Unit 3 Theory of Quadratic Functions

Introduction

This unit explores methods to find the maximum and minimum values of quadratic functions using completing the square and graphical analysis. It also covers the inverse of quadratic functions, determining their domain and range. Additionally, students will learn to solve absolute value quadratic equations and inequalities, as well as equations of rational, radical and exponential forms that can be reduced to quadratic equations. Finally, the unit demonstrates the practical applications of quadratic equations and inequalities in solving real-world problems, providing a strong foundation for problem solving and analysis.

Quadratic Function

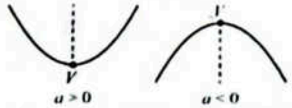
A quadratic function is a polynomial function of degree two. It is typically expressed in the standard form:

$f(x) = ax^2 + bx + c$

where a, b and c are real numbers, and $a \neq 0$.

Analyzing Quadratic Function by Sketching

As we know shape of the graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola. The parabola opens upward or downward, depending on the sign of the leading coefficient a, as shown in the given figure.



The tip of the parabola, labeled as V in the diagrams above, is known as the **vertex** having coordinates (h, k). The vertical line passing through the vertex serves as the **axis of symmetry** for the parabola. The vertex represents a turning point, where the graph changes direction.

- If $a > 0$, then the vertex is a minimum point.
- If $a < 0$, then the vertex is a maximum point.

For sketching the quadratic function, we need to find the x-intercept, y-intercept and the vertex. For analyzing the sketch of quadratic function, we find whether the vertex is a minimum or a maximum point and indicate the intervals where the function is increasing or decreasing.

Example 1: Sketch and analyze $y = -x^2 - 2x + 3$.

Solution:

Given that: $y = -x^2 - 2x + 3$

For y-intercept, put $x = 0$

$y = -(0)^2 - 2(0) + 3 = 3$

$\Rightarrow$ Parabola cuts y-axis at (0, 3)

For x-intercepts, $y = 0$

$-x^2 - 2x + 3 = 0$

$x^2 + 2x - 3 = 0$ Multiply by (-1)

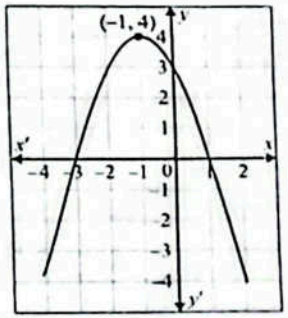
$x^2 + 3x - x - 3 = 0$

$x(x+3) - 1(x+3) = 0$

$(x+3)(x-1) = 0$

$x + 3 = 0, x - 1 = 0$

$x = -3, x = 1$



⇒ Parabola cuts x-axis at $(-3, 0)$ and $(1, 0)$

Now, we find the vertex (h, k) of the parabola.

$$h = \frac{-b}{2a} = -\frac{(-2)}{2(-1)} = -1$$

$$k = -(-1)^2 - 2(-1) + 3 = -1 + 2 + 3 = 4$$

So, the vertex is $(-1, 4)$

Since $a = -1 < 0$, so the vertex $(-1, 4)$ is a maximum point.

The function y is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$

Finding Maximum and Minimum Values of Quadratic Functions by Completing Square

Completing the square is a technique used to rewrite a quadratic function $f(x) = ax^2 + bx + c$ in the following vertex form: $f(x) = a(x - h)^2 + k$

where vertex = (h, k), $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$

- If $a > 0$, the minimum value of $f(x)$ at $x = h$ is k.
- If $a < 0$, the maximum value of $f(x)$ at $x = h$ is k.

Example 2: Find the maximum or minimum value of $f(x) = -2x^2 + 4x + 3$ by completing square.

Solution:

Given that: $f(x) = -2(x^2 - 2x) + 3$

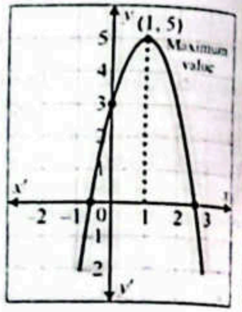
Add and subtract $(1)^2$

$$f(x) = -2(x^2 - 2x + 1^2 - 1^2) + 3$$

$$f(x) = -2\left[(x-1)^2 - 1\right] + 3$$

$$f(x) = -2(x-1)^2 + 2 + 3$$

$$f(x) = -2(x-1)^2 + 5$$



Compare it with

$$f(x) = a(x-h)^2 + k$$

Here $a = -2 < 0, h = 1, k = 5$

As we know

If $a < 0$, the maximum value of $f(x)$ at $x = h$ is k.

Hence, the maximum value of $f(x)$ at $x = 1$ is 5.

Example 3: Find the maximum or minimum value of $f(x) = x^2 - 2x - 3$.

Solution:

Given that: $f(x) = x^2 - 2x - 3$

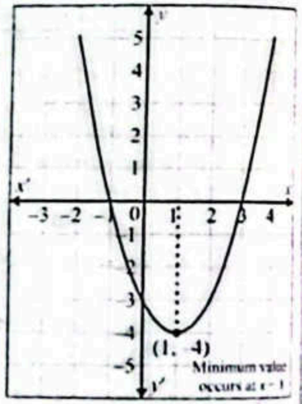
Here $a = 1, b = -2, c = -3$

$$h = -\frac{b}{2a} = -\frac{(-2)}{2(1)} = 1$$

and $k = c - \frac{b^2}{4a} = -3 - \frac{(-2)^2}{4(1)} = -4$

Here $a = 1 > 0$

Therefore, the minimum value of $f(x)$ at $x = 1$ is -4.



Inverse of Quadratic Function

Quadratic functions are typically not one-to-one over their entire domain. To find an inverse for a quadratic function, we must restrict its domain to a portion where it is one-to-one. Commonly, we restrict the domain to either $x \geq h$ (where h is the x coordinate of the vertex) or $x \leq h$.

Example 4: Find the inverse of $f(x) = x^2 + 4x + 3, x \geq -2$. Also find its domain and range.

Solution:

Given that: $f(x) = x^2 + 4x + 3, x \geq -2$

Let $y = x^2 + 4x + 3$ $\quad \because f(x) = y$

$x = y^2 + 4y + 3$ (Interchange x and y)

$y^2 + 4y + 3 - x = 0$

Here $a = 1, b = 4, c = 3 - x$

$$y = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(3-x)}}{2(1)}$$ (Using the quadratic formula)

$$y = \frac{-4 \pm \sqrt{16 - 12 + 4x}}{2}$$

$$y = \frac{-4 \pm \sqrt{4 + 4x}}{2}$$

$$y = \frac{-4 \pm 2\sqrt{1+x}}{2}$$

$f^{-1}(x) = -2 \pm \sqrt{1+x}$ (Replace y with $f^{-1}(x)$)

⇒ $f^{-1}(x) = -2 + \sqrt{1+x}$ or $f^{-1}(x) = -2 - \sqrt{1+x}$

To determine which is the inverse, we find domain and range of the given function.

As $f(x) = x^2 + 4x + 3, x \geq -2$

⇒ Domain $f = [-2, \infty)$

To find range, we proceed as

$f(x) = x^2 + 4x + 3$

⇒ $f(x) = (x+2)^2 - 1$

Therefore, minimum value of $f(x)$ is -1 and hence

Range $f = [-1, \infty)$

By definition of inverse function, we have

Domain $f^{-1} = [-1, \infty)$, Range $f^{-1} = [-2, \infty)$

<u>**Check:**</u>

Let $x = 0 \in$ Domain f^{-1}

⇒ $f^{-1}(0) = -2 + \sqrt{1+0} = -2 + 1 = -1 \in$ Range f^{-1}

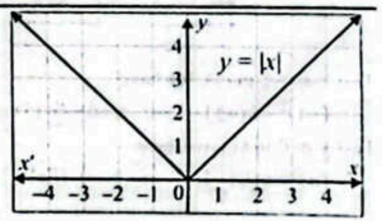
$f^{-1}(0) = -2 - \sqrt{1-0} = -2 - 1 = -3 \notin$ Range f^{-1}

Hence, $f^{-1}(x) = -2 + \sqrt{1+x}$

Absolute Value

The absolute value of x, is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Absolute Value Quadratic Equations

To solve the absolute value quadratic equations, all answers must be substituted back into the original equation to verify whether they are valid or not. Sometimes, "extraneous" solutions may appear which are not valid and must be eliminated from the final answer.

Example 5: Solve $|x^2-4|=5$

Solution:

Given that: $|x^2-4| = 5$

$x^2-4 = \pm 5$

$x^2-4 = 5$ or $x^2-4=-5$

$x^2 = 9$ or $x^2=-1$

$x = \pm 3$ or $x=\pm\sqrt{-1}$ (imaginary)

Check:

For $x = 3$

$|3^2-4| = 5$

$|5| = 5$

$5 = 5$ (True)

For $x=-3$

$|(-3)^2-4| = 5$

$|5| = 5$

$5 = 5$ (True)

Hence solution set $= \{\pm 3\}$

Absolute Value Quadratic Inequalities

Absolute value quadratic inequalities are inequalities that involve a quadratic expression within absolute value bars. They are generally of the following form:

$|ax^2+bx+c|<d, |ax^2+bx+c|>d, |ax^2+bx+c|\le d, |ax^2+bx+c|\ge d$

Example 6: Solve $|x^2-6x-4|<3$

Solution:

Given that: $|x^2-6x-4|<3$

$-3<x^2-6x-4<3$

$-3<x^2-6x-4$ and $x^2-6x-4<3$

$x^2-6x-4-3>0$ and $x^2-6x-4-3<0$

$x^2-6x-1>0$ ...(1) and $x^2-6x-7<0$ ...(2)

Here we solve $x^2-6x-1=0$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-1)}}{2(1)}$$ (Using the quadratic formula)

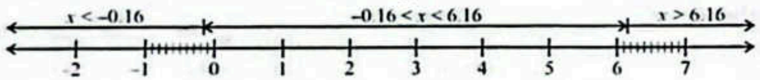
$$x=\frac{6\pm\sqrt{36+4}}{2}=\frac{6\pm\sqrt{40}}{2}=\frac{6\pm2\sqrt{10}}{2}$$

$x=3\pm\sqrt{10}$

$x=3-\sqrt{10}, 3+\sqrt{10}$

$x=-0.16,\ 6.16$

Hence critical numbers divide the number line into three regions.



Test $x=-1$ in (1), we have

$(-1)^2-6(-1)-1>0 \Rightarrow +6>0$ (True)

Test $x=0$ in (1), we have

$(0)^2-6(0)-1>0 \Rightarrow -1>0$ (False)

Test $x=7$ in (1), we have

$(7)^2-6(7)-1>0 \Rightarrow 6>0$ (True)

Solution set is $(-\infty, 3-\sqrt{10})\cup(3+\sqrt{10},\infty)$

Now, we take (2) and solve

$x^2-6x-7=0$

$x^2+x-7x-7=0$

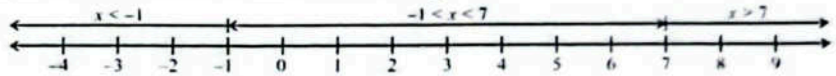
$x(x+1)-7(x+1)=0$

$(x+1)(x-7)=0$

$x+1=0$, $x-7=0$

$x=-1$, $x=7$

These critical numbers divide the number line into three regions.



Test $x=-2, x=0$ and $x=10$ in (2), we have

$(-2)^2-6(-2)-7<0 \Rightarrow 9<0$ (False)

$(0)^2-6(0)-7<0 \Rightarrow -7<0$ (True)

$(10)^2-6(10)-7<0 \Rightarrow 33<0$ (False)

Solution set is $(-1,7)$

Hence the solution set of the given absolute value quadratic inequality is

$$(-\infty,3-\sqrt{10})\cup(3+\sqrt{10},\infty))\cap((-1,7))=(-1,3-\sqrt{10})\cup(3+\sqrt{10},7)$$

Exercise 3.1

1. Find the maximum or minimum value of the following quadratic functions by completing square:

(i) $f(x)=x^2+6x+13$

Solution:

$f(x)=x^2+6x+13$

$=(x^2+6x)+13$

Add and subtract $(3)^2$

$f(x)=(x^2+6x+3^2-3^2)+13$

$=(x+3)^2-9+13$

$=1(x+3)^2+4$

$f(x)=1(x-(-3))^2+4$

Compare it with

$f(x)=a(x-h)^2+k$

$\Rightarrow a=1>0, h=-3, k=4$

As we know

If $a>0$, the minimum value of $f(x)$ at $x=h$ is k.

$\Rightarrow$ Min. value of $f(x)$ at $x=-3$ is 4

(ii) $f(x)=x^2+4x$

Solution:

$f(x)=x^2+4x$

Add and subtract $(2)^2$

$f(x)=x^2+4x+(2)^2-(2)^2$

$=(x+2)^2-4$

$f(x)=1(x-(-2))^2-4$

Compare it with

$f(x)=a(x-h)^2+k$

$\Rightarrow a=1>0, h=-2, k=-4$

As we know

If $a>0$, the minimum value of $f(x)$ at $x=h$ is k.

$\Rightarrow$ Min. value of $f(x)$ at $x=-2$ is -4

(iii) $f(x)=-x^2+8x+13$

Solution:

$f(x)=-x^2+8x+13$

$=-(x^2-8x)+13$

Add and subtract $(4)^2$

$f(x)=-(x^2-8x+4^2-4^2)+13$

$=-((x-4)^2-16)+13$

$=-(x-4)^2+16+13$

$f(x)=-1(x-4)^2+29$

Compare it with

$f(x)=a(x-h)^2+k$

$\Rightarrow a=-1<0, h=4, k=29$

As we know

If $a<0$, the maximum value of $f(x)$ at $x=h$ is k.

$\Rightarrow$ Max. value of $f(x)$ at $x=4$ is 29.

(iv) $f(x) = -x^2 - 3x - 5$

Solution:

$$f(x) = -x^2 - 3x - 5$$
$$= -1(x^2 + 3x) - 5$$

Add and subtract $\left(\frac{3}{2}\right)^2$

$$f(x) = -1\left(x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right) - 5$$
$$= -1\left(\left(x+\frac{3}{2}\right)^2 - \frac{9}{4}\right) - 5$$
$$= -1\left(x+\frac{3}{2}\right)^2 + \frac{9}{4} - 5$$
$$f(x) = -1\left(x - \left(-\frac{3}{2}\right)\right)^2 - \frac{11}{4}$$

Compare it with

$$f(x) = a(x-h)^2 + k$$

$\Rightarrow$ $a = -1 <, h = -\frac{3}{2}, k = -\frac{11}{4}$

As we know

If $a < 0$, the maximum value of $f(x)$ at $x = h$ is k.

$\Rightarrow$ Max. value of $f(x)$ at $x = \frac{-3}{2}$ is $\frac{-11}{4}$.

(v) $f(x) = 3x^2 + 6x - 13$

Solution:

$$f(x) = 3x^2 + 6x - 3$$
$$= 3(x^2 + 2x) - 3$$

Add and subtract $(1)^2$

$$f(x) = 3(x^2 + 2x + 1^2 - 1^2) - 3$$
$$= 3((x+1)^2 - 1) - 3$$
$$= 3(x+1)^2 - 3 - 3$$
$$f(x) = 3(x - (-1))^2 - 6$$

Compare it with

$$f(x) = a(x-h)^2 + k$$

$\Rightarrow$ $a = 3 > 0, h = -1, k = -6$

As we know

If $a > 0$, the minimum value of $f(x)$ at $x = h$ is k

$\Rightarrow$ Min. value of $f(x)$ at $x = -1$ is -6.

(vi) $f(x) = -2x^2 - x + 21$

Solution:

$$f(x) = -2x^2 - x + 21$$
$$= -2\left(x^2 + \frac{1}{2}x\right) + 21$$

Add and subtract $\left(\frac{1}{4}\right)^2$

$$f(x) = -2\left(x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right) + 21$$
$$= -2\left(\left(x+\frac{1}{4}\right)^2 - \frac{1}{16}\right) + 21$$
$$= -2\left(x+\frac{1}{4}\right)^2 + \frac{1}{8} + 21$$
$$= -2\left(x+\frac{1}{4}\right)^2 + \frac{1+168}{8}$$
$$f(x) = -2\left(x - \left(-\frac{1}{4}\right)\right)^2 + \frac{169}{8}$$

Compare it with

$$f(x) = a(x-h)^2 + k$$

$\Rightarrow$ $a = -2 < 0, h = -\frac{1}{4}, k = \frac{169}{8}$

As we know

If $a < 0$, the maximum value of $f(x)$ at $x = h$ is k.

$\Rightarrow$ Max. value of $f(x)$ at $x = -\frac{1}{4}$ is $\frac{169}{8}$.

2. Find the maximum or minimum point by sketching the following quadratic functions. Also find their domain and range:

(i) $f(x) = x^2 - 4x$

Solution:

$$f(x) = x^2 - 4x$$

Let $y = x^2 - 4x$ $\quad \because f(x) = y$

For y-intercept, put $x = 0$

$$y = 0^2 - 4(0) = 0$$

$\Rightarrow$ Parabola cuts y-axis at $(0, 0)$

For x-intercepts, put $y = 0$

$$0 = x^2 - 4x$$
$$x(x-4) = 0$$
$$x = 0; x - 4 = 0 \Rightarrow x = 4$$

$\Rightarrow$ Parabola cuts x-axis at $(0, 0)$ and $(4, 0)$

Now, we find vertex (h, k) of the parabola.

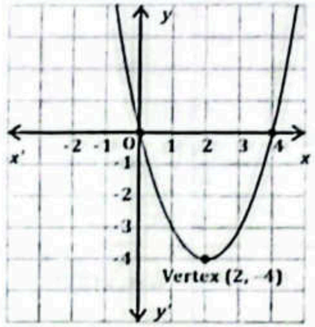
$$f(x) = x^2 - 4x + 0$$

Here $a = 1, b = -4, c = 0$

$$h = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = 2$$
$$k = f(h) = f(2) = (2)^2 - 4(2)$$
$$= 4 - 8 = -4$$

So, the vertex is $(2, -4)$

Hence, $a = 1 > 0$, so the vertex $(2, -4)$ is a minimum point.



$f(x) = x^2 - 4x$

Domain $f = (-\infty, \infty)$

Since minimum value of y is -4, so Range $f = [-4, \infty)$

(ii) $f(x) = x^2 - 5x + 6$

Solution:

$$f(x) = x^2 - 5x + 6$$

Let $y = x^2 - 5x + 6$ $\quad \because f(x) = y$

For y-intercept, put $x = 0$

$$y = 0^2 - 5(0) + 6 = 6$$

$\Rightarrow$ Parabola cuts y-axis at $(0, 6)$

For x-intercepts, put $y = 0$

$$0 = x^2 - 5x + 6$$
$$x^2 - 3x - 2x + 6 = 0$$
$$x(x-3) - 2(x-3) = 0$$
$$(x-3)(x-2) = 0$$
$$x - 3 = 0 \quad ; \quad x - 2 = 0$$
$$x = 3 \quad ; \quad x = 2$$

$\Rightarrow$ Parabola cuts x-axis at $(3, 0)$ and $(2, 0)$

Now, we find vertex (h, k) of the parabola

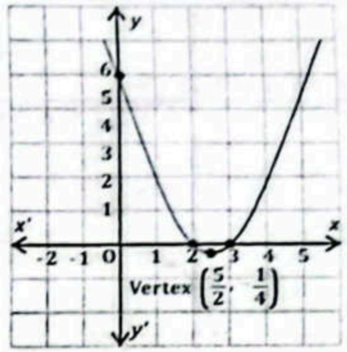
$$f(x) = x^2 - 5x + 6$$

Here $a = 1, b = -5, c = 6$

$$h = \frac{-b}{2a} = \frac{-(-5)}{2(1)} = \frac{5}{2}$$
$$k = f(h) = f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 6$$
$$= \frac{25}{4} - \frac{25}{2} + 6 = \frac{25 - 50 + 24}{4} = -\frac{1}{4}$$

So, the vertex is $\left(\frac{5}{2}, -\frac{1}{4}\right)$

Since $a = 1 > 0$, so the vertex $\left(\frac{5}{2}, -\frac{1}{4}\right)$ is a minimum point.



$f(x) = x^2 - 5x + 6$

Domain $f = (-\infty, \infty)$

Since minimum value of y is $-\frac{1}{4}$, so Range $f = \left[-\frac{1}{4}, \infty\right)$

(iii) $f(x) = -x^2 + 2x - 8$

Solution:

$$f(x) = -x^2 + 2x - 8$$

Let $y = -x^2 + 2x - 8$ $\quad \because f(x) = y$

For y-intercept, put $x = 0$

$$y = -0^2 + 2(0) - 8 = -8$$

$\Rightarrow$ Parabola cuts y-axis at $(0, -8)$

For x-intercepts, put $y = 0$

$$0 = -x^2 + 2x - 8$$

Multiply by -1, we have

$$x^2 - 2x + 8 = 0$$

Here $a = 1, b = -2, c = 8$

Disc $= b^2 - 4ac = (-2)^2 - 4(1)(8)$

$= 4 - 32 = -28 < 0$

Roots will be imaginary.

Thus, The parabola does not intersect the x-axis. Now, we find vertex (h, k) of the parabola.

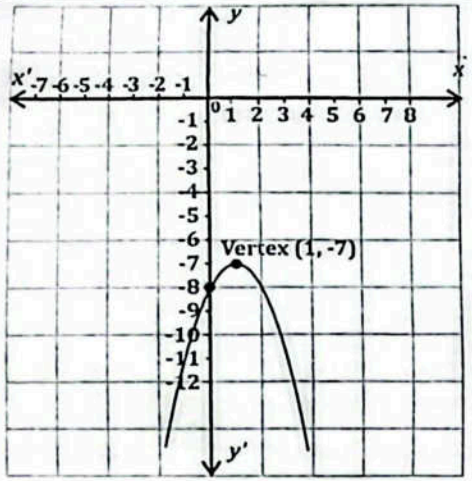
$$f(x) = -x^2 + 2x - 8$$

Here $a = -1, b = 2, c = -8$

$$h = \frac{-b}{2a} = \frac{-2}{2(-1)} = 1$$
$$k = f(h) = f(1) = -1^2 + 2(1) - 8 = -7$$

So, the vertex is $(1, -7)$

Since $a = -1 < 0$, so the vertex $(1, -7)$ is a maximum point.



$f(x) = -x^2 + 2x - 8$

Domain $f = (-\infty, \infty)$

Since maximum value of y is -7, so

Range $f = (-\infty, -7]$

(iv) $f(x) = x^2 - 4x + 4$

Solution:

$f(x) = x^2 - 4x + 4$

Let $y = x^2 - 4x + 4$ $\qquad \because f(x) = y$

For y-intercept, put $x = 0$

$y = 0^2 - 4(0) + 4 = 4$

$\Rightarrow$ Parabola cuts y-axis at $(0, 4)$

For x-intercepts, put $y = 0$

$0 = x^2 - 4x + 4$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$\Rightarrow \quad x - 2 = 0 \quad \Rightarrow x = 2$

$\Rightarrow$ Parabola touches x-axis at $(2, 0)$

Now, we find the vertex (h, k) of the parabola.

$f(x) = 1x^2 - 4x + 4$

Here $a = 1, b = -4, c = 4$

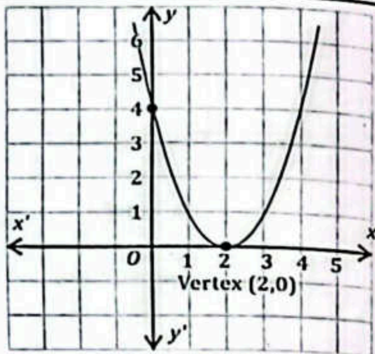
$h = -\dfrac{b}{2a} = \dfrac{-(-4)}{2(1)} = 2$

$k = f(h) = f(2) = (2)^2 - 4(2) + 4$

$= 4 - 8 + 4 = 0$

So, the vertex is $(2, 0)$

Since $a = 1 > 0$, so the vertex $(2, 0)$ is a minimum point.



$f(x) = x^2 - 4x + 4$

Domain $f = (-\infty, \infty)$

Since minimum value of y is 0, so

Range $f = [0, \infty)$

(v) $f(x) = x^2 + 2x - 8.3$

Solution:

$f(x) = x^2 + 2x - 8.3$

Let $y = x^2 + 2x - 8.3$ $\qquad \because f(x) =$

For y-intercept, put $x = 0$

$y = 0^2 + 2(0) - 8\cdot 3 = -8\cdot 3$

$\Rightarrow$ Parabola cuts y-axis at $(0, -8.3)$

For x-intercepts, put $y = 0$

$0 = x^2 + 2x - 8.3$

Here $a = 1, b = 2, c = -8.3$

By quadratic formula

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-8.3)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{37.2}}{2} = \frac{-2 \pm 6.10}{2}$$

$x = \dfrac{-2 + 6.10}{2} \quad ; \quad x = \dfrac{-2 - 6.10}{2}$

$x = 2.05 \quad ; \quad x = -4.05$

$\Rightarrow$ Parabola cuts x-axis at $(2.05, 0)$ and $(-4.05, 0$

Now, we find vertex (h, k) of the parabola

$f(x) = x^2 + 2x - 8.3$

Here $a = 1, b = 2, c = -8.3$

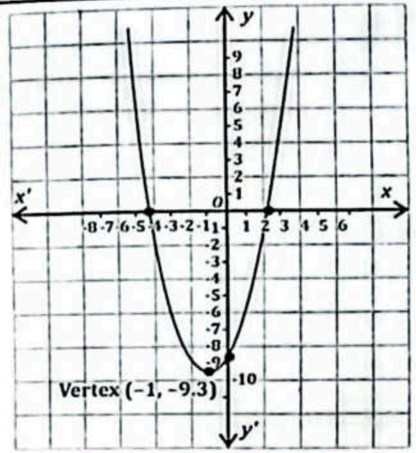
$h = \dfrac{-b}{2a} = \dfrac{-2}{2(1)} = -1$

$k = f(h) = f(-1)$

$= (-1)^2 + 2(-1) - 8\cdot 3 = 1 - 2 - 8\cdot 3 = -9\cdot 3$

So, the vertex is $(-1, -9.3)$

Since $a = 1 > 0$, so the vertex $(-1, -9.3)$ is a minimum point.



$f(x) = x^2 + 2x - 8.3$

Domain $f = (-\infty, \infty)$

since minimum value of y is -9.3, so

Range $f = [-9.3, \infty)$

(vi) $f(x) = 6 - x - x^2$

Solution:

$f(x) = 6 - x - x^2$

Let $y = 6 - x - x^2$ $\qquad \because f(x) = y$

For y-intercept, put $x = 0$

$y = 6 - 0 - 0^2 = 6$

$\Rightarrow$ Parabola cuts y-axis at $(0,6)$

For x-intercepts, put $y = 0$

$0 = 6 - x - x^2$

Multiply by -1

$x^2 + x - 6 = 0$

$x^2 + 3x - 2x - 6 = 0$

$x(x+3) - 2(x+3) = 0$

$(x-2)(x+3) = 0$

$x - 2 = 0 \quad ; \quad x + 3 = 0$

$x = 2 \quad ; \quad x = -3$

$\Rightarrow$ Parabola cuts x-axis at $(2, 0)$ and $(-3, 0)$

Now, we find vertex (h, k) of the parabola

$f(x) = -x^2 - x + 6$

Here $a = -1, b = -1, c = 6$

$h = -\dfrac{b}{2a} = \dfrac{-(-1)}{2(-1)} = -\dfrac{1}{2}$

$k = f(h) = f\left(-\dfrac{1}{2}\right)$

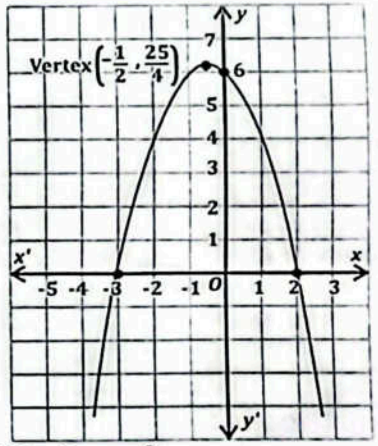
$= -\left(-\dfrac{1}{2}\right)^2 - \left(-\dfrac{1}{2}\right) + 6$

$= -\dfrac{1}{4} + \dfrac{1}{2} + 6$

$= \dfrac{-1 + 2 + 24}{4} = \dfrac{25}{4}$

So, the vertex is $\left(-\dfrac{1}{2}, \dfrac{25}{4}\right)$

Since $a = -1 < 0$, so the vertex $\left(-\dfrac{1}{2}, \dfrac{25}{4}\right)$ is a maximum point.



$f(x) = 6 - x - x^2$

Domain $f = (-\infty, \infty)$

Since maximum value of y is $\dfrac{25}{4}$, so

Range $f = \left(-\infty, \dfrac{25}{4}\right]$

---

3. Find the inverse of the following quadratic functions. Also find their domain and range:

(i) $f(x) = x^2 - 3, x \le 0$

Solution:

$f(x) = x^2 - 3, x \le 0$

Let $y = x^2 - 3$ $\qquad \because f(x) = y$

Interchange x and y.

$x = y^2 - 3$

$y^2 = x + 3$

$y = \pm\sqrt{x+3}$

Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \pm\sqrt{x+3}$

$\Rightarrow \quad f^{-1}(x) = \sqrt{x+3}$

or $\quad f^{-1}(x) = -\sqrt{x+3}$

To determine which is the inverse, we find domain and range of the given function.

As $f(x) = x^2 - 3, x \le 0$

$\Rightarrow$ Domain $f = (-\infty, 0]$

At $x = 0$:

$$f(0) = (0)^2 - 3 = -3 \text{ (Min. value)}$$

Since $a = 1 > 0$, so the parabola opens upward.

$\Rightarrow$ Range $f = [-3, \infty)$

By definition of inverse function

Domain $f^{-1} = [-3, \infty)$, Range $f^{-1} = (-\infty, 0]$

CHECK:

Let $x = 0 \in$ Domain f^{-1}

$\Rightarrow$ $f^{-1}(0) = \sqrt{0+3} = \sqrt{3} \notin$ Range f^{-1}

and $f^{-1}(0) = -\sqrt{0+3} = -\sqrt{3} \in$ Range f^{-1}

Hence $f^{-1}(x) = -\sqrt{x+3}$

(ii) $f(x) = x^2 + 6x + 4, x < -3$

Solution:

$$f(x) = x^2 + 6x + 4, x < -3$$

Let $y = x^2 + 6x + 4$ $\qquad \because f(x) = y$

Interchange x and y.

$$x = y^2 + 6y + 4$$

$$y^2 + 6y + (4 - x) = 0$$

Here $a = 1, b = 6, c = 4 - x$

By using quadratic formula

$$y = \frac{-6 \pm \sqrt{6^2 - 4(1)(4-x)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 16 + 4x}}{2}$$

$$= \frac{-6 \pm \sqrt{20 + 4x}}{2} = \frac{-6 \pm \sqrt{4(5+x)}}{2}$$

$$= \frac{-6 \pm 2\sqrt{5+x}}{2}$$

$$y = -3 \pm \sqrt{5+x}$$

Replace y with $f^{-1}(x)$

$\Rightarrow$ $f^{-1}(x) = -3 + \sqrt{5+x}$ or $f^{-1}(x) = -3 - \sqrt{5+x}$

To determine which is the inverse, we find domain and range of the given function.

As $f(x) = x^2 + 6x + 4, x < -3$

$\Rightarrow$ Domain $f = (-\infty, -3)$

At $x = -3$:

$$f(-3) = (-3)^2 + 6(-3) + 4 = 9 - 18 + 4 = -5$$

Since $a = 1 > 0$, so the parabola opens upward.

$\Rightarrow$ Rang $f = (-5, \infty)$

By definition of inverse function

Domain $f^{-1} = (-5, \infty)$, Range $f^{-1} = (-\infty, -3)$

CHECK:

Let $x = 0 \in$ Domain f^{-1}

$\Rightarrow$ $f^{-1}(0) = -3 + \sqrt{5+0} = -3 + \sqrt{5} = -0.76 \notin$ Range f^{-1}

and $f^{-1}(0) = -3 - \sqrt{5+0} = -3 - \sqrt{5} = -5.24 \in$ Range f^{-1}

Hence $f^{-1}(x) = -3 - \sqrt{5+x}$

(iii) $f(x) = 2x^2 - 8x + 11, x \ge 2$

Solution:

$$f(x) = 2x^2 - 8x + 11, x \ge 2$$

Let $y = 2x^2 - 8x + 11$ $\qquad \because f(x) = y$

Interchange x and y

$$x = 2y^2 - 8y + 11$$

$$2y^2 - 8y + (11 - x) = 0$$

Here $a = 2, b = -8, c = 11 - x$

By using quadratic formula

$$y = \frac{8 \pm \sqrt{(-8)^2 - 4(2)(11-x)}}{2(2)}$$

$$= \frac{8 \pm \sqrt{64 - 88 + 8x}}{4}$$

$$= \frac{8 \pm \sqrt{8x - 24}}{4}$$

$$= \frac{8 \pm \sqrt{8(x-3)}}{4}$$

$$= \frac{8 \pm 2\sqrt{2}\sqrt{x-3}}{4}$$

$$= 2 \pm \frac{\sqrt{2}\sqrt{x-3}}{2}$$

$$= 2 \pm \frac{1}{\sqrt{2}} \cdot \sqrt{x-3}$$

$$y = 2 \pm \sqrt{\frac{x-3}{2}}$$

Replace y with $f^{-1}(x)$

$$f^{-1}(x) = 2 \pm \sqrt{\frac{x-3}{2}}$$

$\Rightarrow$ $f^{-1}(x) = 2 + \sqrt{\frac{x-3}{2}}$ or $f^{-1}(x) = 2 - \sqrt{\frac{x-3}{2}}$

To determine which is the inverse, we find domain and range of the given function.

As $f(x) = 2x^2 - 8x + 11, x \ge 2$

$\Rightarrow$ Domain $f = [2, \infty)$

At $x = 2$:

$$f(2) = 2(2)^2 - 8(2) + 11 = 8 - 16 + 11 = 3$$

Since $a = 270$, so the parabola opens upward.

$\Rightarrow$ Range $f = [3, \infty)$.

By definition of inverse function

Domain $f^{-1} = [3, \infty)$, Range $f^{-1} = [2, \infty)$

CHECK:

Let $x = 5 \in$ Domain f^{-1}

$\Rightarrow$ $f^{-1}(5) = 2 + \sqrt{\frac{5-3}{2}} = 2 + \sqrt{1} = 2 + 1 = 3 \in$ Range f^{-1}

and $f^{-1}(5) = 2 - \sqrt{\frac{5-3}{2}} = 2 - \sqrt{1} = 2 - 1 = 1 \notin$ Range f^{-1}

Hence $f^{-1}(x) = 2 + \sqrt{\frac{x-3}{2}}$

(iv) $f(x) = 3x^2 - 2x + 6, x \ge 5$

Solution:

$$f(x) = 3x^2 - 2x + 6, x \ge 5$$

Let $y = 3x^2 - 2x + 6$

Interchange x and y.

$$x = 3y^2 - 2y + 6$$

$$3y^2 - 2y + (6 - x) = 0$$

Here $a = 3, b = -2, c = 6 - x$

By using quadratic formula

$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(6-x)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 - 72 + 12x}}{6}$$

$$= \frac{2 \pm \sqrt{12x - 68}}{6}$$

$$= \frac{2 \pm \sqrt{4(3x - 17)}}{6}$$

$$= \frac{2 \pm 2\sqrt{3x-17}}{6}$$

$$y = \frac{1 \pm \sqrt{3x-17}}{3}$$

Replace y with $f^{-1}(x)$

$$f^{-1}(x) = \frac{1 \pm \sqrt{3x-17}}{3}$$

$$f^{-1}(x) = \frac{1 + \sqrt{3x-17}}{3} \quad \text{or} \quad f^{-1}(x) = \frac{1 - \sqrt{3x-17}}{3}$$

To determine which is the inverse, we find domain and range of the given function.

As $f(x) = 3x^2 - 2x + 6, x \ge 5$

$\Rightarrow$ Domain $f = [5, \infty)$

At $x = 5$:

$$f(5) = 3(5)^2 - 2(5) + 6 = 75 - 10 + 6 = 71$$

Since $a = 3 > 0$, so the parabola opens upward.

$\Rightarrow$ Range $f = [71, \infty)$

By definition of inverse function

Domain $f^{-1} = [71, \infty)$, Range $f^{-1} = [5, \infty)$

CHECK:

Let $x = 71 \in$ Domain f^{-1}

$\Rightarrow$ $f^{-1}(71) = \frac{1 + \sqrt{3(71) - 17}}{3} = \frac{1 + 14}{3} = 5 \in$ Range f^{-1}

and $f^{-1}(71) = \frac{1 - \sqrt{3(71) - 17}}{3} = \frac{1 - 14}{3} = -4.3 \notin$ Range f^{-1}

Hence $f^{-1}(x) = \frac{1 + \sqrt{3x - 17}}{3}$

(v) $f(x) = 2(x-3)^2 + 1, x \ge 3$

Solution:

$$f(x) = 2(x-3)^2 + 1, x \ge 3$$

Let $y = 2(x-3)^2 + 1$

Interchange x and y

$$x = 2(y-3)^2 + 1$$

$$x - 1 = 2(y-3)^2$$

$$(y-3)^2 = \frac{x-1}{2}$$

$\Rightarrow$ $y - 3 = \pm\sqrt{\frac{x-1}{2}}$

$$y = 3 \pm \sqrt{\frac{x-1}{2}}$$

Replace y with $f^{-1}(x)$

$$f^{-1}(x) = 3 \pm \sqrt{\frac{x-1}{2}}$$

$\Rightarrow$ $f^{-1}(x) = 3 + \sqrt{\frac{x-1}{2}}$ or $f^{-1}(x) = 3 - \sqrt{\frac{x-1}{2}}$

To determine which is the inverse, we find domain and range of the given function.

As $f(x) = 2(x-3)^2 + 1, x \ge 3$

$\Rightarrow$ Domain $f = [3, \infty)$

At $x = 3$:

$$f(3) = 2(3-3)^2 + 1 = 0 + 1 = 1$$

Since $a = 2 > 0$, so the parabola opens upward.

$\Rightarrow$ Range $f = [1, \infty)$

By definition of inverse function

Domain $f^{-1} = [1, \infty)$, Range $f^{-1} = [3, \infty)$

CHECK:

Let $x = 3 \in$ Domain f^{-1}

$f^{-1}(3) = 3 + \sqrt{\frac{3-1}{2}} = 3 + 1 = 4 \in$ Range f^{-1}

and $f^{-1}(3) = 3 - \sqrt{\frac{3-1}{2}} = 3 - 1 = 2 \notin$ Range f^{-1}

Hence $f^{-1}(x) = 3 + \sqrt{\frac{x-1}{2}}$

CHECK:

For $x=\frac{3}{2}$

$$\left|3\left(\frac{3}{2}\right)^2-7\left(\frac{3}{2}\right)+2\right|=\left(\frac{3}{2}\right)^2-\frac{3}{2}+1$$
$$\left|\frac{27}{4}-\frac{21}{2}+2\right|=\frac{9}{4}-\frac{3}{2}+1$$
$$\left|\frac{27-42+8}{4}\right|=\frac{9-6+4}{4}$$
$$\left|-\frac{7}{4}\right|=\frac{7}{4}$$
$$\frac{7}{4}=\frac{7}{4}\text{ (True)}$$

For $x=\frac{1}{2}$

$$\left|3\left(\frac{1}{2}\right)^2-7\left(\frac{1}{2}\right)+2\right|=\left(\frac{1}{2}\right)^2-\frac{1}{2}+1$$
$$\left|\frac{3}{4}-\frac{7}{2}+2\right|=\frac{1}{4}-\frac{1}{2}+1$$
$$\left|\frac{3-14+8}{4}\right|=\frac{1-2+4}{4}$$
$$\left|-\frac{3}{4}\right|=\frac{3}{4}$$
$$\frac{3}{4}=\frac{3}{4}\text{ (True)}$$

Hence, solution set $=\left\{\frac{3\pm\sqrt{7}}{2},\frac{3}{2},\frac{1}{2}\right\}$

(v) $|x^2-4|<5$

Solution:

$|x^2-4|<5$

$-5<x^2-4<5$

$-5<x^2-4$ or $x^2-4<5$

$0<x^2-4+5$ or $x^2-4-5<0$

$x^2+1>0$ or $x^2-9<0$

We take $x^2+1>0$

$\Rightarrow$ $x^2>-1$ true $\forall x\in R$

Solution set $=\{(-\infty,\infty)\}$

Now, we take

$x^2-9<0$

$x^2<9$

$|x|<3$

$-3<x<3$

Solution set $=\{(-3,-3)\}$

Required solution set $=\{(-\infty,\infty)\}\cap\{(-3,3)\}$

$=\{(-3,3)\}$

(vi) $|x^2-3x+2|>4$

Solution:

$|x^2-3x+2|>4$

$x^2-3x+2<-4$ or $x^2-3x+2>4$

$x^2-3x+2+4<0$ or $x^2-3x+2-4>0$

$x^2-3x+6<0\ldots(1)$ or $x^2-3x-2>0\ldots(2)$

Here we solve $x^2-3x+6=0$

Disc $=b^2-4ac$

$=(-3)^2-4(1)(6)$

$=-15$ (roots will be imaginary)

Since $a=1>0$ and $x^2-3x+6<0$,

Therefore $x^2-3x+6<0$ has no solution.

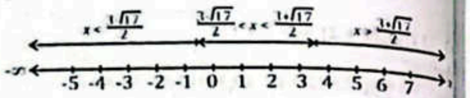
Now, we take (2) and solve

$$x^2-3x-2=0$$

By using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{3\pm\sqrt{(-3)^2-4(1)(-2)}}{2(1)}$$
$$x=\frac{3\pm\sqrt{9+8}}{2}$$
$$x=\frac{3\pm\sqrt{17}}{2}$$
$$x=\frac{3+\sqrt{17}}{2},\frac{3-\sqrt{17}}{2}$$

These critical numbers divide the number line into three regions.



Test $x=-1, x=0$ and $x=4$ in (2), we have.

$(-1)^2-3(-1)-2>0\Rightarrow 2>0$ (True)

$(0)^2-3(0)-2>0\Rightarrow -2>0$ (False)

$(4)^2-3(4)-2>0\Rightarrow 2>0$ (True)

Required solution set $=\left\{\left(-\infty,\frac{3-\sqrt{17}}{2}\right)\cup\left(\frac{3+\sqrt{17}}{2},\infty\right)\right\}$

(vii) $|x^2-5x+6|\le x+2$

Solution:

$$|x^2-5x+6|\le x+2$$
$$-(x+2)\le x^2-5x+6\le+(x+2)$$

$-x-2\le x^2-5x+6$ or $x^2-5x+6\le x+2$

$0\le x^2-5x+6+x+2$ or $x^2-5x+6-x-2\le0$

$x^2-4x+8\ge0\cdots(1)$ or $x^2-6x+4\le0\cdots(2)$

We take (1) and solve

$x^2-4x+8=0$

Disc $=b^2-4ac$

$=(-4)^2-4(1)(8)$

$=-16<0$ (roots will be imaginary)

As $a=1>0$ and $x^2-4x+8\ge0$, so $x^2-4x+8\ge0$ is true for all $x\in R$.

Solution set $=\{(-\infty,\infty)\}$

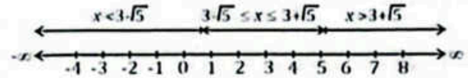
Now, we take (2) and solve

$x^2-6x+4=0$

By using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{6\pm\sqrt{(-6)^2-4(1)(4)}}{2(1)}$$
$$x=\frac{6\pm\sqrt{36-16}}{2}=\frac{6\pm\sqrt{20}}{2}$$
$$x=\frac{6\pm2\sqrt{5}}{2}=3\pm\sqrt{5}$$
$$x=3+\sqrt{5},x=3-\sqrt{5}$$

These critical numbers divide the number line into three regions.



Test $x=0, x=1$ and $x=6$ in (2), we have

$(0)^2-6(0)+4\le0\Rightarrow4\le0$ (False)

$(1)^2-6(1)+4\le0\Rightarrow-2\le0$ (True)

$(6)^2-6(6)+4\le0\Rightarrow4\le0$ (False)

Solution set $=\{[3-\sqrt{5},3+\sqrt{5}]\}$

Required solution set $=\{(-\infty,\infty)\}\cap\{[3-\sqrt{5},3+\sqrt{5}]\}$

$=\{[3-\sqrt{5},3+\sqrt{5}]\}$

(viii) $|2x^2-3x-5|<4$

Solution:

$|2x^2-3x-5|<4$

$-4<2x^2-3x-5<4$

$-4<2x^2-3x-5$ or $2x^2-3x-5<4$

$0<2x^2-3x-5+4$ or $2x^2-3x-5-4<0$

$2x^2-3x-1>0\cdots(1)$ or $2x^2-3x-9<0\cdots(2)$

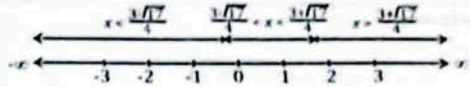
We take (1), and solve

$$2x^2-3x-1=0$$

By using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{3\pm\sqrt{(-3)^2-4(2)(-1)}}{2(2)}$$
$$x=\frac{3\pm\sqrt{9+8}}{4}=\frac{3\pm\sqrt{17}}{4}$$
$$x=\frac{3-\sqrt{17}}{4},x=\frac{3+\sqrt{17}}{4}$$

These critical numbers divide the number line into three regions



Test $x=-1, x=0$ and $x=2$ in (1), we have

$2(-1)^2-3(-1)-1>0\Rightarrow4>0$ (True)

$2(0)^2-3(0)-1>0\Rightarrow-1>0$ (False)

$2(2)^2-3(2)-1>0\Rightarrow1>0$ (True)

Solution set $=\left\{\left(-\infty,\frac{3-\sqrt{17}}{4}\right)\cup\left(\frac{3+\sqrt{17}}{4},\infty\right)\right\}$

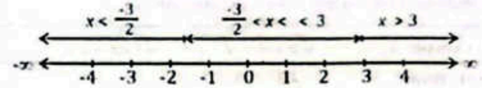
Now, we take (2) and solve

$$2x^2-3x-9=0$$
$$2x^2-6x+3x-9=0$$
$$2x(x-3)+3(x-3)=0$$
$$(2x+3)(x-3)=0$$

$2x+3=0$, $x-3=0$

$x=-\frac{3}{2}$, $x=3$

These critical numbers divide the number line into three regions.



Test $x=-2, x=0$ and $x=4$ in (2), we have

$2(-2)^2-3(-2)-9<0\Rightarrow5<0$ (False)

$2(0)^2-3(0)-9<0\Rightarrow-9<0$ (True)

$2(4)^2-3(4)-9<0\Rightarrow11<0$ (False)

Solution set $=\left\{\left(-\frac{3}{2},3\right)\right\}$

Required solution set

$$=\left\{\left(-\infty,\frac{3-\sqrt{17}}{4}\right)\cap\left(\frac{3+\sqrt{17}}{4},\infty\right)\right\}\cap\left\{\left(-\frac{3}{2},3\right)\right\}$$
$$=\left\{\left(-\frac{3}{2},\frac{3-\sqrt{17}}{4}\right)\cup\left(\frac{3+\sqrt{17}}{4},3\right)\right\}$$

Solutions of Equations Reducible to the Quadratic Equation

There are certain types of equations, which do not look to be of degree 2, but they can be reduced to the quadratic equation. We shall discuss the solutions of the rational, radical and radical equations.

Rational Equations Reducible to the Quadratic Equation

A rational equation is an equation containing one or more rational expressions, where rational expressions typically contain a variable in the denominator.

Example 7: Solve $\frac{1}{x}+\frac{2}{x+1}=1, x\neq 0, x\neq -1$

Solution:

$$\frac{1}{x}+\frac{2}{x+1}=1$$

Multiplying both sides by $x(x+1)$, we have

$$(x+1)+2x = x(x+1)$$
$$x+1+2x = x^2+x$$
$$3x+1 = x^2+x$$
$$x^2+x-3x-1 = 0$$
$$x^2-2x-1 = 0$$

Here $a=1, b=-2, c=-1$

$$x = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)} \quad \text{(Using the quadratic formula)}$$
$$= \frac{2\pm\sqrt{4+4}}{2} = \frac{2\pm\sqrt{8}}{2} = \frac{2\pm 2\sqrt{2}}{2} = 1\pm\sqrt{2}$$

Hence Solution Set = $\{1\pm\sqrt{2}\}$

Radical Equations Reducible to the Quadratic Equation

Equations involving radical expressions of the variable are called radical equations. To solve a radical equation, we first obtain an equation free from radicals. Every solution of radical equation is also a solution of the radical-free equation but the new equation has solutions that are not solutions of the original radical equation. Such extra solutions (roots) are called extraneous roots.

Example 8: Solve $\sqrt{x+8}+\sqrt{x+3}=\sqrt{12x+13}$

Solution:

$$\sqrt{x+8}+\sqrt{x+3}=\sqrt{12x+13} \quad \ldots (A)$$

Squaring both sides, we get

$$\left(\sqrt{x+8}+\sqrt{x+3}\right)^2=\left(\sqrt{12x+13}\right)^2$$
$$\left(\sqrt{x+8}\right)^2+\left(\sqrt{x+3}\right)^2+2\sqrt{(x+8)(x+3)}=12x+13$$
$$x+8+x+3+2\sqrt{(x+8)(x+3)} = 12x+13$$
$$2\sqrt{(x+8)(x+3)} = 12x+13-2x-11$$
$$2\sqrt{x+8}\sqrt{x+3} = 10x+2$$
$$\Rightarrow \sqrt{(x+8)(x+3)} = 5x+1 \quad \text{Dividing by '2'}$$

Squaring again, we have

$$x^2+11x+24 = 25x^2+10x+1$$

$$\Rightarrow 24x^2-x-23=0$$
$$24x^2-24x+23x-23=0$$
$$24x(x-1)+23(x-1)=0$$
$$\Rightarrow (24x+23)(x-1)=0$$

Either $24x+23=0$ or $x-1=0$

$$x=-\frac{23}{24} \quad \text{or} \quad x=1$$

CHECKING

Put $x=1$ in equation (A), we have

$$\sqrt{1+8}+\sqrt{1+3}=\sqrt{12(1)+13} \Rightarrow \sqrt{9}+\sqrt{4}=\sqrt{25} \Rightarrow 3+2=5 \Rightarrow 5=5 \text{ (True)}$$

So, $x=1$ is a root.

Put $x=\frac{-23}{24}$ in equation (A), we have

$$\sqrt{\frac{-23}{24}+8}+\sqrt{\frac{-23}{24}+3}=\sqrt{12\left(\frac{-23}{24}\right)+13} \Rightarrow \frac{20}{\sqrt{24}}=\frac{6}{\sqrt{24}} \text{ (False)}$$

So, $x=\frac{-23}{24}$ is an extraneous root.

Hence solution set = $\{1\}$

Real World Problems of Quadratic Equations and Inequalities

We shall now proceed to solve the problems which, when expressed symbolically, lead to quadratic equations in one variables.

In order to solve such problems, we must:

(i) Suppose the unknown quantities to be x or y etc.
(ii) Translate the problem into symbols and form the equation or inequality satisfying the given conditions.

The method of solving the problems will be illustrated through the following examples:

Example 9: The length of a room is 3 metres greater than its breadth. If the area of the room is 180 square metres, find length and the breadth of the room.

Solution:

Let the breadth of room = x metres
length of room = $(x+3)$ metres
Area of the room = $x(x+3)$ square metres

By the given condition, we have

$$x(x+3) = 180 \quad \because \text{Area of room} = 180 \text{ square metres}$$
$$\Rightarrow x^2+3x-180 = 0$$
$$x^2+15x-12x-180 = 0$$
$$x(x+15)-12(x+15) = 0$$
$$(x+15)(x-12) = 0$$

Either $x=-15$ or $x=12$

As breadth cannot be negative so $x=-15$ is not admissible.

Hence breadth of the room = x = 12 metres

and length of the room = $(x+3)$ = 12 + 3 = 15 metres.

Example 10: A company manufactures laptops and its weekly profit function (in thousands of dollars) is $P(x) = -x^2 + 2x + 3$, where x is the number of laptops produced (in hundreds). Find the range of production levels where the company makes at least \$4,000 profit.

Solution:

Profit Function: $P(x) = -x^2 + 2x + 3$

Here $P(x) \geq 4$ (in thousands of dollars)

$$-x^2 + 2x + 3 \geq 4$$
$$-x^2 + 2x + 3 - 4 \geq 0$$
$$-x^2 + 2x - 1 \geq 0$$
$$x^2 - 2x + 1 \leq 0 \quad \text{Multiply by } (-1)$$
$$(x-1)^2 \leq 0$$

This only holds true when $(x - 1)^2 = 0 \Rightarrow x = 1$(in hundreds)

The company makes exactly \$4,000 profit when 100 laptops are produced (since $x = 1$ means 100 laptops). There is no production level where profit is more than \$4,000.

Exercise 3.2

1. Solve the following equations:

(i) $\dfrac{1}{3x} + \dfrac{4x}{6} = 1, x \neq 0$

Solution:

$$\frac{1}{3x} + \frac{4x}{6} = 1, x \neq 0$$

Multiply each term by LCM 6(3x).

$$1(6) + 4x(3x) = 6(3x)$$
$$6 + 12x^2 - 18x = 0$$

Driving both sides by 6

$$2x^2 - 3x + 1 = 0$$
$$2x^2 - 2x - x + 1 = 0$$
$$2x(x-1) - 1(x-1) = 0$$
$$(x-1)(2x-1) = 0$$

Either $x - 1 = 0$ or $2x - 1 = 0$

$x = 1$; $x = \dfrac{1}{2}$

Hence S.S = $\left\{1, \dfrac{1}{2}\right\}$

(ii) $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{5}{2}; x \neq -1, 0$

Solution:

$$\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2}$$

Multiplying each term by L.C.M $2x(x+1)$

$$2x(x) + 2(x+1)(x+1) = 5x(x+1)$$
$$2x^2 + 2(x^2 + x + x + 1) = 5x^2 + 5x$$
$$2x^2 + 2x^2 + 4x + 2 = 5x^2 + 5x$$
$$4x^2 + 4x + 2 = 5x^2 + 5x$$
$$0 = 5x^2 + 5x - 4x^2 - 4x - 2$$
$$x^2 + x - 2 = 0$$
$$x^2 + 2x - x - 2 = 0$$
$$x(x+2) - 1(x+2) = 0$$
$$(x+2)(x-1) = 0$$

Either $x + 2 = 0$ or $x - 1 = 0$

$x = -2$; $x = 1$

Hence S.S = $(-2, 1)$

(iii) $\dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{7}{x+5}; x \neq -1, -2, -5$

Solution:

$$\frac{1}{x+1} + \frac{2}{x+2} = \frac{7}{x+5}$$

Multiplying each term by L.C.M $(x+1)(x+2)(x+5)$

$$1(x+2)(x+5) + 2(x+1)(x+5) = 7(x+1)(x+2)$$
$$1(x^2 + 2x + 5x + 10) + 2(x^2 + x + 5x + 5) = 7(x^2 + x + 2x + 2)$$
$$x^2 + 7x + 10 + 2(x^2 + 6x + 5) = 7(x^2 + 3x + 2)$$
$$x^2 + 7x + 10 + 2x^2 + 12x + 10 = 7x^2 + 21x + 14$$
$$3x^2 + 19x + 20 = 7x^2 + 21x + 14$$
$$0 = 7x^2 + 21x + 14 - 3x^2 - 19x - 20$$
$$4x^2 + 2x - 6 = 0$$

$\Rightarrow$ $2x^2 + x - 3 = 0$ Dividing by '2'

$\Rightarrow$ $2x^2 + 3x - 2x - 3 = 0$

$\Rightarrow$ $x(2x+3) - 1(2x+3) = 0$

$\Rightarrow$ $(2x+3)(x-1) = 0$

Either $2x + 3 = 0$ or $x - 1 = 0$

$x = -\dfrac{3}{2}$; $x = 1$

Hence S.S = $\left\{-\dfrac{3}{2}, 1\right\}$

(iv) $\dfrac{a}{ax-1} + \dfrac{b}{bx-1} = a + b; x \neq \dfrac{1}{a}, \dfrac{1}{b}$

Solution:

$$\frac{a}{ax-1} + \frac{b}{bx-1} = a + b$$
$$\left\{\frac{a}{ax-1} - b\right\} + \left\{\frac{b}{bx-1} - a\right\} = 0$$
$$\frac{a - b(ax-1)}{ax-1} + \frac{b - a(bx-1)}{bx-1} = 0$$
$$\frac{a - abx + b}{ax-1} + \frac{b - abx + a}{bx-1} = 0$$
$$\frac{a + b - abx}{ax-1} + \frac{a + b - abx}{bx-1} = 0$$
$$(a + b - abx)\left[\frac{1}{ax-1} + \frac{1}{bx-1}\right] = 0$$
$$(a + b - abx)\left[\frac{bx - 1 + ax - 1}{(ax-1)(bx-1)}\right] = 0$$
$$(a - abx + b)(bx - 1 + ax - 1) = 0$$

$\Rightarrow$ $(a - abx + b)(bx + ax - 2) = 0$

Either $(a - abx + b) = 0$ or $(bx - 1 + ax - 1) = 0$

$abx = a + b$; $x(a+b) - 2 = 0$

$x = \dfrac{a+b}{ab}$; $x(a+b) = 2$

$\Rightarrow x = \dfrac{2}{a+b}$

Hence S.S = $\left\{\dfrac{a+b}{ab}, \dfrac{2}{a+b}\right\}$

(v) $\dfrac{x+1}{x-1} + \dfrac{x-1}{x+1} = 2, x \neq 1, x \neq -1$

Solution:

$$\frac{x+1}{x-1} + \frac{x-1}{x+1} = 2, x \neq 1, x \neq -1$$

Multiply each term by LCM $(x-1)(x+1)$.

$$(x+1)^2 + (x-1)^2 = 2(x-1)(x+1)$$
$$x^2 + 1 + 2x + x^2 + 1 - 2x = 2(x^2 - 1^2)$$
$$2x^2 + 2 = 2x^2 - 2$$
$$+2 = -2 \text{ (Not possible)}$$

Hence S.S = { }

(vi) $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2$

Solution:

$$3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2 \quad \text{...(A)}$$
$$3(x^2 + 5x) - 2\sqrt{(x^2 + 5x) + 1} = 2$$

Put $x^2 + 5x = y$ ...(1)

$$3y - 2\sqrt{y+1} = 2$$
$$3y - 2 = \sqrt{2y+1}$$
$$(3y-2)^2 = (2\sqrt{y+1})^2 \quad \text{squaring both sides}$$
$$9y^2 + 4 - 12y = 4(y+1)$$
$$9y^2 - 16y + 4 - 4 = 0$$
$$y(9y - 16) = 0$$

Either $y = 0$ or $9y - 16 = 0 \Rightarrow y = \dfrac{16}{9}$

Put $y = 0$ in equation (1)

$x^2 + 5x = 0 \Rightarrow x(x+5) = 0$

$x = 0$ or $x = -5$

Put $y = \dfrac{16}{9}$ in equation (1)

$$x^2 + 5x = \frac{16}{9}$$
$$9x^2 + 45x - 16 = 0$$
$$9x^2 + 48x - 3x - 16 = 0$$
$$3x(3x+16) - 1(3x+16) = 0$$
$$(3x+16)(3x-1) = 0$$

$\Rightarrow$ $3x + 16 = 0$ or $3x - 1 = 0$

$\Rightarrow$ $x = \dfrac{-16}{3}$ or $x = \dfrac{1}{3}$

S.S = $\left\{\dfrac{1}{3}, \dfrac{-16}{3}\right\}$

(vii) $\sqrt{2x+8} + \sqrt{x+5} = 7$

Solution:

$$\sqrt{2x+8} + \sqrt{x+5} = 7 \quad \text{...(A)}$$

Squaring both sides

$$(\sqrt{2x+8} + \sqrt{x+5})^2 = (7)^2$$
$$(\sqrt{2x+8})^2 + (\sqrt{x+5})^2 + 2\sqrt{(2x+8)(x+5)} = 49$$
$$2x + 8 + x + 5 + 2\sqrt{2x^2 + 10x + 8x + 40} = 49$$
$$3x + 13 + 2\sqrt{2x^2 + 18x + 40} = 49$$
$$2\sqrt{2x^2 + 18x + 40} = 49 - 3x - 13$$
$$2\sqrt{2x^2 + 18x + 40} = 36 - 3x$$

Again squaring both sides

$$(2\sqrt{2x^2 + 18x + 40})^2 = (36 - 3x)^2$$
$$4(2x^2 + 18x + 40) = 1296 + 9x^2 - 216x$$
$$8x^2 + 72x + 160 = 1296 + 9x^2 - 216x$$
$$1296 + 9x^2 - 216x - 8x^2 - 72x - 160 = 0$$
$$x^2 - 288x + 1136 = 0$$
$$x^2 - 284x - 4x + 1136 = 0$$

$$x(x-284)-4(x-284)=0$$
$$(x-284)(x-4)=0$$
Either $x-284=0$ or $x-4=0$
$x=284$; $x=4$

Checking for Extraneous Roots

Put $x=284$ in equation (A)
$$\sqrt{2(284)+8}+\sqrt{284+5}=7$$
$$41=7 \quad \text{(False)}$$
Thus $x=284$ is an extraneous root.

Put $x=4$ in equation (A)
$$\sqrt{2(4)+8}+\sqrt{4+5}=7$$
$$7=7 \quad \text{(True)}$$
Thus $x=4$ is not an extraneous root.

Hence S.S = {4}

(viii) $\sqrt{3x+4}=2+\sqrt{2x-4}$

Solution:
$$\sqrt{3x+4}=2+\sqrt{2x-4} \quad \text{(A)}$$
Squaring both sides
$$(\sqrt{3x+4})^2=(2+\sqrt{2x-4})^2$$
$$3x+4=4+(\sqrt{2x-4})^2+4\sqrt{2x-4}$$
$$3x+4=4+2x-4+4\sqrt{2x-4}$$
$$3x+4-2x=4\sqrt{2x-4}$$
$$x+4=4\sqrt{2x-4}$$
Again squaring both sides
$$(x+4)^2=(4\sqrt{2x-4})^2$$
$$x^2+16+8x=16(2x-4)$$
$$x^2+16+8x=32x-64$$
$$x^2+16+8x-32x+64=0$$
$$x^2-24x+80=0$$
$$x^2-20x-4x+80=0$$
$$x(x-20)-4(x-20)=0$$
$$(x-20)(x-4)=0$$
Either $x-20=0$ or $x-4=0$
$x=20$; $x=4$

Checking for Extraneous Roots

Put $x=20$ in equation (A)
$$\sqrt{3(20)+4}=2+\sqrt{2(20)-4}$$
$$8=8 \quad \text{(True)}$$
Thus $x=20$ is not an extraneous root.

Put $x=4$ in equation (A)
$$\sqrt{3(4)+4}=2+\sqrt{2(4)-4}$$
$$4=4 \quad \text{(True)}$$
Thus $x=4$ is not an extraneous root.

Hence S.S = {4, 20}

(ix) $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

Solution:
$$\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$$
Squaring both sides
$$(\sqrt{x+7}+\sqrt{x+2})^2=(\sqrt{6x+13})^2$$
$$(\sqrt{x+7})^2+(\sqrt{x+2})^2+2\sqrt{(x+7)(x+2)}=6x+13$$
$$x+7+x+2+2\sqrt{x^2+7x+2x+14}=6x+13$$
$$2x+9+2\sqrt{x^2+9x+14}=6x+13$$
$$2\sqrt{x^2+9x+14}=6x+13 \ldots$$
$$2\sqrt{x^2+9x+14}=4x+4$$
Dividing both sides by "2"
$$\sqrt{x^2+9x+14}=2x+2$$
Again squaring both sides
$$(\sqrt{x^2+9x+14})^2=(2x+2)^2$$
$$x^2+9x+14=4x^2+8x \ldots$$
$$4x^2+8x+4-x^2-9x-14=0$$
$$3x^2-x-10=0$$
$$3x^2-6x+5x-10=0$$
$$3x(x-2)+5(x-2)=0$$
$$(x-2)(3x+5)=0$$
Either $x-2=0$ or $3x+5=0$
$x=2$; $x=-\frac{5}{3}$

Checking for Extraneous Roots

Put $x=2$ in equation (A)
$$\sqrt{2+7}+\sqrt{2+2}=\sqrt{6(2)+13}$$
$$5=5$$
Thus $x=2$ is not an extraneous root.

Put $x=-\frac{5}{3}$ in equation (A)
$$\sqrt{-\frac{5}{3}+7}+\sqrt{-\frac{5}{3}+2}=\sqrt{6\left(-\frac{5}{3}\right)+13}$$
$$\frac{5}{\sqrt{3}}=\sqrt{3}$$
Thus $x=-\frac{5}{3}$ is an extraneous root.

Hence S.S = {2}

(x) $\sqrt{x+5}-\sqrt{x-3}=2$

Solution:
$$\sqrt{x+5}-\sqrt{x-3}=2$$
Squaring both sides, we have
$$(\sqrt{x+5}-\sqrt{x-3})^2=(2)^2$$

$$(\sqrt{x+5})^2+(\sqrt{x-3})^2-2\sqrt{x+5}\,\sqrt{x-3}=4$$
$$x+5+x-3-2\sqrt{x^2+2x-15}=4$$
$$-2\sqrt{x^2+2x-15}=4-2x-2$$
$$-2\sqrt{x^2+2x-15}=2-2x$$
Squaring again, we have
$$(-2\sqrt{x^2+2x-15})^2=(2-2x)^2$$
$$4(x^2+2x-15)=4+4x^2-8x$$
$$4x^2+8x-60=4+4x^2-8x$$
$$8x+8x=4+60$$
$$16x=64$$
$$x=\frac{64}{16}$$
$$x=4$$

CHECK.

Put $x=4$ in eq (A), we have
$\sqrt{4+5}-\sqrt{4-3}=2 \Rightarrow 2=2$ (True)
Thus $x=4$ is a root.
Hence S.S = {4}

2. A farmer bought some sheep for Rs. 9000. If he had paid Rs. 100 less for each, he would have got 3 sheep more for the same money. How many sheep did he buy, when the rate in each case is uniform?

Solution:

Let Number of sheep $= x$
Amount for x sheep $= 9000$
$$\text{Amount for one sheep} = \frac{9000}{x} \quad \ldots(i)$$
Amount for $(x+3)$ sheep $= 9000$
$$\text{Amount for one sheep} = \frac{9000}{x+3} \quad \ldots(ii)$$
According to given condition
$$\frac{9000}{x}-100=\frac{9000}{x+3}$$
Multiplying each term by L.C.M "$x(x+3)$"
$$9000(x+3)-100x(x+3)=9000x$$
Dividing each term by "100"
$$90(x+3)-x(x+3)=90x$$
$$90x+270-x^2-3x-90x=0$$
$$-x^2-3x+270=0$$
$$x^2+3x-270=0$$
$$x^2+18x-15x-270=0$$
$$x(x+18)-15(x+18)=0$$
$$(x+18)(x-15)=0$$
Either $x-15=0$ or $x+18=0$
$x=15$; $x=-18$ (Impossible being negative)

Hence required number of sheep are 15.

3. A man sold his stock of eggs for Rs. 240. If he had 2 dozen more, he would have got the same money by selling the whole for Rs. 0.50 per dozen cheaper. How many dozen eggs did he sell?

Solution:

Let Number of eggs $= x$ dozen
Amount for x dozen eggs $= 240$
$$\text{Amount for one dozen eggs} = \frac{240}{x} \quad (i)$$
Now
Amount for $(x+2)$ dozen eggs $= 240$
$$\text{Amount for one dozen eggs} = \frac{240}{x+2} \quad (ii)$$
According to given condition
$$\frac{240}{x}-0.50=\frac{240}{x+2}$$
$$\frac{240}{x}-\frac{1}{2}=\frac{240}{x+2}$$
Multiplying each term by L.C.M "$2x(x+2)$"
$$480(x+2)-x(x+2)=480x$$
$$480x+960-x^2-2x-480x=0$$
$$-x^2+2x-960=0$$
$$x^2-2x+960=0$$
$$x^2+32x-30x+960=0$$
$$x(x+32)-30(x+32)=0$$
$$(x-30)(x+32)=0$$
Either $x-30=0$ or $x+32=0$
$x=30$; $x=-32$ (Impossible)

Hence 30 dozen eggs were sold by the stockist.

4. A cyclist travelled 48 km at a uniform speed. If he had travelled 2 km/hour slower, he would have taken 2 hours more to perform the journey. How long did he take to cover 48 km?

Solution:

Let Speed $= v$ km/hr
Time taken $= t$ hours
As distance = speed × time
So $48 = vt$
$$v=\frac{48}{t} \quad \ldots(i)$$
If he had travelled 2 km/hr slower, then
Speed $= (v-2)$ km / hr
Time taken $= (t+2)$ hours
As distance = speed × time
So $48=(v-2)\cdot(t+2)$
$$v-2=\frac{48}{t+2}$$
$$v=\frac{48}{t+2}+2 \quad \ldots(ii)$$

Equation (ii) – Equation (i)

$$0 = \frac{48}{t+2} + 2 - \frac{48}{t}$$

Multiplying each term by L.C.M '$t(t+2)$'

$$48t + 2t(t+2) - 48(t+2) = 0$$
$$48t + 2t^2 + 4t - 48t - 96 = 0$$
$$48t + 2t^2 + 4t - 48t - 96 = 0$$
$$2t^2 + 4t - 96 = 0$$
$$t^2 + 2t - 48 = 0 \quad \text{Dividing by '2'}$$
$$t^2 + 8t - 6t - 48 = 0$$
$$t(t+8) - 6(t+8) = 0$$
$$(t+8)(t-6) = 0$$

Either $t - 6 = 0$ or $t + 8 = 0$

$t = 6$; $t = -8$ (Not possible)

Required time = 6 hours

5. To do a piece of work, Abdullah takes 10 days more than Abdul Hadi. Together they finish the work in 12 days. How long would Abdul Hadi take to finish it alone?

Solution:

Let Abdul Hadi finishes a job in days = x

One day's work of Abdul Hadi = $\frac{1}{x}$

Abdullah finishes a job in days = $x + 10$

One day's work of Abdullah = $\frac{1}{x+10}$

Together they finished a job in days = 12

one day work of both Abdullah and Abdul Hadi = $\frac{1}{12}$

By using above information, we have an equation

$$\frac{1}{x} + \frac{1}{x+10} = \frac{1}{12}$$

Multiply each Term by LCM $12x(x+10)$

$$12(x+10) + 12x = x(x+10)$$
$$12x + 120 + 12x = x^2 + 10x$$
$$0 = x^2 + 10x - 24x - 120$$
$$x^2 - 14x - 120 = 0$$
$$x^2 - 20x + 6x - 120 = 0$$
$$x(x-20) + 6(x-20) = 0$$
$$(x-20)(x+6) = 0$$

Either $x - 20 = 0$ or $x + 6 = 0$

$x = 20$; $x = -6$ (Impossible)

Abdul Hadi finish work alone in days = $x = 20$

6. The braking distance (in metres) of a car is modeled by:
$d(s) = 0.02s^2 + 0.1s$, where s is the speed of car in km/h
If the maximum safe braking distance is 50 metres, find the range of speed where braking is safe.

Solution:

Braking Distance Function: $d(s) = 0.02s^2 + 0.1s$

where s is the speed of car in km/h and d is the braking distance in metres. since the maximum safe braking distance is 50 m, so for safe braking $d(s) \le 50$

$\Rightarrow \; 0.02s^2 + 0.1s \le 50$

Multiply both sides by 50

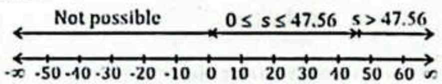
$$s^2 + 5s \le 2500$$
$$s^2 + 5s - 2500 \le 0 \quad \ldots(1)$$

Here we solve $s^2 + 5s - 2500 = 0$

By using quadratic formula

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$s = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(-2500)}}{2(1)}$$
$$= \frac{-5 \pm \sqrt{25 + 10000}}{2} = \frac{-5 \pm \sqrt{10025}}{2}$$
$$= \frac{-5 \pm 100.12}{2}$$
$$= \frac{-5 - 100.12}{2}, \quad s = \frac{-5 + 100.12}{2}$$
$$s = -52.56, \quad s = 47.56$$

These critical numbers divide the number line into three regions.



Since the speed cannot be negative, so we consider only the non-negative part of interval.

Test $s = 10$ and $s = 48$ in (1), we have

$(10)^2 + 5(10) - 2500 \le 0 \Rightarrow -2350 \le 0$ (True)

$(48)^2 + 5(48) - 2500 \le 0 \Rightarrow 44 \le 0$ (False)

Hence, the range of safe speed is $0 \le S \le 47.56$ km/h

7. A rocket follows the height function $h(t) = -5t^2 + 20t + 30$, where $h(t)$ is the height in metres and t is the time in seconds. Find the time interval during which the rocket is at least 40 metres above the ground.

Solution:

Height Function: $h(t) = -5t^2 + 20t + 30$

where h is height in metres and t is the time in seconds.

Since the rocket is at least 40 m above the ground, therefore $h(t) \ge 40$

$\Rightarrow -5t^2 + 20t + 30 \ge 40$

Divide by -5 on both sides

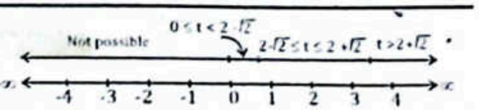
$$t^2 - 4t + 2 \le 0 \; \ldots(1)$$

Here we solve $t^2 - 4t + 2 = 0$

By using quadratic formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$
$$t = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2}$$
$$t = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$
$$t = 2 - \sqrt{2}, \; t = 2 + \sqrt{2}$$

These critical numbers divide the number line in to three regions



Since the time cannot be negative, so we consider only the non-negative part of interval.

test $t = 0.1, t = 1$ and $t = 4$ in(1), we have.

$(0.1)^2 - 4(0.1) + 2 \le 0 \Rightarrow 1.61 \le 0$ (False)

$(1)^2 - 4(1) + 2 \le 0 \Rightarrow -1 \le 0$ (True)

$(4)^2 - 4(4) + 2 \le 0 \Rightarrow 2 \le 0$ (False)

Hence, the rocket is at least 40 metres above the ground during the time interval $2 - \sqrt{2} \le t \le 2 + \sqrt{2}$

or $0.586 \le t \le 3.414$

Formula Sheet

1. If we rewrite the quadratic function $f(x) = ax^2 + bx + c$ by completing the square technique as:

$$f(x) = a(x-h)^2 + k$$

Then, vertex = (h,k), where $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$

- If $a > 0$, the minimum value of $f(x)$ at $x = h$ is k.
- If $a < 0$, the maximum value of $f(x)$ at $x = h$ is k.

2. The absolute value of x, is defined as: $|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

Multiple Choice Questions (MCQs)

Exercise 3.1

1. An equation of the form $ax^2 + bx + c = 0$ is called quadratic if -----
(A) a = 0 (B) a ≠ 0 (C) b = 0 (D) b ≠ 0

2. A quadratic equation has degree -----
(A) 0 (B) 1 (C) 2 (D) 3

3. The parabola $f(x) = ax^2 + bx + c, a \ne 0$ opens upward if-----
(A) $a \le 0$ (B) $a < 0$ (C) $a \ge 0$ (D) $a > 0$

4. For the quadratic function $f(x) = ax^2 + bx + c$ if $a < 0$, then the vertex is a ----- point.
(A) minimum (B) maximum (C) both(A) & (B) (D) none of these

5. Maximum value of $f(x) = -2x^2 + 4x + 3$ is ----
(A) −5 (B) 6 (C) 5 (D) ∞

6. Minimum value of $f(x) = -2x^2 + 4x + 3$ is ----
(A) −5 (B) −∞ (C) 5 (D) ∞

7. The range of inverse of a quadratic function $f(x) = x^2 + 4x + 3, x \ge -2$ is -----
(A) $(-2,\infty)$ (B) $(-2,\infty]$ (C) $[-2,\infty)$ (D) $[-2,\infty]$

8. Solution set of the absolute value quadratic equation $|x^2 - 4| = 5$ is -----
(A) {3} (B) {−3} (C) {±3} (D) { }

Exercise 3.2

9. An equation involving radical expression of the variable is called -----
(A) radical equation (B) algebraic equation (C) exponential equation (D) none of these

10. A radical equation has some solutions which do not satisfy the given radical equation are called:
(A) simple roots (B) extraneous roots (C) rational roots (D) none of these

11. Solution of equation $\sqrt{x+8}+\sqrt{x+3}=\sqrt{12x+13}$ is/are -----
(A) $-\frac{23}{24}$ (B) 1 (C) $-\frac{23}{24}, 1$ (D) none of these

12. The length of a room is 3 metres greater than its breadth. If the area of the room is 180 square metres, find length and the breadth of the room.
(A) 9, 20 (B) 9, 12 (C) 15, 18 (D) 12, 15

ANSWER KEY

1.	B	2.	C	3.	D	4.	B	5.	C	6.	B	7.	D	8.	C	9.	A	10.	B	11.	B	12.	D

Unit 4 Matrices & Determinants

Introduction

This unit introduces the fundamental concepts and operations of matrices, equipping students with the skills to perform matrix addition, subtraction and multiplication involving both real and complex entries. It explores the essential properties of determinants and provides techniques for evaluating the determinant of a 3 × 3 matrix using cofactors and determinant properties. Students will learn to apply row operations to determine the inverse and rank of matrices, as well as distinguish between consistent and inconsistent systems of linear equations through practical examples. The unit further explores into solving systems of linear equations, both homogeneous and non-homogeneous, using advanced methods such as matrix inversion, Cramer's Rule and Gaussian elimination. Emphasis is placed on the real world applications of matrices in diverse fields such as graphic design, cryptography, data encryption, geometric transformations and highlighting the importance and versatility of matrix algebra in solving complex, practical problems.

Matrix

While solving linear systems of equations, a new notation was introduced to reduce the amount of writing. For this new notation the word matrix was first used by the English mathematician James Sylvester (1814-1897). Arthur Cayley (1821-1895) developed the theory of matrices and used them in the linear transformations. Nowadays, matrices are used in high speed computers and also in other various disciplines. The concept of determinants was used by Chinese and Japanese mathematicians but the Japanese mathematician Seki Kowa (1642-1708) and the German Mathematician Gottfried Wilhelm Leibniz (1646-1716) are credited for the invention of determinants. G. Cramer (1704-1752) employed the determinants successfully for solving the systems of linear equations.

Matrix: A rectangular array of numbers enclosed by a pair of bracket is called a matrix.

e.g., $\begin{bmatrix}2 & -1 & 3\\-5 & 4 & 7\end{bmatrix}$ or $\begin{bmatrix}2 & 3 & 0\\1 & -1 & 4\\3 & 2 & 6\\4 & 1 & -1\end{bmatrix}$

Rows: The horizontal lines of numbers are called **rows** of the matrix.

For example the matrix $\begin{bmatrix}2 & -1 & 3\\-5 & 4 & 7\end{bmatrix}$ has two rows and $\begin{bmatrix}2 & 3 & 0\\1 & -1 & 4\\3 & 2 & 6\\4 & 1 & -1\end{bmatrix}$ has four rows.

Columns: The vertical lines of numbers are called **columns** of the matrix.

For example the matrix $\begin{bmatrix}2 & -1 & 3\\-5 & 4 & 7\end{bmatrix}$ and $\begin{bmatrix}2 & 3 & 0\\1 & -1 & 4\\3 & 2 & 6\\4 & 1 & -1\end{bmatrix}$ have three columns.

Entries: The numbers used in rows or columns are said to be the **entries** or **elements** of the matrix.