 \end{vmatrix}$$

By taking $(\ell + m + n)$ common from C_2

$$= (\ell+m+n)\begin{vmatrix} \ell & 1 & 1 \\ m & 1 & 1 \\ n & 1 & 1 \end{vmatrix}$$

$= (\ell + m + n)(0)$ $\quad \because C_2$ and C_3 are identical.

$= 0 = \text{R.H.S}$ (Proved)

(v) $\begin{vmatrix} 2 & 1 & 3x \\ 2 & 3 & 9x \\ 3 & 5 & 15x \end{vmatrix} = 0$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 2 & 1 & 3x \\ 2 & 3 & 9x \\ 3 & 5 & 15x \end{vmatrix}$$

By taking $3x$ common from C_3

$$= 3x\begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 5 \end{vmatrix}$$

$= 3x(0)$ $\quad \because C_2$ and C_3 are identical.

$= 0 = \text{R.H.S}$ (Proved)

(vi) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Solution:

$$\text{L.H.S} = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$$

Multiply and divide R_1 by a, R_2 by b, R_3 by c

$$= \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$$

By taking abc common from C_1

$$= \frac{1}{abc}(abc)\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$= \text{R.H.S}$ (Proved)

3. Using properties of determinants, show that:

(i) $\begin{vmatrix} 3 & 5 & 0 \\ 5 & 25 & 10 \\ 7 & 25 & 1 \end{vmatrix} = 25\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix}$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 3 & 5 & 0 \\ 5 & 25 & 10 \\ 7 & 25 & 1 \end{vmatrix}$$

By taking 5 common from C_2

$$= 5\begin{vmatrix} 3 & 5 & 0 \\ 5 & 5 & 10 \\ 7 & 5 & 1 \end{vmatrix}$$

By taking 5 common from R_2

$$= 5.5\begin{vmatrix} 3 & 5 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix}$$

$$= 25\begin{vmatrix} 3 & 5 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix}$$

= R.H.S (Proved)

(ii) $\begin{vmatrix} a+x & a & a \\ a & a+x & a \\ a & a & a+x \end{vmatrix} = x^2(3a+x)$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a+x & a & a \\ a & a+x & a \\ a & a & a+x \end{vmatrix}$$

By $C_1 + (C_2 + C_3)$

$$= \begin{vmatrix} 3a+x & a & a \\ 3a+x & a+x & a \\ 3a+x & a & a+x \end{vmatrix}$$

By taking $(3a + x)$ common from C_1

$$= (3a+x)\begin{vmatrix} 1 & a & a \\ 1 & a+x & a \\ 1 & a & a+x \end{vmatrix}$$

By $R_2 - R_1$ and $R_3 - R_1$

$$= (3a+x)\begin{vmatrix} 1 & a & a \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$$

By using property of determinant of a triangular matrix.

$$= (3a+x)(1.x.x)$$
$$= x^2(3a+x)$$

= R.H.S (Proved)

(iii) $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Multiply and divide R_1 by x, R_2 by y, R_3 by z

$$= \frac{1}{xyz}\begin{vmatrix} x & x^2 & xyz \\ y & y^2 & xyz \\ z & z^2 & xyz \end{vmatrix}$$

By taking xyz common from C_3

$$= \frac{1}{xyz}(xyz)\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

By interchanging C_1 and C_3

$$= -1\begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$$

By interchanging C_2 and C_3

$$= (-1)(-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \text{R.H.S (Proved)}$$

(iv) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

By $R_2 - R_1$ and $R_3 - R_1$

$$= \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$

Expanding by C_1

$$= 1\begin{vmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{vmatrix} - 0 + 0$$

$$= 1\begin{vmatrix} y-x & (y-x)(y+x) \\ z-x & (z-x)(z+x) \end{vmatrix}$$

By taking $(y - x)$ and $(z - x)$ common from R_1 and R_2 respectively.

$$= (y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$$
$$= (y-x)(z-x)(z+x-y-x)$$
$$= (y-x)(z-x)(z-y)$$
$$= (-1)(x-y)(z-x)(-1)(y-z)$$
$$= (x-y)(y-z)(z-x)$$

= R.H.S (Proved)

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a+1 & b+1 & c+1 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & 1 & 1 \\ a+1 & b+1 & c+1 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a+1 & b+1 & c+1 \\ a^2+2a+1 & b^2+2b+1 & c^2+2c+1 \end{vmatrix}$$

By $R_2 - R_1$, $R_3 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2+2a & b^2+2b & c^2+2c \end{vmatrix}$$

By $R_3 - 2R_2$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

By $C_2 - C_1$, $C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$$

Taking $(b - a)$ and $(c - a)$ common from C_2 and C_3 respectively.

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

Expanding by R_1

$$= (b-a)(c-a)\left\{1\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} - 0 + 0\right\}$$
$$= (b-a)(c-a)(c+a-b-a)$$
$$= (b-a)(c-a)(c-b)$$
$$= (-1)(a-b)(c-a)(-1)(b-c)$$
$$= (a-b)(b-c)(c-a) = \text{R.H.S (Proved)}$$

(vi) $\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

By $R_1 + (R_2 + R_3)$

$$= \begin{vmatrix} 2a^2+2b^2 & 2b^2+2c^2 & 2a^2+2c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

By taking 2 common from R_1

$$= 2\begin{vmatrix} a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

By $R_2 - R_1$ and $R_3 - R_1$

$$= 2\begin{vmatrix} a^2+b^2 & b^2+c^2 & c^2+a^2 \\ -b^2 & 0 & -c^2 \\ -a^2 & -c^2 & 0 \end{vmatrix}$$

By $R_1 + (R_2 + R_3)$

$$= 2\begin{vmatrix} 0 & b^2 & a^2 \\ -b^2 & 0 & -c^2 \\ -a^2 & -c^2 & 0 \end{vmatrix}$$

Expanding by R_1

$$= 2\left\{0 - b^2\begin{vmatrix} -b^2 & -c^2 \\ -a^2 & 0 \end{vmatrix} + a^2\begin{vmatrix} -b^2 & 0 \\ -a^2 & -c^2 \end{vmatrix}\right\}$$
$$= 2(-b^2(0-a^2c^2)+a^2(b^2c^2-0))$$
$$= 2(a^2b^2c^2+a^2b^2c^2)$$
$$= 2(2a^2b^2c^2) = 4a^2b^2c^2 = \text{R.H.S (Proved)}$$

(vii) $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a+b & b+c & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a+b & b+c & c+a \end{vmatrix}$$

By $R_1 + R_2$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b+c & c+a & a+b \\ a+b & b+c & c+a \end{vmatrix}$$

By taking $(a + b + c)$ common from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ a+b & b+c & c+a \end{vmatrix}$$

By $C_2 - C_1$ and $C_3 - C_1$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding by R_1

$$= (a+b+c)\left\{1\begin{vmatrix} a-b & a-c \\ c-a & c-b \end{vmatrix} - 0 + 0\right\}$$

$$= (a+b+c)\{(a-b)(c-b)-(c-a)(a-c)\}$$

$$= (a+b+c)(ac-ab-bc+b^2-ac+c^2+a^2-ac)$$

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$= a^3+b^3+c^3-3abc$ = R.H.S (Proved)

(viii) $\begin{vmatrix} a+t & a & a \\ b & b+t & b \\ c & c & c+t \end{vmatrix} = t^2(a+b+c+t)$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a+t & a & a \\ b & b+t & b \\ c & c & c+t \end{vmatrix}$$

By $R_1 + (R_2 + R_3)$

$$= \begin{vmatrix} a+b+c+t & a+b+c+t & a+b+c+t \\ b & b+c & b \\ c & c & c+t \end{vmatrix}$$

By taking $(a+b+c+t)$ common R_1

$$= (a+b+c+t)\begin{vmatrix} 1 & 1 & 1 \\ b & b+t & b \\ c & c & c+t \end{vmatrix}$$

By $C_2 - C_1$ and $C_3 - C_1$

$$= (a+b+c+d)\begin{vmatrix} 1 & 0 & 0 \\ b & t & 0 \\ c & 0 & t \end{vmatrix}$$

By using property of determinant of a triangular matrix.

$$= (a+b+c+t)(1 \cdot t \cdot t)$$

$$= t^2(a+b+c+t)$$

= R.H.S (Proved)

(ix) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

By $R_1 + (R_2 + R_3)$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

By $C_2 - C_1$, $C_3 - C_1$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -a-b-c & 0 \\ 2c & 0 & -a-b-c \end{vmatrix}$$

By determinant of a triangular matrix.

$$= (a+b+c)(1(-a-b-c)(-a-b-c))$$

$$= (a+b+c)(-1)(a+b+c)(-1)(a+b+c)$$

$$= (a+b+c)^3$$

= R.H.S (Proved)

(x) $\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x+y+z)(x-y)(y-z)(z-x)$

Solution:

$$\text{L.H.S} = \begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

By $R_1 + R_2$

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

By taking $(x+y+z)$ common from R_1

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

By $C_2 - C_1$ and $C_3 - C_1$

$$= (x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

Expanding by R_1

$$= (x+y+z)\left\{1\begin{vmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{vmatrix} - 0 + 0\right\}$$

$$= (x+y+z)\begin{vmatrix} y-x & z-x \\ (y-x)(y+z) & (z-x)(z+x) \end{vmatrix}$$

By taking $(y-x)$ and $(z-x)$ common from C_1 and C_2

$$= (x+y+z)(y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix}$$

$$= (x+y+z)(y-x)(z-x)(z+x-y-x)$$

$$= (x+y+z)(y-x)(z-x)(z-y)$$

$$= (x+y+z)(-1)(x-y)(z-x)(-1)(y-z)$$

$$= (x+y+z)(x-y)(y-z)(z-x)$$

= R.H.S (Proved)

(xi) $\begin{vmatrix} 1 & 1 & 1 \\ a^2+1 & b^2+1 & c^2+1 \\ a^3+a & b^3+b & c^3+c \end{vmatrix} =$

$(a-b)(b-c)(c-a)(ab+bc+ca-1)$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & 1 & 1 \\ a^2+1 & b^2+1 & c^2+1 \\ a^3+a & b^3+b & c^3+c \end{vmatrix}$$

By $R_2 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3+a & b^3+b & c^3+c \end{vmatrix}$$

By $C_2 - C_1$, $C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3+a & b^3+b-a^3-a & c^3+c-a^3-a \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \\ a^3+a & (b^3-a^3)+(b-a) & (c^3-a^3)+(c-a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \\ a^3+a & (b-a)(b^2+ab+a^2)+(b-a) & (c-a)(c^2+ac+a^2)+(c-a) \end{vmatrix}$$

Taking $(b-a)$ and $(c-a)$ common from C_2 and C_3 respectively

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & b+a & c+a \\ a^3+a & b^2+ab+a^2+1 & c^2+ac+a^2+1 \end{vmatrix}$$

Expanding by R_1

$$= (b-a)(c-a)\begin{vmatrix} b+a & c+a \\ b^2+ab+a^2+1 & c^2+ca+a^2+1 \end{vmatrix}$$

By $C_1 - C_2$

$$= (b-a)(c-a)\begin{vmatrix} b-c & c+a \\ b^2-c^2+ab-ac & c^2+ca+a^2+1 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} b-c & c+a \\ (b+c)(b-c)+a(b-c) & c^2+ca+a^2+1 \end{vmatrix}$$

Taking $(b-c)$ common from C_1

$$= (b-a)(c-a)(b-c)\begin{vmatrix} 1 & c+a \\ b+c+a & c^2+ca+a^2+1 \end{vmatrix}$$

$$= (-1)(a-b)(c-a)(b-c)(c^2+ca+a^2+1-bc-$$
$$ba-c^2-ca-ac-a^2)$$

$$= -(a-b)(b-c)(c-a)(-ab-bc-ca+1)$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca-1)$$

= R.H.S (Proved)

(xii) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc+ab+bc+ca$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

By $R_2 - R_1$, $R_3 - R_1$

$$= \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$

By $C_3 - C_2$

$$= \begin{vmatrix} 1+a & 1 & 0 \\ -a & b & -b \\ -a & 0 & c \end{vmatrix}$$

Expanding by R_1

$$= (1+a)\begin{vmatrix} b & -b \\ 0 & c \end{vmatrix} - 1\begin{vmatrix} -a & -b \\ -a & c \end{vmatrix} - 0$$

$$= (1+a)(bc-0) - 1(-ac-ab) + 0$$

$$= bc(1+a) + ac + ab$$

$$= abc + ab + bc + ca$$

= R.H.S (Proved)

(xiii) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$$

By $R_2 - R_1$ and $R_3 - R_1$

$$= \begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & b^2-a^2-ca+bc \\ 0 & c-a & c^2-a^2+bc-ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 - bc \\ 0 & b-a & (b+a)(b-a)+c(b-a) \\ 0 & c-a & (c+a)(c-a)+b(c-a) \end{vmatrix}$$

Taking $(b-a), (c-a)$ common from R_2 and R_3 respectively

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 - bc \\ 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \end{vmatrix}$$

$= (b-a)(c-a)(0)$ ∵ R_2 and R_3 are identical

$= 0$

= R.H.S (Proved)

(xiv) $\begin{vmatrix} 2x+3 & x+2 & x+a \\ 2x+5 & x+3 & x+b \\ 2x+7 & x+4 & x+c \end{vmatrix} = 0$, **where $2b = a + c$ in A.P.**

Solution:

$$\text{L.H.S} = \begin{vmatrix} 2x+3 & x+2 & x+a \\ 2x+5 & x+3 & x+b \\ 2x+7 & x+4 & x+c \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} 2x+3 & x+2 & x+2b-c \\ 2x+5 & x+3 & x+b \\ 2x+7 & x+4 & x+c \end{vmatrix} \qquad \because \begin{cases} 2b = a+c \\ a = 2b-c \end{cases}$$

By $R_1 - R_2$ and $R_3 - R_2$

$$= \begin{vmatrix} -2 & -1 & b-c \\ 2x+5 & x+3 & x+b \\ 2 & 1 & c-b \end{vmatrix}$$

By $R_1 + R_3$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 2x+5 & x+3 & x+b \\ 2 & 1 & c-b \end{vmatrix}$$

$= 0$ ∵ R_1 is zero

= R.H.S (Proved)

(xv) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a+b\omega+c\omega^2)$

$(a+b\omega^2+c\omega)$, where ω is an imaginary cube root of unity.

Solution:

$$\text{L.H.S} = \begin{vmatrix} a & b & c \\ c & d & b \\ b & c & a \end{vmatrix}$$

By $C_1 + (C_2 + C_3)$

$$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix}$$

Taking $(a+b+c)$ common from C_1

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}$$

By $R_2 - R_1$ and $R_3 - R_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & a-b & b-c \\ 0 & c-b & a-c \end{vmatrix}$$

Expanding by C_1

$= (a+b+c)\left\{1\begin{vmatrix} a-b & b-c \\ c-b & a-c \end{vmatrix} - 0 + 0\right\}$

$= (a+b+c)[(a-b)(a-c) - (c-b)(b-c)]$

$= (a+b+c)(a^2 - ac - ab + bc - bc + c^2 + b^2 - b$

$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$

$\text{L.H.S} = a^3 + b^3 + c^3 - 3abc$

$\text{R.H.S} = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

$= (a+b+c)(a^2 + ab\omega^2 + ac\omega + ab\omega + b^2\omega^3 +$

$bc\omega^2 + ac\omega^2 + bc\omega^4 + c^2\omega^3$

$= (a+b+c)(a^2 + b^2\omega^3 + c^2\omega^3 + ab(\omega^2+\omega) +$

$bc(\omega^2+\omega^4) + ac(\omega+\omega^2$

Since ω is an imaginary cube root of unity, so

$$\omega + \omega^2 = -1, \omega^3 = 1, \omega^4 = \omega$$

$\text{R.H.S} = (a+b+c)(a^2 + b^2(1) + c^2(1) +$

$ab(-1) + bc(-1) + ac(-1$

$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + b^3 + c^3 - 3abc$

Hence proved L.H.S = R.H.S

---

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 0 \\ -2 & -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -2 & 5 \\ -3 & -1 & 4 \\ -2 & -1 & 2 \end{bmatrix}$

then find:

(i) A_{13}, A_{23}, A_{33} and $|A|$

Solution:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 0 \\ -2 & -2 & 7 \end{bmatrix}$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -5 \\ -2 & -2 \end{vmatrix} = 1(0-10) = -10$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} = (-1)(-2+4) = -2$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix} = 1(-5-0) = -5$$

Now,

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -5 & 0 \\ -2 & -2 & 7 \end{vmatrix}$$

Here $a_{13} = -3, a_{23} = 0, a_{33} = 7$

Expanding by C_3, we have

$|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$= (-3)(-10) + (0)(-2) + 7(-5)$

$= 30 - 0 - 35 = -5$

(ii) B_{31}, B_{32}, B_{33} and $|B|$

Solution:

$$B = \begin{bmatrix} -5 & -2 & 5 \\ -3 & -1 & 4 \\ -2 & -1 & 2 \end{bmatrix}$$

$$B_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 5 \\ -1 & 4 \end{vmatrix} = 1(-8+5) = -3$$

$$B_{32} = (-1)^{3+2}\begin{vmatrix} -5 & 5 \\ -3 & 4 \end{vmatrix} = -1(-20+15) = 5$$

$$B_{33} = (-1)^{3+3}\begin{vmatrix} -5 & -2 \\ -3 & -1 \end{vmatrix} = 1(5-6) = -1$$

Now,

$$|B| = \begin{vmatrix} -5 & -2 & 5 \\ -3 & -1 & 4 \\ -2 & -1 & 2 \end{vmatrix}$$

Here $b_{31} = -2, b_{32} = -1, b_{33} = 2$

Expanding by R_3

$|B| = b_{31}B_{31} + b_{32}B_{32} + b_{33}B_{33}$

$= (-2)(-3) + (-1)(5) + (2)(-1)$

$= 6 - 5 - 2 = -1$

5. Find values of x if:

(i) $\begin{vmatrix} 2 & 1 & x \\ -1 & -4 & -3 \\ x & 1 & 0 \end{vmatrix} = 5$

Solution:

$$\begin{vmatrix} 2 & 1 & x \\ -1 & -4 & -3 \\ x & 1 & 0 \end{vmatrix} = 5$$

Expanding by R_1

$$2\begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & -4 \\ x & 1 \end{vmatrix} - 5 = 0$$

$2(0+3) - 1(0+3x) + x(-1+4x) - 5 = 0$

$6 - 3x - x + 4x^2 - 5 = 0$

$4x^2 - 4x + 1 = 0$

$(2x-1)^2 = 0$

Taking square root on both sides

$2x - 1 = 0 \quad \Rightarrow \quad \boxed{x = \frac{1}{2}}$

(ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -3 & x \end{vmatrix} = 9$

Solution:

$$\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -3 & x \end{vmatrix} = 9$$

Expanding by R_1

$$1\begin{vmatrix} x+1 & 2 \\ -3 & x \end{vmatrix} - (x-1)\begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3\begin{vmatrix} -1 & x+1 \\ 2 & -3 \end{vmatrix} - 9 = 0$$

$1(x^2 + x + 6) - (x-1)(-x-4) + 3(3 - 2x - 2) - 9 = 0$

$1(x^2 + x + 6) + (x-1)(x+4) + 3(1 - 2x) - 9 = 0$

$x^2 + x + 6 + x^2 + 3x - 4 + 3 - 6x - 9 = 0$

$2x^2 - 2x - 4 = 0$

Dividing by '2', we have

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$(x+1)(x-2) = 0$

Either $x + 1 = 0$ or $x - 2 = 0$

$\Rightarrow$ $x = -1$ or $x = 2$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$$

Expanding by R_1

$$1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$$

$1(x^2 - 12) - 1(2x - 6) + 1(12 - 3x) = 0$

$x^2 - 12 - 2x + 6 + 12 - 3x = 0$

$x^2 - 5x + 6 = 0$

$x^2 - 3x - 2x + 6 = 0$

$$x(x-3)-2(x-3)=0$$

$$(x-2)(x-3)=0$$

Either $x-2=0$ or $x-3=0$

$x=2$; $x=3$

6. Find $|AA^t|$ and $|A^tA|$ if:

(i) $A=\begin{bmatrix}-3 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}$

Solution:

$$A=\begin{bmatrix}-3 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}$$

$$\Rightarrow \quad A^t=\begin{bmatrix}-3 & 2\\ 2 & 1\\ -1 & 3\end{bmatrix}$$

$$AA^t=\begin{bmatrix}-3 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}-3 & 2\\ 2 & 1\\ -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}9+4+1 & -6+2-3\\ -6+2-3 & 4+1+9\end{bmatrix}$$

$$=\begin{bmatrix}14 & -7\\ -7 & 14\end{bmatrix}$$

$$|AA^t|=\begin{bmatrix}14 & -7\\ -7 & 14\end{bmatrix}$$

$$=196-49=147$$

$$A^tA=\begin{bmatrix}-3 & 2\\ 2 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}-3 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}9+4 & -6+2 & 3+6\\ -6+2 & 4+1 & -2+3\\ 3+6 & -2+3 & 1+9\end{bmatrix}$$

$$=\begin{bmatrix}13 & -4 & 9\\ -4 & 5 & 1\\ 9 & 1 & 10\end{bmatrix}$$

$$|A^tA|=\begin{vmatrix}13 & -4 & 9\\ -4 & 5 & 1\\ 9 & 1 & 10\end{vmatrix}$$

Expanding by R_1

$$=13\begin{vmatrix}5 & 1\\ 1 & 10\end{vmatrix}-(-4)\begin{vmatrix}-4 & 1\\ 9 & 10\end{vmatrix}+9\begin{vmatrix}-4 & 5\\ 9 & 1\end{vmatrix}$$

$$=13(50-1)+4(-40-9)+9(-4-45)$$

$$=13(49)+4(-49)+9(-49)$$

$$=637-196-441=0$$

(ii) $A=\begin{bmatrix}3 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}$

Solution:

$$A=\begin{bmatrix}3 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}$$

$$\Rightarrow \quad A^t=\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$$

$$AA^t=\begin{bmatrix}3 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}9+1 & 6+2 & 3+3\\ 6+2 & 4+4 & 2+6\\ 3+3 & 2+6 & 1+9\end{bmatrix}$$

$$=\begin{bmatrix}10 & 8 & 6\\ 8 & 8 & 8\\ 6 & 8 & 10\end{bmatrix}$$

$$|AA^t|=\begin{vmatrix}10 & 8 & 6\\ 8 & 8 & 8\\ 6 & 8 & 10\end{vmatrix}$$

Expanding by R_1

$$=10\begin{vmatrix}8 & 8\\ 8 & 10\end{vmatrix}-8\begin{vmatrix}8 & 8\\ 6 & 10\end{vmatrix}+6\begin{vmatrix}8 & 8\\ 6 & 8\end{vmatrix}$$

$$=10(80-64)-8(80-48)+6(64-48)$$

$$=10(16)-8(32)+6(16)$$

$$=160-256+96$$

$$=256-256=0$$

$$A^tA=\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}9+4+1 & 3+4+3\\ 3+4+3 & 1+4+9\end{bmatrix}$$

$$=\begin{bmatrix}14 & 10\\ 10 & 14\end{bmatrix}$$

$$|A^tA|=\begin{vmatrix}14 & 10\\ 10 & 14\end{vmatrix}$$

$$=196-100=96$$

7. If A is a square matrix of order 3, then show tha $|kA|=k^3|A|$.

Solution:

$$\text{Let} \quad A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$\Rightarrow \quad kA=\begin{bmatrix}ka_{11} & ka_{12} & ka_{13}\\ ka_{21} & ka_{22} & ka_{23}\\ ka_{31} & ka_{32} & ka_{33}\end{bmatrix}$$

$$\text{L.H.S}=|kA|$$

$$=\begin{vmatrix}ka_{11} & ka_{12} & ka_{13}\\ ka_{21} & ka_{22} & ka_{23}\\ ka_{31} & ka_{32} & ka_{33}\end{vmatrix}$$

By taking 'k' as common from R_1, R_2, R_3

$$=k.k.k\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$=k^3|A|=\text{R.H.S (Proved)}$$

8. Find the values of λ if A and B are singular $A=\begin{bmatrix}4 & 2 & 3\\ 7 & \lambda & 6\\ 2 & 3 & 1\end{bmatrix}, B=\begin{bmatrix}-2 & 4 & 5\\ 1 & -2 & 1\\ 2 & \lambda & 0\end{bmatrix}$

Solution:

$$A=\begin{bmatrix}4 & 2 & 3\\ 7 & \lambda & 6\\ 2 & 3 & 1\end{bmatrix}$$

Since the matrix A is singular, therefore

$$|A|=0$$

$$\Rightarrow \quad \begin{vmatrix}4 & 2 & 3\\ 7 & \lambda & 6\\ 2 & 3 & 1\end{vmatrix}=0$$

Expanding by R_1

$$4\begin{vmatrix}\lambda & 6\\ 3 & 1\end{vmatrix}-2\begin{vmatrix}7 & 6\\ 2 & 1\end{vmatrix}+3\begin{vmatrix}7 & \lambda\\ 2 & 3\end{vmatrix}=0$$

$$4(\lambda-18)-2(7-12)+3(21-2\lambda)=0$$

$$4\lambda-72+10+63-6\lambda=0$$

$$-2\lambda+1=0$$

$$-2\lambda=-1 \qquad \Rightarrow \lambda=\frac{1}{2}$$

Since the matrix B is singular, therefore

$$|B|=0$$

$$\Rightarrow \begin{vmatrix}-2 & 4 & 5\\ 1 & -2 & 1\\ 2 & \lambda & 0\end{vmatrix}=0$$

Expanding by R_1

$$-2\begin{vmatrix}-2 & 1\\ \lambda & 0\end{vmatrix}-4\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}+5\begin{vmatrix}1 & -2\\ 2 & \lambda\end{vmatrix}=0$$

$$-2(0-\lambda)-4(0-2)+5(\lambda+4)=0$$

$$2\lambda+8+5\lambda+20=0$$

$$7\lambda=-28$$

$$\lambda=\frac{-28}{7} \qquad \Rightarrow \lambda=-4$$

9. Find the inverse of $A=\begin{bmatrix}1 & 2 & 1\\ -5 & 0 & 4\\ 5 & 4 & 0\end{bmatrix}$ and show that $A^{-1}A=I_3$

Solution:

$$A=\begin{bmatrix}1 & 2 & 1\\ -5 & 0 & 4\\ 5 & 4 & 0\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & 2 & 1\\ -5 & 0 & 4\\ 5 & 4 & 0\end{vmatrix}$$

Expanding by R_1

$$|A|=1\begin{vmatrix}0 & 4\\ 4 & 0\end{vmatrix}-2\begin{vmatrix}-5 & 4\\ 5 & 0\end{vmatrix}+1\begin{vmatrix}-5 & 0\\ 5 & 4\end{vmatrix}$$

$$=1(0-16)-2(0-20)+1(-20-0)$$

$$=-16+40-20=4$$

Since $|A|\neq 0$, so we can find A^{-1}.

Cofactors of elements of A are

$$A_{11}=(-1)^{1+1}\begin{vmatrix}0 & 4\\ 4 & 0\end{vmatrix}=1(0-16)=-16$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}-5 & 4\\ 5 & 0\end{vmatrix}=-1(0-20)=20$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}-5 & 0\\ 5 & 4\end{vmatrix}=1(-20-0)=-20$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}2 & 1\\ 4 & 0\end{vmatrix}=-1(0-4)=4$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\ 5 & 0\end{vmatrix}=1(0-5)=-5$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 5 & 4\end{vmatrix}=-1(4-10)=6$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}2 & 1\\ 0 & 4\end{vmatrix}=1(8-0)=8$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\ -5 & 4\end{vmatrix}=-1(4+5)=-9$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 2\\ -5 & 0\end{vmatrix}=1(0+10)=10$$

$$\text{Matrix of cofactors of } A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$$

$$=\begin{bmatrix}-16 & 20 & -20\\ 4 & -5 & 6\\ 8 & -9 & 10\end{bmatrix}$$

$$\text{adj}\,(A) = (\text{Matrix of cofactors of } A)^t$$

$$\text{adj}\,A=\begin{bmatrix}-16 & 4 & 8\\ 20 & -5 & -9\\ -20 & 6 & 10\end{bmatrix}$$

As we know $\quad A^{-1}=\dfrac{1}{|A|}\text{adj}A$

$$=\frac{1}{4}\begin{bmatrix}-16 & 4 & 8\\ 20 & -5 & -9\\ -20 & 6 & 10\end{bmatrix}$$

We want to show that: $AA^{-1}=I_3$

L.H.S $= AA^{-1}$

$$=\begin{bmatrix}1 & 2 & 1\\ -5 & 0 & 4\\ 5 & 4 & 0\end{bmatrix}\frac{1}{4}\begin{bmatrix}-16 & 4 & 8\\ 20 & -5 & -9\\ -20 & 6 & 10\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}-16+40-20 & 4-10+6 & 8-18+10\\ 80+0-80 & -20+0+24 & -40+0+40\\ -80+80+0 & 20-20+0 & 40-36+0\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$= I_3 =$ R.H.S (Proved)

---

10. Verify that $(AB)^t = B^tA^t$ if:

(i) $A=\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}$

Solution:

$$A=\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix} \text{ and } B=\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}$$

L.H.S $= (AB)^t$

$$=\left(\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}\right)^t$$

$$=\begin{bmatrix}1+3+0 & 1+2+2\\ 0+9+0 & 0+6+1\end{bmatrix}^t$$

$$=\begin{bmatrix}4 & 5\\ 9 & 7\end{bmatrix}^t=\begin{bmatrix}4 & 9\\ 5 & 7\end{bmatrix}$$

R.H.S $= B^tA^t$

$$=\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}^t\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix}^t$$

$$=\begin{bmatrix}1 & -3 & 0\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & -3\\ 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1+3+0 & 0+9+0\\ 1+2+2 & 0+6+1\end{bmatrix}=\begin{bmatrix}4 & 9\\ 5 & 7\end{bmatrix}$$

Hence proved L.H.S = R.H.S

i.e., $(AB)^t = B^tA^t$

(ii) $A=\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}$

Solution:

$$A=\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix} \text{ and } B=\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}$$

L.H.S $= (AB)^t$

$$=\left(\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}\right)^t$$

$$=\begin{bmatrix}1-4 & -3+2\\ 1-8 & -3+4\\ 2-2 & -6+1\end{bmatrix}^t=\begin{bmatrix}-3 & -1\\ -7 & 1\\ 0 & -5\end{bmatrix}^t=\begin{bmatrix}-3 & -7 & 0\\ -1 & 1 & -5\end{bmatrix}$$

R.H.S $= B^tA^t$

$$=\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}^t\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix}^t$$

$$=\begin{bmatrix}1 & -2\\ -3 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1-4 & 1-8 & 2-2\\ -3+2 & -3+4 & -6+1\end{bmatrix}=\begin{bmatrix}-3 & -7 & 0\\ -1 & 1 & -5\end{bmatrix}$$

Hence proved L.H.S = R.H.S

i.e., $(AB)^t = B^tA^t$

Elementary Row Operations on a Matrix

Usually, a given system of linear equations is reduced to a simple equivalent system by applying elementary operations which are stated as below:

(i) Interchanging two equations.

(ii) Multiplying an equation by a non-zero number.

(iii) Adding a multiple of one equation to another equation.

Corresponding to these three elementary operations, the following elementary row operations are applied to matrices to obtain equivalent matrices.

(i) Interchanging two rows

(ii) Multiplying a row by a non-zero number

(iii) Adding a multiple of one row to another row.

Notations that are used to represent row operations for (i) to (iii) are given below:

- Interchanging R_i and R_j is expressed as $R_i \leftrightarrow R_j$.
- k times R_i is denoted by $kR_i \rightarrow R_i'$
- Adding k times R_j to R_i is expressed as $R_i + kR_j \rightarrow R_i'$

  (R_i' is the new row obtained after applying the row operation).

Note:
Matrices A and B are equivalent if B can be obtained by applying in turn a finite number of row operations on A.

For equivalent matrices A and B, we write $A \underset{\sim}{R} B$.

➢ If $A \underset{\sim}{R} B$ then $B \underset{\sim}{R} A$.

Upper Triangular Matrix: A square matrix A is called an upper triangular matrix if all elements below the principal diagonal are zero.

Or

A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for all $i > j$

Lower Triangular Matrix: A square matrix A is said to be lower triangular matrix if all elements above the principal diagonal are zero.

Or

A square matrix $A = [a_{ij}]$ is called an lower triangular matrix if $a_{ij} = 0$ for all $i < j$

Triangular Matrix: A square matrix A is named as triangular matrix whether it is upper triangular or lower triangular.

For example, the matrices $\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 6\end{bmatrix}$ and $\begin{bmatrix}1 & 0 & 0 & 0\\ 3 & 2 & 0 & 0\\ 4 & 1 & 5 & 0\\ -1 & 2 & 3 & 1\end{bmatrix}$ are triangular matrices of order 3 and 4 respectively. The first matrix is upper triangular while the second is lower triangular.

Remember! Diagonal matrices are both upper triangular and lower triangular.

Echelon and Reduced Echelon Forms of Matrices

In any non-zero row of a matrix, the first non-zero entry is called the **leading entry** of that row.

Echelon Form of a Matrix

An $m \times n$ matrix A is called in echelon form if:

(i) In each successive non-zero row the number of zeros before the leading entry is greater than the number of such zeros in the preceding row.

(ii) Every non-zero row in A precedes every zero row (if any).

(iii) The first non-zero entry (or leading entry) in each row is 1.

For examples, the matrices $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in echelon form

Reduced Echelon Form of a Matrix: An $m \times n$ matrix A is said to be in reduced (row) echelon form if the non-zero entry (or leading entry) in R_i lies in C_j, then all other entries of C_j are zero.

For examples, the matrices $\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in (row) reduced echelon form.

Example 8: Reduce $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ to (row) echelon and reduced (row) echelon form.

Solution:

Let $A = \begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 3 & -1 & 9 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ By $R_1 \leftrightarrow R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -5 & 15 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ By $R_2 + (-2)R_1 \to R_2'$ and $R_3 + (-3)R_1 \to R_3'$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ By $\frac{1}{5}R_2 \to R_2'$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ By $R_3 + (-4)R_2 \to R_3'$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ By $R_1 + 1.R_2 \to R_1'$ (Echelon form of A)

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ By $R_1 + (-1)R_3 \to R_1'$ and $R_2 + 1.R_3 \to R_2'$ (Reduced Echelon form of A)

Thus, $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ are (row) echelon and reduced (row) echelon forms of the matrix respectively.

Inverse of a Matrix: Let A be a non-singular matrix. If the application of elementary row operations on A succession reduces A to I, then the resulting matrix is $I \vdots A^{-1}$.

Example 9: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \end{bmatrix}$

Solution:

$$|A| = \begin{vmatrix} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \end{vmatrix} = 2(-8-4) - 5(-6-2) - 1(6-4) = -24 + 40 - 2 = 40 - 26 = 14$$

As $|A| \neq 0$, so A is non-singular.

Consider $[A \vdots I] = \left[\begin{array}{ccc:ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ 1 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$

$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 2 & -2 & 0 & 0 & 1 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ 2 & 5 & -1 & 1 & 0 & 0 \end{array}\right]$ By $R_1 \leftrightarrow R_3$

$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 2 & -2 & 0 & 0 & 1 \\ 0 & -2 & 8 & 0 & 1 & -3 \\ 0 & 1 & 3 & 1 & 0 & -2 \end{array}\right]$ By $R_2 + (-3)R_1 \to R_2'$ and $R_3 + (-2)R_1 \to R_3'$

$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 2 & -2 & 0 & 0 & 1 \\ 0 & 1 & -4 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & 1 & 0 & -2 \end{array}\right]$ By $-\frac{1}{2}R_2 \to R_2'$

$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 0 & 1 & -2 \\ 0 & 1 & -4 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 7 & 1 & \frac{1}{2} & -\frac{7}{2} \end{array}\right]$ By $R_3 + (-1)R_2 \to R_3'$ and $R_1 + (-2)R_2 \to R_1'$

$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 0 & 1 & -2 \\ 0 & 1 & -4 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{array}\right]$ By $\frac{1}{7}R_3 \to R_3'$

$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{6}{7} & \frac{4}{7} & 1 \\ 0 & 1 & 0 & \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{array}\right]$ By $R_1 + (-6)R_3 \to R_1'$ and $R_2 + 4R_3 \to R_2'$

$= [I \vdots A^{-1}]$

Thus, the inverse of A is $\begin{bmatrix} -\frac{6}{7} & \frac{4}{7} & 1 \\ \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix}$

Rank of a Matrix: Let A be a non-zero matrix. If r is the number of non-zero rows when it is reduced to the ech form, then r is called the rank of the matrix A.

Example 10: Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 3 & -1 \\ 0 & 4 & 6 & -2 \end{bmatrix} \quad \text{By } R_2+(-2)R_1 \to R_2' \text{ and } R_3+(-3)R_1 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 4 & 6 & -2 \end{bmatrix} \quad \text{By } \frac{1}{2}R_2 \to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } R_3+(-4)R_2 \to R_3'$$

As the number of non-zero rows is 2, so

Rank of matrix = 2

System of Non-Homogeneous Linear Equations

Three linear equations in three variables such as:

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\} \quad \ldots\text{(i)}$$

is called a system of non-homogeneous linear equations in the three variables x, y and z, if constant terms and d_3 are not all zero.

Consistent: A system of linear equations is said to be consistent if the system has a unique solution infinitely many solutions.

Inconsistent: A system of linear equations is said to be inconsistent if the system has no solution.

Solution of System of Non-Homogeneous Linear Equations:

Now we will solve the system of non-homogeneous linear equations with the help of the following method

(i) Using reduced echelon form

(ii) Using matrix inversion method

(iii) Using Cramer's rule

Method-1: Reduced Echelon Form

There are following steps to solve a system of non-homogeneous linear equations.

(i) Convert to augmented matrix i.e., $A_b = \begin{bmatrix} a_1 & b_1 & c_1 & \vdots & d_1 \\ a_2 & b_2 & c_2 & \vdots & d_2 \\ a_3 & b_3 & c_3 & \vdots & d_3 \end{bmatrix}$

(ii) Convert to reduced echelon form

(iii) Solve by back substitution

Example 11: Solve the following and explain a consistent and inconsistent system:

(i) $\begin{aligned} 2x+5y-z &= 5 \\ 3x+4y+2z &= 11 \\ x+2y-2z &= -3 \end{aligned}$ (ii) $\begin{aligned} x+y+2z &= 1 \\ 2x-y+7z &= 11 \\ 3x+5y+4z &= -3 \end{aligned}$ (iii) $\begin{aligned} x-y+2z &= 1 \\ 2x-6y+5z &= 7 \\ 3x+5y+4z &= -3 \end{aligned}$

Solution:

(i) Augmented matrix = $A_b = \begin{bmatrix} 2 & 5 & -1 & \vdots & 5 \\ 3 & 4 & 2 & \vdots & 11 \\ 1 & 2 & -2 & \vdots & -3 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 3 & 4 & 2 & \vdots & 11 \\ 2 & 5 & -1 & \vdots & 5 \end{bmatrix} \quad \text{By } R_1 \leftrightarrow R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 0 & -2 & 8 & \vdots & 20 \\ 0 & 1 & 3 & \vdots & 11 \end{bmatrix} \quad \text{By } R_2+(-3)R_1 \to R_2' \text{ and By } R_3+(-2)R_1 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 1 & 3 & \vdots & 11 \end{bmatrix} \quad \text{By } -\frac{1}{2}R_2 \to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 6 & \vdots & 17 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 7 & \vdots & 21 \end{bmatrix} \quad \text{By } R_1+(-2)R_2 \to R_1' \text{ and } R_3+(-1)R_2 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 6 & \vdots & 17 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \quad \text{By } \frac{1}{7}R_3 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \quad \text{By } R_1+(-6)\text{R}_3 \to R_1' \text{ and } R_2+4R_3 \to R_2'$$

Thus, the solution is $x=-1, y=2$ and $z=3$, therefore the given system of linear equations has unique solution and it is consistent.

(ii) Augmented matrix = $A_b = \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 2 & -1 & 7 & \vdots & 11 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & -3 & 3 & \vdots & 9 \\ 0 & 2 & -2 & \vdots & -6 \end{bmatrix} \quad \text{By } R_2+(-2)R_1 \to R_2' \text{ and By } R_3+(-3)R_1 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 1 & 2 & \vdots & 1\\ 0 & 1 & -1 & \vdots & -3\\ 0 & 2 & -2 & \vdots & -6\end{bmatrix} \qquad \text{By } -\frac{1}{3}R_2 \to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 3 & \vdots & 4\\ 0 & 1 & -1 & \vdots & -3\\ 0 & 0 & 0 & \vdots & 0\end{bmatrix} \qquad \text{By } R_1+(-1)R_2 \to R_1' \text{ and } R_3+(-2)R_2 \to R_3'$$

The given system is reduced to equivalent system

$$x+3z \;=4 \quad \ldots(1)$$

$$y-z \;=-3 \quad \ldots(2)$$

$$0z \;=0 \Rightarrow z=t \text{ where } t \text{ is any real number.}$$

Put $z=t$ in equation (1) and (2), we have

$$x+3t=4 \Rightarrow x=4-3t$$

$$y-t=-3 \Rightarrow y=t-3$$

Since the given system, is satisfied by $x=4-3t, y=t-3$ and $z=t$ for any real value of t.

Thus, the given system has infinitely many solutions and it is consistent.

(ii) Augmented matrix $= A_b = \begin{bmatrix}1 & -1 & 2 & \vdots & 1\\ 2 & -6 & 5 & \vdots & 7\\ 3 & 5 & 4 & \vdots & -3\end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix}1 & -1 & 2 & \vdots & 1\\ 0 & -4 & 1 & \vdots & 5\\ 0 & 8 & -2 & \vdots & -6\end{bmatrix} \qquad \text{By } R_2+(-2)R_1 \to R_2' \text{ and By } R_3+(-3)R_1 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & -1 & 2 & \vdots & 1\\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4}\\ 0 & 8 & -2 & \vdots & -6\end{bmatrix} \qquad \text{By } -\frac{1}{4}R_2 \to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & \frac{7}{4} & \vdots & -\frac{1}{4}\\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4}\\ 0 & 0 & 0 & \vdots & 4\end{bmatrix} \qquad \text{By } R_1+1.R_2 \to R_1' \text{ and } R_3+(-8)R_2 \to R_3'$$

Thus, the given system is reduced to the equivalent system

$$x+\frac{7}{4}z \;=-\frac{1}{4}$$

$$y-\frac{1}{4}z \;=-\frac{5}{4}$$

$$0z \;=4$$

The third equation $0z = 4$ has no solution, so the system as a whole has no solution. Thus, the system is inconsistent.

Note:

We see that in the case of the system (i), the (row) rank of the augmented matrix and the coefficient matrix of the system is the same, that is, 3 which is equal to the number of the variables in the system (i).

Thus, we observe that a linear system is consistent and has a unique solution if:

Rank of the coefficient matrix (A) = Rank of the augmented matrix (A_b) = Number of variables

In the case of the system (ii), the rank of the coefficient matrix is the same as that of the augmented matrix of the system but it is 2 which is less than the number of variables in the system (ii).

Thus, we observe that a system is consistent and has infinitely many solutions if:

Rank of the coefficient matrix (A) = Rank of the augmented matrix (A_b) < Number of variables

In the case of the system (iii), we see that the rank of the coefficient matrix is not equal to the rank of the augmented matrix of the system.

Thus, we observe that a system is inconsistent if: Rank of the coefficient matrix $(A) \neq$ Rank of the augmented matrix (A_b)

Method-2: Matrix Inversion Method

The matrix inversion method is a way to solve a system of linear equations using the inverse of a matrix.

Example 12: Use matrix inversion method to solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -4\\ 2x_1 - 3x_2 + 2x_3 &= -6\\ 2x_1 + 2x_2 + x_3 &= 5\end{aligned}$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix}1 & -2 & 1\\ 2 & -3 & 2\\ 2 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-4\\ -6\\ 5\end{bmatrix}$$

Let $\qquad AX=B \qquad \ldots(i)$

Where $A=\begin{bmatrix}1 & -2 & 1\\ 2 & -3 & 2\\ 2 & 2 & 1\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ and $B=\begin{bmatrix}-4\\ -6\\ 5\end{bmatrix}$

$$|A|=\begin{vmatrix}1 & -2 & 1\\ 2 & -3 & 2\\ 2 & 2 & 1\end{vmatrix}=1\begin{vmatrix}-3 & 2\\ 2 & 1\end{vmatrix}+2\begin{vmatrix}2 & 2\\ 2 & 1\end{vmatrix}+1\begin{vmatrix}2 & -3\\ 2 & 2\end{vmatrix} \text{ Expanding by } \mathrm{R}_1$$

$$=1(-3-4)+2(2-4)+1(4+6)=1(-7)+2(-2)+1(10)$$

$$=-7-4+10=-1\neq 0, \text{ so } A^{-1} \text{ exists and (i) can be written as}$$

$$\mathrm{X}=A^{-1}B \qquad \ldots(ii)$$

Cofactor of the elements of A.

$$A_{11}=(-1)^{1+1}\begin{vmatrix}-3 & 2\\ 2 & 1\end{vmatrix}=1\cdot(-3-4)=-7$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}2 & 2\\ 2 & 1\end{vmatrix}=(-1)(2-4)=2$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}2 & -3\\ 2 & 2\end{vmatrix}=1.(4+6)=10$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}-2 & 1\\ 2 & 1\end{vmatrix}=(-1)(-2-2)=4$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}=1.(1-2)=-1$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & -2\\ 2 & 2\end{vmatrix}=(-1)(2+4)=-6$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}-2 & 1\\ -3 & 2\end{vmatrix}=1\cdot(-4+3)=-1$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\ 2 & 2\end{vmatrix}=(-1)(2-2)=0$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & -2\\ 2 & -3\end{vmatrix}=1\cdot(-3+4)=1$$

Matrix of cofactors of A $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -7 & 2 & 10 \\ 4 & -1 & -6 \\ -1 & 0 & 1 \end{bmatrix}$

adj A = (matrix of cofactors of $A)^t = \begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}$

Using: $A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{-1}\begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$

Putting values in equation (ii), we have

$$X = A^{-1}B$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -28+24+5 \\ 8-6+0 \\ 40-36-5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow x_1 = 1,\ x_2 = 2 \text{ and } x_3 = -1$$

Thus, the solution set is $\{(x_1, x_2, x_3)\} = \{(1, 2, -1)\}$

Method-3: Cramer's Rule

Consider the system of equations,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{aligned}\right\} \qquad \text{...(i)}$$

These are three linear equations in three variables x_1, x_2, x_3 with coefficients and constant terms in the real field R. We write the above system of equations in matrix form as:

$$AX = B \qquad \text{...(ii)}$$

where $A = [a_{ij}]_{3\times 3}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

We know that the matrix equation (ii) can be written as: $X = A^{-1}B$ (if A^{-1} exists)

We have already proved that $A^{-1} = \frac{1}{|A|}\text{adj}A$

and adj A = (matrix of cofactors of $A)^t = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11}b_1 + A_{21}b_2 + A_{31}b_3 \\ A_{12}b_1 + A_{22}b_2 + A_{32}b_3 \\ A_{13}b_1 + A_{23}b_2 + A_{33}b_3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{A_{11}b_1 + A_{21}b_2 + A_{31}b_3}{|A|} \\ \dfrac{A_{12}b_1 + A_{22}b_2 + A_{32}b_3}{|A|} \\ \dfrac{A_{13}b_1 + A_{23}b_2 + A_{33}b_3}{|A|} \end{bmatrix}$$

Hence $x_1 = \dfrac{b_1A_{11} + b_2A_{21} + b_3A_{31}}{|A|} = \dfrac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$ ... (iii)

$x_2 = \dfrac{b_1A_{12} + b_2A_{22} + b_3A_{32}}{|A|} = \dfrac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|}$ ... (iv)

$x_3 = \dfrac{b_1A_{13} + b_2A_{23} + b_3A_{33}}{|A|} = \dfrac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|}$ ... (v)

The method of solving the system with the help of results (iii), (iv) and (v) is often referred to as Cramer's Rule.

Example 13: Use Cramer's rule to solve the system. $\left.\begin{aligned} 3x_1 + x_2 - x_3 = -4 \\ x_1 + x_2 - 2x_3 = -4 \\ -x_1 + 2x_2 - x_3 = 1 \end{aligned}\right\}$.

Solution:

Here $|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$ Expanding by R_1

$$|A| = 3(-1+4) - 1(-1-2) - 1(2+1) = 9+3-3 = 9$$

$$|A_{x_1}| = \begin{vmatrix} -4 & 1 & -1 \\ -4 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = -4\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} -4 & -2 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} -4 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= -4(-1+4) - 1(4+2) - 1(-8-1) = -4(3) - 1(6) - 1(-9) = -12-6+9 = -9$$

$\Rightarrow \quad x_1 = \dfrac{|A_{x_1}|}{|A|} = \dfrac{-9}{9} = -1$

$$|A_{x_2}| = \begin{vmatrix} 3 & -4 & -1 \\ 1 & -4 & -2 \\ -1 & 1 & -1 \end{vmatrix} = 3\begin{vmatrix} -4 & -2 \\ 1 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -4 \\ -1 & 1 \end{vmatrix}$$

$$= 3(4+2) + 4(-1-2) - 1(1-4) = 3(6) + 4(-3) - 1(-3) = 18-12+3 = 9$$

$\Rightarrow \quad x_2 = \dfrac{|A_{x_2}|}{|A|} = \dfrac{9}{9} = 1$

$$|A_{x_3}| = \begin{vmatrix} 3 & 1 & -4 \\ 1 & 1 & -4 \\ -1 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -4 \\ -1 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= 3(1+8) - 1(1-4) - 4(2+1) = 3(9) - 1(-3) - 4(3) = 27+3-12 = 18$$

$$\Rightarrow \qquad x_3 = \frac{|A_{x_3}|}{|A|} = \frac{18}{9} = 2$$

Hence $x_1 = -1, x_2 = 1, x_3 = 2$

Thus, the solution set is $\{(x_1, x_2, x_3)\} = \{(-1, 1, 2)\}$

System of Homogeneous Linear Equations

The system of following homogeneous linear equations:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 \end{aligned}\right\} \qquad \ldots (i)$$

is always satisfied by $x_1 = 0, x_2 = 0$ and $x_3 = 0$, so such a system is always consistent.

Trivial Solution: The solution (0,0,0) of the above homogeneous system is called the trivial solution.

Non-Trivial Solution: Any other solution of system (i) other than the trivial solution is called a non-trivial solution.

Solution of System of Homogeneous Linear Equations by Gaussian Elimination Method

Gaussian Elimination is a systematic method for solving systems of linear equations, named after the German mathematician Carl Friedrich Gauss. It involves performing a series of row operations on the system's augmented matrix to transform it into row-echelon form. Once the matrix is in this simplified form, the solution to the system can be determined through back substitution. This method is widely used due to its efficiency and clarity in solving linear systems.

Example 14: Solve the following system of equations by Gaussian Elimination method:

$$x + 2y + z = 0$$
$$2x + 3y + 4z = 0$$
$$4x + 3y + 2z = 0$$

Solution:

$$\text{Augmented matrix} = A_b = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -5 & -2 & 0 \end{array}\right] \quad \text{By } R_2 + (-2)R_1 \to R_2' \text{ and } R_3 + (-4)R_1 \to R_3'$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -5 & -2 & 0 \end{array}\right] \quad \text{By } (-1)R_2 \to R_2'$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -12 & 0 \end{array}\right] \quad \text{By } R_3 + 5R_2 \to R_3'$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{By } \frac{-1}{12}R_3 \to R_3' \quad (\text{Rank of } A = 3 = \text{ number of variables})$$

The given system is reduced to equivalent system

$$x + 2y + z = 0 \qquad \ldots (i)$$
$$y - 2z = 0 \qquad \ldots (ii)$$
$$z = 0$$

Put $z = 0$ in equation (2), we have

$$y - 2(0) = 0 \Rightarrow y = 0$$

Put y = 0 and $z = 0$ in equation (2), we have

$$x + 2(0) + 0 = 0 \Rightarrow x = 0$$

Thus, the system has only trivial solution, i.e., $(x, y, z) = (0, 0, 0)$.

Example 15: Solve the following system of equations using Gaussian Elimination Method.

$$x_1 + x_2 + x_3 = 0$$
$$x_1 - x_2 + 3x_3 = 0$$
$$x_1 + 3x_2 - x_3 = 0$$

Solution:

$$\text{Augmented matrix} = A_b = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 \\ 1 & 3 & -1 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \quad \text{By } R_2 + (-1)R_1 \to R_2' \text{ and } R_3 + (-1)R_1 \to R_3'$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \quad \text{By} \left(-\frac{1}{2}\right) R_2 \to R_2'$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{By } R_3 + (-2)R_2 \to R_3' \quad (\text{Rank of } A < \text{ number of variables})$$

The given system is reduced to equivalent system

$$x_1 + x_2 + x_3 = 0 \qquad \ldots (i)$$
$$x_2 - x_3 = 0 \qquad \ldots (ii)$$
$$0x_3 = 0 \Rightarrow x_3 = t \text{, where } t \text{ is any real number.}$$

Put $x_3 = t$ in equation (2), we have

$$x_2 - t = 0 \qquad \Rightarrow x_2 = t$$

Put $x_2 = t$ and $x_3 = t$ in equation (1), we have

$$x_1 + t + t = 0 \qquad \Rightarrow x_1 = -2t$$

Hence $x_1 = -2t$, $x_2 = t$ and $x_3 = t$ for any value of t.

From above examples we observe that:

Rule - I: Homogeneous system of linear equation has only trivial solution if rank of $A =$ number of variables.

Rule - II: Homogeneous system of linear equation has non-trivial solution if rank of $A <$ number of variables.

Applications of Matrices in Real World

Matrices play a crucial role in solving real-world problems across various fields. In graphic design, they help manipulate images through transformations like scaling, rotation, and reflection. Data encryption and cryptography use matrices for secure communication by encoding and decoding messages. In seismic analysis, engineers use matrices to model and predict earthquake wave behavior. Geometric transformations, such as translation and dilation, rely on matrices to modify shapes in computer graphics. Additionally, social network analysis leverages matrices to represent and analyze relationships between individuals, identifying key influencers and connections in a network.

Transformation or Reflection Matrix is a mathematical tool that represents the reflection of a point or object across a mirror line in a coordinate plane. It's a matrix representation of a reflection transformation. In two dimensions, this typically means reflecting across the x-axis, y-axis or a line such as $y = x$.

To reflect a matrix over the x-axis, we have multiply it by $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

To reflect a matrix over the y-axis, we have multiply it by $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$

To reflect a matrix over the line $y = x$, we have multiply it by $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

Example 16: A triangle has the vertices $A(2,3), B(-1,4)$ and $C(3,-2)$. Find the vertices of the reflected triangle over the x-axis by using transformation matrix.

Solution:

Given vertices of a triangle are $A(2,3), B(-1,4)$ and $C(3,-2)$

Column matrices of given vertices are $A=\begin{bmatrix}2\\3\end{bmatrix}, B=\begin{bmatrix}-1\\4\end{bmatrix}, C=\begin{bmatrix}3\\-2\end{bmatrix}$

To reflect the given points over the x-axis, we use the transformation matrix $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

The vertices A', B' and C' of the reflected triangle are

$$A' = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2+0\\0-3\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix} = (2,-3)$$

$$B' = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-1\\4\end{bmatrix} = \begin{bmatrix}-1+0\\0-4\end{bmatrix} = \begin{bmatrix}-1\\-4\end{bmatrix} = (-1,-4)$$

$$C' = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}3-0\\0+2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix} = (3,2)$$

Thus, the vertices of the reflected triangle are $A'(2,-3), B'(-1,-4)$ and $C'(3,2)$.

Coding: Coding is the process of converting a message into a specific format using a code. A code is a system of symbols, words or signals used to represent other words or meanings. It's often used to hide the actual meaning of a message.

To **decode** a message, we multiply coded matrix by the inverse of the given matrix.

Example 17: Use matrix $A=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}$ to encode the message: ATTACK, where letters A to Z are corresponding to the numbers 1 to 26.

Solution: Here

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*
1	2	3	4	5	6	7	8	9	10	11	12	13
N	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*
14	15	16	17	18	19	20	21	22	23	24	25	26

Message: ATTACK

Divide the letters of the message into groups of two.

AT TA CK

Column matrices with assigned number are

$$\begin{bmatrix}A\\T\end{bmatrix}=\begin{bmatrix}1\\20\end{bmatrix} \;;\; \begin{bmatrix}T\\A\end{bmatrix}=\begin{bmatrix}20\\1\end{bmatrix} \;;\; \begin{bmatrix}C\\K\end{bmatrix}=\begin{bmatrix}3\\11\end{bmatrix}$$

So, the message in 2×1 matrices is $\begin{bmatrix}1\\20\end{bmatrix}\begin{bmatrix}20\\1\end{bmatrix}\begin{bmatrix}3\\11\end{bmatrix}$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A i.e.,

$$\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}1\\20\end{bmatrix} = \begin{bmatrix}1 & + & 40\\ 3 & + & 20\end{bmatrix} = \begin{bmatrix}41\\23\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}20\\1\end{bmatrix} = \begin{bmatrix}20+2\\60+1\end{bmatrix} = \begin{bmatrix}22\\61\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}3\\11\end{bmatrix} = \begin{bmatrix}3+22\\9+11\end{bmatrix} = \begin{bmatrix}25\\20\end{bmatrix}$$

So, the desired coded message is $\begin{bmatrix}41\\23\end{bmatrix}\begin{bmatrix}22\\61\end{bmatrix}\begin{bmatrix}25\\20\end{bmatrix}$

Exercise 4.3

1. Find the inverses of the following matrices by using row operations:

(i) $\begin{bmatrix}2 & 6 & -3\\ 0 & -2 & 0\\ -2 & 5 & 6\end{bmatrix}$

Solution:

Let $A = \begin{bmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & 4 & 6\end{bmatrix}$

$$|A| = \begin{vmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & 4 & 6\end{vmatrix}$$

Expanding by R_1

$$|A| = 1\begin{vmatrix}-2 & 0\\ 4 & 6\end{vmatrix} - 2\begin{vmatrix}0 & 0\\ -2 & 6\end{vmatrix} - 3\begin{vmatrix}0 & -2\\ -2 & 4\end{vmatrix}$$

$$|A| = 1(-12-0) - 2(0-0) - 3(0-4)$$
$$= -12 + 0 + 12 = 0$$

As $|A| = 0$, so A^{-1} does not exist.

(ii) $\begin{bmatrix}1 & 2 & -1\\ 0 & -2 & 8\\ 1 & 0 & 2\end{bmatrix}$

Solution:

Let $A = \begin{bmatrix}1 & 2 & -1\\ 0 & -2 & 8\\ 1 & 0 & 2\end{bmatrix}$

$$|A| = \begin{vmatrix}1 & 2 & -1\\ 0 & -2 & 8\\ 1 & 0 & 2\end{vmatrix}$$

Expanding by R_1

$$|A| = 1(-4-0) - 2(0-8) - 1(0+2)$$
$$= -4 + 16 - 2 = 10$$

As $|A| \neq 0$, so A^{-1} exists.

Consider $[A \vdots I] = \left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -2 & 8 & 0 & 1 & 0\\ 1 & 0 & 2 & 0 & 0 & 1\end{array}\right]$

By $R_3 + (-1)R_1 \rightarrow R_3'$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -2 & 8 & 0 & 1 & 0\\ 0 & -2 & 3 & -1 & 0 & 1\end{array}\right]$$

By $-\frac{1}{2}R_2 \rightarrow R_2'$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & -4 & 0 & -\frac{1}{2} & 0\\ 0 & -2 & 3 & -1 & 0 & 1\end{array}\right]$$

By $R_1 + (-2)R_2 \rightarrow R_1'$ and $R_3 + 2R_2 \rightarrow R_3'$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & 0 & 7 & 1 & 1 & 0\\ 0 & 1 & -4 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & -5 & -1 & -1 & 1\end{array}\right]$$

By $-\frac{1}{5}R_3 \rightarrow R_3'$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & 0 & 7 & 1 & 1 & 0\\ 0 & 1 & -4 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5}\end{array}\right]$$

By $R_1+(-7)R_3\to R_1'$ and $R_2+4R_3\to R_2'$

$$R\begin{bmatrix}1&0&0&\vdots&-\frac{2}{5}&-\frac{2}{5}&\frac{7}{5}\\0&1&0&\vdots&\frac{4}{5}&\frac{3}{10}&-\frac{4}{5}\\0&0&1&\vdots&\frac{1}{5}&\frac{1}{5}&-\frac{1}{5}\end{bmatrix}=[I\ \vdots\ A^{-1}]$$

Thus $A^{-1}=\begin{bmatrix}-\frac{2}{5}&-\frac{2}{5}&\frac{7}{5}\\\frac{4}{5}&\frac{3}{10}&-\frac{4}{5}\\\frac{1}{5}&\frac{1}{5}&-\frac{1}{5}\end{bmatrix}$

(iii) $\begin{bmatrix}1&6&2\\2&13&0\\0&-1&1\end{bmatrix}$

Solution:

Let $A=\begin{bmatrix}1&6&2\\2&13&0\\0&-1&1\end{bmatrix}$

$$|A|=\begin{vmatrix}1&6&2\\2&13&0\\0&-1&1\end{vmatrix}$$

Expanding by R_1

$$|A|=1(13-0)-6(2-0)+2(-2-0)$$
$$=13-12-4=-3$$

As $|A|\neq 0$, so A^{-1} exists.

Consider $[A\ \vdots\ I]=\begin{bmatrix}1&6&2&\vdots&1&0&0\\2&13&0&\vdots&0&1&0\\0&-1&1&\vdots&0&0&1\end{bmatrix}$

By $R_2+(-2)R_1\to R_2'$

$$R\begin{bmatrix}1&6&2&\vdots&1&0&0\\0&1&-4&\vdots&-2&1&0\\0&-1&1&\vdots&0&0&1\end{bmatrix}$$

By $R_1+(-6)R_2\to R_1'$ and $R_3+R_2\to R_3'$

$$R\begin{bmatrix}1&0&26&\vdots&13&-6&0\\0&1&-4&\vdots&-2&1&0\\0&0&-3&\vdots&-2&1&1\end{bmatrix}$$

By $-\frac{1}{3}R_3\to R_3'$

$$R\begin{bmatrix}1&0&26&\vdots&13&-6&0\\0&1&-4&\vdots&-2&1&0\\0&0&1&\vdots&\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}\end{bmatrix}$$

By $R_1+(-26)R_3\to R_1'$ and $R_2+4R_3\to R_2'$

$$R\begin{bmatrix}1&0&0&\vdots&-\frac{13}{3}&\frac{8}{3}&\frac{26}{3}\\0&1&0&\vdots&\frac{2}{3}&-\frac{1}{3}&-\frac{4}{3}\\0&0&1&\vdots&\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}\end{bmatrix}=[I\ \vdots\ A^{-1}]$$

Thus $A^{-1}=\begin{bmatrix}-\frac{13}{3}&\frac{8}{3}&\frac{26}{3}\\\frac{2}{3}&-\frac{1}{3}&-\frac{4}{3}\\\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}\end{bmatrix}$

2. Find the rank of the following matrices:

(i) $\begin{bmatrix}1&-1&3&1\\-2&-6&1&-1\\3&1&4&-2\end{bmatrix}$

Solution:

$$\begin{bmatrix}1&-1&3&1\\-2&-6&1&-1\\3&1&4&-2\end{bmatrix}$$

By $R_2+2R_1\to R_2'$ and $R_3+(-3)R_1\to R_3'$

$$R\begin{bmatrix}1&-1&3&1\\0&-8&7&1\\0&4&-5&-5\end{bmatrix}$$

By $-\frac{1}{8}R_2\to R_2'$

$$R\begin{bmatrix}1&-1&3&1\\0&1&-\frac{7}{8}&-\frac{1}{8}\\0&4&-5&-5\end{bmatrix}$$

By $R_3+(-4)R_2\to R_3'$

$$R\begin{bmatrix}1&-1&3&1\\0&1&-\frac{7}{8}&-\frac{1}{8}\\0&0&-\frac{3}{2}&-\frac{9}{2}\end{bmatrix}$$

By $\frac{-2}{3}R_3\to R_3'$

$$R\begin{bmatrix}1&-1&3&1\\0&1&-\frac{7}{8}&-\frac{1}{8}\\0&0&1&3\end{bmatrix}\text{ (Echelon Form)}$$

As the number of non-zero rows is 3, so
Rank of matrix = 3

(ii) $\begin{bmatrix}1&-2&3\\2&-4&6\\-1&0&2\\0&1&-1\end{bmatrix}$

Solution:

$$\begin{bmatrix}1&-2&3\\2&-4&6\\-1&0&2\\0&1&-1\end{bmatrix}$$

By $R_2+(-2)R_1\to R_2', R_3+R_1\to R_3'$

$$R\begin{bmatrix}1&-2&3\\0&0&0\\0&-2&5\\0&1&-1\end{bmatrix}$$

By $R_2\leftrightarrow R_4$, we have

$$R\begin{bmatrix}1&-2&3\\0&1&-1\\0&-2&5\\0&0&0\end{bmatrix}$$

By $R_3+2R_2\to R_3'$

$$R\begin{bmatrix}1&-2&3\\0&1&-1\\0&0&3\\0&0&0\end{bmatrix}\text{ (Echelon Form)}$$

As the number of non-zero rows is 3, so
Rank of matrix = 3

(iii) $\begin{bmatrix}3&-1&3&0&1\\1&2&-1&-3&-2\\2&3&4&2&1\\2&5&-2&-3&3\end{bmatrix}$

Solution:

$$\begin{bmatrix}3&-1&3&0&1\\1&2&-1&-3&-2\\2&3&4&2&1\\2&5&-2&-3&3\end{bmatrix}$$

By $R_1\leftrightarrow R_2$, we have

$$R\begin{bmatrix}1&2&-1&-3&-2\\3&-1&3&0&1\\2&3&4&2&1\\2&5&-2&-3&3\end{bmatrix}$$

By $R_2+(-3)R_1\to R_2', R_3+(-2)R_1\to R_3',\ R_4+(-2)R_1\to R_4'$

$$R\begin{bmatrix}1&2&-1&-3&-2\\0&-7&6&9&7\\0&-1&6&8&5\\0&1&0&3&7\end{bmatrix}$$

By $R_2\leftrightarrow R_4$, we have

$$R\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&-1&6&8&5\\0&-7&6&9&7\end{bmatrix}$$

By $R_3+R_2\to R_3'$ and $R_4+7R_2\to R_4'$

$$R\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&0&6&11&12\\0&0&6&30&56\end{bmatrix}$$

By $\frac{1}{6}R_3\to R_3'$

$$R\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&0&1&\frac{11}{6}&2\\0&0&6&30&56\end{bmatrix}$$

By $R_4+(-6)R_3\to R_4'$

$$R\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&0&1&\frac{11}{6}&2\\0&0&0&19&44\end{bmatrix}$$

By $\frac{1}{19}R_4\to R_4'$

$$R\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&0&1&\frac{11}{6}&2\\0&0&0&1&\frac{44}{19}\end{bmatrix}$$

As the number of non-zero rows is 4, so
Rank of matrix = 4

3. Solve the following systems of linear equations by Cramer's rule:

(i) $\left.\begin{aligned}2x+y-z&=1\\x-y+2z&=3\\3x+2y+z&=4\end{aligned}\right\}$

Solution:

$$2x+y-z=1$$
$$x-y+2z=3$$
$$3x+2y+z=4$$

Here $|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$

Expanding by R_1

$$|A| = 2\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= 2(-1-4) - 1(1-6) - 1(2+3)$$

$$= 2(-5) - 1(-5) - 1(5)$$

$$|A| = -10 + 5 - 5 = -10 \neq 0$$

Now,

$$|A_x| = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -1 & 2 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix}$$

$$= 1(-1-4) - 1(3-8) - 1(6+4)$$

$$= -5 + 5 - 10$$

$\Rightarrow \quad x = \dfrac{|A_x|}{|A|} = \dfrac{-10}{-10} = 1$

$$|A_y| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= 2(3-8) - 1(1-6) - 1(4-9)$$

$$= -10 + 5 + 5 = 0$$

$\Rightarrow \quad y = \dfrac{|A_y|}{|A|} = \dfrac{0}{-10} = 0$

$$|A_z| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= 2(-4-6) - 1(4-9) + 1(2+3)$$

$$= -20 + 5 + 5 = -10$$

$\Rightarrow \quad z = \dfrac{|A_z|}{|A|} = \dfrac{-10}{-10} = 1$

Hence $x = 1, y = 0, z = 1$

Thus, S.S = $\{(1, 0, 1)\}$

(ii) $\left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 5 \\ -2x_1 + 3x_2 + 2x_3 &= 3 \end{aligned}\right\}$

Solution:

$$x_1 + 2x_2 - 3x_3 = 0$$
$$4x_1 - x_2 + x_3 = 5$$
$$-2x_1 + 3x_2 + 2x_3 = 3$$

Here $|A| = \begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ -2 & 3 & 2 \end{vmatrix}$

Expanding by R_1

$$|A| = 1\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ -2 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix}$$

$$= 1(-2-3) - 2(8+2) - 3(12-2)$$

$$|A| = -5 - 20 - 30 = -55 \neq 0$$

Now,

$$|A_{x_1}| = \begin{vmatrix} 0 & 2 & -3 \\ 5 & -1 & 1 \\ 3 & 3 & 2 \end{vmatrix}$$

$$= 0\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 5 & 1 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 5 & -1 \\ 3 & 3 \end{vmatrix}$$

$$= 0 - 2(10-3) - 3(15+3)$$

$$= -14 - 54 = -68$$

$\Rightarrow \quad x_1 = \dfrac{|A_{x_1}|}{|A|} = \dfrac{-68}{-55} = \dfrac{68}{55}$

$$|A_{x_2}| = \begin{vmatrix} 1 & 0 & -3 \\ 4 & 5 & 1 \\ -2 & 3 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & 1 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 1 \\ -2 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ -2 & 3 \end{vmatrix}$$

$$= 1(10-3) - 0 - 3(12+10)$$

$$= 7 - 66 = -59$$

$\Rightarrow \quad x_2 = \dfrac{|A_{x_2}|}{|A|} = \dfrac{-59}{-55} = \dfrac{59}{55}$

$$|A_{x_3}| = \begin{vmatrix} 1 & 2 & 0 \\ 4 & -1 & 5 \\ -2 & 3 & 3 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 5 \\ 3 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ -2 & 3 \end{vmatrix} + 0\begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix}$$

$$= 1(-3-15) - 2(12+10) + 0$$

$$= -18 - 44 = -62$$

$\Rightarrow \quad x_3 = \dfrac{|A_{x_3}|}{|A|} = \dfrac{-62}{-55} = \dfrac{62}{55}$

Hence $x_1 = \dfrac{68}{55}, x_2 = \dfrac{59}{55}, x_3 = \dfrac{62}{55}$

Thus, S.S = $\left\{\left(\dfrac{68}{55}, \dfrac{59}{55}, \dfrac{62}{55}\right)\right\}$

(iii) $\left.\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ x_1 + 2x_2 + 2x_3 &= 2 \\ x_1 - 2x_2 - x_3 &= 1 \end{aligned}\right\}$

Solution:

$$2x_1 - x_2 + x_3 = 1$$
$$x_1 + 2x_2 + 2x_3 = 2$$
$$x_1 - 2x_2 - x_3 = 1$$

Here $|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$

Expanding by R_1

$$|A| = 2\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 2(-2+4) + 1(-1-2) + 1(-2-2)$$

$$= 4 - 3 - 4 = -3 \neq 0$$

Now,

$$|A_{x_1}| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 1(-2+4) + 1(-2-2) + 1(-4-2)$$

$$= 2 - 4 - 6 = -8$$

$\Rightarrow \quad x_1 = \dfrac{|A_{x_1}|}{|A|} = \dfrac{-8}{-3} = \dfrac{8}{3}$

$$|A_{x_2}| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= 2(-2-2) - 1(-1-2) + 1(1-2)$$

$$= -8 + 3 - 1 = -6$$

$\Rightarrow \quad x_2 = \dfrac{|A_{x_2}|}{|A|} = \dfrac{-6}{-3} = 2$

$$|A_{x_3}| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 2(2+4) + 1(1-2) + 1(-2-2)$$

$$= 12 - 1 - 4 = 7$$

$\Rightarrow \quad x_3 = \dfrac{|A_{x_3}|}{|A|} = \dfrac{7}{-3} = \dfrac{-7}{3}$

Hence $x_1 = \dfrac{8}{3}, x_2 = 2, x_3 = -\dfrac{7}{3}$

Thus, S.S = $\left\{\left(\dfrac{8}{3}, 2, \dfrac{-7}{3}\right)\right\}$

---

4. Solve the following systems of linear equations by matrix inversion method:

(i) $\left.\begin{aligned} x - 2y + z &= -1 \\ 3x + y - 2z &= 4 \\ y - z &= 1 \end{aligned}\right\}$

Solution:

In matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

Let $AX = B$,

where $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

$\Rightarrow \quad X = A^{-1}B \qquad \ldots(1)$

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 1(-1+2) + 2(-3+0) + 1(3-0)$ Expanding by R_1

$= 1 - 6 + 3 = -2$

Since $|A| \neq 0$, so we can find A^{-1}

adj A = (Matrix of co-factors)^t = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = 1(-1+2) = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = -1(-3-0) = 3$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = 1(3-0) = 3$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}-2 & 1\\ 1 & -1\end{vmatrix}=-1(2-1)=-1$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\ 0 & -1\end{vmatrix}=1(-1-0)=-1$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix}=-1(1-0)=-1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}-2 & 1\\ 1 & -2\end{vmatrix}=1(4-1)=3$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\ 3 & -2\end{vmatrix}=-1(-2-3)=5$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix}=1(1+6)=7$$

Now, $\text{adj}A=\begin{bmatrix}1 & 3 & 3\\ -1 & -1 & -1\\ 3 & 5 & 7\end{bmatrix}^t=\begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}$

We know that

$$A^{-1}=\frac{1}{|A|}.(\text{adj}\,A)=\frac{1}{-2}\begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}$$

Putting values in equation (1)

$$X=\frac{1}{-2}\begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}\begin{bmatrix}-1\\ 4\\ 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-1-4+3\\ -3-4+5\\ -3-4+7\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-2\\ -2\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$\Rightarrow \quad x=1,\ y=1,\ z=0$

Thus, S.S = {(1, 1, 0)}

$$\text{(ii)}\quad \left.\begin{aligned}2x_1+x_2+3x_3&=3\\ x_1+3x_2-2x_3&=0\\ -3x_1-x_2+2x_3&=4\end{aligned}\right\}$$

Solution:

In matrix form

$$\begin{bmatrix}2 & 1 & 3\\ 1 & 3 & -2\\ -3 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}3\\ 0\\ 4\end{bmatrix}$$

Let $AX=B$ …(i)

where $A=\begin{bmatrix}2 & 1 & 3\\ 1 & 3 & -2\\ -3 & -1 & 2\end{bmatrix}$, $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 0\\ 4\end{bmatrix}$

$$|A|=\begin{vmatrix}2 & 1 & 3\\ 1 & 3 & -2\\ -3 & -1 & 2\end{vmatrix}$$

Expanding by R_1

$$|A|=2\begin{vmatrix}3 & -2\\ -1 & 2\end{vmatrix}-1\begin{vmatrix}1 & -2\\ -3 & 2\end{vmatrix}+3\begin{vmatrix}1 & 3\\ -3 & -1\end{vmatrix}$$

$$=2(6-2)-1(2-6)+3(-1+9)$$

$$=8+4+24=36$$

As $|A|\neq 0$, so A^{-1} exists

Equation (i) $\Rightarrow X=A^{-1}B$ …(ii)

Cofactors of elements of A are

$$A_{11}=(-1)^{1+1}\begin{vmatrix}3 & -2\\ -1 & 2\end{vmatrix}=1(6-2)=4$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}1 & -2\\ -3 & 2\end{vmatrix}=-1(2-6)=4$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}1 & 3\\ -3 & -1\end{vmatrix}=1(-1+9)=8$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}1 & 3\\ -1 & 2\end{vmatrix}=-1(2+3)=-5$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}2 & 3\\ -3 & 2\end{vmatrix}=1(4+9)=13$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}2 & 1\\ -3 & -1\end{vmatrix}=-1(-2+3)=-1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}1 & 3\\ 3 & -2\end{vmatrix}=1(-2-9)=-11$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}=-1(-4-3)=7$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix}=1(6-1)=5$$

$$\text{Matrix of cofactors of } A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$$

$$=\begin{bmatrix}4 & 4 & 8\\ -5 & 13 & -1\\ -11 & 7 & 5\end{bmatrix}$$

$$\text{adj}\,(A)=(\text{Matrix of cofactors of } A)^t$$

$$=\begin{bmatrix}4 & -5 & -11\\ 4 & 13 & 7\\ 8 & -1 & 5\end{bmatrix}$$

As we know

$$A^{-1}=\frac{1}{|A|}\text{adj}(A)$$

$$=\frac{1}{36}\begin{bmatrix}4 & -5 & -11\\ 4 & 13 & 7\\ 8 & -1 & 5\end{bmatrix}$$

Putting values in eq. (ii), we have

$$X=\frac{1}{36}\begin{bmatrix}4 & -5 & -11\\ 4 & 13 & 7\\ 8 & -1 & 5\end{bmatrix}\begin{bmatrix}3\\ 0\\ 4\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\frac{1}{36}\begin{bmatrix}12+0-44\\ 12+0+28\\ 24+0+20\end{bmatrix}=\frac{1}{36}\begin{bmatrix}-32\\ 40\\ 44\end{bmatrix}=\begin{bmatrix}-\frac{8}{9}\\ \frac{10}{9}\\ \frac{11}{9}\end{bmatrix}$$

Hence $x_1=-\frac{8}{9}, x_2=\frac{10}{9}, x_3=\frac{11}{9}$

Thus, $\text{S.S}=\left\{\left(-\frac{8}{9},\frac{10}{9},\frac{11}{9}\right)\right\}$

$$\text{il)}\quad \left.\begin{aligned}x+y\ &=2\\ 2x-z\ &=1\\ 2y-3z\ &=-1\end{aligned}\right\}$$

olution:

matrix form

$$\begin{bmatrix}1 & 1 & 0\\ 2 & 0 & -1\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$$

et $AX=B$,

here $A=\begin{bmatrix}1 & 1 & 0\\ 2 & 0 & -1\\ 0 & 2 & -3\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, $B=\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$

$X=A^{-1}B$ …(1)

$$|A|=\begin{vmatrix}1 & 1 & 0\\ 2 & 0 & -1\\ 0 & 2 & -3\end{vmatrix}=1(0+2)-1(-6+0)+0(4-0)$$

Expanding by R_1

$|A|=2+6+0=8$

nce $|A|\neq 0$, so we can find A^{-1}

j A = (Matrix of co-factors)^t

$$=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^t$$

$$_{11}=(-1)^{1+1}\begin{vmatrix}0 & -1\\ 2 & -3\end{vmatrix}=1(0+2)=2$$

$$_{12}=(-1)^{1+2}\begin{vmatrix}2 & -1\\ 0 & -3\end{vmatrix}=-1(-6-0)\ 6$$

$$_{13}=(-1)^{1+3}\begin{vmatrix}2 & 0\\ 0 & 2\end{vmatrix}=1(4-0)=4$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}1 & 0\\ 2 & -3\end{vmatrix}=-1(-3-0)=3$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 0\\ 0 & -3\end{vmatrix}=1(-3-0)=-3$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}=-1(2-0)=-2$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix}=1(-1-0)=-1$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 0\\ 2 & -1\end{vmatrix}=-1(-1-0)=1$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}=1(0-2)=-2$$

Now,

$$\text{adj}A=\begin{bmatrix}2 & 6 & 4\\ 3 & -3 & -2\\ -1 & 1 & -2\end{bmatrix}^t=\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}$$

We know that $A^{-1}=\frac{1}{|A|}.\text{adj}A=\frac{1}{8}\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}$

Putting values in equation (1)

$$X=\frac{1}{8}\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{8}\begin{bmatrix}4+3+1\\ 12-3-1\\ 8-2+2\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}8\\ 8\\ 8\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$\Rightarrow \quad x=1, y=1, z=1$

Thus S.S = {(1, 1, 1)}

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5. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms:

$$\text{(i)}\quad \left.\begin{aligned}x_1+2x_2-2x_3\ &=-1\\ 2x_1+3x_2+x_3\ &=1\\ 5x_1+4x_2-3x_3\ &=1\end{aligned}\right\}$$

Solution:

$$x_1+2x_2-2x_3=-1$$

$$2x_1+3x_2+x_3=1$$

$$5x_1+4x_2-3x_3=1$$

Augmented matrix = $A_b = \begin{bmatrix} 1 & 2 & -2 & \vdots & -1 \\ 2 & 3 & 1 & \vdots & 1 \\ 5 & 4 & -3 & \vdots & 1 \end{bmatrix}$

Solution by echelon form:

By $R_2+(-2)R_1 \to R_2'$ and $R_3+(-5)R_1 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & \vdots & -1 \\ 0 & -1 & 5 & \vdots & 3 \\ 0 & -6 & 7 & \vdots & 6 \end{bmatrix}$$

By $(-1)R_2 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & \vdots & -1 \\ 0 & 1 & -5 & \vdots & -3 \\ 0 & -6 & 7 & \vdots & 6 \end{bmatrix}$$

By $R_3+6R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & \vdots & -1 \\ 0 & 1 & -5 & \vdots & -3 \\ 0 & 0 & -23 & \vdots & -12 \end{bmatrix}$$

By $-\frac{1}{23}R_3 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & \vdots & -1 \\ 0 & 1 & -5 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & \frac{12}{23} \end{bmatrix} \quad \ldots(a)$$

Implies that

$$x_1+2x_2-2x_3 = -1 \quad \ldots(1)$$

$$x_2-5x_3 = -3 \quad \ldots(2)$$

$$x_3 = \frac{12}{23}$$

Put $x_3 = \frac{12}{23}$ in equation (2)

$$x_2 - 5\left(\frac{12}{23}\right) = -3$$

$$x_2 = -3+\frac{60}{23} = \frac{69+60}{23}$$

$$\Rightarrow \quad x_2 = -\frac{9}{23}$$

Put $x_2 = -\frac{9}{23}, x_3 = \frac{12}{23}$ in equation (1)

$$x_1 + 2\left(-\frac{9}{23}\right) - 2\left(\frac{12}{23}\right) = -1$$

$$x_1 = -1+\frac{18}{23}+\frac{24}{23}$$

$$= \frac{-23+18+24}{23}$$

$$\Rightarrow \quad x_1 = \frac{19}{23}$$

Thus S.S = $\left\{\left(\frac{19}{23}, -\frac{9}{23}, \frac{12}{23}\right)\right\}$

Solution by reduced echelon form:

From (a), we have

$$\begin{bmatrix} 1 & 2 & -2 & \vdots & -1 \\ 0 & 1 & -5 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & \frac{12}{23} \end{bmatrix}$$

By $R_1+(-2)R_2 \to R_1$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 8 & \vdots & 5 \\ 0 & 1 & -5 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & \frac{12}{23} \end{bmatrix}$$

By $R_1+(-8)R_3 \to R_1'$t and $R_2+5R_3 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{19}{23} \\ 0 & 1 & 0 & \vdots & -\frac{9}{23} \\ 0 & 0 & 1 & \vdots & \frac{12}{23} \end{bmatrix}$$

$$\Rightarrow \quad x_1 = \frac{19}{23}, x_2 = \frac{-9}{23} \text{ and } x_3 = \frac{12}{23}$$

Thus, S.S = $\left\{\left(\frac{19}{23}, -\frac{9}{23}, \frac{12}{23}\right)\right\}$

$$\textbf{(ii)} \quad \left.\begin{aligned} x+2y+z &= 2 \\ 2x+y+2z &= 3 \\ 2x+3y-z &= 7 \end{aligned}\right\}$$

Solution:

$$x+2y+z = 2$$

$$2x+y+2z = 3$$

$$2x+3y-z = 7$$

Augmented matrix = $A_b = \begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 2 & 1 & 2 & \vdots & 3 \\ 2 & 3 & -1 & \vdots & 7 \end{bmatrix}$

Solution by echelon form:

By $R_2+(-2)R_1 \to R_2'$ and $R_3+(-2)R_1 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & -3 & 0 & \vdots & -1 \\ 0 & -1 & -3 & \vdots & 3 \end{bmatrix}$$

By $-\frac{1}{3}R_2 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & \frac{1}{3} \\ 0 & -1 & -3 & \vdots & 3 \end{bmatrix}$$

By $R_3+R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & \frac{1}{3} \\ 0 & 0 & -3 & \vdots & \frac{10}{3} \end{bmatrix}$$

By $-\frac{1}{3}R_3 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & \frac{1}{3} \\ 0 & 0 & 1 & \vdots & -\frac{10}{9} \end{bmatrix} \quad \ldots(a)$$

Implies that

$$x+2y+z = 2 \quad \ldots(1)$$

$$y = \frac{1}{3} \text{ and } z = -\frac{10}{9}$$

Put $y = \frac{1}{3}$ and $z = -\frac{10}{9}$ in equation (1)

$$x + 2\left(\frac{1}{3}\right) - \frac{10}{9} = 2$$

$$x = 2 - \frac{2}{3} + \frac{10}{9}$$

$$= \frac{18-6+10}{9}$$

$$\Rightarrow \quad x = \frac{22}{9}$$

Thus, S.S = $\left\{\left(\frac{22}{9}, \frac{1}{3}, -\frac{10}{9}\right)\right\}$

Solution by reduced echelon form:

From (a), we have

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & \frac{1}{3} \\ 0 & 0 & 1 & \vdots & -\frac{10}{9} \end{bmatrix}$$

By $R_1+(-2)R_2 \to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 & \vdots & \frac{4}{3} \\ 0 & 1 & 0 & \vdots & \frac{1}{3} \\ 0 & 0 & 1 & \vdots & -\frac{10}{9} \end{bmatrix}$$

By $R_1+(-1)R_3 \to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{22}{9} \\ 0 & 1 & 0 & \vdots & \frac{1}{3} \\ 0 & 0 & 1 & \vdots & -\frac{10}{9} \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{22}{9}, y = \frac{1}{3} \text{ and } z = -\frac{10}{9}$$

Thus, S.S = $\left\{\left(\frac{22}{9}, \frac{1}{3}, -\frac{10}{9}\right)\right\}$

$$\textbf{(iii)} \quad \left.\begin{aligned} x_1+4x_2+x_3 &= 2 \\ 2x_1+x_2-2x_3 &= 9 \\ 3x_1+x_2-x_3 &= 12 \end{aligned}\right\}$$

Solution:

$$x_1+4x_2+x_3 = 2$$

$$2x_1+x_2-2x_3 = 9$$

$$3x_1+x_2-x_3 = 12$$

Augmented matrix = $A_b = \begin{bmatrix} 1 & 4 & 1 & \vdots & 2 \\ 2 & 1 & -2 & \vdots & 9 \\ 3 & 1 & -1 & \vdots & 12 \end{bmatrix}$

Solution by echelon form:

By $R_2+(-2)R_1 \to R_2'$ and $R_3+(-3)R_1 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & \vdots & 2 \\ 0 & -7 & -4 & \vdots & 5 \\ 0 & -11 & -4 & \vdots & 6 \end{bmatrix}$$

By $-\frac{1}{7}R_2 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & \vdots & 2 \\ 0 & 1 & \frac{4}{7} & \vdots & -\frac{5}{7} \\ 0 & -11 & -4 & \vdots & 6 \end{bmatrix}$$

By $R_3+11R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & \vdots & 2 \\ 0 & 1 & \frac{4}{7} & \vdots & -\frac{5}{7} \\ 0 & 0 & \frac{16}{7} & \vdots & -\frac{13}{7} \end{bmatrix}$$

By $\frac{7}{16}R_3 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & \vdots & 2 \\ 0 & 1 & \frac{4}{7} & \vdots & -\frac{5}{7} \\ 0 & 0 & 1 & \vdots & -\frac{13}{16} \end{bmatrix} \qquad \ldots(a)$$

Implies that

$$x_1 + 4x_2 + x_3 = 2 \qquad \ldots(1)$$

$$x_2 + \frac{4}{7}x_3 = -\frac{5}{7} \qquad \ldots(2)$$

$$x_3 = -\frac{13}{16}$$

Put $x_3 = -\frac{13}{16}$ in equation (2)

$$x_2 + \frac{4}{7}\left(-\frac{13}{16}\right) = -\frac{5}{7}$$

$$x_2 = -\frac{5}{7} + \frac{13}{28}$$

$$= \frac{-20+13}{28} = \frac{-7}{28}$$

$$\Rightarrow \qquad x_2 = \frac{-1}{4}$$

Put $x_2 = \frac{-1}{4}$ and $x_3 = \frac{-13}{16}$ in equation (1)

$$x_1 + 4\left(-\frac{1}{4}\right) - \frac{13}{16} = 2$$

$$x_1 = 2 + 1 + \frac{13}{16}$$

$$= 3 + \frac{13}{16} = \frac{48+13}{16}$$

$$\Rightarrow \qquad x_1 = \frac{61}{16}$$

Thus, S.S = $\left\{\left(\frac{61}{16}, \frac{-1}{4}, \frac{13}{16}\right)\right\}$

Solution by reduced echelon form:
From (a), we have

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & \frac{4}{7} & -\frac{5}{7} \\ 0 & 0 & 1 & -\frac{13}{16} \end{bmatrix}$$

By $R_1 + (-4)R_2 \to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{-9}{7} & \vdots & \frac{34}{7} \\ 0 & 1 & \frac{4}{7} & \vdots & \frac{-5}{7} \\ 0 & 0 & 1 & \vdots & \frac{-13}{16} \end{bmatrix}$$

By $R_1 + \frac{9}{7}R_3 \to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{61}{16} \\ 0 & 1 & 0 & \vdots & -\frac{1}{4} \\ 0 & 0 & 1 & \vdots & \frac{13}{16} \end{bmatrix}$$

$\Rightarrow \quad x_1 = \frac{61}{16}, x_2 = \frac{-1}{4}$ and $x_3 = \frac{13}{16}$

Thus, S.S = $\left\{\left(\frac{61}{16}, -\frac{1}{4}, \frac{13}{16}\right)\right\}$

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6. Solve the following systems of homogeneous equations by using Gaussian elimination met

$$\text{(i)} \quad \left.\begin{aligned} x+4x-2z &= 0 \\ 2x+y+5z &= 0 \\ 5x+2y+8z &= 0 \end{aligned}\right\}$$

Solution:

$$\begin{aligned} x+4x-2z &= 0 \\ 2x+y+5z &= 0 \\ 5x+2y+8z &= 0 \end{aligned}$$

$$\text{Augmented matrix} = A_b = \begin{bmatrix} 1 & 4 & -2 & \vdots & 0 \\ 2 & 1 & 5 & \vdots & 0 \\ 5 & 2 & 8 & \vdots & 0 \end{bmatrix}$$

By $R_2 + (-2)R_1 \to R_2'$ and $R_3 + (-5)R_1 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & -2 & \vdots & 0 \\ 0 & -7 & 9 & \vdots & 0 \\ 0 & -18 & 18 & \vdots & 0 \end{bmatrix}$$

By $R_2 \leftrightarrow R_3$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & -2 & \vdots & 0 \\ 0 & -18 & 18 & \vdots & 0 \\ 0 & -7 & 9 & \vdots & 0 \end{bmatrix}$$

By $-\frac{1}{18}R_2 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & -2 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & -7 & 9 & \vdots & 0 \end{bmatrix}$$

By $R_3 + 7R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & -2 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 2 & \vdots & 0 \end{bmatrix}$$

By $\frac{1}{2}R_3 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & -2 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

Implies that

$$x + 4y - 2z = 0 \qquad \ldots(1)$$

$$y - z = 0 \qquad \ldots(2)$$

$$z = 0$$

Put $z = 0$ in equation (2)

$$y - 0 = 0 \qquad \Rightarrow y = 0$$

Put $y = 0$, $z = 0$ in equation (1)

$$x + 4(0) - 2(0) = 0$$

$$x = 0$$

$\Rightarrow \quad x = 0, y = 0, z = 0$

Thus, the system has only trivial solution. i.e.,
S.S = ((0, 0, 0))

$$\text{(ii)} \quad \left.\begin{aligned} x_1+4x_2+2x_3 &= 0 \\ 2x_1+x_2-3x_3 &= 0 \\ 3x_1+2x_2-4x_3 &= 0 \end{aligned}\right\}$$

Solution:

$$\begin{aligned} x_1+4x_2+2x_3 &= 0 \\ 2x_1+x_2-3x_3 &= 0 \\ 3x_1+2x_2-4x_3 &= 0 \end{aligned}$$

$$\text{Augmented matrix} = A_b = \begin{bmatrix} 1 & 4 & 2 & \vdots & 0 \\ 2 & 1 & -3 & \vdots & 0 \\ 3 & 2 & -4 & \vdots & 0 \end{bmatrix}$$

By $R_2 + (-2)R_1 \to R_2'$ and $R_3 + (-3)R_1 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & \vdots & 0 \\ 0 & -7 & -7 & \vdots & 0 \\ 0 & -10 & -10 & \vdots & 0 \end{bmatrix}$$

By $-\frac{1}{7}R_2 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & -10 & -10 & \vdots & 0 \end{bmatrix}$$

By $R_3 + 10R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Implies that

$$x_1 + 4x_2 + 2x_3 = 0 \qquad \ldots(1)$$

$$x_2 + x_3 = 0 \qquad \ldots(2)$$

$$0x_3 = 0$$

Let $x_3 = t$, where t is arbitrary

Put $x_3 = t$ in equation (2)

$$x_2 + t = 0 \qquad \Rightarrow x_2 = -t$$

Put $x_2 = -t$ and $x_3 = t$ in equation (1)

$$x_1 + 4(-t) + 2t = 0$$

$$\Rightarrow \qquad x_1 = 2t$$

Hence $x_1 = 2t, x_2 = -t, x_3 = t$ for any value of t.

$$\text{(iii)} \quad \left.\begin{aligned} x_1+2x_2-x_3 &= 0 \\ x_1-x_2+5x_3 &= 0 \\ 2x_1+x_2+4x_3 &= 0 \end{aligned}\right\}$$

Solution:

$$\begin{aligned} x_1+2x_2-x_3 &= 0 \\ x_1-x_2+5x_3 &= 0 \\ 2x_1+x_2+4x_3 &= 0 \end{aligned}$$

$$\text{Augmented matrix } A_b = \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 1 & -1 & 5 & \vdots & 0 \\ 2 & 1 & 4 & \vdots & 0 \end{bmatrix}$$

By $R_2 + (-1)R_1 \to R_2'$ and $R_3 + (-2)R_1 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -3 & 6 & \vdots & 0 \\ 0 & -3 & 6 & \vdots & 0 \end{bmatrix}$$

By $-\frac{1}{3}R_2 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & -3 & 6 & \vdots & 0 \end{bmatrix}$$

By $R_3 + 3R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Implies that

$$x_1 + 2x_2 - x_3 = 0 \qquad \ldots$$

$$x_2 - 2x_3 = 0 \qquad \ldots$$

$$0x_3 = 0$$

Let $x_3 = t$, where t is arbitrary

Put $x_3 = t$ in equation (2)

$$x_2 - 2t = 0 \qquad \Rightarrow x_2 =$$

Put $x_2 = 2t$ and $x_3 = t$ in equation (1)

$$x_1 + 2(2t) - t = 0 \qquad \Rightarrow x_1 = -$$

Hence $x_1 = -3t, x_2 = 2t, x_3 = t$ for any value of t.

7. A triangle has vertices at $A(4, 1)$, $B(-2, 5)$ and $C(0, -3)$. Find the vertices of the reflected triangle over the y-axis using a transformation matrix.

Solution:

Given vertices of triangle are

$$A(4,1), B(-2,5), C(0,-3)$$

Column matrices of given vertices are

$$A=\begin{bmatrix}4\\1\end{bmatrix}, B=\begin{bmatrix}-2\\5\end{bmatrix}, C=\begin{bmatrix}0\\-3\end{bmatrix}$$

To reflect the given points over the y-axis, we use the transformation matrix $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

The vertices A', B' and C' of reflected triangle are

$$A' = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}-4+0\\0+1\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}=(-4,1)$$

$$B' = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}2+0\\0+5\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}=(2,5)$$

$$C' = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\-3\end{bmatrix}=\begin{bmatrix}0+0\\0-3\end{bmatrix}=\begin{bmatrix}0\\-3\end{bmatrix}=(0,-3)$$

Hence $A'(-4, 1)$, $B'(2, 5)$ and $C'(0, -3)$

8. The point A is mapped to $(30, 20, -5)$ by the scaling matrix $P=\begin{bmatrix}-5 & 0 & 0\\0 & -5 & 0\\0 & 0 & -5\end{bmatrix}$.

Find the coordinates of A.

[Hint: If A is mapped to A' by scaling matrix P, then $AP = A'$]

Solution:

Given that:

Scaling matrix: $P=\begin{bmatrix}-5 & 0 & 0\\0 & -5 & 0\\0 & 0 & -5\end{bmatrix}$

Required point $= A(x, y, z)=\begin{bmatrix}x\\y\\z\end{bmatrix}$

Let scaled point: $A' = (30,20,-5)=\begin{bmatrix}30\\20\\-5\end{bmatrix}$

Apply the scaling:

$$\Rightarrow \quad PA = A'$$

$$\begin{bmatrix}-5 & 0 & 0\\0 & -5 & 0\\0 & 0 & -5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}30\\20\\-5\end{bmatrix}$$

$$\begin{bmatrix}-5x+0+0\\0-5y+0\\0+0-5z\end{bmatrix}=\begin{bmatrix}30\\20\\-5\end{bmatrix}$$

$$\begin{bmatrix}-5x\\-5y\\-5z\end{bmatrix}=\begin{bmatrix}30\\20\\-5\end{bmatrix}$$

On comparing, we have

$$-5x = 30 \quad \Rightarrow \quad x = -6$$
$$-5y = 20 \quad \Rightarrow \quad y = -4$$
$$-5z = -5 \quad \Rightarrow \quad z = 1$$

Hence, $A(-6, -4, 1)$

9. Find the equation of the image of the curve with equation $y = x^2$ under the transformation with associated matrix $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$.

Solution:

Given Curve: $y = x^2$

Transformation matrix: $P=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

Let $A(x, y)$ be the point which is transformed to $A'(X, Y)$ by the matrix P.

Apply the transformation, we have

$$PA = A'$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}X\\Y\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}x\\x^2\end{bmatrix}=\begin{bmatrix}X\\Y\end{bmatrix} \qquad \because y=x^2$$

$$\begin{bmatrix}x+2x^2\\3x+4x^2\end{bmatrix}=\begin{bmatrix}X\\Y\end{bmatrix}$$

$$\Rightarrow \quad x+2x^2 = X \qquad \text{...(1)}$$
$$3x+4x^2 = Y \qquad \text{...(2)}$$

Equation (2) − 2 × Equation (1)

$$\begin{array}{r}3x+4x^2 = \ \ Y\\ \pm 2x \pm 4x^2 = \pm 2X\\ \hline x = Y-2X\end{array} \qquad \text{Put in eq. (1), we have}$$

$$X = Y-2X+2(Y-2X)^2$$

$$3X-Y = 2(Y-2X)^2 \text{ (transformed equation)}$$

10. Use the matrix $A=\begin{bmatrix}1 & 0 & 1\\2 & -1 & 1\\0 & 1 & 2\end{bmatrix}$ to encode the message: KEEP IT UP, where letters A to Z are corresponding to the numbers 1 to 26.

Solution:

$$A=\begin{bmatrix}1 & 0 & 1\\2 & -1 & 1\\0 & 1 & 2\end{bmatrix}$$

Message: KEEP IT UP

Divide the letters of the message into groups of three.

KEE PIT UP—

Column matrices with assigned numbers are

$$\begin{bmatrix}K\\E\\E\end{bmatrix}=\begin{bmatrix}11\\5\\5\end{bmatrix}, \begin{bmatrix}P\\I\\T\end{bmatrix}=\begin{bmatrix}16\\9\\20\end{bmatrix}, \begin{bmatrix}U\\P\\\_\end{bmatrix}=\begin{bmatrix}21\\16\\0\end{bmatrix}$$

Now, $\begin{bmatrix}1 & 0 & 1\\2 & -1 & 1\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}11\\5\\5\end{bmatrix}=\begin{bmatrix}11+0+5\\22-5+5\\0+5+10\end{bmatrix}=\begin{bmatrix}16\\22\\15\end{bmatrix}$

$$\begin{bmatrix}1 & 0 & 1\\2 & -1 & 1\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}16\\9\\20\end{bmatrix}=\begin{bmatrix}16+0+20\\32-9+20\\0+9+40\end{bmatrix}=\begin{bmatrix}36\\43\\49\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 1\\2 & -1 & 1\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}21\\16\\0\end{bmatrix}=\begin{bmatrix}21+0+0\\42-16+0\\0+16+0\end{bmatrix}=\begin{bmatrix}21\\26\\16\end{bmatrix}$$

So, required encoded message is $\begin{bmatrix}16\\22\\15\end{bmatrix}\begin{bmatrix}36\\43\\49\end{bmatrix}\begin{bmatrix}21\\26\\16\end{bmatrix}$

11. Decode the message $\begin{bmatrix}11\\20\\43\end{bmatrix}\begin{bmatrix}25\\10\\41\end{bmatrix}\begin{bmatrix}22\\14\\41\end{bmatrix}$ that was encode using matrix $A=\begin{bmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{bmatrix}$, where the numbers 1 to 26 are corresponding to the letters A to Z, and 27 is representing space or –.

Solution:

Encoded Message: $\begin{bmatrix}11\\20\\43\end{bmatrix}\begin{bmatrix}25\\10\\41\end{bmatrix}\begin{bmatrix}22\\14\\41\end{bmatrix}$

$$A=\begin{bmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{bmatrix}$$

To decode the given message, first we find the A^{-1}

$$|A| = \begin{vmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{vmatrix}=1(0-1)-1(1-2)-1(1-0)$$

$$= -1+1-1 = -1 \neq 0, \text{ so } A^{-1} \text{ exists.}$$

Cofactors of elements of matrix A are

$$A_{11}=(-1)^{1+1}\begin{vmatrix}0 & 1\\1 & 1\end{vmatrix}=1(0-1)=-1$$

$$A_{12}=\begin{pmatrix}1\\1\end{pmatrix}^{1+2}\begin{vmatrix}1 & 1\\2 & 1\end{vmatrix}=-1(1-2)=1$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}1 & 0\\2 & 1\end{vmatrix}=1(1-0)=1$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}1 & -1\\1 & 1\end{vmatrix}=-1(1+1)=-2$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & -1\\2 & 1\end{vmatrix}=1(1+2)=3$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 1\\2 & 1\end{vmatrix}=-1(1-2)=1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}1 & -1\\0 & 1\end{vmatrix}=1(1-0)=1$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & -1\\1 & 1\end{vmatrix}=-1(1+1)=-2$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 1\\1 & 0\end{vmatrix}=1(0-1)=-1$$

As we know

$$\text{adj} A = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{bmatrix}^t = \begin{bmatrix}-1 & 1 & 1\\-2 & 3 & 1\\1 & -2 & -1\end{bmatrix}^t$$

$$= \begin{bmatrix}-1 & -2 & 1\\1 & 3 & -2\\1 & 1 & -1\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj} A = \frac{1}{-1}\begin{bmatrix}-1 & -2 & 1\\1 & 3 & -2\\1 & 1 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2 & -1\\-1 & -3 & 2\\-1 & -1 & 1\end{bmatrix}$$

Now,

$$\begin{bmatrix}1 & 2 & -1\\-1 & -3 & 2\\-1 & -1 & 1\end{bmatrix}\begin{bmatrix}11\\20\\43\end{bmatrix}=\begin{bmatrix}11+40-43\\-11-60+86\\-11-20+43\end{bmatrix}=\begin{bmatrix}8\\15\\12\end{bmatrix}=\begin{bmatrix}H\\O\\L\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & -1\\-1 & -3 & 2\\-1 & -1 & 1\end{bmatrix}\begin{bmatrix}25\\10\\41\end{bmatrix}=\begin{bmatrix}25+20-41\\-25-30+82\\-25-10+41\end{bmatrix}=\begin{bmatrix}4\\27\\6\end{bmatrix}=\begin{bmatrix}D\\\_\\F\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & -1\\-1 & -3 & 2\\-1 & -1 & 1\end{bmatrix}\begin{bmatrix}22\\14\\41\end{bmatrix}=\begin{bmatrix}22+28-41\\-22-42+82\\-22-14+41\end{bmatrix}=\begin{bmatrix}9\\18\\5\end{bmatrix}=\begin{bmatrix}I\\R\\E\end{bmatrix}$$

Hence, decoded message is HOLD FIRE.

Formula Sheet

1. $(A+B)^t = A^t + B^t$, $(A-B)^t = A^t - B^t$, $(AB)^t = B^t A^t$, $(A^t)^t = A$
2. Cofactor of an Element a_{ij} : $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}.
3. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:
   - $|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}$ for $i = 1, 2, 3$

     or
   - $|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}$ for $j = 1, 2, 3$
4. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj A = (matrix of cofactors of A)^t = $\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$
5. For a square matrix A of order 3, $|kA| = k^3|A|$.
6. If A is non-singular matrix then $A^{-1} = \frac{1}{|A|}\text{adj}A$.
7. If $\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{aligned}\right\} \Rightarrow AX = B$, where $|A| \neq 0$, then by Cramer's rule

$$x_1 = \frac{|A_{x_1}|}{|A|} = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}, \quad x_2 = \frac{|A_{x_2}|}{|A|} = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|}, \quad x_3 = \frac{|A_{x_3}|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|}$$

Multiple Choice Questions (MCQs)

Exercise 4.1

1. If A is a matrix of order m × n, then number of elements in each row of A is -----

(A) n (B) m (C) m + n (D) m – n

2. If A is a matrix of order 4 × 3, then number of elements in each column of A is:

(A) 2 (B) 3 (C) 4 (D) 5

3. The elements of principal diagonal of a square matrix $A = [a_{ij}]_{3\times3}$ are:

(A) a_{11}, a_{12}, a_{13} (B) a_{21}, a_{22}, a_{23} (C) a_{11}, a_{22}, a_{33} (D) none of these

4. The matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a -----

(A) rectangular matrix (B) diagonal matrix (C) scalar matrix (D) unit matrix

5. If order of a matrix A is 2 × 3 and that of matrix B is 3 × 2, then order of (AB)' is:

(A) 3 × 3 (B) 2 × 2 (C) 3 × 2 (D) 2 × 3

6. If the matrix $\begin{bmatrix} \lambda & 4 \\ 3 & 2 \end{bmatrix}$ is singular then λ equals -----

(A) 2 (B) 4 (C) 6 (D) 8

7. If A & B are square matrices of the same order then -----

(A) $(A-B)^2 = A^2 - 2AB + B^2$ (B) $(A-B)^2 \neq A^2 - 2AB + B^2$

(C) $(A-B)^2 = A^2 + 2BA + B^2$ (D) none of these

Exercise 4.2

8. If A is any matrix of order m × n, then minor of any element of A is a determinant of order:

(A) m × n (B) (m – 1) × (n – 1) (C) (m – 1) × n (D) m × (n – 1)

9. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \\ 4 & 5 & 2 \end{bmatrix}$, then M_{13} = -----

(A) 13 (B) 0 (C) 10 (D) 7

10. If $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ then cofactor of 6 is -----

(A) +1 (B) –1 (C) –6 (D) 3

11. If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2$ then $\begin{vmatrix} c & d \\ a & b \end{vmatrix}$ = -----

(A) 2 (B) –2 (C) ±2 (D) 0

12. A square matrix $A = [a_{ij}]$ is lower-triangular if -----

(A) $a_{ij} = 0\ \forall i = j$ (B) $a_{ij} = 0\ \forall i < j$ (C) $a_{ij} \neq 0\ \forall i < j$ (D) $a_{ij} \neq 0\ \forall i > j$

13. The value of determinant $\begin{vmatrix} 1 & 12 & 25 \\ 0 & 3 & 15 \\ 0 & 0 & 8 \end{vmatrix}$ is -----

(A) 0 (B) 1 (C) 8 (D) 24

14. If the matrices A and B are conformable for multiplication then (AB)' = -----

(A) AB (B) A (C) B (D) B' A'

Exercise 4.3

15. Rank of matrix $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ is -----

(A) 0 (B) 3 (C) 2 (D) 1

16. The equation ax + by = k where a ≠ 0, b ≠ 0, k ≠ 0 is called ----- equation.

(A) conditional (B) homogeneous (C) non-homogeneous (D) consistent

17. The equation ax + by = k where a ≠ 0, b ≠ 0, k = 0 is called ----- equation.

(A) conditional (B) homogeneous (C) non-homogeneous (D) consistent

18. If rank of matrix of coefficient, rank of augmented matrix and number of variables in a system are equal then the system has ------

(A) unique solution (B) finite solution (C) infinite many solutions (D) no solution

19. The trivial solution of homogenous system of linear equation in three variable is ---

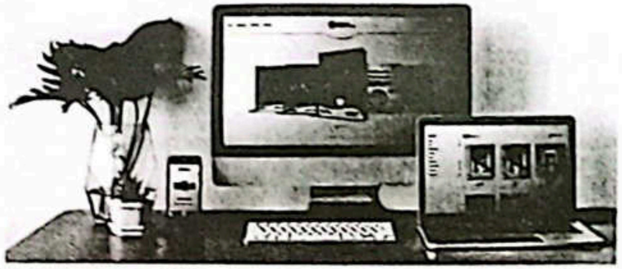
(A) (0, 0, 1) (B) (0, 1, 0) (C) (1, 0, 0) (D) (0, 0, 0)

20. If a system has no solution then the system is called -----

(A) consistent (B) inconsistent (C) unique (D) homogeneous

ANSWER KEY

1.	A	2.	C	3.	C	4.	B	5.	B	6.	C	7.	B	8.	B	9.	B	10.	B
11.	B	12.	B	13.	D	14.	D	15.	C	16.	C	17.	B	18.	A	19.	D	20.	B



Unit 5 Partial Fractions

Introduction

We have learnt in the previous classes how to add two or more rational fractions into a single rational fraction. For example,

(i) $\frac{1}{x-1}+\frac{2}{x+2}=\frac{3x}{(x-1)(x+2)}$

and (ii) $\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x-2}=\frac{5x^2+5x-3}{(x+1)^2(x-2)}$

In this unit we shall learn how to reverse the order in (i) and (ii) that is to express a single rational function as a sum of two or more single rational functions which are called **Partial Fractions**.

Partial Fraction: When we express a single rational function as a sum of two or more single rational functions which are called partial fractions.

Partial Fraction Resolution: Expressing a rational function as a sum of partial fractions is called Partial Fraction Resolution:

- It is an extremely valuable tool in the study of calculus to decompose a complex rational function into a sum of simpler fractions.

Equation: An open sentence formed by using the sign of equality '=' is called an equation. The equations can be divided into the following two kinds:

Conditional equation: It is an equation in which two algebraic expressions are equal for particular values of the variable e.g.,

(a) $2x=3$ is a conditional equation and it is true only if $x=\frac{3}{2}$.

(b) $x^2+x-6=0$ is a conditional equation and it is true for $x=2,-3$ only.

Note:
For simplicity, a conditional equation is called an equation.

Identity: It is an equation which holds good for all values of the variable e.g.,

(a) $(a+b)x\equiv ax+bx$ is an identity and its two sides are equal for all values of x.

(b) $(x+3)(x+4)\equiv x^2+7x+12$ is also an identity which is true for all values of x.

- For convenience, the symbol "=" shall be used both for equation and identity.

Rational Fraction:

An expression of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x with real coefficients and $Q(x)\neq 0$, is called a rational fraction. A rational fraction is of two types.

Proper Rational Fraction:

A rational function $\frac{P(x)}{Q(x)}$ is called a **Proper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator.

For example, $\frac{3}{x+1},\frac{2x-5}{x^2+4}$ and $\frac{9x^2}{x^3-1}$ are proper rational fractions or proper fractions.