Velocity is the derivative of the position function:

$$v(t)=\frac{ds}{dt}=\frac{d}{dt}\left(4t^3-3t^2+2t\right)=12t^2-6t+2$$

Substitute $t=2$:

$$v(2)=12(2)^2-6(2)+2=48-12+2=38$$

So, the velocity at $t=2$ is 38 m/s.

Acceleration is the derivative of the velocity function:

$$a(t)=\frac{dv}{dt}$$

$$=\frac{d}{dt}\left(12t^2-6t+2\right)=24t-6$$

Substitute t = 2

$$a(2)=24(2)-6$$

$$=48-6=42$$

So, the acceleration at $t=2$ is 42 m/s^2.

Financial Investments:

Example 14: A bank offers a compound interest rate on an investment, and the value of the investment after t years is given by $V(t)=5000(1+0.04t)^2$. Find the rate of change of the investment value after 10 years.

Solution:

Investment Growth Function: $V(t)=5000(1+0.04t)^2$

The rate of change of the investment is the derivative of $V(t)$ with respect to t.

$$V'(t)=\frac{dV}{dt}=\frac{d}{dt}\left(5000(1+0.04t)^2\right)$$

$$=5000(2)(1+0.04t)^1(0.04)$$

$$=10000(0.04)(1+0.04t)$$

$$V'(t)=400(1+0.04t)$$

Substitute t = 10 :

$$V'(10)=400(1+0.04\times10)=400(1+0.40)=400\times1.4=560$$

So, the investment is growing at a rate of Rs. 560 per year after 10 years.

Structural Stress:

Example 15: The stress on a beam under a varying load is modeled by $S(x)=500x-2x^3$, where $S(x)$ is the stress in pascals (Pa) and x is the distance (in metres) from the beam's fixed end. Find the rate of change of stress at $x=5$ metres.

Solution:

Stress Function: $S(x)=500x-2x^3$

The rate of change of stress is the derivative of $S(x)$ with respect to x.

$$S'(x)=\frac{dS}{dt}=\frac{d}{dt}(500x-2x^3)=500-6x^2$$

Substitute $x=5$:

$$S'(5)=500-6(5)^2=500-6\times25=500-150=350$$

So, the stress is increasing at a rate of 350Pa per metres at $x=5$ metres.

Exercise 13.3

1. A car's position at time t is given by $s(t)=5t^3-3t^2+t$. Find the velocity by differentiating the position function with respect to time.

Solution:

Position Function: $s(t)=5t^3-3t^2+t$

Velocity is the derivative of the position function:

$$V(t)=\frac{ds}{dt}=\frac{d}{dt}(5t^3-3t^2+t)$$

$$V(t)=5\cdot\frac{d}{dt}(t^3)-3\frac{d}{dt}(t^2)+\frac{d}{dt}(t)$$

$$V(t)=5(3t^{3-1}\cdot1)-3(2t^{2-1}\cdot1)+1$$

$$V(t)=15t^2-6t+1$$

2. Structural stress on a bridge is modeled by the function $S(x)=100-5x^2$, where x is the distance from the center of the bridge. Calculate the rate of change of stress at that point.

Solution:

Stress Function: $S(x)=100-5x^2$

Rate of change of stress is the derivative of $S(x)$ w.r.t 'x'.

$$S'(x)=\frac{ds}{dx}=\frac{d}{dx}(100-5x^2)$$

$$=\frac{d}{dx}(100)-5\frac{d}{dx}(x^2)$$

$$=0-5(2x)=-10x$$

Since the centre of bridge is at $x=0$, therefore rate of change at that point is

$$S'(0)=-10(0)=0$$

Maximum stress $=S(0)=100-5(0)^2=100$

3. A company's revenue function is given by $R(x)=1000x-10x^2$, where is the number of units produced. The cost function is $C(x)=300x+2000$.

(i) Find the profit function $P(x)$

(ii) Determine the marginal profit when $x=15$

Solution:

Revenue Function: $R(x)=1000x-10x^2$

cost Function: $C(x)=300x+2000$

(i) Profit Function = Revenue Function − Cost Function

$$P(x)=R(x)-C(x)$$

$$P(x)=1000x-10x^2-300x-2000$$

$$P(x)=-10x^2+700x-2000$$

(ii) Marginal profit is the derivative of $P(x)$ w.r.t 'x'.

$$P'(x)=\frac{dp}{dx}=\frac{d}{dx}(-10x^2+700x-2000)$$

$$=-10\frac{d}{dx}(x^2)+700\frac{d}{dx}(x)-\frac{d}{dx}(2000)$$

$$=-10(2x)+700(1)-0$$

$$=-20x+700$$

Marginal profit at $x=15$ is

$$P'(5)=-20(15)+700$$

$$=-300+700$$

$$=400$$

4. An investment grows according to the function $A(t)=10000(1+0.05t)^3$, where $A(t)$ is the value of the investment and t is the time in years.

(i) Find the rate of change of the investment after 8 years.

(ii) What is the investment value after 8 years?

Solution:

(i) Investment Grows Function:

$$A(t)=10000(1+0.05t)^3$$

Rate of change of the investment is the derivative of $A(t)$ w.r.t 't'.

$$A'(t)=\frac{dA}{dt}=\frac{d}{dt}10000(1+0.05t)^3$$

$$A'(t)=10000\cdot\frac{d}{dt}(1+0.05t)^3$$

$$A'(t)=10000\cdot3(1+0.05t)^2\cdot(0.05)$$

$$A'(t)=1500(1+0.05t)^2$$

Rate of change of the investment after 8 years is

$$A'(8)=1500(1+0.05(8))^2$$

$$=1500(1.4)^2$$

$$=1500(1.96)$$

$$=2940$$

(ii) Investment value after 8 years is

$$A(8)=10000(1+0.05(8))^3$$

$$=10000(1.4)^3$$

$$=10000(2.744)$$

$$=27440$$

5. The position of a particle moving along a line is given by $s(t)=5t^3-12t^2+8t$, where $s(t)$ is the position in meters and t is the time in seconds.

(i) Determine the velocity of the particle at $t=4$ seconds

Solution:

Position Function: $s(t)=5t^3-12t^2+8t$.

The velocity of the particle is the derivative of $s(t)$ w.r.t. 't'

$$\begin{aligned} V(t) &= \frac{ds}{dt}=\frac{d}{dt}\left(5t^3-12t^2+8t\right)\\ &=5\frac{d}{dt}\left(t^3\right)-12\frac{d}{dt}\left(t^2\right)+8\frac{d}{dt}(t)\\ &=5\left(3t^2\right)-12(2t)+8(1)\\ V(t) &= 15t^2-24t+8 \qquad \ldots(1)\end{aligned}$$

Velocity of the particle at $t = 4$ is

$$\begin{aligned} V(4) &= 15(4)^2-24(4)+8\\ &=240-96+8\\ &=152\ \text{m/sec}\end{aligned}$$

(ii) Find the acceleration at $t=4$ seconds

Solution:

Acceleration is the derivative of the velocity function:

$$\begin{aligned} a(t) &= \frac{dv}{dt}=\frac{d}{dt}\left(15t^2-24t+8\right) \qquad \text{Using (1)}\\ &=15(2t)-24(1)+0\\ &=30t-24\end{aligned}$$

Acceleration at $t = 4$ is

$$\begin{aligned} a(4) &= 30(4)-24=120-24\\ &=96\ m/s^2\end{aligned}$$

(iii) When is the particle at rest?

Solution:

If the particle at rest, Then

$$V(t)=0$$

$$15t^2-24x+8=0$$

Here $a=15, b=-24, c=8$

By using quadratic formula

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$t=\frac{24\pm\sqrt{(-24)^2-4(15)(8)}}{2(15)}$$

$$t=\frac{24\pm\sqrt{576-480}}{30}$$

$$t=\frac{24\pm\sqrt{96}}{30}$$

$$t=\frac{24\pm4\sqrt{6}}{30}$$

$$t=\frac{2(12\pm2\sqrt{6})}{30}$$

$$t=\frac{12\pm2\sqrt{6}}{15}$$

$$\Rightarrow \quad t=\frac{12+2\sqrt{6}}{15} \quad \text{and} \quad t=\frac{12-2\sqrt{6}}{15}$$

$$\Rightarrow \quad t=1.13 \quad \text{and} \quad t=0.47$$

Thus, the particle is at rest when $t=1.13s$ and $t=0.47s$.

6. The position of a car traveling along a straight highway is given by $x(t)=30t^2-4t^3$, where $x(t)$ is the distance traveled in kilometres and t is the time in hours.

(i) Find the car's velocity at $t = 3$ hours.

(ii) Determine the car's acceleration at $t = 3$ hours

Solution:

Position Function: $x(t) = 30t^2 - 4t^3$

(i) The velocity of car is the derivative of $x(t)$ w.r.t. 'x'.

$$\begin{aligned} V(t) &= \frac{dx}{dt}=\frac{d}{dt}\left(30t^2-4t^3\right)\\ &=30(2t)-4(3t^2)\\ &=60t-12t^2 \qquad \ldots(1)\end{aligned}$$

Velocity of car at $t = 3$ is

$$\begin{aligned} V(3) &= 60(3)-12(3)^2\\ &=180-108\\ &=72\ \text{km/hour}\end{aligned}$$

(ii) Acceleration is the derivative of the velocity function:

$$\begin{aligned} a(t) &= \frac{dv}{dt}=\frac{d}{dt}\left(60t-12t^2\right) \qquad \text{Using (1)}\\ &=60(1)-12(2t)\\ &=60-24t\end{aligned}$$

Acceleration of car at $t = 3$ is

$$\begin{aligned} a(3) &= 60-24(3)\\ &=60-72\\ &=-12\ \text{km/h}^2\end{aligned}$$

7. The stress on a beam under a varying load is given by $S(x)=400x-x^3$, where $S(x)$ is the stress in pascals (Pa) and x is the distance from the fixed end in metres. Calculate the rate of change of stress at 6 meters.

Solution:

Stress Function: $S(x) = 400x - x^3$

The rate of change of stress is the derivative of $S(x)$ w.r.t. 'x'.

$$\begin{aligned} S'(x) &= \frac{ds}{dx}=\frac{d}{dx}\left(400x-x^3\right)\\ &=400(1)-3x^2\\ &=400-3x^2\end{aligned}$$

Put $x = 6$, we have

$$\begin{aligned} S'(6) &= 400-3(6)^2\\ &=400-108\\ &=292\end{aligned}$$

Thus, the rate of change of stress at $x = 6$ meters is 292 Pa/m.

8. The cost $C(r)$ to construct a cylindrical tank depends on the radius of the base, and is given by $C(r)=8000\pi r^2+\frac{150000}{r}$, where the first term represents the cost of the base and the second term represents the cost of the walls. Determine the rate of change of the cost at $r=4$ metres.

Solution:

Cost Function: $C(r)=8000\pi r^2+\frac{150000}{r}$

The rate of change of cost is the derivative of $C(r)$ w.r.t. 'r'.

$$\begin{aligned} C'(r) &= \frac{dc}{dr}=\frac{d}{dr}\left(8000\pi r^2+150000r^{-1}\right)\\ &=8000\pi(2r)+150000\left(-1r^{-2}\cdot1\right)\\ &=16000\pi r-\frac{150000}{r^2}\end{aligned}$$

Rate of change of cost at $r = 4$ is

$$\begin{aligned} C'(4) &= 16000\pi(4)-\frac{150000}{4^2}\\ &=64000\pi-9375\\ &=64000(3.1416)-9375\\ &=191687.4\ \text{units/m}\end{aligned}$$

Formula Sheet

1. $m_{sec}=\frac{f(x+\delta x)-f(x)}{\delta x}$	2. $m_{tan}=\lim\limits_{\delta x\to0}\frac{f(x+\delta x)-f(x)}{\delta x}=f'(x)$
3. $v_{avg}=\frac{s(t_1)-s(t)}{t_1-t}$	4. $v_{inst}=\lim\limits_{\delta t\to0}\frac{s(t+\delta t)-s(t)}{\delta t}=\frac{ds}{dt}$
5. $a=\lim\limits_{\delta t\to0}\frac{v(t+\delta t)-v(t)}{\delta t}=\frac{dv}{dt}$	6. $\frac{dy}{dx}=f'(x)=\lim\limits_{\delta x\to0}\frac{f(x+\delta x)-f(x)}{\delta x}$
7. $\frac{d}{dx}(c)=0$ (Derivative of constant is zero)	8. $\frac{d}{dx}(x^n)=nx^{n-1}$, where n is any real number.
9. $\frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)]$, where c is a constant.	
10. $\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$	(Sum Rule)
11. $\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]$	(Difference Rule)
12. $\frac{d}{dx}[f(x)g(x)]=\left[\frac{d}{dx}[f(x)]\right]g(x)+f(x)\left[\frac{d}{dx}[g(x)]\right]$	(Product Rule)
13. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\left[\frac{d}{dx}(f(x))\right]g(x)-f(x)\left[\frac{d}{dx}(g(x))\right]}{[g(x)]^2}$	(Quotient Rule)

Multiple Choice Questions (MCQs)

Exercise 13.1

1. **For the curve $y=f(x)$, the expression $\frac{f(x+\delta x)-f(x)}{\delta x}$ is the slope of:**

(A) secant line (B) tangent line (C) normal line (D) none of these

2. **For the curve $y=f(x)$, the expression $\lim\limits_{\delta x\to0}\frac{f(x+\delta x)-f(x)}{\delta x}$ is the slope of:**

(A) secant line (B) tangent line (C) normal line (D) none of these

3. **The gradient of the curve $f(x)=x^2-2$ at the point $P(-1,-1)$ is -----**

(A) 1 (B) 2 (C) -2 (D) 3

4. The notation used by Lagrange for derivative is -----

(A) df/dx (B) $f''(x)$ (C) $f'(x)$ (D) Df(x)

5. $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$ is equal to -----

(A) $f'(x)$ (B) f(x) (C) $f''(x)$ (D) $f'(a)$

6. If $x \in D_f$ and $f'(x)$ exists, then f is said to be ----- at x

(A) differentiable (B) discontinuous (C) increasing (D) decreasing

7. The derivative of $\sqrt{x}$ at x = a is -----

(A) $\frac{1}{2\sqrt{a}}$ (B) $2\sqrt{a}$ (C) $\frac{1}{\sqrt{a}}$ (D) $\frac{-1}{2\sqrt{a}}$

8. If $f(x) = x^{\frac{2}{3}}$, then $f'(8)$ = -----

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{3}$ (D) 3

Exercise 13.2

9. $\frac{d}{dx}(x)^n = n(x)^{n-1}$ where -----

(A) n∈ C {Complex numbers} (B) n ∈ R {Real numbers}

(C) n∈ Q {Rational numbers} (D) n ∈N {Natural numbers}

10. $\frac{d}{dx}(x - 5)(3 - x)$ = -----

(A) 2x + 8 (B) −2x + 8 (C) 2x − 8 (D) x + 8

11. $\frac{d}{dx}(x^2 + 1)^2$ equals -----

(A) $2(x^2 + 1)$ (B) $\frac{(x^2 + 1)^2}{3}$ (C) $2x(x^2 + 1)$ (D) $4x(x^2 + 1)$

12. $\frac{d}{dx}\left(\frac{x^2 - 4}{x - 2}\right)$ equals -----

(A) 0 (B) 1 (C) x + 2 (D) x − 2

13. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ = -----

(A) $\frac{f(x)\,g'(x) - g(x)\,f'(x)}{[g(x)]^2}$ (B) $\frac{f'(x)\,g'(x) - f(x)\,g'(x)}{[g(x)]^2}$

(C) $\frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$ (D) $\frac{g'(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$

14. If $y = \sqrt{1 - x^2}$, $0 < x < 1$, then $\frac{dy}{dx}$ = -----

(A) $\sqrt{x^2 - 1}$ (B) $\frac{1}{\sqrt{1 - x^2}}$ (C) $\frac{x}{\sqrt{1 - x^2}}$ (D) $\frac{-x}{\sqrt{1 - x^2}}$

ANSWER KEY

1.	A	2.	B	3.	C	4.	C	5.	D	6.	A	7.	A	8.	C	9.	B	10.	B
11.	D	12.	B	13.	C	14.	D												

◆

Unit 14

Vectors in Space

Introduction

In this unit, we will look into the rectangular coordinate system in three-dimensional space and explore the fundamental mathematical operations involving vectors in space. We will begin by understanding the dot product (or scalar product) and the cross product (or vector product) of two vectors and learn about their geometric interpretation. Further, we emphasize their practical applications. For example, we will see how these concepts can be used to calculate the area of a triangle and the area of a parallelogram. Finally, we will explore the extensive use of vectors in three-dimensional space, particularly in physics, where they play an important role in determining forces, velocities, and other essential physical quantities. For example, determining the work done by a constant force when moving an object along a specified vector.

Vectors (Recall)

In previous classes, we learned about two fundamental quantities: scalars and vectors.

Scalar Quantity (Scalar):

A scalar is a quantity that has only magnitude or size, such as mass, time, density, temperature, length, volume, speed work etc.

Vector Quantity (Vector):

A vector is a quantity that has both magnitude and direction, for example displacement, velocity, acceleration, weight, force, momentum, electric and magnetic fields, etc.

Geometric Interpretation of Vector:

Geometrically, a vector is represented as a directed line segment $\overline{AB}$ with A as its initial point and B as the terminal point.

Vector in Two-Dimension:

In two-dimension (R^2) a vector has components that can be represented by an ordered pair $[x, y]$ of real numbers. For the vector $\underline{u} = [x, y]$, x and y represent the components of $\underline{u}$.

Addition of Vectors:

For any two vectors $\underline{u} = [x_1, y_1]$ and $\underline{v} = [x_2, y_2]$, we have

$$\underline{u} + \underline{v} = [x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$$

Scalar Multiplication of a Vector:

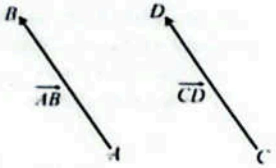
For $\underline{u} = [x, y]$ and $a \in R$, we have $a\underline{u} = a[x, y] = [ax, ay]$

Equal Vectors:

Two vectors $\underline{u} = [x_1, y_1]$ and $\underline{v} = [x_2, y_2]$ of R^2 are said to be equal if and only if they have the same components. That is, $[x_1, y_1] = [x_2, y_2]$ if and only if $x_1 = x_2$ and $y_1 = y_2$ and we write $\underline{u} = \underline{v}$.

Or

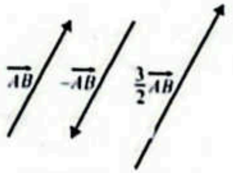
In other words, two vector $\underline{u}$ and $\underline{v}$ are said to be equal, if they have same magnitude and same direction.



Parallel Vectors:

Two vectors are parallel if and only if they are non-zero scalar multiple of each other.

For example, vectors $\overrightarrow{AB}$ − $\overrightarrow{AB}$ and $\frac{3}{2}AB$ are parallel.

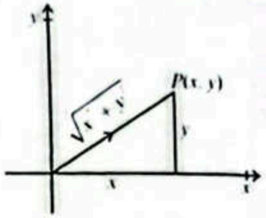


Magnitude of a Vector:

The magnitude (or norm or length) of a vector in 2D represents the length of the vector from the origin to the point represented by the vector.

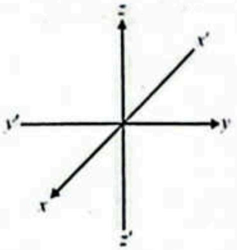
For any vector $\underline{u} = [x, y]$ in R^2, we define the magnitude, as the distance of the point $P(x, y)$ from the origin O.

$$\text{Magnitude of } \overrightarrow{OP} = |\overrightarrow{OP}| = |\underline{u}| = \sqrt{x^2 + y^2}$$



Now, we will learn some mathematical operations involving vectors in three-dimensional space.

Rectangular Coordinate System in Space:

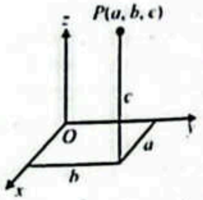


In space a rectangular coordinate system is constructed using three mutually orthogonal (perpendicular) axes, which have origin as their common point of intersection. When sketching figures, we follow the convention that the positive x-axis points towards the reader, the positive y-axis to the right and the positive z-axis points upwards.

These axes are also labeled in accordance with the right-hand rule. The fingers of the right hand, pointing in the direction of the positive x-axis, curled images toward the positive y-axis, and the thumb will point in the direction of the positive z-axis. A point P in space has three coordinates, one along x-axis, the second along y-axis and the third along z-axis.

Right hand rule

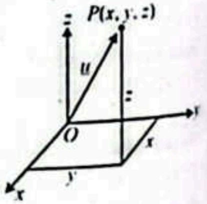
If the directed distances along x-axis, y- axis and z-axis respectively are a, and c, then the point P is written with a unique triple of real numbers as $P(a, b, c)$ (see figure).



Concept of a Vector in Space:

The set $R^3 = \{(x, y, z) : x, y, z \in R\}$ is called 3-dimensional space. An element (x, y, z) of R^3 represents a point $P(x, y, z)$, which is uniquely determined by its coordinates x, y and z.

Given a vector $\underline{u}$ in space, there exists a unique point $P(x, y, z)$ in space such that the vector $\overrightarrow{OP}$ is equal to $\underline{u}$ (see figure). Now each element $(x, y, z) \in R^3$ is associated with a unique ordered triple (x, y, z), which represents the vector $\underline{u} = \overrightarrow{OP} = [x, y, z]$.



Fundamental Mathematical Operations for Vectors in Space:

We define addition and scalar multiplication in R^3 by:

(i) **Addition of Vectors:**

For any two vectors $\underline{u} = [x, y, z]$ and $\underline{v} = [x', y', z']$ we have

$$\underline{u} + \underline{v} = [x, y, z] + [x', y', z'] = [x + x', y + y', z + z']$$

(ii) **Scalar Multiplication of a Vector:**

For $\underline{u} = [x, y, z]$ and $a \in R$, we have $a\underline{u} = a[x, y, z] = [ax, ay, az]$

The set of all ordered triples $[x, y, z]$ of real numbers, together with the rules of addition and scalar multiplication is called the set of vectors in R^3. For the vector $\underline{u} = [x, y, z]$, x, y and z are called the components of $\underline{u}$.

The definition of vectors in R^3 states that vector addition and scalar multiplication are to be carried out also for vectors in space just as for vectors in the plane. Similarly, we define in R^3 :

(a) **Negative of Vector:**

The negative of the vector $\underline{u} = [x, y, z]$ as $\underline{u} = (-1)\underline{u} = [-x, -y, -z]$

(b) **Difference of two vectors:**

The difference of two vectors $\underline{v} = [x', y', z']$ and $\underline{w} = [x'', y'', z'']$ as

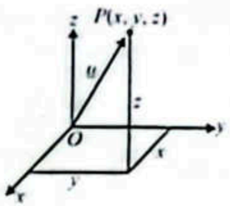
$$\underline{v} - \underline{w} = \underline{v} + (-\underline{w}) = [x' - x'', y' - y'', z' - z'']$$

(c) **Zero Vector:** The zero vector as $\underline{O} = [0, 0, 0]$

(d) **Equality of two vectors:** Two vectors $\underline{v} = [x', y', z']$ and $\underline{w} = [x'', y'', z']$ are equal that is $\underline{v} = \underline{w}$ if and only if $x' = x''$, $y' = y''$ and $z' = z''$.

(e) **Position Vector:**

For any point $P(x, y, z)$ in R^3, a vector $\underline{u} = [x, y, z]$ is represented by a directed line segment $\overrightarrow{OP}$, whose initial point is at origin. Such vectors are called position vectors in R^3.



Magnitude of a Vector in Space:

We define the magnitude, norm, or length of a vector $\underline{u} = [x, y, z]$ in space by the distance of the point $P(x, y, z)$ from the origin O.

$$\therefore \quad |\overrightarrow{OP}| = |\underline{u}| = \sqrt{x^2 + y^2 + z^2}$$

Example 1: For the vectors, $\underline{u} = [1, -2, 3]$, $\underline{v} = [2, 1, 3]$ and $\underline{w} = [-1, 4, 0]$, find the following:

(i) $\underline{v} + \underline{w}$ (ii) $2\underline{w}$ (iii) $|\underline{u}|$ (iv) $|\underline{v} - 2\underline{w}|$ (v) $|2\underline{u} - \underline{v} + 3\underline{w}|$

Solution:

(i) $\underline{v} + \underline{w} = [2 - 1, 1 + 4, 3 + 0] = [1, 5, 3]$

(ii) $2\underline{w} = 2[-1, 4, 0] = [-2, 8, 0]$

(iii) $|\underline{u}| = |[1, -2, 3]| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

(iv) $\underline{v} - 2\underline{w} = [2, 1, 3] - [-1, 4, 0] = [4, -7, 3]$

$|\underline{v} - 2\underline{w}| = \sqrt{(4)^2 + (-7)^2 + (3)^2} = \sqrt{16 + 49 + 9} = \sqrt{74}$

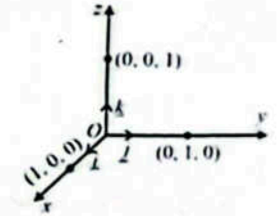
(v) $2\underline{u} - \underline{v} + 3\underline{w} = 2[1, -2, 3] - [2, 1, 3] + 3[-1, 4, 0] = [2, -4, 6] - [2, 1, 3] + [-3, 12, 0] = [-3, 7, 3]$

$|2\underline{u} - \underline{v} + 3\underline{w}| = |[-3, 7, 3]|$

$= \sqrt{(-3)^2 + (7)^2 + (3)^2} = \sqrt{9 + 49 + 9} = \sqrt{67}$

Components of a Vector:

As in plane, we introduce three special vectors $\underline{i} = [1, 0, 0]$, $\underline{j} = [0, 1, 0]$ and $\underline{k} = [0, 0, 1]$ in R^3



As magnitude of $\underline{i} = \sqrt{1^2 + 0^2 + 0^2} = 1$

magnitude of $\underline{j} = \sqrt{0^2 + 1^2 + 0^2} = 1$

and magnitude of $\underline{k} = \sqrt{0^2 + 0^2 + 1^2} = 1$, so

$\underline{i}, \underline{j}$ and $\underline{k}$ are called unit vectors along x-axis, y-axis and z-axis respectively.

Using the definition of addition and scalar multiplication, the vector $[x, y, z]$ can be written as:

$$\underline{u} = [x, y, z] = [x, 0, 0] + [0, y, 0] + [0, 0, z]$$
$$= x[1, 0, 0] + y[0, 1, 0] + z[0, 0, 1] = x\underline{i} + y\underline{j} + z\underline{k}$$

Thus, each vector $[x, y, z]$ in R^3 can be uniquely represented by $x\underline{i} + y\underline{j} + z\underline{k}$.

Unit Vector:

A unit vector is defined as a vector whose magnitude is unity. In three-dimensional space the unit vector of the vector $\underline{u} = x\underline{i} + y\underline{j} + z\underline{k}$ is written as $\hat{u}$ (read as u hat) and is defined by

$$\hat{u} = \frac{\underline{u}}{|\underline{u}|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\underline{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\underline{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\underline{k}$$

In terms of unit vector $\underline{i}, \underline{j}$, and $\underline{k}$, the sum $\underline{u}+\underline{v}$ of two vectors. $\underline{u}=[x_1, y_1, z_1]$ and $\underline{v}=[x_2, y_2, z_2]$ is written as:

$$\begin{aligned}\underline{u}+\underline{v} &= [x_1+x_2, y_1+y_2, z_1+z_2]\\ &= (x_1+x_2)\underline{i}+(y_1+y_2)\underline{j}+(z_1+z_2)\underline{k}\end{aligned}$$

Example 2: Find the unit vector of $\underline{u}=2\underline{i}+5\underline{j}-\underline{k}$.

Solution:

Given vector $\underline{u}=2\underline{i}+5\underline{j}-\underline{k}$ $\Rightarrow |\underline{u}|=\sqrt{(2)^2+(5)^2+(-1)^2}=\sqrt{30}$

The unit vector is:

$$\hat{u}=\frac{\underline{u}}{|\underline{u}|}=\frac{2\underline{i}+5\underline{j}-\underline{k}}{\sqrt{30}}=\frac{1}{\sqrt{30}}(2\underline{i}+5\underline{j}-\underline{k})$$

Example 3: If $\underline{v}=2\underline{i}+3\underline{j}+\underline{k}, \underline{v}=4\underline{i}+6\underline{j}+2\underline{k}$ and $\underline{w}=-6\underline{i}-9\underline{j}-3\underline{k}$, then show that $\underline{u}, \underline{v}$ and $\underline{w}$ are parallel to each other.

Solution:

As $\underline{v}=4\underline{i}+6\underline{j}+2\underline{k}$

$\underline{v}=2(2\underline{i}+3\underline{j}+\underline{k})$

$\underline{v}=2\underline{u}$

$\Rightarrow$ $\underline{u}$ and $\underline{v}$ are parallel vectors.

As $\underline{w}=-6\underline{i}-9\underline{j}-3\underline{k}$

$\underline{w}=-3(2\underline{i}+3\underline{j}+\underline{k})$

$\underline{w}=-3\underline{u}$

$\Rightarrow$ $\underline{u}$ and $\underline{w}$ are parallel vectors.

Hence $\underline{u}, \underline{v}$ and $\underline{w}$ are parallel to each other.

Properties of Vectors:

Let $\underline{u}, \underline{v}$ and $\underline{w}$ be vectors in the plane or in space and let $a, b \in R$, then they have the following properties:

(i) $\underline{u}+\underline{v}=\underline{v}+\underline{u}$ (Commutative property)

(ii) $(\underline{u}+\underline{v})+\underline{w}=\underline{u}+(\underline{v}+\underline{w})$ (Associative property)

(iii) $\underline{u}+\underline{o}=\underline{u}$ (Additive Identity)

(iv) $\underline{u}+(-1)\underline{u}=\underline{u}-\underline{u}=\underline{o}$ (Inverse for vector addition)

(v) $a(\underline{v}+\underline{w})=a\underline{v}+a\underline{w}$ (Distributive property)

(vi) $a(b\underline{u})=(ab)\underline{u}$ (Scalar multiplication)

Proof:

(i) Since for any two real numbers $a, b \in R, a+b=b+a$, it follows that for any two vectors $\underline{u}=[x, y, z]$ and $\underline{v}=[x', y', z']$ in R^3, where components of $\underline{u}$ and $\underline{v}$ belong to R.

We have
$$\begin{aligned}\underline{u}+\underline{v} &= [x,y,z]+[x',y',z']\\ &= [x+x', y+y', z+z']\\ &= [x'+x, y'+y, z'+z] \qquad \because a+b=b+a\\ &= [x',y',z']+[x,y,z]\\ &= \underline{v}+\underline{u}\end{aligned}$$

So, addition of vectors in R^3 is commutative.

(ii) Since for any three real numbers $a, b, c \in R, (a+b)+c=a+(b+c)$, it follows that for any three vectors $\underline{u}=[x,y,z], \underline{v}=[x',y',z']$ and $\underline{w}=[x'',y'',z'']$ in R^3. Where components of $\underline{u}, \underline{v}$ and $\underline{w}$ belong to R.

We have
$$\begin{aligned}(\underline{u}+\underline{v})+\underline{w} &= [x+x', y+y', z+z']+[x'', y'', z'']\\ &= [(x+x')+x'', (y+y')+y'', (z+z')+z'']\\ &= [x+(x'+x''), y+(y'+y''), z+(z'+z'')] \qquad \because (a+b)+c=a+(b+c)\\ &= [x,y,z]+[x'+x'', y'+y'', z'+z'']\\ &= \underline{u}+(\underline{v}+\underline{w})\end{aligned}$$

So, addition of vectors in R^3 is associative.

(iii) Since for any real number a and 0, $a+0=a$, it follows that for any vectors, $\underline{u}=[x, y, z]$, and $\underline{o}=[0,0,0]$, where $\underline{o}$ is the zero vector in R^3.

We have
$$\begin{aligned}\underline{u}+\underline{o} &= [x,y,z]+[0,0,0]\\ &= [x+0, y+0, z+0]\\ &= [x,y,z]=\underline{u}\\ \underline{u}+\underline{o} &= \underline{u}\end{aligned}$$

Thus, $\underline{o}$ is the additive identity in R^3

(iv) Since for any real number a, there exist $-a$ such that $a+(-a)=a-a=0$, it follows that for any vector, $\underline{u}=[x, y, z]$, there exists $-\underline{u}=[-x, -y, -z]$ in R^3

Such that
$$\begin{aligned}\underline{u}+(-\underline{u}) &= [x,y,z]+[-x,-y,-z]=[x+(-x), y+(-y), z+(-z)]\\ &= [x-x, y-y, z-z]\\ &= [0,0,0]=\underline{o}, \text{ where } \underline{o} \text{ is the additive identity}\\ \underline{u}+(-\underline{u}) &= \underline{o}\\ \underline{u}+(-\underline{u}) &= [x,y,z]+[-x,-y,-z]=[x+(-x), y+(-y), z+(-z)]\\ &= [x-x, y-y, z-z]\\ &= [0,0,0]=\underline{o}, \text{ where } \underline{o} \text{ is the additive identity}\\ \underline{u}+(-\underline{u}) &= \underline{o}\end{aligned}$$

Thus $\underline{u}$ is the additive inverse of $\underline{u}$ in R^3

(v) Since for any three real numbers $a, b, c \in R$, $a(b+c)=ab+ac$, it follows that for any $a \in R$ and for any two vectors $\underline{v}=[x, y, z]$ and $\underline{w}=[x', y', z']$ in R^3, where components of $\underline{v}$ and $\underline{w}$ belong to R.

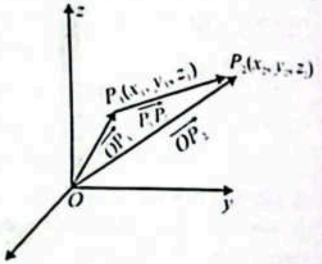
We have
$$\begin{aligned}a(\underline{v}+\underline{w}) &= a([x,y,z]+[x',y',z'])\\ &= a[x+x', y+y', z+z']\\ &= [a(x+x'), a(y+y'), a(z+z')]\\ &= [ax+ax', ay+ay', az+az'] \qquad \because a(b+c)=ab+ac\\ &= [ax, ay, az]+[ax', ay', az']\\ &= a[x,y,z]+a[x',y',z']\\ &= a\underline{v}+a\underline{w}\end{aligned}$$

So, addition of vectors in R^3 is distributive.

(vi) Since for any three real numbers $a, b, c \in R$, $a(bc)=(ab)c$, it follows that for any $a, b \in R$ and for any vector $\underline{u}=[x, y, z]$ in R^3, where components of $\underline{u}$ belong to R.

We have
$$\begin{aligned}a(b\underline{u}) &= a(b[x,y,z])\\ &= a[bx, by, bz]\\ &= [a(bx), a(by), a(bz)]\\ &= [(ab)x, (ab)y, (ab)z] \qquad \because a(bc)=(ab)c\\ &= (ab)[x,y,z]\\ &= (ab)\underline{u} \text{ (Proved)}\end{aligned}$$

Distance Between Two Points in Space:



If $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are the position vectors of the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

The vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

Distance between P_1 and P_2 is

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

This is called distance formula between two points P_1 and P_2 in R^3.

Example 4: Suppose a butterfly's flight path passed through points (2,4,7) and (6,1,3), where each unit represents a metre. What is the magnitude of the displacement the butterfly experienced in traveling between these two points?

Solution:

Given points: $(2, 4, 7)$ and $(6, 1, 3)$

By using distance formula, we have

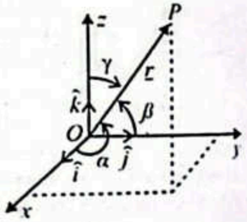
$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$d = \sqrt{(6-2)^2 + (1-4)^2 + (3-7)^2}$$

$$d = \sqrt{16+9+16} = \sqrt{41} = 6.40$$

The magnitude of the displacement the butterfly experienced in traveling between these two points is approximately 6.40 metres.

Direction Angles and Direction Cosines of a Vector:

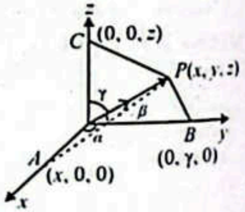


Let $\underline{r} = \overrightarrow{OP} = x\underline{i} + y\underline{j} + z\underline{k}$ be a non-zero vector, let α, β and γ denote the angles formed between $\underline{r}$ and the unit coordinate vectors $\underline{i}, \underline{j}$ and $\underline{k}$ respectively, where $0 \le \alpha \le \pi, 0 \le \beta \le \pi$ and $0 \le \gamma \le \pi$

(i) The angles α, β and γ are called the direction angles of the vector $\underline{r}$.

(ii) The numbers $\cos\ \alpha, \cos\ \beta$ and $\cos\ \gamma$ are called direction cosines of the vector $\underline{r}$.

Important Result: Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$



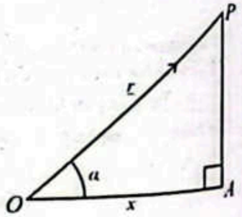
Proof:

Let $\underline{r} = [x, y, z] = x\underline{i} + y\underline{j} + z\underline{k}$

Therefore $|\underline{r}| = \sqrt{x^2 + y^2 + z^2} = r$

From the above figure it is clear that the triangle OAP is a right triangle with $m\angle A = 90°$. Therefore

In right triangle OAP



$$\cos\alpha = \frac{|\overline{OA}|}{|\overline{OP}|} = \frac{x}{r},$$

Similarly, $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$

$$\text{L.H.S} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$= \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1 = \text{R.H.S (Proved)}$$

Remember That: The numbers $\cos\alpha = \frac{x}{r}, \cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$ are called the direction cosines of $\overrightarrow{OP}$

Exercise 14.1

1. Let $\underline{u} = 3\underline{i} + 2\underline{j} - 5\underline{k}$, $\underline{v} = \underline{i} - 5\underline{j} - \underline{k}$ and $w = -4\underline{i} - \underline{j} + 7\underline{k}$. Find the following:

(i) $\underline{u} + 2\underline{v} + \underline{w}$

Solution:

$$\underline{u} + 2\underline{v} + \underline{w} = ?$$

$$= (3\underline{i} + 2\underline{j} - 5\underline{k}) + 2(\underline{i} - 5\underline{j} - \underline{k}) + (-4\underline{i} - \underline{j} + 7\underline{k})$$

$$= 3\underline{i} + 2\underline{j} - 5\underline{k} + 2\underline{i} - 10\underline{j} - 2\underline{k} - 4\underline{i} - \underline{j} + 7\underline{k}$$

$$= \underline{i} - 9\underline{j} + 0\underline{k} = \underline{i} - 9\underline{j}$$

(ii) $\underline{v} - 3\underline{w}$

Solution:

$$v - 3\underline{w} = ?$$

$$= (\underline{i} - 5\underline{j} - \underline{k}) - 3(-4\underline{i} - \underline{j} + 7\underline{k})$$

$$= \underline{i} - 5\underline{j} - \underline{k} + 12\underline{i} + 3\underline{j} - 21\underline{k}$$

$$= 13\underline{i} - 2\underline{j} - 22\underline{k}$$

(iii) $|3\underline{v} + \underline{w}|$.

Solution:

$$|3\underline{v} + \underline{w}| = ?$$

$$3\underline{v} + \underline{w} = 3(\underline{i} - 5\underline{j} - \underline{k}) + (-4\underline{i} - \underline{j} + 7\underline{k})$$

$$= 3\underline{i} - 15\underline{j} - 3\underline{k} - 4\underline{i} - \underline{j} + 7\underline{k}$$

$$= -1\underline{i} - 16\underline{j} + 4\underline{k}$$

Now,

$$|3\underline{v} + \underline{w}| = \sqrt{(-1)^2 + (-16)^2 + (4)^2}$$

$$= \sqrt{1 + 256 + 16} = \sqrt{273}$$

2. Find the magnitude of the vector $\underline{v}$ and write the direction cosines of $\underline{v}$.

(i) $\underline{v} = 3\underline{i} - 2\underline{j} + 6\underline{k}$

Solution:

$$\underline{v} = 3\underline{i} - 2\underline{j} + 6\underline{k}$$

$$|\underline{v}| = \sqrt{(3)^2 + (-2)^2 + (6)^2}$$

$$|\underline{v}| = \sqrt{9 + 4 + 36}$$

$$|\underline{v}| = \sqrt{49} = 7$$

Direction ratios of $\underline{v}$ are

$$3, -2, 6$$

Direction cosines of $\underline{v}$ are

$$\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$$

(ii) $\underline{v} = -4\underline{i} + 4\underline{j} + 2\underline{k}$

Solution:

$$\underline{v} = -4\underline{i} + 4\underline{j} + 2\underline{k}$$

$$|\underline{v}| = \sqrt{(-4)^2 + (4)^2 + (2)^2}$$

$$|\underline{v}| = \sqrt{16 + 16 + 4}$$

$$|\underline{v}| = \sqrt{36} = 6$$

Direction ratios of $\underline{v}$ are

$$-4, 4, 2$$

Direction cosines of $\underline{v}$ are

$$-\frac{4}{6}, \frac{4}{6}, \frac{2}{6}$$

$$= -\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$$

(iii) $\underline{v} = -6\underline{i} + 8\underline{j}$

Solution:

$$\underline{v} = -6\underline{i} + 8\underline{j} + 0\underline{k}$$

$$|\underline{v}| = \sqrt{(-6)^2 + (8)^2 + (0)^2}$$

$$|\underline{v}| = \sqrt{36 + 64 + 0}$$

$$|\underline{v}| = \sqrt{100} = 10$$

Direction ratios of $\underline{v}$ are

$$-6, 8, 0$$

Direction cosines of $\underline{v}$ are

$$-\frac{6}{10}, \frac{8}{10}, \frac{0}{10}$$

$$= -\frac{3}{5}, \frac{4}{5}, 0$$

3. Find t, so that $|2\underline{i} + (t-1)\underline{j} + t\underline{k}| = \sqrt{13}$

Solution:

$$|2\underline{i} + (t-1)\underline{j} + t\underline{k}| = \sqrt{13}$$

$$\sqrt{2^2 + (t-1)^2 + t^2} = \sqrt{13}$$

Squaring both sides, we have

$$4 + (t-1)^2 + t^2 = 13$$

$$4 + t^2 + 1 - 2t + t^2 - 13 = 0$$

$$2t^2 - 2t - 8 = 0$$

$$t^2 - t - 4 = 0 \qquad \text{Dividing by '2'}$$

Here $a = 1, b = -1, c = -4$

By using quadratic formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)}$$
$$t = \frac{1 \pm \sqrt{1+16}}{2}$$
$$t = \frac{1 \pm \sqrt{17}}{2}$$

4. Find a unit vector in the direction of $\underline{v} = -\underline{i} + 4\underline{j} - 8\underline{k}$

Solution:

Let $\hat{v}$ be a unit vector in the direction of $\underline{v}$, so

$$\underline{v} = -\underline{i} + 4\underline{j} - 8\underline{k}$$
$$|\underline{v}| = \sqrt{(-1)^2 + (4)^2 + (-8)^2}$$
$$= \sqrt{1+16+64}$$
$$= \sqrt{81} = 9$$

Required unit vector is

$$\hat{v} = \frac{v}{|\underline{v}|}$$
$$= \frac{-\underline{i} + 4\underline{j} - 8\underline{k}}{9}$$
$$\hat{v} = -\frac{1}{9}\underline{i} + \frac{4}{9}\underline{j} - \frac{8}{9}\underline{k}$$

5. If $\underline{u} = 2\underline{i} + \underline{j} - 3\underline{k}$, $\underline{v} = -\underline{i} + 4\underline{j} + 2\underline{k}$ and $\underline{w} = 3\underline{i} - 2\underline{j} + \underline{k}$, Find a unit vector parallel to $4\underline{u} - 3\underline{v} + 2\underline{w}$.

Solution:

Let

$$\underline{a} = 4\underline{u} - 3\underline{v} + 2\underline{w}$$
$$= 4(2\underline{i} + \underline{j} - 3\underline{k}) - 3(-\underline{i} + 4\underline{j} + 2\underline{k}) + 2(3\underline{i} - 2\underline{j} + \underline{k})$$
$$= 8\underline{i} + 4\underline{j} - 12\underline{k} + 3\underline{i} - 12\underline{j} - 6\underline{k} + 6\underline{i} - 4\underline{j} + 2\underline{k}$$
$$\underline{a} = 17\underline{i} - 12\underline{j} - 16\underline{k}$$
$$|\underline{a}| = \sqrt{(17)^2 + (-12)^2 + (-16)^2}$$
$$= \sqrt{289+144+196} = \sqrt{689}$$

Let $\hat{b}$ be a unit vector parallel to a, Then

$$= \frac{\underline{a}}{|\underline{a}|}$$
$$= \frac{17i - 12\underline{j} - 16\underline{k}}{\sqrt{689}}$$
$$= \frac{17}{\sqrt{689}}\underline{i} - \frac{12}{\sqrt{689}}\underline{j} - \frac{16}{\sqrt{689}}\underline{k}$$

6. Find a vector whose:

(i) magnitude is 5 and is parallel to $3\underline{i} + 4\underline{j} - \underline{k}$

Solution:

Let required vector = $\underline{u}$

$$|\underline{u}| = 5$$

Given vector: $\underline{v} = 3\underline{i} + 4\underline{j} - \underline{k}$

As $\underline{u} \parallel \underline{v}$, so

$$\underline{u} = |\underline{u}|\hat{v}$$
$$= 5\left(\frac{\underline{v}}{|\underline{v}|}\right)$$
$$= 5\left(\frac{3\underline{i} + 4\underline{j} - \underline{k}}{\sqrt{(3)^2 + (4)^2 + (-1)^2}}\right)$$
$$= \frac{15\underline{i} + 20\underline{j} - 5\underline{k}}{\sqrt{9+16+1}}$$
$$= \frac{15\underline{i} + 20\underline{j} - 5\underline{k}}{\sqrt{26}}$$
$$\underline{u} = \frac{15}{\sqrt{26}}\underline{i} + \frac{20}{\sqrt{26}}\underline{j} - \frac{5}{\sqrt{26}}\underline{k}$$

(ii) magnitude is 7 and is parallel to $-\underline{i} + \underline{j} + \underline{k}$.

Solution:

Let required rector = $\underline{u}$

$$|\underline{u}| = 7$$

Given vector: $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$

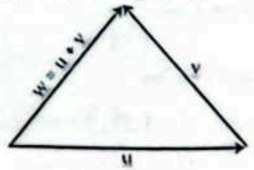
As $\underline{u} \parallel \underline{v}$, So

$$\underline{u} = |\underline{u}|\hat{v}$$
$$= 7\left(\frac{\underline{v}}{|\underline{v}|}\right)$$
$$= 7\left(\frac{-\underline{i} + \underline{j} + 1\underline{k}}{\sqrt{(-1)^2 + 1^2 + 1^2}}\right)$$
$$= \frac{-7\underline{i} + 7\underline{j} + 7\underline{k}}{\sqrt{1+1+1}}$$
$$= \frac{-7\underline{i} + 7\underline{j} + 7\underline{k}}{\sqrt{3}}$$
$$= \frac{-7}{\sqrt{3}}\underline{i} + \frac{7}{\sqrt{3}}\underline{j} + \frac{7}{\sqrt{3}}\underline{k}$$

7. If $\underline{u} = x\underline{i} + 2\underline{j} + 3\underline{k}$, $\underline{v} = \underline{i} + y\underline{j} - 3\underline{k}$ and $\underline{w} = -2\underline{i} - 3\underline{j}$ represent the sides of a triangle. Find the values of x and y.

Solution:

$$\underline{u} = x\underline{i} + 2\underline{j} + 3\underline{k}, \underline{v} = \underline{i} + y\underline{j} - 3\underline{k}, \underline{w} = -2\underline{i} - 3\underline{j} + \underline{k}$$



Since $\underline{u}$, $\underline{v}$ and $\underline{w}$ are the sides of a triangle, therefore

$$\underline{u} + \underline{v} = \underline{w}$$
$$(x\underline{i} + 2\underline{j} + 3\underline{k}) + (\underline{i} + y\underline{j} - 3\underline{k}) = -2\underline{i} - 3\underline{j} + 0\underline{k}$$
$$(x+1)\underline{i} + (2+y)\underline{j} + (3-3)\underline{k} = -2\underline{i} - 3\underline{j} + 0\underline{k}$$
$$(x+1)\underline{i} + (2+y)\underline{j} + 0\underline{k} = -2\underline{i} - 3\underline{j} + 0\underline{k}$$

Comparing both sides

$$x + 1 = -2 \Rightarrow x = -2 - 1 \Rightarrow \boxed{x = -3}$$

and $2 + y = -3 \Rightarrow y = -3 - 2 \Rightarrow \boxed{y = -5}$

Hence $x = -3$ and $y = -5$

8. The position vectors of the points A, B, C and D are $\underline{u} = \underline{i} + 2\underline{j} + \underline{k}$, $\underline{v} = 7\underline{i} + 8\underline{j} + 4\underline{k}$, $\underline{w} = -\underline{i} + \underline{k}$ and $\underline{z} = \underline{i} + 2\underline{j} + 2\underline{k}$ respectively. Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

Solution:

The position vectors of A, B and D are

$$\overrightarrow{OA} = \underline{u} = \underline{i} + 2\underline{j} + \underline{k}$$
$$\overrightarrow{OB} = \underline{v} = 7\underline{i} + 8\underline{j} + 4\underline{k}$$
$$\overrightarrow{OC} = \underline{w} = -\underline{i} + \underline{k}$$
$$\overrightarrow{OD} = \underline{z} = \underline{i} + 2\underline{j} + 2\underline{k}$$
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$\overrightarrow{AB} = \underline{v} - \underline{u}$$
$$= 7\underline{i} + 8\underline{j} + 4\underline{k} - \underline{i} - 2\underline{j} - \underline{k}$$
$$\overrightarrow{AB} = 6\underline{i} + 6\underline{j} + 3\underline{k}$$
$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$
$$\overrightarrow{CD} = \underline{z} - \underline{w}$$
$$= \underline{i} + 2\underline{j} + 2\underline{k} + \underline{i} - \underline{k}$$
$$\overrightarrow{CD} = 2\underline{i} + 2\underline{j} + \underline{k}$$

As

$$\overrightarrow{AB} = 3(2\underline{i} + 2\underline{j} + \underline{k})$$
$$\overrightarrow{AB} = 3\overrightarrow{CD}$$

$\Rightarrow$ $\overrightarrow{AB} \parallel \overrightarrow{CD}$ with same direction.

9. We say that two vectors $\underline{v}$ and $\underline{w}$ in space are parallel if there is a scalar c such that $\underline{v} = c\underline{w}$. The vectors point in the same direction if $c > 0$ and the vectors point in the opposite direction if $c < 0$

(a) Find two vectors of length 2 parallel to the vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$.

(b) Find the constant a so that the vectors $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.

(c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$.

(d) Find a and b so that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $a\underline{i} + b\underline{j} - 2\underline{k}$ are parallel.

(a) Find two vectors of length 2 parallel to the vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$.

Solution:

Let required vector = $\underline{u}$

$$|\underline{u}| = 2$$

Given vector: $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$

As $\underline{u} \parallel \underline{v}$, so

$$\underline{u} = |\underline{u}|(\pm\hat{v})$$
$$= \pm 2\frac{v}{|\underline{v}|}$$
$$= \pm 2\frac{2\underline{i} - 4\underline{j} + 4\underline{k}}{\sqrt{(2)^2 + (-4)^2 + (4)^2}}$$
$$= +2\frac{2\underline{i} - 4\underline{j} + 4\underline{k}}{\sqrt{36}}$$
$$= \pm\frac{2}{6}(2\underline{i} - 4\underline{j} + 4\underline{k})$$
$$\underline{u} = \pm\frac{1}{3}(2\underline{i} - 4\underline{j} + 4\underline{k})$$

$$\underline{u} = +\frac{1}{3}(2\underline{i} - 4\underline{j} + 4\underline{k})$$
$$\underline{u} = \frac{2}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{4}{3}\underline{k}$$

(Same direction)

$$\underline{u} = -\frac{1}{3}(2\underline{i} - 4\underline{j} + 4\underline{k})$$
$$\underline{u} = -\frac{2}{3}\underline{i} + \frac{4}{3}\underline{j} - \frac{4}{3}\underline{k}$$

(Opposite direction)

(b) Find the constant a so that the vectors $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.

Solution:

$$\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$$
$$\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$$

As $\underline{u} \parallel \underline{v}$, so

Direction ratios are proportional

i.e.,
$$\frac{1}{a} = -\frac{3}{9} = \frac{4}{-12}$$
$$\frac{1}{a} = -\frac{1}{3} = -\frac{1}{3}$$
$$\Rightarrow \quad \frac{1}{a} = -\frac{1}{3}$$
$$\Rightarrow \quad a = -3$$

(c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$.

Solution:

Letrequired vector = $\underline{u}$

$$|\underline{u}| = 5$$

Given vector = $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

As $\underline{u}$ lies in the direction opposite that of $\underline{v}$, so

$$\underline{u} = |\underline{u}|(-\hat{v})$$

$$= -5\frac{\underline{v}}{|\underline{v}|}$$

$$= -5\frac{\underline{i}-2\underline{j}+3\underline{k}}{\sqrt{(1)^2+(-2)^2+(3)^2}} = \frac{-5\underline{i}+10\underline{j}-15\underline{k}}{\sqrt{1+4+9}}$$

$$\underline{u} = \frac{-5}{\sqrt{14}}\underline{i}+\frac{10}{\sqrt{14}}\underline{j}-\frac{15}{\sqrt{14}}\underline{k}$$

(d) Find a and b so that the vectors $3\underline{i}-\underline{j}+4\underline{k}$ and $a\underline{i}+b\underline{j}-2\underline{k}$ are parallel.

Solution:

Let
$$\underline{u} = 3\underline{i}-\underline{j}+4\underline{k}$$
$$\underline{v} = a\underline{i}+b\underline{j}-2\underline{k}$$

As $\underline{u} \parallel \underline{v}$, so

Direction ratios are proportional

i.e.,
$$\frac{3}{a} = -\frac{1}{b} = \frac{4}{-2}$$
$$\frac{3}{a} = -\frac{1}{b} = -2$$

$$\frac{3}{a} = -2$$
$$3 = -2a$$
$$\Rightarrow \quad a = -\frac{3}{2}$$

$$-\frac{1}{b} = -2$$
$$1 = 2b$$
$$\Rightarrow \quad b = \frac{1}{2}$$

10. A spacecraft moves from point $(120, 240, -50)$ to point $(130, 210, 80)$ in kilometres. What is the magnitude of the displacement vector in kilometres?

Solution:

Let $A(120,240,-50), B(130,210,80)$

The displacement vector is:

$$\vec{d} = \overline{AB}$$
$$= (x_2-x_1)\underline{i}+(y_2-y_1)\underline{j}+(z_2-z_1)\underline{k}$$
$$= (130-120)\underline{i}+(210-240)\underline{j}+(80+50)\underline{k}$$
$$= 10\underline{i}-30\underline{j}+130\underline{k}$$

Now,

$$|\vec{d}| = \sqrt{(10)^2+(-30)^2+(130)^2}$$
$$= \sqrt{100+900+16900}$$
$$= \sqrt{17900} = \sqrt{100\times 179}$$
$$|\vec{d}| = 10\sqrt{179} \text{ km}$$

11. Find the direction cosines for the given vector:

(i) $\underline{u} = -6\underline{i}+3\underline{j}+2\underline{k}$

Solution:

$$\underline{u} = -6\underline{i}+3\underline{j}+2\underline{k}$$
$$|\underline{u}| = \sqrt{(-6)^2+(3)^2+(2)^2}$$
$$|\underline{u}| = \sqrt{36+9+4}$$
$$= \sqrt{49} = 7$$

Direction ratios of $\underline{u}$ are

$$-6, 3, 2$$

Direction cosines of y are

$$-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$$

(ii) $\underline{v} = 4\underline{i}+2\underline{j}-5\underline{k}$

Solution:

$$\underline{v} = 4\underline{i}+2\underline{j}-5\underline{k}$$
$$|\underline{v}| = \sqrt{(4)^2+(2)^2+(-5)^2}$$
$$= \sqrt{16+4+25}$$
$$= \sqrt{45} = 3\sqrt{5}$$

Direction ratios of $\underline{v}$ are

$$4, 2, -5$$

Direction cosines of $\underline{v}$ are

$$\frac{4}{3\sqrt{5}}, \frac{2}{3\sqrt{5}}, \frac{-5}{3\sqrt{5}} = \frac{4}{3\sqrt{5}}, \frac{2}{3\sqrt{5}}, \frac{-\sqrt{5}}{3}$$

(iii) PQ, where $P(9,3,13)$ and $Q(11,6,19)$.

Solution:

$\overline{PQ}$, where $P(9,3,13), Q(11,6,19)$

$$\overline{PQ} = (x_2-x_1)\underline{i}+(y_2-y_1)\underline{j}+(z_2-z_1)\underline{k}$$
$$\overline{PQ} = (11-9)\underline{i}+(6-3)\underline{j}+(19-13)\underline{k}$$
$$\overline{PQ} = 2\underline{i}+3\underline{j}+6\underline{k}$$
$$|\overline{PQ}| = \sqrt{(2)^2+(3)^2+(6)^2}$$
$$|\overline{PQ}| = \sqrt{4+9+36}$$
$$= \sqrt{49} = 7$$

Direction ratios of $\overline{PQ}$ are

$$2, 3, 6$$

Direction cosines of $\overline{PQ}$ are

$$\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$$

12. Which of the following triple can be the direction angles of a single vector?

(i) 45°, 45°, 60° **(ii) 30°, 45°, 60°**

(iii) 45°, 60°, 60°

Solution:

(i) 45°, 45°, 60°

Let $\alpha = 45°, \beta = 45°, \gamma = 60°$

If α, β, γ are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$
$$= \cos^2 45° + \cos^2 45° + \cos^2 60°$$
$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$
$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{4} = \frac{2+2+1}{4}$$
$$= \frac{5}{4} \neq \text{R.H.S}$$

Thus α, β, γ are not the direction angles of a single vector.

(ii) 30°, 45°, 60°

Solution:

Let $\alpha = 30°, \beta = 45°, \gamma = 60°$

If α, β, γ are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$
$$= \cos^2 30° + \cos^2 45° + \cos^2 60°$$
$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$
$$= \frac{3}{4}+\frac{1}{2}+\frac{1}{4} = \frac{3+2+1}{4}$$
$$= \frac{3}{2} \neq \text{R.H.S}$$

Thus α, β, γ are not the direction angles of a single vector.

(iii) 45°, 60°, 60°

Solution:

Let $\alpha = 45°, \beta = 60°, \gamma = 60°$

If α, β, γ are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$
$$= \cos^2 45° + \cos^2 60° + \cos^2 60°$$
$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$
$$= \frac{1}{2}+\frac{1}{4}+\frac{1}{4} = \frac{2+1+1}{4}$$
$$= \frac{4}{4} = 1 = \text{R.H.S}$$

Thus α, β, γ are the direction angles of a single vector.

Product of Two Vectors:

Multiplication of two vectors is an important algebraic operation in vector algebra. This algebraic operation plays a fundamental role for understanding various physical and mathematical real-life situation. Unlike the multiplication of numbers, product of vector can be performed in two distinct ways. The two primary types of vector multiplication are the dot product and the cross product. The dot product is a scalar number while cross product is a vector quantity.

Dot or Scalar Product

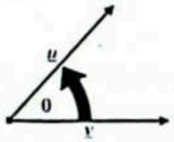
Dot or Scalar Product of Two Vectors and Its Geometrical Interpretation:

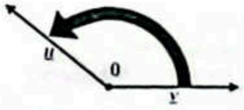
We shall now consider products of two vectors that originated in the study of physics and engineering. The concept of angle between two vectors is expressed in terms of a scalar product of two vectors.

Definition 1:

Let two non-zero vectors $\underline{u}$ and $\underline{v}$, in the plane or in space, have some initial point. The dot product of $\underline{u}$ and $\underline{v}$, written as $\underline{u}\cdot\underline{v}$, is defined by

$$\underline{u}\cdot\underline{v} = |\underline{u}||\underline{v}|\cos\theta$$





Where θ in the angle between $\underline{u}$ and $\underline{v}$ and $0 \le \theta \le \pi$

Definition 2:

(a) If $\underline{u} = a_1\underline{i}+b_1\underline{j}$ and $\underline{v} = a_2\underline{i}+b_2\underline{j}$ are two non-zero vectors in the plane. The dot product $\underline{u}\cdot\underline{v}$ is defined by:

$$\underline{u}\cdot\underline{v} = a_1a_2+b_1b_2$$

(b) If $\underline{u} = a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v} = a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ are two non-zero vectors in space. The dot product $\underline{u}\cdot\underline{v}$ is defined by

$$\underline{u}\cdot\underline{v} = a_1a_2+b_1b_2+c_1c_2$$

Note:
The dot product is also referred as the scalar product or the inner product.

Example 5: Prove the equivalence of above two definitions of dot product of two vectors.

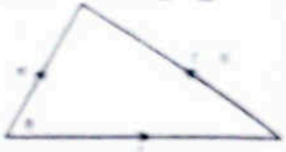
(i) If $\underline{v} = [x_1, y_1]$ and $\underline{w} = [x_2, y_2]$ are two vectors in the plane, then $\underline{v}\cdot\underline{w} = x_1x_2 + y_1y_2$

(ii) If $\underline{v}$ and $\underline{w}$ are two non-zero vectors in the plane, then $\underline{v}\cdot\underline{w} = |\underline{v}||\underline{w}|\cos\theta$, where θ is the angle between $\underline{v}$ and $\underline{w}$ and $0 \le \theta \le \pi$

Proof: Let $\underline{v}$ and $\underline{w}$ be the sides of a triangle then the third side opposite to the angle θ has length $|\underline{v} - \underline{w}|$

By law of cosines, we have

$$|\underline{v} - \underline{w}|^2 = |\underline{v}|^2 + |\underline{w}|^2 - 2|\underline{v}||\underline{w}|\cos\theta \qquad \ldots(1)$$



Given That: If $\underline{v} = [x_1, y_1]$ and $\underline{w} = [x_2, y_2]$, then

$$\underline{v} - \underline{w} = [x_1 - x_2, y_1 - y_2]$$

Putting values in equation (1), we have

$$\|[x_1 - x_2, y_1 - y_2]\|^2 = \|[x_1, y_1]\|^2 + \|[x_2, y_2]\|^2 - 2|\underline{v}||\underline{w}|\cos\theta$$

$$\left[\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right]^2 = \left[\sqrt{x_1^2 + y_1^2}\right]^2 + \left[\sqrt{x_2^2 + y_2^2}\right]^2 - 2|\underline{v}||\underline{w}|\cos\theta$$

The law of cosine:
$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2|\underline{v}||\underline{w}|\cos\theta$$

$$x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2|\underline{v}||\underline{w}|\cos\theta$$

$$-2x_1x_2 - 2y_1y_2 = -2|\underline{v}||\underline{w}|\cos\theta$$

Dividing both sides by -2, we have

$$x_1x_2 + y_1y_2 = |\underline{v}||\underline{w}|\cos\theta$$

$$\Rightarrow \qquad x_1x_2 + y_1y_2 = |\underline{v}||\underline{w}|\cos\theta = \underline{v}\cdot\underline{w}$$

Hence, $\underline{v}\cdot\underline{w} = x_1x_2 + y_1y_2$ and $\underline{v}\cdot\underline{w} = |\underline{v}||\underline{w}|\cos\theta$

Deduction of the Important Results:

By applying the definition of dot product to unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$, we have

(a) $\underline{i}\cdot\underline{i} = |\underline{i}||\underline{i}|\cos 0^\circ = 1$
$\underline{j}\cdot\underline{j} = |\underline{j}||\underline{j}|\cos 0^\circ = 1$
$\underline{k}\cdot\underline{k} = |\underline{k}||\underline{k}|\cos 0^\circ = 1$

(b) $\underline{i}\cdot\underline{j} = |\underline{i}||\underline{j}|\cos 90^\circ = 0$
$\underline{j}\cdot\underline{k} = |\underline{j}||\underline{k}|\cos 90^\circ = 0$
$\underline{k}\cdot\underline{i} = |\underline{k}||\underline{i}|\cos 90^\circ = 0$

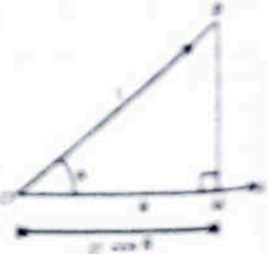
Projection of a Vector along Another Vector:

In many physical applications, it is required to know "how much" of a vector is applied along a given direction. For this purpose, we find the projection of one vector along the other vector.

Let $\overline{OA} = \underline{u}$ and $\overline{OB} = \underline{v}$

Let θ be the angle between them, such that $0 \le \theta \le \pi$

Draw $\overline{BM} \perp \overline{OA}$. Then $\overline{OM}$ is called the projection of $\underline{v}$ along $\underline{u}$.



From the figure $\dfrac{\overline{OM}}{\overline{OB}} = \cos\theta$, that is,

$$\overline{OM} = |\overline{OB}|\cos\theta = |\underline{v}|\cos\theta \qquad \ldots(1)$$

Now, $\underline{u}\cdot\underline{v} = |\underline{u}||\underline{v}|\cos\theta = |\underline{u}|(|\underline{v}|\cos\theta) = |\underline{u}|(\overline{OM})$

$= (\text{magnitude of } \underline{u})(\text{projection of } \underline{v} \text{ along } \underline{u})$

Thus, **geometrically**, the dot product of two vectors represents the product of the magnitude of one vector and the projection of the other vector onto it. In other words, the dot product of two vectors shows how much one vector extends in the direction of another.

Now by definition, $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|} \qquad \ldots(2)$

From (1) and (2), $\overline{OM} = |\underline{v}|\dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}|}$

Therefore Projection of $\underline{v}$ along $\underline{u} = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}|} = \underline{v}\cdot\hat{\underline{u}}$

Similarly, Projection of $\underline{u}$ along $\underline{v} = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{v}|} = \underline{u}\cdot\hat{\underline{v}}$

Properties of Dot Product:

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be vectors and let c be any real number, then

(i) $\underline{u}\cdot\underline{v} = 0 \Rightarrow \underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$ or $\underline{u} \perp \underline{v}$
(ii) $\underline{u}\cdot\underline{v} = \underline{v}\cdot\underline{u}$ (Commutative property)
(iii) $\underline{u}\cdot(\underline{v} + \underline{w}) = \underline{u}\cdot\underline{v} + \underline{u}\cdot\underline{w}$ (Distributive property)
(iv) $(c\underline{u})\cdot\underline{v} = c(\underline{u}\cdot\underline{v})$ (c is scalar)
(v) $\underline{u}\cdot\underline{u} = |\underline{u}|^2$

Dot Product of Vectors in terms of their components:

Let $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ be two non-zero vectors.

From distributive law we can write

$$\underline{u}\cdot\underline{v} = (a_1\underline{i} + b_1\underline{j} + c_1\underline{k})\cdot(a_2\underline{i} + b_2\underline{j} + c_2\underline{k})$$

$$= a_1a_2(\underline{i}\cdot\underline{i}) + a_1b_2(\underline{i}\cdot\underline{j}) + a_1c_2(\underline{i}\cdot\underline{k}) + b_1a_2(\underline{j}\cdot\underline{i}) + b_1b_2(\underline{j}\cdot\underline{j}) + b_1c_2(\underline{j}\cdot\underline{k})$$

$$+ c_1a_2(\underline{k}\cdot\underline{i}) + c_1b_2(\underline{k}\cdot\underline{j}) + c_1c_2(\underline{k}\cdot\underline{k})$$

$\Rightarrow \qquad \underline{u}\cdot\underline{v} = a_1a_2 + b_1b_2 + c_1c_2 \qquad \underline{i}\cdot\underline{i} = \underline{j}\cdot\underline{j} = \underline{k}\cdot\underline{k} = 1$ and $\underline{i}\cdot\underline{j} = \underline{j}\cdot\underline{k} = \underline{k}\cdot\underline{i} = 0$

Hence the dot product of two vectors is the sum of the product of their corresponding components.

Example 6: Show that the components of a vector are the projections of that vector along $\underline{i}$, $\underline{j}$ and $\underline{k}$ respectively.

Proof: Let $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$, then

Projection of $\underline{v}$ along $\underline{i} = \underline{v}\cdot\hat{\underline{i}} = \dfrac{\underline{v}\cdot\underline{i}}{|\underline{i}|} = (a\underline{i} + b\underline{j} + c\underline{k})\cdot\underline{i} = a + 0 + 0 = a$

Projection of $\underline{v}$ along $\underline{j} = \underline{v}\cdot\hat{\underline{j}} = \dfrac{\underline{v}\cdot\underline{j}}{|\underline{j}|} = (a\underline{i} + b\underline{j} + c\underline{k})\cdot\underline{j} = 0 + b + 0 = b$

Projection of $\underline{v}$ along $\underline{k} = \underline{v}\cdot\hat{\underline{k}} = \dfrac{\underline{v}\cdot\underline{k}}{|\underline{k}|} = (a\underline{i} + b\underline{j} + c\underline{k})\cdot\underline{k} = 0 + 0 + c = c$

Hence components a, b and c of vector $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$ are projections of vector $\underline{v}$ along $\underline{i}$, $\underline{j}$ and $\underline{k}$ respectively.

Example 7: Prove that in any triangle ABC

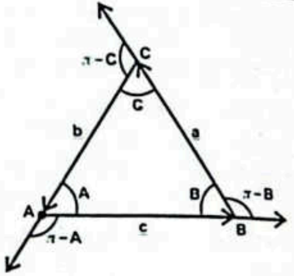
(i) $a^2 = b^2 + c^2 - 2bc\cos A$ (Cosine Law) (ii) $a = b\cos C + c\cos B$ (Projection Law)

Proof:

Consider a triangle ABC such that

$\overline{BC} = \underline{a}$, $\overline{CA} = \underline{b}$, $\overline{AB} = \underline{c}$

$\therefore \quad \underline{a}+\underline{b}+\underline{c}=0 \qquad \ldots(1)$



(i) $\underline{a} = -\underline{b}-\underline{c}$

$\underline{a} = -(\underline{b}+\underline{c})$

Squaring on both sides

$$(\underline{a})^2 = (\underline{b}+\underline{c})^2$$

$$\underline{a}.\underline{a} = (\underline{b}+\underline{c}).(\underline{b}+\underline{c})$$

$$a^2 = \underline{b}.\underline{b}+\underline{b}.\underline{c}+\underline{c}.\underline{b}+\underline{c}.\underline{c}$$

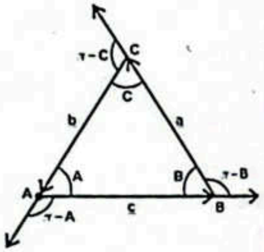
$$a^2 = b^2+2\underline{b}.\underline{c}+c^2 \qquad \because \underline{b}.\underline{c}=\underline{c}.\underline{b}$$

$$a^2 = b^2+c^2+2bc\cos(\pi-A)$$

$$a^2 = b^2+c^2+2bc(-\cos A) \qquad \because \cos(\pi-A)=-\cos A$$

$$a^2 = b^2+c^2-2ab\cos A$$

(ii) $\underline{a}+\underline{b}+\underline{c} = 0$ using eq. (1)



$\underline{a} = -\underline{b}-\underline{c}$

$\underline{a} = -(\underline{b}+\underline{c})$

Taking dot product with $\underline{a}$

$$\underline{a}.\underline{a} = -(\underline{b}+\underline{c}).\underline{a}$$

$$a^2 = -\{\underline{b}.\underline{a}+\underline{c}.\underline{a}\}$$

$$a^2 = -\{ba\cos(\pi-C)+ca\cos(\pi-B)\}$$

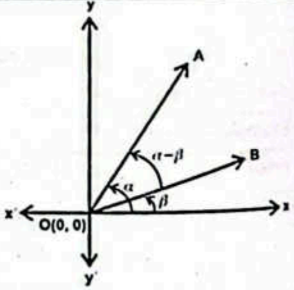
$$a^2 = -\{-ba\cos C-ca\cos B\} \qquad \because \cos(\pi-C)=-\cos C$$

$$a^2 = ba\cos C+ca\cos B \qquad \cos(\pi-B)=-\cos B$$

$$a = b\cos C+c\cos B \qquad \text{Dividing by 'a'}$$

Example 8: Prove that: $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Proof:



Consider two unit vectors $\overline{OA}$ and $\overline{OB}$ in *xy*-plane making angles α, β with +ve *x*-axis respectively.

So that $m\angle AOB = \alpha-\beta$

$$\overline{OA} = \cos\alpha\,\underline{i}+\sin\alpha\,\underline{j}$$

$$\overline{OB} = \cos\beta\,\underline{i}+\sin\beta\,\underline{j}$$

Taking dot product of $\overline{OA}$ and $\overline{OB}$

$$\overline{OA}.\overline{OB} = (\cos\alpha\,\underline{i}+\sin\alpha\,\underline{j}).(\cos\beta\,\underline{i}+\sin\beta\,\underline{j})$$

$$|\overline{OA}|\,|\overline{OB}|\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$(1)(1)\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta \qquad \because |\overline{OA}|=|\overline{OB}|=1$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Orthogonality of Two Vectors:

Definition: Two non-zero vectors $\underline{u}$ and $\underline{v}$ are orthogonal (perpendicular) if and only if $\underline{u}\cdot\underline{v}=0$.

The dot product of two vectors $\underline{u}$ and $\underline{v}$ becomes zero when $\underline{u}\perp\underline{v}, \theta=90°$ or $\frac{\pi}{2}$ radians.

$$\underline{u}\cdot\underline{v}=|\underline{u}||\underline{v}|\cos 90° = 0$$

Thus, $\underline{u}\cdot\underline{v}=0 \Leftrightarrow \underline{u}\perp\underline{v}$

> **Note:**
> The zero vector $\underline{o}$ is orthogonal to every vector because:
> $\underline{o}\cdot\underline{b}=0 \quad \forall\underline{b}$

Example 9: If $\underline{u}=3\underline{i}-\underline{j}-2\underline{k}$ and $\underline{v}=\underline{i}+2\underline{j}-\underline{k}$, then find $\underline{u}\cdot\underline{v}$.

Solution: $\underline{u}\cdot\underline{v} = (3\underline{i}-\underline{j}-2\underline{k}).(\underline{i}+2\underline{j}-\underline{k})$

$$= (3)(1)+(-1)(2)+(-2)(-1) = 3$$

Example 10: If $\underline{u}=2\underline{i}-4\underline{j}+5\underline{k}$ and $\underline{v}=4\underline{i}-3\underline{j}-4\underline{k}$, then prove that $\underline{u}$ and $\underline{v}$ are orthogonal.

Solution: $\underline{u}\cdot\underline{v} = \left(2\underline{i}-4\underline{j}+5\underline{k}\right)\cdot\left(4\underline{i}-3\underline{j}-4\underline{k}\right)$

$$= (2)(4)+(-4)(-3)+(5)(-4) = 0$$

$\Rightarrow$ $\underline{u}$ and $\underline{v}$ are perpendicular

Example 11: Find a scalar α so that the vectors $2\underline{i}+\alpha\underline{j}+5\underline{k}$ and $3\underline{i}+\underline{j}+\alpha\underline{k}$ are orthogonal.

Solution: Let $\underline{u}=2\underline{i}+\alpha\underline{j}+5\underline{k}$ and $\underline{v}=3\underline{i}+\underline{j}\neq\underline{k}$

It is given that $\underline{u}$ and $\underline{v}$ are orthogonal. Therefore

$$\underline{u}\cdot\underline{v} = 0$$

$$\Rightarrow \left(2\underline{i}+\alpha\underline{j}+5\underline{k}\right)\cdot\left(3\underline{i}+\underline{j}+\alpha\underline{k}\right)=0$$

$$(2)(3)+(\alpha)(1)+(5)(\alpha)=0 \qquad \Rightarrow 6+\alpha+5\alpha=0 \qquad \Rightarrow \boxed{\alpha=-1}$$

Angle Between Two Vectors:

The angle between two vectors $\underline{u}$ and $\underline{v}$ is determined from the definition of dot product, that is

(a) $\underline{u}\cdot\underline{v}=|\underline{u}||\underline{v}|\cos\theta$, where $0\le\theta\le\pi \qquad \Rightarrow \quad \cos\theta=\dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$

(b) If $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$, then

$$\underline{u}\cdot\underline{v} = a_1a_2+b_1b_2+c_1c_2$$

$$|\underline{u}| = \sqrt{a_1^2+b_1^2+c_1^2} \text{ and } |\underline{v}|=\sqrt{a_2^2+b_2^2+c_2^2}$$

$$\because \quad \cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\therefore \quad \cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Example 12: Find the angle between the vectors.

$\underline{u}=2\underline{i}-\underline{j}+\underline{k}$ and $\underline{v}=-\underline{i}+\underline{j}$

Solution: $\underline{u}\cdot\underline{v} = \left(2\underline{i}-\underline{j}+\underline{k}\right)\cdot\left(-\underline{i}+\underline{j}+0\underline{k}\right)$

$$= (2)(-1)+(-1)(1)+(1)(0) = -3$$

and $|\underline{u}| = |2\underline{i}-\underline{j}+\underline{k}|=\sqrt{(2)^2+(-1)^2+(1)^2}=\sqrt{6}$

$$|\underline{v}| = |-\underline{i}+\underline{j}+0\underline{k}|=\sqrt{(-1)^2+(1)^2+(0)^2}=\sqrt{2}$$

Now, $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{-3}{\sqrt{6}\cdot\sqrt{2}}$

$$\Rightarrow \quad \cos\theta = -\frac{\sqrt{3}}{2}$$

$$\theta = \pi-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \qquad \because \theta \text{ lies in II-quadrant}$$

$$\theta = \pi-\frac{\pi}{6} = \frac{5\pi}{6}$$

Example 13: Show that the vectors $\overline{AB}=2\underline{i}-\underline{j}+\underline{k}, \overline{BC}=\underline{i}-3\underline{j}-5\underline{k}$ and $\overline{AC}=3\underline{i}-4\underline{j}-4\underline{k}$ are the sides of a right triangle.

Solution: Given $\overline{AB} = 2\underline{i}-\underline{j}+\underline{k}$, $\overline{BC}=\underline{i}-3\underline{j}-5\underline{k}$ and $\overline{AC} = 3\underline{i}-4\underline{j}-4\underline{k}$

Since $\overline{AB}+\overline{BC} = \left(2\underline{i}-\underline{j}+\underline{k}\right)+\left(\underline{i}-3\underline{j}-5\underline{k}\right)$

$= 3\underline{i}-4\underline{j}-4\underline{k}=\overline{AC}$ (third side)

Therefore $\overline{AB}, \overline{BC}$ and $\overline{AC}$ form a triangle ABC.
Further, we prove that $\Delta\, ABC$ is a right triangle

Since $\overline{AB}\cdot\overline{BC} = \left(2\underline{i}-\underline{j}+\underline{k}\right)\cdot\left(\underline{i}-3\underline{j}-5\underline{k}\right)$

$= (2)(1)+(-1)(-3)+(1)(-5)=2+3-5=0$

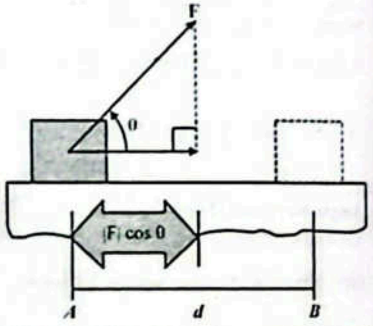
Therefore $\overline{AB} \perp \overline{BC}$
Hence, ΔABC is a right triangle.

Work Done By a Constant Force:

If a constant force F, applied to a body, acts at an angle θ to the direction of motion, then the work done by $\underline{F}$ is defined to be the product of the component of $\underline{F}$ in the direction of the displacement and the distance that the body moves.
In figure, a constant force $\underline{F}$ acting on a body, displaces it from A to B.

$\therefore$ Work done = (component of $\underline{F}$ along AB)(displacement)

$= (F\cos\theta)(AB)=\underline{F}\cdot\underline{AB}=\underline{F}\cdot\underline{d}$



Example 14: The constant forces $2\underline{i}+5\underline{j}+6\underline{k}$ and $\underline{i}-2\underline{j}-\underline{k}$ act on a body displaced form position $P(4,-3,-2)$ to $Q(6,1,-3)$. Find the total work done.

Solution:

$P(4,-3,-2)$, $Q(6,1,-3)$

Let $\overline{F_1}=2\underline{i}+5\underline{j}+6\underline{k}$ and $\overline{F_2}=\underline{i}-2\underline{j}-\underline{k}$

Total force: $\underline{F}=\overline{F_1}+\overline{F_2}=\left(2\underline{i}+5\underline{j}+6\underline{k}\right)+\left(-\underline{i}-2\underline{j}-\underline{k}\right)=2\underline{i}+5\underline{j}+6\underline{k}-\underline{i}-2\underline{j}-\underline{k}$

$\underline{F}=\underline{i}+3\underline{j}+5\underline{k}$

The displacement of the body $=\overline{PQ}=(6-4)\underline{i}+(1+3)\underline{j}+(-3+2)\underline{k}$

$\Rightarrow$ $\underline{d} = 2\underline{i}+4\underline{j}-\underline{k}$

$\therefore$ work done $= \underline{F}\cdot\underline{d}$

$= (\underline{i}+3\underline{j}+5\underline{k}).(2\underline{i}+4\underline{j}-\underline{k})=2+12-5=9$ units

Exercise 14.2

1. Find the cosines of the angle θ between $\underline{u}$ and $\underline{v}$:

(i) $\underline{u}=2\underline{i}+3\underline{j}+\underline{k}, \underline{v}=-\underline{i}+2\underline{j}+2\underline{k}$

Solution:

$\underline{u}=2\underline{i}+3\underline{j}+\underline{k}, \underline{v}=-\underline{i}+2\underline{j}+2\underline{k}$

If θ is the angle between $\underline{u}$ and $\underline{v}$, Then

$$\cos\theta=\frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\cos\theta=\frac{(2\underline{i}+3\underline{j}+\underline{k}).(-1\underline{i}+2\underline{j}+2\underline{k})}{\sqrt{(2)^2+(3)^2+(1)^2}\sqrt{(-1)^2+(2)^2+(2)^2}}$$

$$\cos\theta=\frac{(2)(-1)+(3)(2)+(1)(2)}{\sqrt{4+9+1}\sqrt{1+4+4}}$$

$$\cos\theta=\frac{-2+6+2}{\sqrt{14}\cdot\sqrt{9}}=\frac{6}{3\sqrt{14}}$$

$$\cos\theta=\frac{2}{\sqrt{14}}$$

(ii) $\underline{u}=[-3,2,5], \underline{v}=[1,6,-2]$

Solution:

$\underline{u} = [-3,2,5]=-3\underline{i}+2\underline{j}+5\underline{k}$

$\underline{v} = [1,6,-2]=\underline{i}+6\underline{j}-2\underline{k}$

If θ is the angle between $\underline{u}$ and $\underline{v}$, then.

$$\cos\theta=\frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\cos\theta=\frac{\left(-3\underline{i}+2\underline{j}+5\underline{k}\right)\cdot\left(1\underline{i}+6\underline{j}-2\underline{k}\right)}{\sqrt{(-3)^2+(2)^2+(5)^2}\sqrt{(1)^2+(6)^2+(-2)^2}}$$

$$\cos\theta=\frac{(-3)(1)+(2)(6)+(5)(-2)}{\sqrt{9+4+25}\cdot\sqrt{1+36+4}}$$

$$\cos\theta=\frac{-3+12-10}{\sqrt{38}\sqrt{41}}=\frac{-1}{\sqrt{1558}}$$

2. If $\underline{a}+\underline{b}+\underline{c}=\underline{0}$ and $|\underline{a}|=3, |\underline{b}|=5$ and $|\underline{c}|=7$. Find the angle between $\underline{a}$ and $\underline{b}$.

Solution:

Given that: $\underline{a}+\underline{b}+\underline{c}=\underline{0}, |\underline{a}|=3, |\underline{b}|=5, |\underline{c}|=7$

As $\underline{a}+\underline{b}+\underline{c}=\underline{0}$

So $\underline{c}=-(\underline{a}+\underline{b})$

Squaring both sides, we have

$(\underline{c})^2 = (\underline{a}+\underline{b})^2$

$|\underline{c}|^2 = |\underline{a}|^2+|\underline{b}|^2+2\underline{a}.\underline{b}$

Put $|\underline{a}|=3, |\underline{b}|=5, |\underline{c}|=7$, we have

$(7)^2 = (3)^2+(5)^2+2|9||6|\cos\theta$

$49-9-25 = 2(3)(5)\cos\theta$

$\frac{15}{30} = \cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\boxed{\theta = 60^\circ}$

3. If $|\underline{a}|=3, |\underline{b}|=4$ and $|\underline{a}+\underline{b}|=5$. Find the angle between $\underline{a}$ and $\underline{b}$.

Solution:

Given that: $|\underline{a}|=3, |\underline{b}|=4, |\underline{a}+\underline{b}|=5$

$|\underline{a}+\underline{b}| = 5$

Squaring both sides, we have

$|\underline{a}+\underline{b}|^2 = 5$

$|\underline{a}|^2+|\underline{b}|^2+2\underline{a}.\underline{b} = 25$

$|\underline{a}|^2+|\underline{b}|^2+2|\underline{a}||\underline{b}|\cos\theta = 25$

Put $|\underline{a}|=3, |\underline{b}|=4$, we have

$3^2+4^2+2(3)(4)\cos\theta = 25$

$24\cos\theta = 25-25$

$24\cos\theta = 0$

$\cos\theta = 0$

$\theta = \cos^{-1}(0)$

$\boxed{\theta = 90^\circ}$

4. Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when:

(i) $\underline{a}=2\underline{i}+3\underline{j}-\underline{k}, \underline{b}=\underline{i}-2\underline{j}+4\underline{k}$

Solution:

$\underline{a} = 2\underline{i}+3\underline{j}-1\underline{k}, \underline{b}=\underline{i}-2\underline{j}+4\underline{k}$

$\underline{a}.\underline{b} = \left(2\underline{i}+3\underline{j}-1\underline{k}\right)\cdot\left(1\underline{i}-2\underline{j}+4\underline{k}\right)$

$= (2)(1)+(3)(-2)+(-1)(4)$

$= 2-6-4=-8$

$|\underline{a}| = \sqrt{(2)^2+(3)^2+(-1)^2}$

$= \sqrt{4+9+1}=\sqrt{14}$

$|\underline{b}| = \sqrt{(1)^2+(-2)^2+(4)^2}$

$= \sqrt{1+4+16}=\sqrt{21}$

Projection of $\underline{a}$ along $\underline{b}=\underline{a}.\hat{\underline{b}}=\frac{\underline{a}\cdot\underline{b}}{|\underline{b}|}=\frac{-8}{\sqrt{21}}$

Projection of $\underline{b}$ along $\underline{a}=\underline{b}.\hat{\underline{a}}=\frac{\underline{a}\cdot\underline{b}}{|\underline{a}|}=\frac{-8}{\sqrt{14}}$

<u>In vector form</u>

Projection of $\underline{a}$ and $\underline{b}$ $=\frac{-8}{\sqrt{21}}\hat{\underline{b}}$

$$=\frac{-8}{\sqrt{21}}\frac{\underline{b}}{|\underline{b}|}=\frac{-8}{\sqrt{21}}\frac{\underline{b}}{\sqrt{21}}=\frac{-8}{21}\underline{b}$$

Projection of $\underline{b}$ and $\underline{a}$ $=\frac{-8}{\sqrt{14}}\hat{\underline{a}}$

$$=\frac{-8}{\sqrt{14}}\frac{\underline{a}}{|\underline{a}|}=\frac{-8}{\sqrt{14}}\frac{\underline{a}}{\sqrt{14}}=\frac{-8}{14}\underline{a}=\frac{-4}{7}\underline{a}$$

(ii) $\underline{a}=4\underline{i}-2\underline{j}+3\underline{k}, \underline{b}=\underline{i}+\underline{j}+\underline{k}$

Solution:

$\underline{a} = 4\underline{i}-2\underline{j}+3\underline{k}, \underline{b}=\underline{i}+\underline{j}+\underline{k}$

$\underline{a}.\underline{b} = \left(4\underline{i}-2\underline{j}+3\underline{k}\right)\cdot\left(1\underline{i}+1\underline{j}+1\underline{k}\right)$

$= (4)(1)+(-2)(1)+(3)(1)$

$= 4-2+3=5$

$|\underline{a}| = \sqrt{(4)^2+(-2)^2+(3)^2}$

$= \sqrt{16+4+9}=\sqrt{29}$

$|\underline{b}| = \sqrt{1^2+1^2+1^2}$

$= \sqrt{1+1+1}=\sqrt{3}$

Projection of $\underline{a}$ along $\underline{b}=\underline{a}\cdot\hat{\underline{b}}=\frac{\underline{a}\cdot\underline{b}}{|\underline{b}|}=\frac{5}{\sqrt{3}}$

Projection of $\underline{b}$ along $\underline{a} = \underline{b}\cdot\hat{\underline{a}} = \dfrac{\underline{a}\cdot\underline{b}}{|\underline{a}|} = \dfrac{5}{\sqrt{29}}$

In vector form

$$\text{Projection of } \underline{a} \text{ and } \underline{b} = \frac{5}{\sqrt{3}}\hat{\underline{b}}$$

$$= \frac{5}{\sqrt{3}}\frac{\underline{b}}{|\underline{b}|} = \frac{5}{\sqrt{3}}\frac{\underline{b}}{\sqrt{3}} = \frac{5}{3}\underline{b}$$

$$\text{Projection of } \underline{b} \text{ and } \underline{a} = \frac{5}{\sqrt{29}}\hat{\underline{a}}$$

$$= \frac{5}{\sqrt{29}}\frac{\underline{a}}{|\underline{a}|} = \frac{5}{\sqrt{29}}\frac{\underline{a}}{\sqrt{29}} = \frac{5}{29}\underline{a}$$

5. Find a real number α so that the vectors $\underline{u}$ and $\underline{v}$ are perpendicular:

(i) $\underline{u} = \alpha\underline{i} + 3\underline{j} + \underline{k}, \underline{v} = \underline{i} - 2\underline{j} + \alpha\underline{k}$

Solution:

$$\underline{u} = \alpha\underline{i} + 3\underline{j} + 1\underline{k}, \underline{v} = \underline{i} - 2\underline{j} + \alpha\underline{k}$$

Since $\underline{u}$ and $\underline{v}$ are perpendicular, therefore

$$\begin{aligned}
\underline{u}\cdot\underline{v} &= 0\\
(\alpha\underline{i} + 3\underline{j} + 1\underline{k})\cdot(1\underline{i} - 2\underline{j} + \alpha\underline{k}) &= 0\\
(\alpha)(1) + (3)(-2) + (1)(\alpha) &= 0\\
\alpha - 6 + \alpha &= 0\\
2\alpha &= 6\\
\alpha &= \frac{6}{2}\\
&\boxed{\alpha = 3}
\end{aligned}$$

(ii) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}, \underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$

Solution:

$$\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - 1\underline{k}, \underline{v} = 1\underline{i} + \alpha\underline{j} + 3\underline{k}$$

Since $\underline{u}$ and $\underline{v}$ are perpendicular, therefore

$$\begin{aligned}
\underline{u}\cdot\underline{v} &= 0\\
(\alpha\underline{i} + 2\alpha\underline{j} - 1\underline{k})\cdot(1\underline{i} + \alpha\underline{j} + 3\underline{k}) &= 0\\
(\alpha)(1) + (2\alpha)(\alpha) + (-1)(3) &= 0\\
\alpha + 2\alpha^2 - 3 &= 0\\
2\alpha^2 + \alpha - 3 &= 0\\
2\alpha^2 + 3\alpha - 2\alpha - 3 &= 0\\
\alpha(2\alpha + 3) - 1(2\alpha + 3) &= 0\\
(\alpha - 1)(2\alpha + 3) &= 0
\end{aligned}$$

Either $\alpha - 1 = 0$ or $2\alpha + 3 = 0$

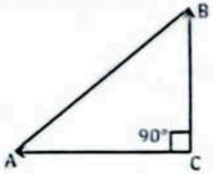
$$\alpha = 1 \text{ or } \alpha = -\frac{3}{2}$$

6. Find the number z so that the triangle with vertices $A(3, 0, -2), B(0, 3, 1)$ and $C(1, 1, z)$ is a right triangle with right at C.

Solution:

Vertices of a right triangle are

$$A(3,0,-2), B(0,3,1), C(1,1,z), m\angle C = 90^\circ$$



$$\begin{aligned}
\overline{CA} &= (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}\\
\overline{CA} &= (3-1)\underline{i} + (0-1)\underline{j} + (-2-z)\underline{k}\\
&= 2\underline{i} - 1\underline{j} - (2+z)\underline{k}\\
\overline{CB} &= (0-1)\underline{i} + (3-1)\underline{j} + (1-z)\underline{k}\\
&= -\underline{i} + 2\underline{j} + (1-z)\underline{k}
\end{aligned}$$

As m$\angle C = 90^\circ$, so $\overline{CA} \perp \overline{CB}$

$\Rightarrow \quad \overline{CA}\cdot\overline{CB} = 0$

$$\begin{aligned}
(2\underline{i} - \underline{j} - (2+z)\underline{k})\cdot(-\underline{i} + 2\underline{j} + (1-z)\underline{k}) &= 0\\
(2)(-1) + (-1)(2) + (-(2+z))(1-z) &= 0\\
-2 - 2 - (2 - 2z + z - z^2) &= 0\\
-4 - 2 + z + z^2 &= 0\\
z^2 + z - 6 &= 0\\
z^2 + 3z - 2z - 6 &= 0\\
z(z+3) - 2(z+3) &= 0\\
(z-2)(z+3) &= 0
\end{aligned}$$

Either $z - 2 = 0$ or $z + 3 = 0$

$$z = 2 \text{ or } z = -3$$

7. If $\hat{a}$ and $\hat{b}$ are unit vectors and 2θ is the angle between them, show that $\sin\theta = \frac{1}{2}|\hat{a} - \hat{b}|$.

Solution:

Given that: $|\underline{a}| = 1, |\underline{b}| = 1$ and angle between $\underline{a}$ and $\underline{b}$ is 2θ.

As we know

$$\begin{aligned}
|\underline{a} - \underline{b}|^2 &= |\underline{a}|^2 + |\underline{b}|^2 - 2\underline{a}\cdot\underline{b}\\
|\underline{a} - \underline{b}|^2 &= 1^2 + 1^2 - 2|\underline{a}||\underline{b}|\cos 2\theta\\
|\underline{a} - \underline{b}|^2 &= 2 - 2(1)(1)\cos 2\theta\\
|\underline{a} - \underline{b}|^2 &= 2 - 2(1 - 2\sin^2\theta)\\
|\underline{a} - \underline{b}|^2 &= 2 - 2 + 4\sin^2\theta\\
4\sin^2\theta &= |\underline{a} - \underline{b}|^2\\
\sin^2\theta &= \frac{1}{4}|\underline{a} - \underline{b}|^2
\end{aligned}$$

By taking square root of both sides

$$\sin\theta = \frac{1}{2}|\underline{a} - \underline{b}| \text{ As required.}$$

8. If $|\underline{a} + \underline{b}| = |\underline{a} - \underline{b}|$, then show that $\underline{a}$ and $\underline{b}$ are perpendicular.

Solution:

Given that: $|\underline{a} + \underline{b}| = |\underline{a} - \underline{b}|$

Squaring both sides, we have

$$\begin{aligned}
|\underline{a} + \underline{b}|^2 &= |\underline{a} - \underline{b}|^2\\
|\underline{a}|^2 + |\underline{b}|^2 + 2\underline{a}\cdot\underline{b} &= |\underline{a}|^2 + |\underline{b}|^2 - 2\underline{a}\cdot\underline{b}\\
2\underline{a}\cdot\underline{b} + 2\underline{a}\cdot\underline{b} &= |\underline{a}|^2 + |\underline{b}|^2 - |\underline{a}|^2 - |\underline{b}|^2\\
4\underline{a}\cdot\underline{b} &= 0\\
\underline{a}\cdot\underline{b} &= 0
\end{aligned}$$

Thus, the vectors $\underline{a}$ and $\underline{b}$ are perpendicular.

9. (i) Show that the vectors $3\underline{i} - 2\underline{j} + \underline{k}, \underline{i} - 3\underline{j} + 5\underline{k}$ and $2\underline{i} + \underline{j} - 4\underline{k}$ form a right triangle.

Solution:

Let $\underline{u} = 3\underline{i} - 2\underline{j} + \underline{k}$

$\underline{v} = \underline{i} - 3\underline{j} + 5\underline{k}$

$\underline{w} = 2\underline{i} + \underline{j} - 4\underline{k}$

$$\begin{aligned}
\text{Since } \underline{v} + \underline{w} &= \underline{i} - 3\underline{j} + 5\underline{k} + 2\underline{i} + \underline{j} - 4\underline{k}\\
&= 3\underline{i} - 2\underline{j} + \underline{k}\\
&= \underline{u}
\end{aligned}$$

Therefore $\underline{u}$, $\underline{v}$ and $\underline{w}$ are the sides of a triangle.

$$\begin{aligned}
\text{Since } \underline{u}\cdot\underline{w} &= (3\underline{i} - 2\underline{j} + \underline{k})\cdot(2\underline{i} + \underline{j} - 4\underline{k})\\
&= 6 - 2 - 4\\
\underline{u}\cdot\underline{w} &= 0 \quad \Rightarrow \underline{u} \perp \underline{w}
\end{aligned}$$

Therefore $\underline{u}$, $\underline{v}$ and $\underline{w}$ are the sides of a right triangle.

(ii) Show that the set of points $P(4, -1, 2), Q(1, 3, -1)$ and $R(-2, 4, 6)$ form a right triangle.

Solution:

Given points are $P(4,-1,2), Q(1,3,-1), R(-2,4,6)$

$$\begin{aligned}
\overrightarrow{PQ} &= (1-4)\underline{i} + (3+1)\underline{j} + (-1-2)\underline{k}\\
&= -3\underline{i} + 4\underline{j} - 3\underline{k}\\
\overrightarrow{QR} &= (-2-1)\underline{i} + (4-3)\underline{j} + (6+1)\underline{k}\\
&= -3\underline{i} + 1\underline{j} + 7\underline{k}\\
\overrightarrow{PR} &= (-2-4)\underline{i} + (4+1)\underline{j} + (6-2)\underline{k}\\
&= -6\underline{i} + 5\underline{j} + 4\underline{k}\\
\text{Since } \overrightarrow{PQ} + \overrightarrow{QR} &= -3\underline{i} + 4\underline{j} - 3\underline{k} - 3\underline{i} + 1\underline{j} + 7\underline{k}\\
&= -6\underline{i} + 5\underline{j} + 4\underline{k}\\
&= \overrightarrow{PR}
\end{aligned}$$

Therefore P, Q and R are the vertices of a triangle

$$\begin{aligned}
\text{As} \quad \overrightarrow{PQ}\cdot\overrightarrow{QR} &= (-3\underline{i} + 4\underline{j} - 3\underline{k})\cdot(-3\underline{i} + 1\underline{j} + 7\underline{k})\\
&= 9 + 4 - 21 = -8 \neq 0
\end{aligned}$$

$\Rightarrow \quad \overrightarrow{PQ} \perp \overrightarrow{QR}$

$$\begin{aligned}
\text{As} \quad \overrightarrow{QR}\cdot\overrightarrow{PR} &= (-3\underline{i} + 1\underline{j} + 7\underline{k})\cdot(-6\underline{i} + 5\underline{j} + 4\underline{k})\\
&= 18 + 5 + 28 = 57 \neq 0
\end{aligned}$$

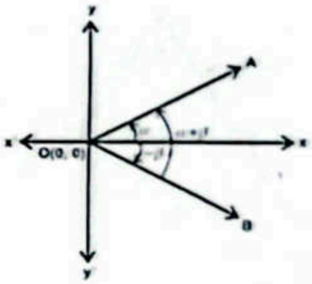
$\Rightarrow \quad \overrightarrow{QR} \perp \overrightarrow{PR}$

$$\begin{aligned}
\text{As} \quad \overrightarrow{PQ}\cdot\overrightarrow{PR} &= (-3\underline{i} + 4\underline{j} - 3\underline{k})\cdot(-6\underline{i} + 5\underline{j} + 4\underline{k})\\
&= 18 + 20 - 12 = 26 \neq 0
\end{aligned}$$

Thus, P, Q and R are the vertices of a triangle, but does not form a right triangle.

10. Prove that the $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Solution:



Consider two unit vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in xy-plane making angles α, $-\beta$ with +ve x-axis respectively.

So that m$\angle AOB = \alpha + \beta$

$$\begin{aligned}
\overrightarrow{OA} &= \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}\\
\overrightarrow{OB} &= \cos(-\beta)\,\underline{i} + \sin(-\beta)\,\underline{j} = \cos\beta\,\underline{i} - \sin\beta\,\underline{j}
\end{aligned}$$

Taking dot product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$\overrightarrow{OA}\cdot\overrightarrow{OB} = (\cos\alpha\,\underline{i} + \sin\alpha\,\underline{j})\cdot(\cos\beta\,\underline{i} - \sin\beta\,\underline{j})$

$|\overrightarrow{OA}|\,|\overrightarrow{OB}|\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

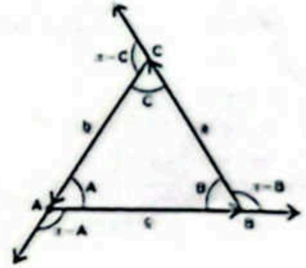
$(1)(1)\cdot\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \because |\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

11. Prove that in any triangle ABC.

(i) $b = c\cos A + a\cos C$

Solution:



Consider a triangle ABC such that

$$\overline{BC}=\underline{a},\ \overline{CA}=\underline{b},\ \overline{AB}=c$$

$$\therefore\quad \underline{a}+\underline{b}+\underline{c}=0$$

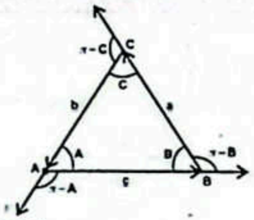
$$\underline{b}=-\underline{a}-\underline{c}$$

Taking dot product with $\underline{b}$

$$\begin{aligned}
\underline{b}\cdot\underline{b} &= -(\underline{a}+\underline{c})\cdot\underline{b}\\
b^2 &= -\underline{a}\cdot\underline{b}-\underline{c}\cdot\underline{b}\\
b^2 &= -ab\cos(\pi-C)-cb\cos(\pi-A)\\
b^2 &= -ab(-\cos C)-bc(-\cos A) && \because \cos(\pi-C)=-\cos C\\
b^2 &= ab\cos C+bc\cos A && \because \cos(\pi-A)=-\cos A\\
\Rightarrow\quad b &= a\cos C+c\cos A && \text{(Dividing by } b\text{)}
\end{aligned}$$

(ii) $c=a\cos B+b\cos A$

Solution:



As $\underline{a}+\underline{b}+\underline{c}=0$

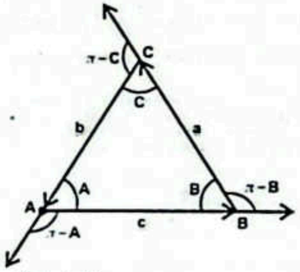
$$\underline{c}=-\underline{a}-\underline{b}$$

Taking dot product with $\underline{c}$

$$\begin{aligned}
\underline{c}\cdot\underline{c} &= -(\underline{a}+\underline{b})\cdot\underline{c}\\
c^2 &= -\underline{a}\cdot\underline{c}-\underline{b}\cdot\underline{c}\\
&= -ac\cos(\pi-B)-bc\cos(\pi-A)\\
&= -ac(-\cos B)-bc(-\cos A) && \because \cos(\pi-B)=-\cos B\\
c^2 &= ac\cos B+bc\cos A && \because \cos(\pi-A)=-\cos A\\
\Rightarrow\quad c &= a\cos B+b\cos A && \text{(Dividing by } c\text{)}
\end{aligned}$$

(iii) $b^2=c^2+a^2-2ca\cos B$

Solution:



As $\underline{a}+\underline{b}+\underline{c}=0$

$$\underline{b}=-\underline{a}-\underline{c}$$

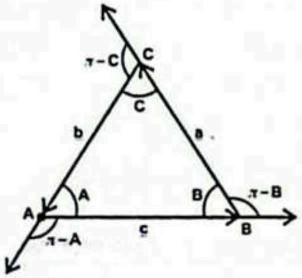
$$\underline{b}=-(\underline{a}+\underline{c})$$

Squaring on both sides

$$\begin{aligned}
(\underline{b})^2 &= (\underline{a}+\underline{c})^2\\
\underline{b}\cdot\underline{b} &= (\underline{a}+\underline{c})\cdot(\underline{a}+\underline{c})\\
b^2 &= \underline{a}\cdot\underline{a}+\underline{a}\cdot\underline{c}+\underline{c}\cdot\underline{a}+\underline{c}\cdot\underline{c}\\
b^2 &= a^2+\underline{a}\cdot\underline{c}+\underline{a}\cdot\underline{c}+c^2 && \because \underline{a}\cdot\underline{c}=\underline{c}\cdot\underline{a}\\
b^2 &= a^2+c^2+2\,\underline{a}\cdot\underline{c}\\
b^2 &= a^2+c^2+2ac\cos(\pi-B)\\
b^2 &= a^2+c^2+2ac(-\cos B) && \because \cos(\pi-B)=-\cos B\\
b^2 &= a^2+c^2-2ac\cos B
\end{aligned}$$

(iv) $c^2=a^2+b^2-2ab\cos C$

Solution:



As $\underline{a}+\underline{b}+\underline{c}=0$

$$\underline{c}=-\underline{a}-\underline{b}$$

$$\underline{c}=-(\underline{a}+\underline{b})$$

Squaring on both sides

$$\begin{aligned}
(\underline{c})^2 &= (\underline{a}+\underline{b})^2\\
\underline{c}\cdot\underline{c} &= (\underline{a}+\underline{b})\cdot(\underline{a}+\underline{b})\\
c^2 &= \underline{a}\cdot\underline{a}+\underline{a}\cdot\underline{b}+\underline{b}\cdot\underline{a}+\underline{b}\cdot\underline{b}\\
&= a^2+\underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{b}+b^2 && \because \underline{a}\cdot\underline{b}=\underline{b}\cdot\underline{a}\\
&= a^2+b^2+2\,\underline{a}\cdot\underline{b}\\
&= a^2+b^2+2ab\cos(\pi-C)\\
&= a^2+b^2+2ab(-\cos C) && \because \cos(\pi-C)=-\cos C\\
c^2 &= a^2+b^2-2ab\cos C
\end{aligned}$$

12. Show that for any vectors $\underline{a}$ and $\underline{b}$, $||\underline{a}|-|\underline{b}||\le|\underline{a}+\underline{b}|\le|\underline{a}|+|\underline{b}|$

Solution:

To show: $||\underline{a}|-|\underline{b}||\le|\underline{a}+\underline{b}|\le|\underline{a}|+|\underline{b}|$

First, we will prove that:

$$|\underline{a}+\underline{b}|\le|\underline{a}|+|\underline{b}|$$

Consider:

$$\begin{aligned}
|\underline{a}+\underline{b}|^2 &= (\underline{a}+\underline{b})\cdot(\underline{a}+\underline{b})\\
|\underline{a}+\underline{b}|^2 &= \underline{a}\cdot\underline{a}+\underline{a}\cdot\underline{b}+\underline{b}\cdot\underline{a}+\underline{b}\cdot\underline{b}\\
|\underline{a}+\underline{b}|^2 &= |\underline{a}|^2+2\underline{a}\cdot\underline{b}+|\underline{b}|^2 && \because \underline{a}\cdot\underline{b}=\underline{b}\cdot\underline{a}\\
|\underline{a}+\underline{b}|^2 &\le |\underline{a}|^2+2|\underline{a}||\underline{b}|+|\underline{b}|^2 && \because \underline{a}\cdot\underline{b}\le|\underline{a}||\underline{b}|\\
|\underline{a}+\underline{b}|^2 &\le (|\underline{a}|+|\underline{b}|)^2
\end{aligned}$$

By taking square root on both sides

$$|\underline{a}+\underline{b}|\le|\underline{a}|+|\underline{b}| \qquad ...(1)$$

Now, we will prove that:

$$|\underline{a}+\underline{b}|\ge||\underline{a}|-|\underline{b}||$$

Take

$$|\underline{a}|=|(\underline{a}+\underline{b})-\underline{b}|$$

$$|\underline{a}|=|(\underline{a}+\underline{b})+(-\underline{b})|$$

By using inequality (1), we have

$$\begin{aligned}
|\underline{a}| &\le |\underline{a}+\underline{b}|+|-\underline{b}|\\
|\underline{a}| &\le |\underline{a}+\underline{b}|+|\underline{b}|\\
|\underline{a}|-|\underline{b}| &\le |\underline{a}+\underline{b}| && ...(2)
\end{aligned}$$

Similarly, $|\underline{b}|-|\underline{a}|\le|\underline{a}+\underline{b}|$

$$\Rightarrow\quad -(|\underline{a}|-|\underline{b}|)\le|\underline{a}+\underline{b}| \qquad ...(3)$$

Combining (2) and (3), we get

$$||\underline{a}|-|\underline{b}||\le|\underline{a}+\underline{b}| \qquad ...(4)$$

From (1) and (3), it is clear That

$$||\underline{a}|-|\underline{b}||\le|\underline{a}|+|\underline{b}| \text{ (As required)}$$

13. Find the work done, if the point at which the constant force $\underline{F}=2\underline{i}+5\underline{j}+3\underline{k}$ is applied to an object, moves it from $P_1(2,-3,1)$ to $P(7,5,3)$.

Solution:

$$\vec{F}=2\underline{i}+5\underline{j}+3\underline{k},\ P_1(2,-3,1),\ P_2(7,5,3)$$

Displacement: $\vec{d}=\overline{P_1P_2}$

$$\begin{aligned}
\vec{d} &= (7-2)\underline{i}+(5+3)\underline{j}+(3-1)\underline{k}\\
&= 5\underline{i}+8\underline{j}+2\underline{k}
\end{aligned}$$

As we know

$$\begin{aligned}
\text{Work done} &= \vec{F}\cdot\vec{d}\\
&= (2\underline{i}+5\underline{j}+3\underline{k})\cdot(5\underline{i}+8\underline{j}+2\underline{k})\\
&= (2)(5)+(5)(8)+(3)(2)\\
&= 10+40+6=56 \text{ units}
\end{aligned}$$

14. A particle, acted by constant forces $\underline{F_1}=3\underline{i}+4\underline{j}-3\underline{k}$ and $\underline{F_2}=\underline{i}+4\underline{j}-\underline{k}$, is displaced from $A(2,1,3)$ to $B(5,4,4)$. Find the work done.

Solution:

$$\overline{F_1}=3\underline{i}+4\underline{j}-3\underline{k},\ \overline{F_2}=\underline{i}+4\underline{j}-\underline{k}$$

$$A(2,1,3),\ B(5,4,4)$$

$$\begin{aligned}
\text{Total force} = \vec{F} &= \overline{F_1}+\overline{F_2}\\
\vec{F} &= 3\underline{i}+4\underline{j}-3\underline{k}+\underline{i}+4\underline{j}-\underline{k}\\
\vec{F} &= 4\underline{i}+8\underline{j}-4\underline{k}
\end{aligned}$$

Displacement:

$$\begin{aligned}
\vec{d} &= \overline{AB}\\
&= (5-2)\underline{i}+(4-1)\underline{j}+(4-3)\underline{k}\\
&= 3\underline{i}+3\underline{j}+\underline{k}
\end{aligned}$$

As we know

$$\begin{aligned}
\text{Work done} &= \vec{F}\cdot\vec{d}\\
&= (4i+8i-4k)\cdot(3i+3i+1\underline{k})\\
&= (4)(3)+(8)(3)+(-4)(1)\\
&= 12+24-4=32 \text{ units}
\end{aligned}$$

15. A particle is displaced from the point $A(5,-5,-7)$ to the point $B(6,2,-2)$ under the action of constant forces defined by $10\underline{i}-\underline{j}+11\underline{k}$, $4\underline{i}+5\underline{j}+9\underline{k}$ and $-2\underline{i}+\underline{j}-9\underline{k}$. Show that the total work done by the force is 102 units.

Solution:

$$A(5,-5,-7),\ B(6,2,-2)$$

Let

$$\begin{aligned}
\overline{F_1} &= 10\underline{i}-\underline{j}+11\underline{k}\\
\overline{F_2} &= 4\underline{i}+5\underline{j}+9\underline{k}\\
\overline{F_3} &= -2\underline{i}+\underline{j}-9\underline{k}
\end{aligned}$$

Total force $=\vec{F}=\overline{F_1}+\overline{F_2}+\overline{F_3}$

$$\begin{aligned}
\vec{F} &= 10\underline{i}-\underline{j}+11\underline{k}+4\underline{i}+5\underline{j}+9\underline{k}-2\underline{i}+\underline{j}-9\underline{k}\\
\vec{F} &= 12\underline{i}+5\underline{j}+11\underline{k}
\end{aligned}$$

Displacement: $\vec{d}=\overline{AB}$

$$\begin{aligned}
\vec{d} &= (6-5)\underline{i}+(2+5)\underline{j}+(-2+7)\underline{k}\\
\vec{d} &= \underline{i}+7\underline{j}+5\underline{k}
\end{aligned}$$

As we know

$$\begin{aligned}
\text{Work done} &= \vec{F}\cdot\vec{d}\\
&= (12\underline{i}+5\underline{j}+11\underline{k})\cdot(1\underline{i}+7\underline{j}+5\underline{k})\\
&= (12)(1)+(5)(7)+(11)(5)\\
&= 12+35+55=102 \text{ unit}
\end{aligned}$$

16. A force of magnitude 6 units acting parallel to $4\underline{i}+3\underline{j}-\underline{k}$ displace the point of application from $A(2,-1,3)$ to $B(7,3,2)$. Find the work done.

Solution:

Let required force $=\vec{F}$ and $|\vec{F}|=6$ units

Given vector: $\underline{v}=4\underline{i}+3\underline{j}-\underline{k}$

$$A(2,-1,3),\ B(7,3,2)$$

As $\vec{F}\parallel\underline{v}$, so

$$\begin{aligned}
\vec{F} &= |\vec{F}|\hat{v}\\
\vec{F} &= 6\left(\frac{\underline{v}}{|\underline{v}|}\right)\\
\vec{F} &= 6\left(\frac{4\underline{i}+3\underline{j}-1\underline{k}}{\sqrt{(4)^2+(3)^2+(-1)^2}}\right)\\
\vec{F} &= \frac{6}{\sqrt{26}}(4\underline{i}+3\underline{i}-1\underline{k})
\end{aligned}$$

Displacement: $\vec{d}=\overline{AB}$

$$\begin{aligned}
\vec{d} &= (7-2)\underline{i}+(3+1)\underline{j}+(2-3)\underline{k}\\
\vec{d} &= 5\underline{i}+4\underline{j}-1\underline{k}
\end{aligned}$$

As we know

$$\text{Work done}=\vec{F}\cdot\vec{d}$$

$$= \frac{6}{\sqrt{26}}(4\underline{i}+3\underline{j}-1\underline{k})\cdot(5\underline{i}+4\underline{j}-1\underline{k})$$

$$= \frac{6}{\sqrt{26}}((4)(5)+(3)(4)+(-1)(-1))$$

$$= \frac{6}{\sqrt{26}}(20+12+1) = \frac{6}{\sqrt{26}}(33)$$

$$= \frac{198}{\sqrt{26}} \text{ units}$$

Cross Product or Vector Product

The Cross Product or Vector Product of Two Vectors and its Geometrical Interpretation:

One of the key multiplication operations involving vectors in space is the cross product. Unlike the dot product, which results is a scalar, the cross product of two vectors yields a vector quantity. The vector product of two vectors is widely used in Physics, particularly in fields of mechanics and electricity.

- ➢ Cross product is only defined for vectors in space.

Definition:

Let $\underline{u}$ and $\underline{v}$ be two non-zero vectors. The cross or vector product of $\underline{u}$ and $\underline{v}$ gives a vector that is perpendicular to both the vectors $\underline{u}$ and $\underline{v}$, written as $\underline{u} \times \underline{v}$, is defined by

$$\underline{u}\times\underline{v} = (|\underline{u}|\,|\underline{v}|\sin\theta)\underline{n}$$

where θ is the angle between the vectors, such that $0 \le \theta \le \pi$ and $\underline{n}$ is a unit vector perpendicular to the plane of $\underline{u}$ and $\underline{v}$ with direction given by the right-hand rule.

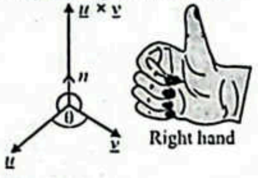


Figure (a)

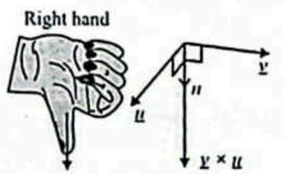


Figure (b)

Right hand rule

(i) If the fingers of the right hand point along the vector $\underline{u}$ and then curl towards the vector $\underline{v}$, then the thumb will give the direction of $\underline{n}$ which is $\underline{u} \times \underline{v}$. It is shown in the figure (a).

(ii) In figure (b), the right hand rule shows the direction of $\underline{v}\times\underline{u}$.

Parallel Vectors:

If $\underline{u}$ and $\underline{v}$ are parallel vectors, then ($\theta = 0 \Rightarrow \sin 0 = 0$).

$$\underline{u}\times\underline{v} = |\underline{u}||\underline{v}|\sin\theta\,\underline{n}$$

$$\underline{u}\times\underline{v} = 0 \quad \text{or} \quad |\underline{u}\times\underline{v}| = 0$$

And if $\underline{u} \times \underline{v} = \underline{0}$, then either $\sin\theta = 0$ or $|\underline{u}| = 0$ or $|\underline{v}| = 0$

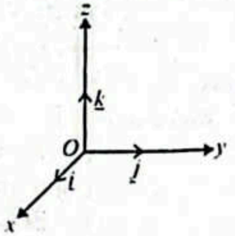
(i) If $\sin\theta = 0 \Rightarrow \theta = 0°$ or $180°$. Which shows that the vectors $\underline{u}$ and $\underline{v}$ are parallel.

(ii) If $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$, then since the zero vector has no specific direction, we adopt the convention that the zero vector is parallel to every vector.

> **Note:**
> Zero vector is both parallel and perpendicular to every vector. This apparent contradiction will cause no trouble, since the angle between two vectors is never applied when one of them is zero vector.

Derivation of Useful Results of Cross Products:

By applying the definition of cross product to unit vectors $\underline{i}, \underline{j}$ and $\underline{k}$, we have:



(a)

$$\underline{i}\times\underline{i} = (|\underline{i}|\,|\underline{i}|\sin 0°)\underline{n} = \underline{0}$$

$$\underline{j}\times\underline{j} = (|\underline{j}|\,|\underline{j}|\sin 0°)\underline{n} = \underline{0}$$

$$\underline{k}\times\underline{k} = (|\underline{k}|\,|\underline{k}|\sin 0°)\underline{n} = \underline{0}$$

(b)

$$\underline{i}\times\underline{j} = (|\underline{i}||\underline{j}|\sin 90°)\underline{k} = \underline{k}$$

$$\underline{j}\times\underline{k} = (|\underline{j}||\underline{k}|\sin 90°)\underline{i} = \underline{i}$$

$$\underline{k}\times\underline{i} = (|\underline{k}||\underline{i}|\sin 90°)\underline{j} = \underline{j}$$

(c)

$$\underline{u}\times\underline{u} = (|\underline{u}||\underline{u}|\sin 0)\underline{n} = \underline{0}$$

> **Note:**
> The cross product of $\underline{i}, \underline{j}$ and $\underline{k}$ are written in the cyclic pattern. The given figure is helpful in remembering this pattern.

Properties of Cross Product:

The cross product possesses the following properties:

(i) $\underline{u}\times\underline{v} = \underline{0}$ if $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$

(ii) $\underline{u}\times\underline{v} = -\underline{v}\times\underline{u}$

(iii) $\underline{u}\times(\underline{v}+\underline{w}) = \underline{u}\times\underline{v} + \underline{u}\times\underline{w}$

(iv) $\underline{u}\times(k\underline{v}) = (k\underline{u})\times\underline{v} = k(\underline{u}\times\underline{v})$

(v) $\underline{u}\times\underline{u} = \underline{0}$

The proofs of these properties are left as an exercise for the students.

Analytical Expressions of $\underline{u}\times\underline{v}$ (Determinant formula for $u \times v$):

Let $\underline{u} = a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v} = a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$, then

$$\underline{u}\times\underline{v} = \left(a_1\underline{i}+b_1\underline{j}+c_1\underline{k}\right)\times\left(a_2\underline{i}+b_2\underline{j}+c_2\underline{k}\right)$$

By using distributive property, we have

$$\underline{u}\times\underline{v} = a_1a_2(\underline{i}\times\underline{i}) + a_1b_2(\underline{i}\times\underline{j}) + a_1c_2(\underline{i}\times\underline{k}) + b_1a_2(\underline{j}\times\underline{i}) + b_1b_2(\underline{j}\times\underline{j}) + b_1c_2(\underline{j}\times\underline{k})$$
$$+c_1a_2(\underline{k}\times\underline{i}) + c_1b_2(\underline{k}\times\underline{j}) + c_1c_2(\underline{k}\times\underline{k})$$

As we know: $\underline{i}\times\underline{j} = \underline{k} = -\underline{j}\times\underline{i}$, $\underline{j}\times\underline{k} = \underline{i} = -\underline{k}\times\underline{j}$, $\underline{k}\times\underline{i} = \underline{j} = -\underline{i}\times\underline{k}$ and $\underline{i}\times\underline{i} = \underline{j}\times\underline{j} = \underline{k}\times\underline{k} = \underline{0}$

$$\underline{u}\times\underline{v} = a_1a_2(\underline{0}) + a_1b_2(\underline{k}) + a_1c_2(-\underline{j}) + b_1a_2(-\underline{k}) + b_1b_2(\underline{0}) + b_1c_2(\underline{i}) + c_1a_2(\underline{j}) + c_1b_2(-\underline{i}) + c_1c_2(\underline{0})$$

$$= a_1b_2\underline{k} - a_1c_2\underline{j} - b_1a_2\underline{k} + b_1c_2\underline{i} + c_1a_2\underline{j} - c_1b_2\underline{i}$$

$$\underline{u}\times\underline{v} = (b_1c_2-c_1b_2)\underline{i} - (a_1c_2-c_1a_2)\underline{j} + (a_1b_2-b_1a_2)\underline{k} \qquad \text{...(i)}$$

The expression of 3 × 3 determinant

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1c_2-c_1b_2)\underline{i} - (a_1c_2-c_1a_2)\underline{j} + (a_1b_2-b_1a_2)\underline{k}$$

The terms on R.H.S of equation (i) are the same as the terms in the expansion of the above determinant.

Hence

$$\underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \qquad \text{...(ii)}$$

which is known as determinant formula for $\underline{u}\times\underline{v}$.

> **Note:**
> The expression on R.H.S. of equation (ii) is not an actual determinant, since its entries are not all scalars. It is simply a way of remembering the complicated expression on R.H.S of equation (i).

Example 15: Find a vector perpendicular to each of the vectors. Also verify that $\underline{a}$ and $\underline{b}$ are ⊥ to $\underline{a}\times\underline{b}$

$$\underline{a} = 2\underline{i}-\underline{j}+\underline{k} \text{ and } \underline{b} = 4\underline{i}+2\underline{j}-\underline{k}$$

Solution: A vector perpendicular to both the vectors $\underline{a}$ and $\underline{b}$ is $\underline{a}\times\underline{b}$.

$$\therefore \quad \underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix} = -\underline{i}+6\underline{j}+8\underline{k}$$

Verification: $\underline{a}\cdot\underline{a}\times\underline{b} = (2\underline{i}-\underline{j}+\underline{k})\cdot(-\underline{i}+6\underline{j}+8\underline{k}) = (2)(-1)+(-1)(6)+(1)(8) = 0 \quad \Rightarrow \underline{a} \perp \underline{a}\times\underline{b}$

and $\underline{b}\cdot\underline{a}\times\underline{b} = (8\underline{i}+2\underline{j}-\underline{k})\cdot(\underline{i}-4\underline{j}+8\underline{k}) = (8)(1)+(2)(-4)+(-1)(8) = 0 \Rightarrow \underline{b}\perp\underline{a}\times\underline{b}$

Hence $\underline{a}\times\underline{b}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

Angle Between Two Vectors (Cross Product):

The sine of the angle between two vectors $\underline{a}$ and $\underline{b}$ is determined from the definition of cross product.

If θ is the angle between $\underline{a}$ and $\underline{b}$, then $|\underline{a}\times\underline{b}| = |\underline{a}||\underline{b}|\sin\theta$ or $\sin\theta = \dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$

Example 16: If $\underline{a} = 4\underline{i}+3\underline{j}+\underline{k}$ and $\underline{b} = 2\underline{i}-\underline{j}+2\underline{k}$. Find a unit vector perpendicular to both $\underline{a}$ and $\underline{b}$. Also find the sine of the angle between the vectors $\underline{a}$ and $\underline{b}$.

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\underline{i}-6\underline{j}-10\underline{k}$$

$$|\underline{a}\times\underline{b}| = \sqrt{(7)^2+(-6)^2+(-10)^2} = \sqrt{185}$$

A unit vector perpendicular to $\underline{a}$ and $\underline{b}$ is

$$\hat{n} = \frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \frac{7\underline{i}-6\underline{j}-10\underline{k}}{\sqrt{185}}$$

$$= \frac{7}{\sqrt{185}}\underline{i} - \frac{6}{\sqrt{185}}\underline{j} - \frac{10}{\sqrt{185}}\underline{k}$$

Now $|\underline{a}| = \sqrt{(4)^2+(3)^2+(1)^2} = \sqrt{26}$

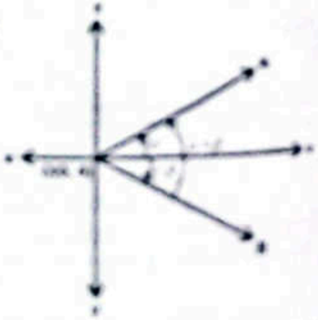
and $|\underline{b}| = \sqrt{(2)^2+(-1)^2+(2)^2} = 3$

If θ is the angle between $\underline{a}$ and $\underline{b}$, then

$$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|} = \frac{\sqrt{185}}{3\sqrt{26}}$$

Example 17: Prove that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Proof:



Consider two unit vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in xy-plane making angles α, $-\beta$ with the x-axis respectively.

Such that $m\angle AOB = \alpha+\beta$

$$\overrightarrow{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$$

$$\overrightarrow{OB} = \cos(-\beta)\underline{i} + \sin(-\beta)\underline{j} = \cos\beta\,\underline{i} - \sin\beta\,\underline{j}$$

Taking cross product of $\overrightarrow{OB}$ and $\overrightarrow{OA}$

$$\overrightarrow{OB}\times\overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$|\overrightarrow{OB}||\overrightarrow{OA}|\sin(\alpha+\beta)\underline{k} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

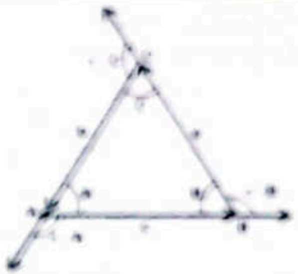
$$1\cdot 1\,\sin(\alpha+\beta)\underline{k} = 0 - 0 + \underline{k}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$\sin(\alpha+\beta)\underline{k} = \underline{k}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\cos\beta$$

Example 18: In any triangle ABC, prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{(Law of Sines)}$$



Proof:

Consider a triangle ABC such that

$\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$, $\overrightarrow{AB} = \underline{c}$

$\underline{a}+\underline{b}+\underline{c} = \underline{0}$

Taking cross product with $\underline{a}$

$\underline{a}\times(\underline{a}+\underline{b}+\underline{c}) = \underline{a}\times\underline{0}$

$\underline{a}\times\underline{a}+\underline{a}\times\underline{b}+\underline{a}\times\underline{c} = \underline{0}$

$\underline{0}+\underline{a}\times\underline{b}+\underline{a}\times\underline{c} = \underline{0}$

$\underline{a}\times\underline{b} = -\underline{a}\times\underline{c}$

$\underline{a}\times\underline{b} = \underline{c}\times\underline{a}$

$|\underline{a}\times\underline{b}| = |\underline{c}\times\underline{a}|$

$ab\sin(\pi - C) = ca\sin(\pi - B)$

$b\sin C = c\sin B$

$\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ ...(1)

Taking cross product with $\underline{b}$

$\underline{b}\times(\underline{a}+\underline{b}+\underline{c}) = \underline{b}\times\underline{0}$

$\underline{b}\times\underline{a}+\underline{b}\times\underline{b}+\underline{b}\times\underline{c} = \underline{0}$

$\underline{b}\times\underline{a}+\underline{0}+\underline{b}\times\underline{c} = \underline{0}$

$\underline{b}\times\underline{a} = -\underline{b}\times\underline{c}$

$\underline{a}\times\underline{b} = \underline{b}\times\underline{c}$

$|\underline{a}\times\underline{b}| = |\underline{b}\times\underline{c}|$

$ab\sin(\pi - C) = bc\sin(\pi - A)$

$a\sin C = c\sin A$

$\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$ ...(2)

From (1) and (2), we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 19: If $\underline{u} = 2\underline{i}-\underline{j}+\underline{k}$ and $\underline{v} = 4\underline{i}+2\underline{j}-\underline{k}$, find by determinant formula

(i) $\underline{u}\times\underline{u}$ (ii) $\underline{u}\times\underline{v}$ (iii) $\underline{v}\times\underline{u}$

Solution:

Given That $\underline{u} = 2\underline{i}-\underline{j}+\underline{k}$ and $\underline{v} = 4\underline{i}+2\underline{j}-\underline{k}$

By determinant formula, we have

(i) $\underline{u}\times\underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = \underline{0}$ (Two rows are same.)

(ii) $\underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix} = (1-2)\underline{i} - (-2-4)\underline{j} + (4+4)\underline{k} = -\underline{i}+6\underline{j}+8\underline{k}$

(iii) $\underline{v}\times\underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = (2-1)\underline{i} - (4+2)\underline{j} + (-4-4)\underline{k} = \underline{i}-6\underline{j}-8\underline{k}$

Real World Applications on Cross or Vector Product:

(a) Area of Parallelogram:

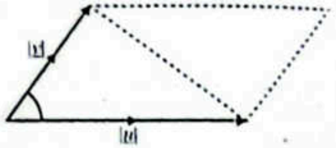
Suppose $\underline{u}$ and $\underline{v}$ are two non-zero vectors and θ is the angle between them, and suppose that $|\underline{u}|$ and $|\underline{v}|$ represent the length of the adjacent sides of a parallelogram, (see figure). We know that

Area of parallelogram = Base × Height
$= (\text{Base})(h) = |\underline{u}||\underline{v}|\sin\theta$

Area of parallelogram $= |\underline{u}\times\underline{v}|$

(b) Area of Triangle:

From figure it is clear that



$$\text{Area of triangle} = \frac{1}{2}(\text{ Area of parallelogram })$$

$$\text{Area of triangle} = \frac{1}{2}|\underline{u}\times\underline{v}|$$

where $\underline{u}$ and $\underline{v}$ are vectors along two adjacent sides of the triangle.

Example 20: Find area of the parallelogram whose vertices are $P(0,0,0), Q(-1,2,4), R(2,-1,4)$ and $S(1,1,8)$.

Solution: $P(0,0,0), Q(-1,2,4), R(2,-1,4), S(1,1,8)$

$$\overline{PQ} = (-1-0)\underline{i}+(2-0)\underline{j}+(4-0)\underline{k} = -1\underline{i}+2\underline{j}+4\underline{k}$$

$$\overline{PR} = (2-0)\underline{i}+(-1-0)\underline{j}+(4-0)\underline{k} = 2\underline{i}-1\underline{j}+4\underline{k}$$

$$\overline{PS} = (1-0)\underline{i}+(1-0)\underline{j}+(8-0)\underline{k} = 1\underline{i}+1\underline{j}+8\underline{k}$$

As $\overline{PQ}+\overline{PR} = \overline{PS}$ (diagonal), so $\overline{PQ}$ and $\overline{PR}$ are adjacent sides of a parallelogram $PQSR$.

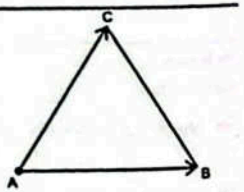
$$\text{Now } \overline{PQ}\times\overline{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 2 & 4 \\ 2 & -1 & 4 \end{vmatrix} = \underline{i}(8+4)-\underline{j}(-4-8)+\underline{k}(1-4) = 12\underline{i}+12\underline{j}-3\underline{k}$$

$$\text{Area of parallelogram} = |\overrightarrow{PQ}+\overrightarrow{PR}|$$

$$= \sqrt{(12)^2+(12)^2+(-3)^2} = \sqrt{144+144+9} = \sqrt{297}$$

Example 21: Find the area of the triangle with vertices $A(1,-1,1), B(2,1,-1)$ and $C(-1,1,2)$. Also find a unit vector perpendicular to the plane of triangle ABC.

Solution: $A(1,-1,-1), B(2,1,-1), C(-1,1,2)$



$$\overline{AB} = (x_2-x_1)\underline{i}+(y_2-y_1)\underline{j}+(z_2-z_1)\underline{k}$$

$$= (2-1)\underline{i}+(1+1)\underline{j}+(-1-1)\underline{k} = \underline{i}+2\underline{j}-2\underline{k}$$

$$\overline{AC} = (-1-1)\underline{i}+(1+1)\underline{j}+(2-1)\underline{k} = -2\underline{i}+2\underline{j}+\underline{k}$$

$$\text{Now, } \overline{AB}\times\overline{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{vmatrix} = (2+4)\underline{i}-(1-4)\underline{j}+(2+4)\underline{k} = 6\underline{i}+3\underline{j}+6\underline{k}$$

$$\text{Area of the triangle} = \frac{1}{2}|\overline{AB}\times\overline{AC}| = \frac{1}{2}|6\underline{i}+3\underline{j}+6\underline{k}|$$

$$= \frac{1}{2}\sqrt{36+9+36} = \frac{1}{2}\sqrt{81} = \frac{9}{2}\text{sq. unit}$$

$$\text{Unit vector} \perp \text{to the plane } ABC = \frac{\overline{AB}\times\overline{AC}}{|\overline{AB}\times\overline{AC}|}$$

$$= \frac{1}{9}(6\underline{i}+3\underline{j}+6\underline{k}) = \frac{1}{3}(2\underline{i}+\underline{j}+2\underline{k})$$

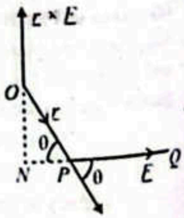
(c) Moment of Force

Let a force $F(\overline{PQ})$ act at a point P, as shown in the figure. The moment of $\underline{F}$ about O

= Product of force $\underline{F}$ and the perpendicular distance $\overline{ON}$ in the direction of $\underline{n}$.

$$= (\overline{PQ})(\overline{ON})(\hat{\underline{n}}) = (PQ)(OP)\sin\theta(\hat{\underline{n}})$$

$$= \overline{OP}\times\overline{PQ} = \underline{r}\times\underline{F}$$



Exercise 14.3

1. Compute the cross product $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$. Check your answer by showing that each $\underline{a}$ and $\underline{b}$ are perpendicular to $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$.

(i) $\underline{a} = 2\underline{i}+\underline{j}-\underline{k}, \underline{b} = \underline{i}-\underline{j}+\underline{k}$

Solution:

$$\underline{a} = 2\underline{i}+\underline{j}-\underline{k}, b = \underline{i}-\underline{j}+\underline{k}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \underline{i}(1-i)-\underline{j}(2+1)+\underline{k}(-2-1)$$

$$\underline{a}\times\underline{b} = 0\underline{i}-3\underline{j}-3\underline{k}$$

$$\Rightarrow \quad \underline{b}\times\underline{a} = 0\underline{i}+3\underline{j}+3\underline{k} \qquad \because \underline{b}\times\underline{a} = -(\underline{a}\times\underline{b})$$

Checking:

As $\underline{a}.(\underline{a}\times\underline{b}) = (2\underline{i}+\underline{j}-\underline{k}).(0\underline{i}-3\underline{j}-3\underline{k})$

$= 0-3+3 = 0$

So $\underline{a}\perp\underline{a}\times\underline{b}$

As $\underline{b}.(\underline{a}\times\underline{b}) = (\underline{i}-\underline{j}+\underline{k}).(0\underline{i}-3\underline{j}-3\underline{k})$

$= 0+3-3 = 0$

So $\underline{b}\perp\underline{a}\times\underline{b}$

As $\underline{a}.(\underline{b}\times\underline{a}) = (2\underline{i}+\underline{j}-\underline{k}).(0\underline{i}+3\underline{j}+3\underline{k})$

$= 0+3-3 = 0$

So $\underline{a}\perp\underline{b}\times\underline{a}$

As $\underline{b}.(\underline{b}\times\underline{a}) = (\underline{i}-\underline{j}+\underline{k}).(0\underline{i}+3\underline{j}+3\underline{k})$

$= 0-3+3 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

(ii) $\underline{a} = \underline{i}+3\underline{j}+2\underline{k}, \underline{b} = 2\underline{i}-\underline{j}+\underline{k}$

Solution:

$$\underline{a} = \underline{i}+3\underline{j}+2\underline{k}, \underline{b} = 2\underline{i}-\underline{j}+\underline{k}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \underline{i}(3+2)-\underline{j}(1-4)+\underline{k}(-1-6)$$

$$\underline{a}\times\underline{b} = 5\underline{i}+3\underline{j}-7\underline{k}$$

$$\Rightarrow \quad \underline{b}\times\underline{a} = -5\underline{i}-3\underline{j}+7\underline{k} \qquad \because \underline{b}\times\underline{a} = -(\underline{a}\times\underline{b})$$

Checking:

As $\underline{a}\cdot(\underline{a}\times\underline{b}) = (1\underline{i}+3\underline{j}+2\underline{k})\cdot(5\underline{i}+3\underline{j}-7\underline{k})$

$= 5+9-14 = 0$

So, $\underline{a}\perp\underline{a}\times\underline{b}$

As $\underline{b}\cdot(\underline{a}\times\underline{b}) = (2\underline{i}-1\underline{j}+1\underline{k})\cdot(5\underline{i}+3\underline{j}-7\underline{k})$

$= 10-3-7 = 0$

So, $b\perp a\times b$

As $\underline{a}\cdot(\underline{b}\times\underline{a}) = (1\underline{i}+3\underline{j}+2\underline{k})\cdot(-5\underline{i}-3\underline{j}+7\underline{k})$

$= -5-9+14 = 0$

So, $\underline{a}\perp\underline{b}\times\underline{a}$

As $\underline{b}\cdot(\underline{b}\times\underline{a}) = (2\underline{i}-1\underline{j}+1\underline{k})\cdot(-5\underline{i}-3\underline{j}+7\underline{k})$

$= -10+3+7 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

(iii) $\underline{a} = 2\underline{i}-2\underline{j}+\underline{k}, \underline{b} = -\underline{i}+\underline{j}+3\underline{k}$

Solution:

$$\underline{a} = 2\underline{i}-2\underline{j}+\underline{k}, \underline{b} = -\underline{i}+\underline{j}+3\underline{k}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 1 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= \underline{i}(-6-1)-\underline{j}(6+1)+\underline{k}(2-2)$$

$$\underline{a}\times\underline{b} = -7\underline{i}-7\underline{j}+0\underline{k}$$

$$\Rightarrow \quad \underline{b}\times\underline{a} = 7\underline{i}+7\underline{j}+0\underline{k} \qquad \because \underline{b}\times\underline{a} = -(\underline{a}\times\underline{b})$$

Checking:

As $\underline{a}\cdot(\underline{a}\times\underline{b}) = (2\underline{i}-2\underline{j}+1\underline{k})\cdot(-7\underline{i}-7\underline{j}+0\underline{k})$

$= -14+14+0 = 0$

So, $\underline{a}\perp\underline{a}\times\underline{b}$

As $\underline{b}\cdot(\underline{a}\times\underline{b}) = (-1\underline{i}+1\underline{j}+3\underline{k})\cdot(-7\underline{i}-7\underline{j}+0\underline{k})$

$= 7-7+0 = 0$

So, $\underline{b}\perp\underline{a}\times\underline{b}$

As $\underline{a}\cdot(\underline{b}\times\underline{a}) = (2\underline{i}-2\underline{j}+1\underline{k})\cdot(7\underline{i}+7\underline{j}+0\underline{k})$

$= 14-14+0 = 0$

So, $\underline{a}\perp\underline{b}\times\underline{a}$

As $\underline{b}\cdot(\underline{b}\times\underline{a}) = (-1\underline{i}+1\underline{j}+3\underline{k})\cdot(7\underline{i}+7\underline{j}+0\underline{k})$

$= -7+7+0 = 0$

So, $\underline{b}\perp\underline{b}\times\underline{a}$

(iv) $\underline{a} = -4\underline{i}+\underline{j}-2\underline{k}, \underline{b} = 2\underline{i}+\underline{j}+\underline{k}$

Solution:

$$\underline{a} = -4\underline{i}+\underline{j}-2\underline{k}, \quad \underline{b} = 2\underline{i}+\underline{j}+\underline{k}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \underline{i}(1+2)-\underline{j}(-4+4)+\underline{k}(-4-2)$$

$$\underline{a}\times\underline{b} = 3\underline{i}+0\underline{j}-6\underline{k}$$

$$\Rightarrow \quad \underline{b}\times\underline{a} = -3\underline{i}+0\underline{j}+6\underline{k} \qquad \because \underline{b}\times\underline{a} = -(\underline{a}\times\underline{b})$$

Checking:

As $\underline{a}\cdot(\underline{a}\times\underline{b}) = (-4\underline{i}+\underline{j}-2\underline{k})\cdot(3\underline{i}+0\underline{j}-6\underline{k})$
$= -12+0+12 = 0$

So $\underline{a}\perp\underline{a}\times\underline{b}$

As $\underline{b}\cdot(\underline{a}\times\underline{b}) = (2\underline{i}+\underline{j}+\underline{k})\cdot(3\underline{i}+0\underline{j}-6\underline{k})$
$= 6+0-6 = 0$

So $\underline{b}\perp\underline{a}\times\underline{b}$

As $\underline{a}\cdot(\underline{b}\times\underline{a}) = (-4\underline{i}+\underline{j}-2\underline{k})\cdot(-3\underline{i}+0\underline{j}+6\underline{k})$
$= 12+0-12 = 0$

So $\underline{a}\perp\underline{b}\times\underline{a}$

As $\underline{b}\cdot(\underline{b}\times\underline{a}) = (2\underline{i}+\underline{j}+\underline{k})\cdot(-3\underline{i}+0\underline{j}+6\underline{k})$
$= -6+0+6 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

2. Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$. Also find sine of the angle between them:

(i) $\underline{a}=\underline{i}+6\underline{j}-3\underline{k}, \underline{b}=2\underline{i}+\underline{j}+3\underline{k}$

Solution:

$$\underline{a} = \underline{i}+6\underline{j}-3\underline{k}, \underline{b}=2\underline{i}+\underline{j}+3\underline{k}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 6 & -3 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= \underline{i}(18+3)-\underline{j}(3+6)+\underline{k}(1-12)$$

$$\underline{a}\times\underline{b} = 21\underline{i}-9\underline{j}-11\underline{k}$$

$$|\underline{a}\times\underline{b}| = \sqrt{(21)^2+(-9)^2+(-11)^2}$$

$$= \sqrt{441+81+121} = \sqrt{643}$$

Required unit vector $= \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$$\Rightarrow \quad \hat{n} = \frac{21\underline{i}-9\underline{j}-11\underline{k}}{\sqrt{643}}$$

$$= \frac{21}{\sqrt{643}}\underline{i}-\frac{9}{\sqrt{643}}\underline{j}-\frac{11}{\sqrt{643}}\underline{k}$$

Let θ be the angle between $\underline{a}$ and $\underline{b}$, then

$$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$= \frac{\sqrt{643}}{\sqrt{(1)^2+(6)^2+(-3)^2}\sqrt{(2)^2+(1)^2+(3)^2}}$$

$$= \frac{\sqrt{643}}{\sqrt{1+36+9}\sqrt{4+1+9}}$$

$$= \frac{\sqrt{643}}{\sqrt{46}\cdot\sqrt{14}} = \frac{\sqrt{643}}{\sqrt{644}}$$

(ii) $\underline{a}=-\underline{i}-\underline{j}-\underline{k}, \underline{b}=2\underline{i}-3\underline{j}+4\underline{k}$

Solution:

$$\underline{a} = -\underline{i}-\underline{j}-\underline{k}, \quad \underline{b}=2\underline{i}-3\underline{j}+4\underline{k}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$$= \underline{i}(-4-3)-\underline{j}(-4+2)+\underline{k}(3+2)$$

$$= -7\underline{i}+2\underline{j}+5\underline{k}$$

$$|\underline{a}\times\underline{b}| = \sqrt{49+4+25} = \sqrt{78}$$

Required unit vector $= \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$$\hat{n} = \frac{-7\underline{i}+2\underline{j}+5\underline{k}}{\sqrt{78}}$$

$$= \frac{-7}{\sqrt{78}}\underline{i}+\frac{2}{\sqrt{78}}\underline{j}+\frac{5}{\sqrt{78}}\underline{k}$$

If θ be angle between $\underline{a}$ and $\underline{b}$, then

$$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$\sin\theta = \frac{\sqrt{78}}{\sqrt{1+1+1}\ \sqrt{4+9+16}}$$

$$= \frac{\sqrt{78}}{\sqrt{3}\cdot\sqrt{29}} = \sqrt{\frac{78}{3}}\cdot\frac{1}{\sqrt{29}}$$

$$\sin\theta = \sqrt{\frac{26}{29}}$$

(iii) $\underline{a}=\underline{i}+\underline{j}+\underline{k}, \underline{b}=\underline{i}-\underline{j}-\underline{k}$

Solution:

$$\underline{a} = \underline{i}+\underline{j}+\underline{k}, \underline{b}=\underline{i}-\underline{j}-\underline{k}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \underline{i}(-1+1)-\underline{j}(-1-1)+\underline{k}(-1-1)$$

$$\underline{a}\times\underline{b} = 0\underline{i}+2\underline{j}-2\underline{k}$$

$$|\underline{a}\times\underline{b}| = \sqrt{(0)^2+(2)^2+(-2)^2}$$

$$= \sqrt{0+4+4} = \sqrt{8} = 2\sqrt{2}$$

Required unit vector $= \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$$\Rightarrow \quad \hat{n} = \frac{0\underline{i}+2\underline{j}-2\underline{k}}{2\sqrt{2}}$$

$$\hat{n} = \frac{0}{2\sqrt{2}}\underline{i}+\frac{2}{2\sqrt{2}}\underline{j}-\frac{2}{2\sqrt{2}}\underline{k}$$

$$\hat{n} = 0\underline{i}+\frac{1}{\sqrt{2}}\underline{j}-\frac{1}{\sqrt{2}}\underline{k}$$

Let θ be the angle between $\underline{a}$ and $\underline{b}$, then

$$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$= \frac{2\sqrt{2}}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-1)^2+(-1)^2}}$$

$$= \frac{2\sqrt{2}}{\sqrt{1+1+1}\sqrt{1+1+1}}$$

$$= \frac{2\sqrt{2}}{\sqrt{3}\sqrt{3}}$$

$$\sin\theta = \frac{2\sqrt{2}}{(\sqrt{3})^2} = \frac{2\sqrt{2}}{3}$$

(iv) $\underline{a}=5\underline{i}+\underline{j}-3\underline{k}, \underline{b}=-2\underline{i}+4\underline{j}+\underline{k}$

Solution:

$$\underline{a} = 5\underline{i}+\underline{j}-3\underline{k}, \underline{b}=-2\underline{i}+4\underline{j}+\underline{k}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 1 & -3 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \underline{i}(1+12)-\underline{j}(5-6)+\underline{k}(20+2)$$

$$\underline{a}\times\underline{b} = 13\underline{i}+\underline{j}+22\underline{k}$$

$$|\underline{a}\times\underline{b}| = \sqrt{(13)^2+(1)^2+(22)^2}$$

$$= \sqrt{169+1+484} = \sqrt{654}$$

Required unit vector $= \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$$= \frac{13\underline{i}+\underline{j}+22\underline{k}}{\sqrt{654}}$$

$$= \frac{13}{\sqrt{654}}\underline{i}+\frac{1}{\sqrt{654}}\underline{j}+\frac{22}{\sqrt{654}}\underline{k}$$

Let θ be the angle between $\underline{a}$ and $\underline{b}$, then

$$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$= \frac{\sqrt{654}}{\sqrt{(5)^2+(1)^2+(-3)^2}\sqrt{(-2)^2+(4)^2+(1)^2}}$$

$$= \frac{\sqrt{654}}{\sqrt{25+1+9}\sqrt{4+16+1}}$$

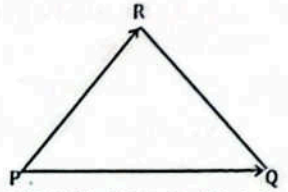
$$= \frac{\sqrt{654}}{\sqrt{35}\sqrt{21}}$$

$$\sin\theta = \frac{\sqrt{654}}{\sqrt{735}}$$

3. Find the area of the triangle, formed by the points P, Q and R.

(i) $P(2,3,5); Q(1,2,0); R(4,1,2)$

Solution:



$$P(2,3,5), Q(1,2,0), R(4,1,2)$$

$$\overline{PQ} = (1-2)\underline{i}+(2-3)\underline{j}+(0-5)\underline{k}$$

$$= -1\underline{i}-1\underline{j}-5\underline{k}$$

$$\overline{PR} = (4-2)\underline{i}+(1-3)\underline{j}+(2-5)\underline{k}$$

$$= 2\underline{i}-2\underline{j}-3\underline{k}$$

$$\overline{PQ}\times\overline{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -5 \\ 2 & -2 & -3 \end{vmatrix}$$

$$= \underline{i}(3-10)-\underline{j}(3+10)+\underline{k}(2+2)$$

$$= -7\underline{i}-13\underline{j}+4\underline{k}$$

$$|\overline{PQ}\times\overline{PR}| = \sqrt{(-7)^2+(-13)^2+(4)^2}$$

$$= \sqrt{49+169+16} = \sqrt{234} = 3\sqrt{26}$$

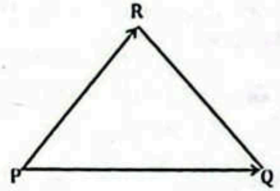
Area of $\Delta PQR = \dfrac{1}{2}|\overline{PQ}\times\overline{PR}|$

$$= \frac{1}{2}(3\sqrt{26})$$

$$= \frac{3\sqrt{26}}{2} \text{ sq. units}$$

(ii) $P(0,0,1); Q(2,-1,2); R(-1,3,2)$

Solution:



$$P(0,0,1), Q(2,-1,2), R(-1,3,2)$$

$$\overline{PQ} = (2-0)\underline{i}+(-1-0)\underline{j}+(2-1)\underline{k}$$

$$= 2\underline{i}-1\underline{j}+1\underline{k}$$

$$\overline{PR} = (-1-0)\underline{i}+(3-0)\underline{j}+(2-1)\underline{k}$$

$$= -1\underline{i}+3\underline{j}+1\underline{k}$$

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ -1 & 3 & 1 \end{vmatrix}$$

$$\begin{aligned} &= \underline{i}(-1-3) - \underline{j}(2+1) + \underline{k}(6-1) \\ &= -4\underline{i} - 3\underline{j} + 5\underline{k} \end{aligned}$$

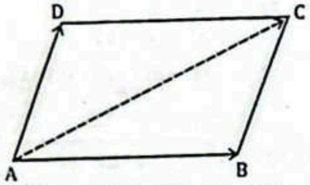
$$\begin{aligned} |\overline{PQ}\times\overline{PR}| &= (-4)^2 + (-3)^2 + (5)^2 \\ &= \sqrt{16+9+25} = \sqrt{50} = 5\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{Area of } \Delta PQR &= \frac{1}{2}|\overline{PQ}\times\overline{PR}| \\ &= \frac{1}{2}(5\sqrt{2}) \\ &= \frac{5\sqrt{2}}{2} \text{ or } \frac{5}{\sqrt{2}} \text{ sq. units} \end{aligned}$$

4. Find the area of a parallelogram, whose vertices are:

(i) $A(1,1,1); B(4,2,3); C(5,6,7); D(2,5,5)$

Solution:



$A(1,1,1), B(4,2,3), C(5,6,7), D(2,5,5)$

$$\begin{aligned} \overline{AB} &= (4-1)\underline{i} + (2-1)\underline{j} + (3-1)\underline{k} \\ &= 3\underline{i} + \underline{j} + 2\underline{k} \\ \overline{AC} &= (5-1)\underline{i} + (6-1)\underline{j} + (7-1)\underline{k} \\ &= 4\underline{i} + 5\underline{j} + 6\underline{k} \\ \overline{AD} &= (2-1)\underline{i} + (5-1)\underline{j} + (5-1)\underline{k} \\ &= \underline{i} + 4\underline{j} + 4\underline{k} \end{aligned}$$

As $\overline{AB} + \overline{AB} = \overline{AC}$ (diagonal), so $\overline{AB}$ and $\overline{AB}$ are adjacent sides of parallelogram $ABCD$.

$$\overline{AB}\times\overline{AB} = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 1 & 4 & 4 \end{vmatrix}$$

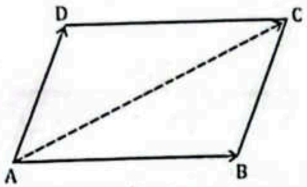
$$= i(4-8) - j(12-2) + k(12-1)$$

$$\overline{AB} \times \overline{AD} = -4i - 10i + 11\underline{k}$$

$$\begin{aligned} \text{Area of parallelogram } ABCD &= \left|\overline{AB}\times\overline{AB}\right| \\ &= \sqrt{(-4)^2 + (-10)^2 + (11)^2} \\ &= \sqrt{16+100+121} \\ &= \sqrt{237} \text{ sq. units.} \end{aligned}$$

(ii) $A(4,5,6); B(1,3,2); C(-2,0,1); D(1,2,5)$

Solution:



$A(4,5,6), B(1,3,2), C(-2,0,1), D(1,2,5)$

$$\begin{aligned} \overline{AB} &= (1-4)\underline{i} + (3-5)\underline{j} + (2-6)\underline{k} \\ &= -3\underline{i} - 2\underline{j} - 4\underline{k} \\ \overline{AC} &= (-2-4)\underline{i} + (0-5)\underline{j} + (1-6)\underline{k} \\ &= -6\underline{i} - 5\underline{j} - 5\underline{k} \\ \overline{AD} &= (1-4)\underline{i} + (2-5)\underline{j} + (5-6)\underline{k} \\ &= -3\underline{i} - 3\underline{j} - \underline{k} \end{aligned}$$

As $\overline{AB} + \overline{AB} = \overline{AC}$ (diagonal), so $\overrightarrow{AB}$ and $\overrightarrow{AB}$ are adjacent sides of parallelogram $ABCD$.

$$\overline{AB} \times \overline{AB} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -3 & -2 & -4 \\ -3 & -3 & -1 \end{vmatrix}$$

$$\begin{aligned} &= \underline{i}(2-12) - \underline{j}(3-12) + \underline{k}(9-6) \\ &= -10\underline{i} + 9\underline{j} + 3\underline{k} \end{aligned}$$

$$\begin{aligned} \text{Area of parallelogram } ABCD &= |\overline{AB}\times\overline{AD}| \\ &= \sqrt{(-10)^2 + (9)^2 + (3)^2} \\ &= \sqrt{100+81+9} \\ &= \sqrt{190} \text{ sq. units.} \end{aligned}$$

5. If the cross product of the vectors $\underline{u} = 7\underline{i} - 4\underline{j} + 5\underline{k}$ and $\underline{v} = a\underline{i} - b\underline{j} + 3\underline{k}$ is zero, then find the values of a and b.

Solution:

$$\underline{u} = 7\underline{i} - 4\underline{j} + 5\underline{k}, \underline{v} = a\underline{i} - b\underline{j} + 3\underline{k}$$

Since $\underline{u}\times\underline{v} = \underline{0}$, therefore and $\underline{v}$ are parallel.

So, their components are proportional.

$$\begin{aligned} \frac{7}{a} &= \frac{-4}{-b} = \frac{5}{3} \\ \frac{7}{a} &= \frac{4}{b} = \frac{5}{3} \\ \Rightarrow \quad \frac{7}{a} &= \frac{5}{3} \\ 7\times\frac{3}{5} &= a \\ a &= \frac{21}{5} \end{aligned}$$

$$\begin{aligned} \frac{4}{b} &= \frac{5}{3} \\ 4\times\frac{3}{5} &= b \\ b &= \frac{12}{5} \end{aligned}$$

Hence $a = \frac{21}{5}$ and $b = \frac{12}{5}$.

6. Which vectors, if any, are perpendicular or parallel

(i) $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}; \underline{v} = \underline{j} - 5\underline{k}; \underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

Solution:

$\underline{u} = 5\underline{i} - \underline{j} + \underline{k}; \underline{v} = \underline{j} - 5\underline{k}; \underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

As $\underline{u}\cdot\underline{v} = (5\underline{i} - \underline{j} + \underline{k})\cdot(0\underline{i} + \underline{j} - 5\underline{k}) = 0 - 1 - 5 = -6 \neq 0$

So $\underline{u}$ and $\underline{v}$ are not perpendicular.

As $\underline{u}\cdot\underline{w} = (5\underline{i} - \underline{j} + \underline{k})\cdot(-15\underline{i} + 3\underline{j} - 3\underline{k})$

$= -75 - 3 - 3 = -81 \neq 0$

So $\underline{u}$ and $\underline{w}$ are not perpendicular.

As $\underline{v}\cdot\underline{w} = (0\underline{i} + \underline{j} - 5\underline{k})\cdot(-15\underline{i} + 3\underline{j} - 3\underline{k})$

$= 0 + 3 + 15 = 18 \neq 0$

So $\underline{v}$ and $\underline{w}$ are not perpendicular.

Now $\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

$= -3(5\underline{i} - \underline{j} + \underline{k})$

$\underline{w} = -3\underline{u}$

$\Rightarrow$ $\underline{u} \parallel \underline{w}$ but opposite in direction

(ii) $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}; \underline{v} = -\underline{i} + \underline{j} + \underline{k}; \underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

Solution:

$\underline{u} = \underline{i} + 2\underline{j} - \underline{k}; \underline{v} = -\underline{i} + \underline{j} + \underline{k}; \underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

As $\underline{u}\cdot\underline{v} = (\underline{i} + 2\underline{j} - \underline{k})\cdot(-\underline{i} + \underline{j} + \underline{k}) = -1 + 2 - 1 = 0$

So $\underline{u}$ and $\underline{v}$ are perpendicular.

As $\underline{u}\cdot\underline{w} = (\underline{i} + 2\underline{j} - \underline{k})\cdot(-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k})$

$= -\frac{\pi}{2} - 2\pi - \frac{\pi}{2} = -3\pi \neq 0$

So $\underline{u}$ and $\underline{w}$ are not perpendicular.

As $\underline{v}\cdot\underline{w} = (-\underline{i} + \underline{j} + \underline{k})\cdot(-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k})$

$= \frac{\pi}{2} - \pi + \frac{\pi}{2} = \frac{\pi - 2\pi + \pi}{2} = 0$

So $\underline{v}$ and $\underline{w}$ are perpendicular.

Now $\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

$= \frac{-\pi\underline{i} - 2\pi\underline{j} + \pi\underline{k}}{2} = -\frac{\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$

$\underline{w} = \frac{\pi}{2}\underline{u}$ $\Rightarrow \underline{u} \parallel \underline{w}$ but opposite in direction.

7. Use the definition of cross product, for any vectors $\underline{u}, \underline{v}, \underline{w}$ and scalar k, prove that

(i) $\underline{u} \times (-\underline{u}) = 0$

Solution:

To prove: $\underline{u}\times(-\underline{u}) = \underline{0}$

L.H.S $= \underline{u}\times(-\underline{u})$

Since $-\underline{u}$ is in opposite direction of $\underline{u}$, therefore

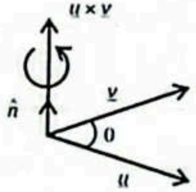
$$\begin{aligned} \theta &= 180^\circ \\ \text{L.H.S} &= (|\underline{u}||-\underline{u}|\sin 180^\circ)\hat{n} \\ &= |\underline{u}||\underline{u}|(0)\hat{n} \qquad \because \sin 180^\circ = 0, |-\underline{u}| = |\underline{u}| \\ &= 0 \\ &= \text{R.H.S (Proved)} \end{aligned}$$

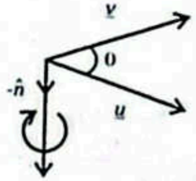
(ii) $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

Solution:

To prove: $\underline{u}\times\underline{v} = -\underline{v}\times\underline{u}$

As we know





$$\begin{aligned} \underline{u}\times\underline{v} &= |\underline{u}||\underline{v}|\sin\theta\,\hat{n} && \text{...(1)} \\ \underline{v}\times\underline{u} &= |\underline{u}||\underline{v}|\sin\theta(-\hat{n}) \\ &= -|\underline{u}||\underline{v}|\sin\theta\,\hat{n} \\ \underline{v}\times\underline{u} &= -(\underline{u}\times\underline{v}) && \text{Using (1)} \\ -(\underline{v}\times\underline{u}) &= \underline{u}\times\underline{v} \\ \underline{u}\times\underline{v} &= -\underline{v}\times\underline{u} \text{ (Proved)} \end{aligned}$$

(iii) $\underline{u} \times (k\underline{v}) = (k\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$

Solution:

To prove: $\underline{u}\times(k\underline{v}) = (k\underline{u})\times\underline{v} = k(\underline{u}\times\underline{v})$

where $\underline{u}$ and $\underline{v}$ are any two vectors and k is a scalar.

$$\begin{aligned} \underline{u}\times(k\underline{v}) &= |\underline{u}||k\underline{v}|\sin\theta\,\hat{n} \\ &= |\underline{u}||k||\underline{v}|\sin\theta\,\hat{n} \\ &= |k||\underline{u}||\underline{v}|\sin\theta\,\hat{n} \\ &= |k\underline{u}||v|\sin\theta\,\hat{n} \\ &= (k\underline{u})\times\underline{v} && \text{...(1)} \\ (k\underline{u})\times\underline{v} &= |k\underline{u}||\underline{v}|\sin\theta\,\hat{n} \\ &= |k||\underline{u}||\underline{v}|\sin\theta\,\hat{n} \\ &= k(|\underline{u}||\underline{v}|\sin\theta\,\hat{n}) \\ &= k(\underline{u}\times\underline{v}) && \text{...(2)} \end{aligned}$$

Combining (1) and (2), we have

$\underline{u}\times(k\underline{v})=(k\underline{u})\times\underline{v}=k(\underline{u}\times\underline{v})$ (Proved)

(iv) $\underline{u}\times(\underline{v}+\underline{w})=(\underline{u}\times\underline{v})+(\underline{u}\times\underline{w})$

Solution:

To prove: $\underline{u}\times(\underline{v}+\underline{w})=(\underline{u}\times\underline{v})+(\underline{u}\times\underline{w})$

Let $\vec{r}=\underline{u}\times(\underline{v}+\underline{w})-(\underline{u}\times\underline{v})-(\underline{u}\times\underline{w})$ and $\vec{d}$ is a non-zero arbitrary vector.

By taking dot product of $\vec{d}$ and $\vec{r}$

$$\vec{d}.\vec{r}=\vec{d}\cdot[\underline{u}\times(\underline{v}+\underline{w})-(\underline{u}\times\underline{v})-(\underline{u}\times\underline{w})]$$
$$=\vec{d}\cdot\underline{u}\times(\underline{v}+\underline{w})-\vec{d}\cdot(\underline{u}\times\underline{v})-\vec{d}\cdot(\underline{u}\times\underline{w})$$

By interchanging dot and cross, we have

$$=\vec{d}\times\underline{u}\cdot(\underline{v}+\underline{w})-\vec{d}\times\underline{u}\cdot\underline{v}-\vec{d}\times\underline{u}\cdot\underline{w}$$

Distributivity of dot product over addition

$$\vec{d}.\vec{r}=\vec{d}\times\cancel{\underline{u}}\cdot\underline{v}+\vec{d}\times\cancel{\underline{u}}\cdot\underline{w}-\vec{d}\times\cancel{\underline{u}}\cdot\underline{v}-\vec{d}\times\cancel{\underline{u}}\cdot\underline{w}$$
$$\vec{d}.\vec{r}=0$$
$$\Rightarrow\quad\vec{r}=\underline{0}\qquad\because\vec{d}\text{ is a non-zero vector.}$$
$$\underline{u}\times(\underline{v}+\underline{w})-(\underline{u}\times\underline{v})-(\underline{u}\times\underline{w})=\underline{0}$$

$\underline{u}\times(\underline{v}+\underline{w})=\underline{u}\times\underline{v}+\underline{u}\times\underline{w}$ (Proved)

8. Prove that:

$\underline{a}\times(\underline{b}+\underline{c})+\underline{b}\times(\underline{c}+\underline{a})+\underline{c}\times(\underline{a}+\underline{b})=\underline{0}$

Solution:

$$\text{L.H.S}=\underline{a}\times(\underline{b}+\underline{c})+\underline{b}\times(\underline{c}+\underline{a})+\underline{c}\times(\underline{a}+\underline{b})$$
$$=\underline{a}\times\underline{b}+\underline{a}\times\underline{c}+\underline{b}\times\underline{c}+\underline{b}\times\underline{a}+\underline{c}\times\underline{a}+\underline{c}\times\underline{b}$$

$\because$ using distributive property

$$=(\underline{a}\times\underline{b})+(\underline{a}\times\underline{c})+(\underline{b}\times\underline{c})-(\underline{a}\times\underline{b})-(\underline{a}\times\underline{c})-(\underline{b}\times\underline{c})$$
$$=\underline{0}$$
$$=\text{R.H.S}$$

Thus $\underline{a}\times(\underline{b}+\underline{c})+\underline{b}\times(\underline{c}\times\underline{a})+\underline{c}\times(\underline{a}+\underline{b})=\underline{0}$

9. If $\underline{a}+\underline{b}+\underline{c}=\underline{0}$, then prove that

$\underline{a}\times\underline{b}=\underline{b}\times\underline{c}=\underline{c}\times\underline{a}$

Solution:

As $\underline{a}+\underline{b}+\underline{c}=\underline{0}$

Taking cross product with $\underline{a}$

$$\underline{a}\times(\underline{a}+\underline{b}+\underline{c})=\underline{a}\times\underline{0}$$
$$\underline{a}\times\underline{a}+\underline{a}\times\underline{b}+\underline{a}\times\underline{c}=\underline{0}$$
$$\underline{0}+\underline{a}\times\underline{b}+\underline{a}\times\underline{c}=\underline{0}\qquad\because\underline{a}\times\underline{a}=\underline{0}$$
$$\underline{a}\times\underline{b}=-\underline{a}\times\underline{c}$$
$$\underline{a}\times\underline{b}=\underline{c}\times\underline{a}\qquad\dots(1)$$

Taking cross product with $\underline{b}$

$$\underline{b}\times(\underline{a}+\underline{b}+\underline{c})=\underline{b}\times\underline{0}$$
$$\underline{b}\times\underline{a}+\underline{b}\times\underline{b}+\underline{b}\times\underline{c}=\underline{0}$$
$$\underline{b}\times\underline{a}+\underline{0}\times\underline{b}\times\underline{c}=\underline{0}\qquad\because\underline{b}\times\underline{b}=\underline{0}$$
$$\underline{b}\times\underline{c}=-\underline{b}\times\underline{a}$$
$$\underline{b}\times\underline{c}=\underline{a}\times\underline{b}\qquad\dots(2)$$

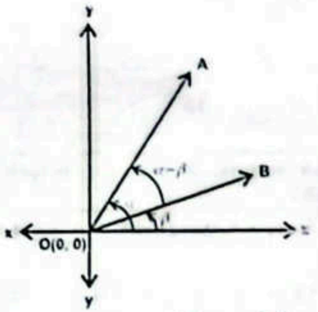
From (1) and (2), we conclude that

$$\underline{a}\times\underline{b}=\underline{b}\times\underline{c}=\underline{c}\times\underline{a}$$

10. Prove that:

$\sin(\alpha-\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Solution:



Consider two unit vectors $\overline{OA}$ and $\overline{OB}$ in xy-plane making angles α, β with +ive x-axis respectively

Such that $m\angle AOB=\alpha-\beta$

$$\overline{OA}=\cos\alpha\underline{i}+\sin\alpha\underline{j}$$
$$\overline{OB}=\cos\beta\underline{i}+\sin\beta\underline{j}$$

Taking cross product of $\overline{OB}$ and $\overline{OA}$

$$\overline{OB}\times\overline{OA}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\\cos\beta&\sin\beta&0\\\cos\alpha&\sin\alpha&0\end{vmatrix}$$
$$|\overrightarrow{OB}||\overrightarrow{OA}|\sin(\alpha-\beta)\underline{k}=\underline{i}(0-0)-\underline{j}(0-0)+\underline{k}(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$$
$$1\,.\,1\,.\sin(\alpha-\beta)\underline{k}=\underline{k}(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\qquad\because|\overrightarrow{OA}|=|\overline{OB}|=1$$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\cos\beta$

11. Show that $|\underline{a}\times\underline{b}|^2=|\underline{a}|^2|\underline{b}|^2-(\underline{a}\cdot\underline{b})^2$

Solution:

$$\text{R.H.S}=|\underline{a}|^2|\underline{b}|^2-(\underline{a}.\underline{b})^2$$
$$=|\underline{a}|^2|\underline{b}|^2-(|\underline{a}||\underline{b}|\cos\theta)^2$$
$$=|\underline{a}|^2|\underline{b}|^2-|\underline{a}|^2|\underline{b}|\cos^2\theta$$
$$=|\underline{a}|^2|\underline{b}|^2(1-\cos^2\theta)$$
$$=|\underline{a}|^2|\underline{b}|^2\sin^2\theta$$
$$=(|\underline{a}||\underline{b}|\sin\theta)^2$$
$$=|\underline{a}\times\underline{b}|^2$$
$$=\text{L.H.S (Proved)}$$

12. Use the definition of cross product, prove that for any vectors $\underline{u}$ and $\underline{v}$ $(\underline{u}+\underline{v})\times(\underline{u}-\underline{v})=-2(\underline{u}\times\underline{v})$.

Solution:

To prove: $(\underline{u}+\underline{v})\times(\underline{u}-\underline{v})=-2(\underline{u}\times\underline{v})$

By using distributive property of cross product over addition.

$$\text{L.H.S}=(\underline{u}+\underline{v})\times(\underline{u}-\underline{v})$$
$$=\underline{u}\times(\underline{u}-\underline{v})+\underline{v}\times(\underline{u}-\underline{v})$$

$$=\underline{u}\times\underline{u}-\underline{u}\times\underline{v}+\underline{v}\times\underline{u}-\underline{v}\times\underline{v}$$

Put $\underline{u}\times\underline{u}=\underline{0}$ and $\underline{v}\times\underline{v}=\underline{0}$

$$\text{L.H.S}=\underline{0}-\underline{u}\times\underline{v}+\underline{v}\times\underline{u}-\underline{0}$$
$$=-\underline{u}\times\underline{v}+\underline{v}\times\underline{u}$$
$$=-(\underline{u}\times\underline{v})-(\underline{u}\times\underline{v})$$
$$=-2(\underline{u}\times\underline{v})$$
$$=\text{R.H.S (Proved)}$$

13. Find the moment about the point $M(1,-3,3)$ of the force represented by $\overrightarrow{AB}$, where the coordinates of points $A(4,3,-1)$ and $B(-1,3,7)$ are given.

Solution:

Given points are $M(1,-3,3), A(4,3,-1), B(-1,3,7)$

$$\vec{F}=\overline{AB}$$
$$=(-1-4)\underline{i}+(3-3)\underline{j}+(7+1)\underline{k}$$
$$=-5\underline{i}+0\underline{j}+8\underline{k}$$
$$\vec{r}=\overline{MA}$$
$$=(4-1)\underline{i}+(3+3)\underline{j}+(-1-3)\underline{k}$$
$$=3\underline{i}+6\underline{j}-4\underline{k}$$

As we know

Moment of $\vec{F}$ about $M=\vec{r}\times\vec{F}$

$$=\begin{vmatrix}i&j&k\\3&6&-4\\-5&0&8\end{vmatrix}$$
$$=\underline{i}(48-0)-\underline{j}(24-20)+\underline{k}(0+30)$$
$$=48\underline{i}-4\underline{j}+30\underline{k}$$

14. A force $\vec{F}=6\underline{i}+4\underline{j}-4\underline{k}$ is applied at the point $A(1,-1,2)$. Find the moment of the force about the point $B(3,-2,3)$.

Solution:

Given that: $\vec{F}=6\underline{i}+4\underline{j}-4\underline{k}$

$$B(3,-2,3),A(1,-1,2)$$
$$\vec{r}=\overline{BA}$$
$$=(1-3)\underline{i}+(-1+2)\underline{j}+(2-3)\underline{k}$$
$$=-2\underline{i}+\underline{j}-\underline{k}$$

As we know

Moment of $\vec{F}$ about $B=\vec{r}\times\vec{F}$

$$=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\-2&1&-1\\6&4&-4\end{vmatrix}$$
$$=0\underline{i}-14\underline{j}-14\underline{k}$$

15. Give a force $\underline{F}=2\underline{i}+\underline{j}-3\underline{k}$ acting at a point $A(1,-2,1)$. Find the moment of $\vec{F}$ about the point $B(2,0,-2)$.

Solution:

Given that $\underline{F}=2\underline{i}+\underline{j}-3\underline{k}$

$$\underline{r}=\overrightarrow{BA}=(1-2)\underline{i}+(-2-0)\underline{j}+(1+2)\underline{k}$$
$$=-\underline{i}-2\underline{j}+3\underline{k}$$

We know that

Moment of $\underline{F}$ about B $=\vec{r}\times\vec{F}$

$$=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\-1&-2&3\\2&1&-3\end{vmatrix}$$
$$=\underline{i}(6-3)-\underline{j}(3-6)+\underline{k}(-1+4)$$
$$=3\underline{i}+3\underline{j}+3\underline{k}$$

16. A force $\underline{F}=-2\underline{i}+\underline{j}-3\underline{k}$ is applied at $P(-1,-3,2)$. Find its moment about the point $Q(4,2,2)$.

Solution:

Given that: $\vec{F}=-2\underline{i}+\underline{j}-3\underline{k}$

$$(4,2,2),\bar{P}(-1,-3,2)$$
$$\vec{r}=\overrightarrow{QP}$$
$$=(-1-4)\underline{i}+(-3-2)\underline{j}+(2-2)\underline{k}$$
$$=-5\underline{i}-5\underline{j}+0\underline{k}$$

As we know

Moment of $\vec{F}$ about $Q=\bar{R}\times\vec{F}$

$$=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\-5&-5&0\\-2&1&-3\end{vmatrix}$$
$$=\underline{i}(15-0)-\underline{j}(15-0)+\underline{k}(-5-10)$$
$$=15\underline{i}-15\underline{j}-15\underline{k}$$

Example 22: Find the moment about the point $M(-2,4,-6)$ of the force represented by $\overline{AB}$, where coordinates of points A and B are $(1,2,-3)$ and $(3,-4,2)$ respectively.

Solution:

Here $A(1,2,-3), B(3,-4,2)$ and $M(-2,4,-6)$

$$\underline{F}=\overline{AB}=(3-1)\underline{i}+(-4-2)\underline{j}+(2+3)\underline{k}=2\underline{i}-6\underline{j}+5\underline{k}$$
$$\underline{r}=\overline{MA}=(1+2)\underline{i}+(2-4)\underline{j}+(-3+6)\underline{k}=3\underline{i}-2\underline{j}+3\underline{k}$$

Moment of $\underline{F}$ about $M = \underline{r}\times\underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 3 \\ 2 & -6 & 5 \end{vmatrix}$$

$$= (-10+18)\underline{i}-(15-6)\underline{j}+(-18+4)\underline{k}$$

$$= 8\underline{i}-9\underline{j}-14\underline{k}$$

Scalar Triple Product:

The scalar triple product is a key concept in vector calculus with wide-ranging applications covering various fields. In three-dimensional space, it provides a significant role in calculating the volume of geometric shapes such as parallelepipeds and tetrahedrons, defined by three vectors, which we will learn later in this chapter. Additionally, it plays as a vital tool for determining the coplanarity of vectors, providing a condition to verify whether three vectors lie within the same plane.

There are two types of triple product of vectors:

(a) **Scalar Triple Product:** $\underline{u}\cdot(\underline{v}\times\underline{w})$

(b) **Vector Triple Product:** $\underline{u}\times(\underline{v}\times\underline{w})$

In this section we shall study the scalar triple product only.

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be three non-zero vectors

Definition: The scalar triple product of vector $\underline{u}$, $\underline{v}$ and $\underline{w}$ is defined by

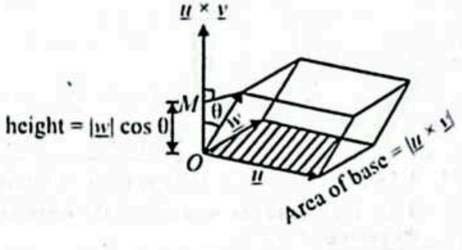
$$\underline{u}\cdot(\underline{v}\times\underline{w}) \text{ or } \underline{v}\cdot(\underline{w}\times\underline{u}) \text{ or } \underline{w}\cdot(\underline{u}\times\underline{v})$$

The scalar triple product $\underline{u}\cdot(\underline{v}\times\underline{w})$ is written as $\underline{u}\cdot(\underline{v}\times\underline{w})=[\underline{u}\quad\underline{v}\quad\underline{w}]$

The Volume of the Parallelepiped:

The triple scalar product $(\underline{u}\times\underline{v})\cdot\underline{w}$ represents the volume of the parallelepiped having $\underline{u}$, $\underline{v}$ and $\underline{w}$ as its conterminous edges. As it is seen from the formula that:

$$(\underline{u}\times\underline{v})\cdot\underline{w}=|\underline{u}\times\underline{v}||\underline{w}|\cos\theta$$



Hence,

(i) $|\underline{u}\times\underline{v}|$ = area of the parallelogram with two adjacent sides $\underline{u}$ and $\underline{v}$.

(ii) $|\underline{w}|\cos\theta$ = height of the parallelepiped

$$(\underline{u}\times\underline{v})\cdot\underline{w} = |\underline{u}\times\underline{v}||\underline{w}|\cos\theta = (\text{Area of Parallelogram})\ (\text{height})$$

$$= \text{Volume of the parallelepiped}$$

Similarly, be taking the base plane formed by $\underline{v}$ and $\underline{w}$, we have

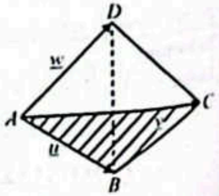
The volume of the parallelepiped = $(\underline{v}\times\underline{w})\cdot\underline{u}$

And by taking the base plane formed by $\underline{w}$ and $\underline{u}$, we have

The volume of the parallelepiped = $(\underline{w}\times\underline{u})\cdot\underline{v}$

So, we have: $(\underline{u}\times\underline{v})\cdot\underline{w}=(\underline{v}\times\underline{w})\cdot\underline{u}=(\underline{w}\times\underline{u})\cdot\underline{v}$

The Volume of the Tetrahedron:



Volume o the tetrahedron $ABCD=\frac{1}{3}$ (area of ΔABC)(height of D above the plane ABC)

$$=\frac{1}{3}\times\frac{1}{2}|\underline{u}\times\underline{v}|(h)$$

$$=\frac{1}{6}(\text{Area of parallelogram with } AB \text{ and } AC \text{ as adjacent sides})\ (h)$$

$$=\frac{1}{3}(\text{Volume of the parallelepiped with } \underline{u}, \underline{v}, \underline{w} \text{ as edges})$$

Thus, volume of tetrahedron $=\frac{1}{6}(\underline{u}\times\underline{v})\cdot\underline{w}=\frac{1}{6}[\underline{u}\underline{v}\underline{w}]$

> **Note:**
> As volume is always positive so ignore negative sign if $(\underline{u}\times\underline{v})\cdot\underline{w}$ is negative

Scalar Triple Product of Vectors in Terms of Components:

Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$

Now $$\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad \underline{v}\times\underline{w} = (b_2c_3-b_2c_3)\underline{i}-(a_2c_3-a_3c_2)\underline{j}+(a_2b_3-a_3b_2)\underline{k}$$

$$\therefore \quad \underline{u}\cdot(\underline{v}\times\underline{w}) = (a_1\underline{i}+b_1\underline{j}+c_1\underline{k}).((b_2c_3-b_2c_3)\underline{i}-(a_2c_3-a_3c_2)\underline{j}+(a_2b_3-a_3b_2)\underline{k})$$

$$= a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$$

$$\Rightarrow \quad \underline{u}\cdot(\underline{v}\times\underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Which is called the determinant formula for scalar triple product of $\underline{u}$, $\underline{v}$ and $\underline{w}$ in component form.

Example 23: Prove that dot and cross product are interchangeable in scalar triple product.

Solution:

Consider $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$ are the arbitrary vectors.

The determinant formula for scalar triple product of vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$ is given by:

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{Interchanging } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \qquad \text{Interchanging } R_1 \text{ and } R_3$$

$$= \underline{w}\cdot(\underline{u}\times\underline{v})=(\underline{u}\times\underline{v})\cdot\underline{w} \qquad (\because \underline{a}\cdot b=\underline{b}\cdot\underline{a})$$

Hence, $\underline{u}\cdot(\underline{v}\times\underline{w})=(\underline{u}\times\underline{v})\cdot\underline{w}$

Thus, the position of dot and cross can be interchanged in scalar triple product.

Example 24: Assuming $\underline{i}$, $\underline{j}$ and $\underline{k}$ are unit vectors in a cartesian coordinate system.

Prove that $\underline{i}\cdot\underline{j}\times\underline{k}=\underline{j}\cdot\underline{k}\times\underline{i}=\underline{k}\cdot\underline{i}\times\cdot\underline{j}$

Solution:

Given that: $\underline{i}$, $\underline{j}$ and $\underline{k}$ are unit vector,

So, we can write $\underline{i}=\underline{i}+0\underline{j}+0\underline{k}$, $\underline{j}=0\underline{i}+\underline{j}+0\underline{k}$, $\underline{k}=0\underline{i}+0\underline{j}+\underline{k}$

By using the determinant form for scalar triple product, we have

$$\underline{i}\cdot\underline{j}\times\underline{k} = \begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = 1(1-0)-0(0-1)+0(0-0)=1$$

$$\underline{j}\cdot\underline{k}\times\underline{i} = \begin{vmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{vmatrix} = 0(0-0)-1(0-1)+0(0-0)=1$$

$$\underline{k}\cdot\underline{i}\times\underline{j} = \begin{vmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix} = 0(0-0)-0(0-0)+1(1-0)=1$$

Therefore $\underline{i}.\underline{j}\times\underline{k} = \underline{j}\underline{k}\times\underline{i} = \underline{k}.\underline{i}\times\underline{j}$

Example 25: Find the volume of the parallelepiped determined by $\underline{u}=\underline{i}+2\underline{j}-\underline{k},\ \underline{v}=\underline{i}-2\underline{j}+3\underline{k},\ \underline{w}=\underline{i}-7\underline{j}-4\underline{k}$

Solution:

Volume of the parallelepiped $= \underline{u}\cdot(\underline{v}\times\underline{w})$

$$= \begin{vmatrix}1 & 2 & -1\\ 1 & -2 & 3\\ 1 & -7 & -4\end{vmatrix} = 1(8+21)-2(-4-3)-1(-7+2)=29+14+5$$

$$= 48 \text{ cubic units}$$

Example 26: Find the volume of the tetrahedron whose vertices are $A(2,1,8)$, $B(3,2,9)$, $C(2,1,4)$ and $D(3,3,0)$.

Solution:

$A(2,1,8)$, $B(3,2,9)$, $C(2,1,4)$ and $D(3,3,0)$

$$\overline{AB}=(3-2)\underline{i}+(2-1)\underline{j}+(9-8)\underline{k}=\underline{i}+\underline{j}+\underline{k}$$

$$\overline{AC}=(2-2)\underline{i}+(1-1)\underline{j}+(4-8)\underline{k}=0\underline{i}-0\underline{j}-4\underline{k}$$

$$\overline{AD}=(+3-2)\underline{i}+(3-1)\underline{j}+(0-8)\_=\underline{i}+2\underline{j}-8\underline{k}$$

Volume of the tetrahedron $=\frac{1}{6}\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right]$

$$=\frac{1}{6}\begin{vmatrix}1 & 1 & 1\\ 0 & 0 & -4\\ 1 & 2 & -8\end{vmatrix}=\frac{1}{6}[1(0+8)-1(0+4)+1(0-0)]$$

$$=\frac{1}{6}[8-4]=\frac{4}{6}=\frac{2}{3} \text{ cubic units}$$

Coplanar Vectors and Condition for Coplanarity of Three Vectors:

Coplanar Vector: Vectors are coplanar if they lie in the same plane or can be combined in the same plane.

Consider the three coplanar vectors $\underline{u}, \underline{v}$ and $\underline{w}$ in a plane as shown in a figure.

The cross product $\underline{v}\times\underline{w}$ gives a vector that is perpendicular to both the vectors $\underline{v}$ and $\underline{w}$,

As $\underline{u}, \underline{v}$ and $\underline{w}$ are coplanar, so $\underline{v}\times\underline{w}$ is also perpendicular to $\underline{u}$

Thus, the dot product of $\underline{u}$ and $\underline{v}\times\underline{w}$ is zero. i.e., $\underline{u}\cdot(\underline{v}\times\underline{w})=0$ $\quad\because$ If $\underline{a}\perp\underline{b}$ then $\underline{a}\cdot\underline{b}=0$

Condition for Coplanarity of Three Vectors:

Thus, we conclude that if the three vectors $\underline{u}, \underline{v}$ and $\underline{w}$ are coplanar then their scalar triple product is zero.

Properties of Scalar Triple Product

1. If $\underline{u}, \underline{v}$ and $\underline{w}$ are coplanar, then the volume of the parallelepiped so formed is zero that is $(\underline{u}\times\underline{v})\cdot\underline{w}=0$ and hence the vectors $\underline{u}, \underline{v}, \underline{w}$ are coplanar $\Leftrightarrow (\underline{u}\times\underline{v})\cdot\underline{w}=0$
2. If any two vectors of scalar triple product are equal, then its value is zero i.e., $[\underline{u}\ \underline{u}\ \underline{w}]=[\underline{u}\ \underline{v}\ \underline{v}]=[\underline{u}\ \underline{w}\ \underline{w}]=0$

Example 27: Prove that four points $A(-3,5,-4)$, $B(-1,1,1)$, $C(-1,2,2)$ and $D(-3,4,-5)$ are coplanar.

Proof:

$A(-3,5,-4)$, $B(-1,1,1)$, $C(-1,2,2)$ and $D(-3,4,-5)$

$$\overline{AB}=(-1+3)\underline{i}+(1-5)\underline{j}+(1+4)\underline{k}=2\underline{i}-4\underline{j}+5\underline{k}$$

$$\overline{AC}=(-1+3)\underline{i}+(2-5)\underline{j}+(2+4)\underline{k}=2\underline{i}-3\underline{j}+6\underline{k}$$

$$\overline{AD}=(-3+3)\underline{i}+(4-5)\underline{j}+(-5+4)\underline{k}=0\underline{i}-\underline{j}-\underline{k}=-\underline{j}-\underline{k}$$

Volume of the parallelepiped formed $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ is

$$[\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix}2 & -4 & 5\\ 2 & -3 & 6\\ 0 & -1 & -1\end{vmatrix} = 2(3+6)+4(-2-0)+5(-2-0)$$

$$= 18-8-10=0$$

As the volume is zero, so the points A, B, C and D are coplaner.

Example 28: Find the value of α, so that $\alpha\underline{i}+\underline{j}$, $\underline{i}+\underline{j}+3\underline{k}$ and $2\underline{i}+\underline{j}-2\underline{k}$ are coplanar.

Solution:

Let $\underline{u}=\alpha\underline{i}+\underline{j}+0\underline{k}, \underline{v}=\underline{i}+\underline{j}+3\underline{k}$ and $\underline{w}=2\underline{i}+\underline{j}-2\underline{k}$ be three given vectors.

Since the vectors $\underline{u}, \underline{v}$ and $\underline{w}$ are coplanar, therefore scalar triple product will be zero, i.e., $[\underline{u}\ \underline{v}\ \underline{w}]=0$

$$[\underline{u}\ \underline{v}\ \underline{w}] = \begin{vmatrix}\alpha & 1 & 0\\ 1 & 1 & 3\\ 2 & 1 & -2\end{vmatrix} = 0$$

$$\Rightarrow \alpha(-2-3)-1(-2-6)+0(1-2)=0$$

$$-5\alpha+8=0 \quad \Rightarrow -5\alpha=-8$$

$$\boxed{\alpha=\frac{8}{5}}$$

Applications of Vectors in Real World:

Example 29: A plumber exerts a force of 30 pounds along the negative y-axis on a lever connected to a machine. The pivot point of the lever is at the origin $(0,0,0)$, and the force is applied at the point $(1.2\text{ft}, 0.5\text{ft}, 0\text{ft})$. Determine the torque produced by this force about the pivot point.

Solution:

Let $O(0,0,0)$ and $A(1.2, 0.5, 0)$, then

$$\underline{r}=\overrightarrow{OA}=(1.2-0)\underline{i}+(0.5-0)\underline{j}+(0-0)\underline{k}$$

$$\underline{r}=1.2\underline{i}+0.5\underline{j}+0\underline{k}$$

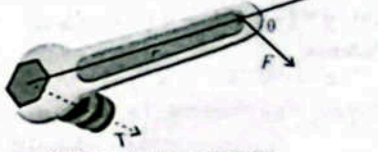
The force $\vec{F}$ is exerted downward along negative y-axis with a magnitude of 30 pounds is

$$\underline{F}=0\underline{i}-30\underline{j}+0\underline{k}$$

Torque τ produced by the force $=\underline{r}\times\underline{F}$

Key Concept:

Torque quantifies the rotational effect of a force applied to an object about a pivot point. It is determined by taking the cross product of the position vector $\underline{r}$ (which extends from the pivot point to the point where the force is applied) and the force vector $\underline{F}$ itself.



Mathematically, $\underline{\tau}=\underline{r}\times\underline{F}$

$$\underline{\tau} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1.2 & 0.5 & 0 \\ 0 & -30 & 0 \end{vmatrix} = 0\underline{i} - 0\underline{j} - 36\underline{k}$$

$$\underline{\tau} = -36\underline{k} \text{ pound-feet}$$

Thus, the torque is 36 feet-pounds in the negative z-direction.

Example 30: During a building construction, a crane exerts a force to pull a concrete block, represented by the vector $\underline{F} = [4500, 3300, 2140]$ Newton. Each component corresponds to the force exerted along the x, y and z axes, respectively. What is the magnitude of this force?

Solution: Using the formula for the magnitude of a vector in three-dimensional space

$$|\underline{F}| = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{4500^2 + 3300^2 + 2140^2} = \sqrt{20250000 + 10890000 + 4579600}$$

$$= \sqrt{35719600} = 5976.59\delta$$

The magnitude of the force exerted by the crane is approximately 5976.59 Newton.

Example 31: The components of $\underline{u} = 300\underline{i} + 250\underline{j} + 180\underline{k}$ represent the respective number of jackets, shoes, and handbags sold at a store. The components of $\underline{v} = 3500\underline{i} + 4200\underline{j} + 6840\underline{k}$ represent the respective prices (in rupees) per unit for each product. Find $\underline{u} \cdot \underline{v}$ and explain what the result tells us in real life.

Solution: The dot product of $\underline{u}$ and $\underline{v}$ is

$$\underline{u} \cdot \underline{v} = (300\underline{i} + 250\underline{j} + 180\underline{k}) \cdot (3500\underline{i} + 4200\underline{j} + 6840\underline{k})$$

$$= 1{,}050{,}000 + 1{,}050{,}000 + 1{,}231{,}200$$

$$= 3{,}331{,}200$$

The result $\underline{u} \cdot \underline{v} = 3{,}331{,}200$ tells us that total revenue generated from selling all the three product is Rs. 3,331,200.

Exercise 14.4

1. Find the volume of parallelepiped for which the given vectors are three edges

(i) $\underline{u} = 3\underline{i} + 2\underline{k};\ \underline{v} = \underline{i} + 2\underline{j} + \underline{k};\ \underline{w} = -\underline{j} + 4\underline{k}$

Solution:

$\underline{u} = 3\underline{i} + 2\underline{k};\quad \underline{v} = \underline{i} + 2\underline{j} + \underline{k};\ \underline{w} = -\underline{j} + 4\underline{k}$

Volume of parallelepiped $= [\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$$= 3(8+1) - 0(4-0) + 2(-1-0)$$

$$= 27 - 2 = 25 \text{ cubic unit}$$

(ii) $\underline{u} = \underline{i} - 4\underline{j} - \underline{k};\ \underline{v} = \underline{i} - \underline{j} - 2\underline{k};\ \underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Solution:

$\underline{u} = \underline{i} - 4\underline{j} - \underline{k};\quad \underline{v} = \underline{i} - \underline{j} - 2\underline{k};\ \underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Volume of parallelepiped $= [\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= 1(-1-6) + 4(1+4) - 1(-3+2)$$

$$= -7 + 20 + 1 = 14 \text{ cubic unit}$$

(iii) $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k};\ \underline{v} = 2\underline{i} - \underline{j} - \underline{k};\ \underline{w} = \underline{j} + \underline{k}$

Solution:

$\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k};\quad \underline{v} = 2\underline{i} - \underline{j} - \underline{k};\ \underline{w} = \underline{j} + \underline{k}$

Volume of parallelepiped $= [\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(-1+1) + 2(2+0) + 3(2+0)$$

$$= 0 + 4 + 6 = 10 \text{ cubic unit}$$

2. Verify that $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$ If $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k};\ \underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$

Solution:

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= 3(3+10) + 1(4+4) + 5(20-6)$$

$$= 39 + 8 + 70 = 117 \qquad \ldots(1)$$

$$\underline{b} \cdot \underline{c} \times \underline{a} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= 4(25+1) - 3(10-3) - 2(-2-15)$$

$$= 104 - 21 + 34 = 117 \qquad \ldots(2)$$

$$\underline{c} \cdot \underline{a} \times \underline{b} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= 2(2-15) - 5(-6-20) + 1(9+4)$$

$$= -26 + 130 + 13 = 117 \qquad \ldots(3)$$

From (1), (2) and (3) we conclude that

$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$

3. Prove that the vectors $\underline{i} - 2\underline{j} + 3\underline{k}, -2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{i} - 3\underline{j} + 5\underline{k}$ are coplanar.

Solution:

Let $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k},\ \underline{v} = -2\underline{i} + 3\underline{j} - 4\underline{k},\ \underline{w} = \underline{i} - 3\underline{j} + 5\underline{k}$

$$[\underline{u}\ \underline{v}\ \underline{w}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

$$= 1(15-12) + 2(-10+4) + 3(6-3)$$

$$= 3 - 12 + 9 = 12 - 12 = 0$$

Thus the vectors $\underline{u}$, $\underline{v}$, $\underline{w}$ are coplanar.

4. Find the constant α such that the vectors are coplanar.

(i) $\underline{i} - \underline{j} + \underline{k}, \underline{i} - 2\underline{j} - 3\underline{k}$ and $3\underline{i} - \alpha\underline{j} + 5\underline{k}$

Solution:

Let $\underline{u} = \underline{i} - \underline{j} + \underline{k},\quad \underline{v} = \underline{i} - 2\underline{j} - 3\underline{k},\quad \underline{w} = 3\underline{i} - \alpha\underline{j} + 5\underline{k}$

The vector $\underline{u}$, $\underline{v}$, $\underline{w}$ are coplanar, if

$$[\underline{u}\ \underline{v}\ \underline{w}] = 0$$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

$$1(-10-3\alpha) + 1(5+9) + 1(-\alpha+6) = 0$$

$$-10 - 3\alpha + 14 - \alpha + 6 = 0$$

$$-4\alpha + 10 = 0$$

$$\Rightarrow \alpha = \frac{10}{4} \Rightarrow \alpha = \frac{5}{2}$$

(ii) $\underline{i} - 2\alpha\underline{j} - \underline{k}, \underline{i} - 2\underline{j} + 2\underline{k}$ and $\alpha\underline{i} - 2\underline{j} + \underline{k}$

Solution:

Let $\underline{u} = \underline{i} - 2\alpha\underline{j} - \underline{k}, \underline{v} = \underline{i} - 2\underline{j} + 2\underline{k}, \underline{w} = \alpha\underline{i} - 2\underline{j} + \underline{k}$

Since $\underline{u}$, $\underline{v}$ and $\underline{w}$ are coplanar, therefore

$$[\underline{u}\ \underline{v}\ \underline{w}] = 0$$

$$\begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -2 & 2 \\ \alpha & -2 & 1 \end{vmatrix} = 0$$

$$1(-2+4) + 2\alpha(1-2\alpha) - 1(-2+2\alpha) = 0$$

$$\alpha + 2\alpha - 4\alpha^2 + 2 - 2\alpha = 0$$

$$-4\alpha^2 + 4 = 0$$

$$-4\alpha^2 = -4$$

$$4\alpha^2 = 4$$

$$\alpha^2 = 1$$

$$\boxed{\alpha = \pm 1}$$

5. Prove that the points whose position vectors are $A(-6\underline{i} + 3\underline{j} + 2\underline{k})$, $B(3\underline{i} - 2\underline{j} + 4\underline{k})$, $C(5\underline{i} + 7\underline{j} + 3\underline{k})$, $D(-13\underline{i} + 17\underline{j} - \underline{k})$ are coplanar.

Solution:

Given position vectors of A, B, C, D are

$$\overline{OA} = -6\underline{i} + 3\underline{j} + 2\underline{k},\ \overline{OB} = 3\underline{i} - 2\underline{j} + 4\underline{k},$$

$$\overline{OC} = 5\underline{i} + 7\underline{j} + 3\underline{k},\ \overline{OD} = -13\underline{i} + 17\underline{j} + \underline{k}$$

Now $\overline{AB} = \overline{OB} - \overline{OA}$

$$= 3\underline{i} - 2\underline{j} + 4\underline{k} + 6\underline{i} - 3\underline{j} - 2\underline{k} = 9\underline{i} - 5\underline{j} + 2\underline{k}$$

$$\overline{AC} = \overline{OC} - \overline{OA}$$

$$= 5\underline{i} + 7\underline{j} + 3\underline{k} + 6\underline{i} - 3\underline{j} - 2\underline{k} = 11\underline{i} + 4\underline{j} + \underline{k}$$

$$\overline{AD} = \overline{OD} - \overline{OA}$$

$$= -13\underline{i} + 17\underline{j} - \underline{k} + 6\underline{i} - 3\underline{j} - 2\underline{k} = -7\underline{i} + 14\underline{j} - 3\underline{k}$$

Consider $[\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} 9 & -5 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix}$

$$= 9(-12-14) + 5(-33+7) + 2(154+28)$$

$$= -234 - 130 + 364 = 0$$

Thus the points A, B, C and D are coplanar.

6. (a) Find the value of :

(i) $2\underline{i} \times 2\underline{j} \cdot \underline{k}$

Solution:

$$2\underline{i} \times 2\underline{j} \cdot \underline{k} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(2-0) - 0(0-0) + 0(0-0)$$

$$= 2(2) - 0 + 0 = 4$$

(ii) $3\underline{j} \cdot \underline{k} \times \underline{i}$

Solution:

$$3\underline{j} \cdot \underline{k} \times \underline{i} = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 0(0-0) - 3(0-1) + 0$$

$$= 0 + 3 + 0 = 3$$

(iii) $[\underline{k}\ \underline{i}\ \underline{j}]$

Solution:

$$[\underline{k}\ \ \underline{i}\ \ \underline{j}] = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$
$$= 0(0-0) - 0\,(0-0) + 1\,(1-0)$$
$$= 0 - 0 + 1 = 1$$

(iv) $[\underline{i}\ \underline{i}\ \underline{k}]$

Solution:

$$[\underline{i}\ \ \underline{i}\ \ \underline{k}] = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= 1(0-0) - 0(1-0) + 0(0-0)$$
$$= 0 - 0 + 0 = 0$$

(b) Prove that $\underline{u}\cdot(\underline{v}\times\underline{w}) + \underline{v}\cdot(\underline{w}\times\underline{u}) + \underline{w}\cdot(\underline{u}\times\underline{v}) = 3\underline{u}\cdot(\underline{v}\times\underline{w})$

Solution:

We know that:

$$\underline{u}\,.\,(\underline{v}\times\underline{w}) = \underline{v}\,.\,(\underline{w}\times\underline{u}) = \underline{w}\,.\,(\underline{u}\times\underline{v}) \qquad \ldots(1)$$

$$\text{L.H.S} = \underline{u}\,.\,(\underline{v}\times\underline{w}) + \underline{v}\,.\,(\underline{w}\times\underline{u}) + \underline{w}\,.\,(\underline{u}\times\underline{v})$$
$$= \underline{u}\,.\,(\underline{v}\times\underline{w}) + \underline{u}\,.\,(\underline{v}\times\underline{w}) + \underline{u}\,.\,(\underline{v}\times\underline{w}) \quad \text{using eq (1)}$$
$$= 3\,\underline{u}\,.\,(\underline{v}\times\underline{w})$$
$$= \text{R.H.S}$$

Thus $\underline{u}\,.\,(\underline{v}\times\underline{w}) + \underline{v}\,.\,(\underline{w}\times\underline{u}) + \underline{w}\,.\,(\underline{u}\times\underline{v}) = 3\underline{u}\,.\,(\underline{v}\times\underline{w})$

7. Find volume of tetrahedron with the vertices

(i) $(0,1,2), (3,2,1), (1,2,1)$ and $(5,5,6)$

Solution:

Let A(0, 1, 2), B(3, 2, 1), C(1, 2, 1), D(5, 5, 6)

$$\overrightarrow{AB} = (3-0)\,\underline{i} + (2-1)\,\underline{j} + (1-2)\,\underline{k} = 3\underline{i} + \underline{j} - \underline{k}$$
$$\overrightarrow{AC} = (1-0)\,\underline{i} + (2-1)\,\underline{j} + (1-2)\,\underline{k} = \underline{i} + \underline{j} - \underline{k}$$
$$\overrightarrow{AD} = (5-0)\,\underline{i} + (5-1)\,\underline{j} + (6-2)\,\underline{k} = 5\underline{i} + 4\underline{j} + 4\underline{k}$$

Volume of tetrahedron $ABCD = \frac{1}{6}[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$$
$$= \frac{1}{6}\{3(4+4) - 1(4+5) - 1(4-5)\}$$
$$= \frac{1}{6}(24 - 9 + 1) = \frac{1}{6}(16) = \frac{8}{3} \text{ cubic units}$$

(ii) $(2,1,8), (3,2,9), (2,1,4)$ and $(3,3,10)$

Solution:

Let A(2, 1, 8), B(3, 2, 9), C(2, 1, 4), D(3, 3, 10)

$$\overrightarrow{AB} = (3-2)\,\underline{i} + (2-1)\,\underline{j} + (9-8)\,\underline{k} = \underline{i} + \underline{j} + \underline{k}$$
$$\overrightarrow{AC} = (2-2)\,\underline{i} + (1-1)\,\underline{j} + (4-8)\,\underline{k} = 0\underline{i} + 0\underline{j} - 4\underline{k}$$
$$\overrightarrow{AD} = (3-2)\,\underline{i} + (3-1)\,\underline{j} + (10-8)\,\underline{k} = \underline{i} + 2\underline{j} + 2\underline{k}$$

Volume of tetrahedron $ABCD = \frac{1}{6}[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$
$$= \frac{1}{6}\{1(0+8) - 1(0+4) + 1(0-0)\}$$
$$= \frac{1}{6}(8 - 4 + 0) = \frac{1}{6}(4) = \frac{2}{3} \text{ cubic units}$$

8. Prove that the points whose position vectors are $A(3\underline{i} + 2\underline{j} - \underline{k}), B(\underline{i} - 2\underline{j} + \underline{k})$, $C(6\underline{i} + 4\underline{j} - 2\underline{k}), D(9\underline{i} + 6\underline{j} - 3\underline{k})$ are coplanar.

Solution:

Position vectors of points A, B, C and D are.

$$\overrightarrow{OA} = 3\underline{i} + 2\underline{j} - \underline{k}$$
$$\overrightarrow{OB} = \underline{i} - 2\underline{j} + \underline{k}$$
$$\overrightarrow{OC} = 6\underline{i} + 4\underline{j} - 2\underline{k}$$
$$\overrightarrow{OD} = 9\underline{i} + 6\underline{j} - 3\underline{k}$$
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \underline{i} - 2\underline{j} + \underline{k} - 3\underline{i} - 2\underline{j} + \underline{k}$$
$$= -2\underline{i} - 4\underline{j} + 2\underline{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= 6\underline{i} + 4\underline{j} - 2\underline{k} - 3\underline{i} - 2\underline{j} + \underline{k}$$
$$= 3\underline{i} + 2\underline{j} - \underline{k}$$
$$\overrightarrow{AD} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= 9\underline{i} + 6\underline{j} - 3\underline{k} - 3\underline{i} - 2\underline{j} + \underline{k}$$
$$= 6\underline{i} + 4\underline{j} - 2\underline{k}$$

Consider

$$\left[\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right] = \begin{vmatrix} -2 & -4 & 2 \\ 3 & 2 & -1 \\ 6 & 4 & -2 \end{vmatrix}$$
$$= -2(-4+4) + 4(-6+6) + 2(12-12)$$
$$= -2(0) + 4(0) + 2(0)$$
$$= 0 + 0 + 0 = 0$$

Thus, the points A, B, C and D are collinear.

9. Prove that for any three non-zero vector $\underline{u}, \underline{v}$ and $\underline{w}$ $(\underline{u} + \underline{v})\cdot[(\underline{v} + \underline{w})\times(\underline{w} + \underline{u})] = 2[\underline{u}\underline{v}\underline{w}]$

Solution:

$$\text{L.H.S} = (\underline{u}+\underline{v})\cdot[(\underline{v}+\underline{w})\times(\underline{w}+\underline{u})]$$

By distributivity of cross product over addition

$$= (\underline{u}+\underline{v})\cdot[\underline{v}\times(\underline{w}+\underline{u}) + \underline{w}\times(\underline{w}+\underline{u})]$$
$$= (\underline{u}+\underline{v})\cdot[\underline{v}\times\underline{w} + \underline{v}\times\underline{u} + \underline{w}\times\underline{w} + \underline{w}\times\underline{u}]$$
$$= (\underline{u}+\underline{v})\cdot[\underline{v}\times\underline{w} + \underline{v}\times\underline{u} + \underline{0} + \underline{w}\times\underline{u}] \quad \because \underline{w}\times\underline{w} = \underline{0}$$
$$= (\underline{u}+\underline{v})\cdot[\underline{v}\times\underline{w} + \underline{v}\times\underline{u} + \underline{w}\times\underline{u}]$$

$$= \underline{u}\cdot(\underline{v}\times\underline{w}) + \underline{u}\cdot(\underline{v}\times\underline{u}) + \underline{u}\cdot(\underline{w}\times\underline{u})$$
$$+ \underline{v}\cdot(\underline{v}\times\underline{w}) + \underline{v}\cdot(\underline{v}\times\underline{u}) + \underline{v}\cdot(\underline{w}\times\underline{u})$$

If any two vectors of scalar triple product are equal, then its value is zero.

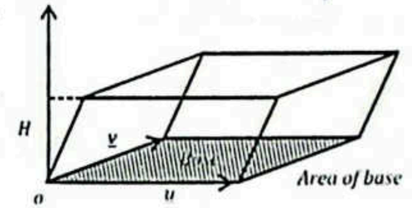
$$= \underline{u}\cdot(\underline{v}\times\underline{w}) + 0 + 0 + 0 + 0 + \underline{v}\cdot(\underline{w}\times\underline{u})$$
$$= \underline{u}\cdot(\underline{v}\times\underline{w}) + \underline{v}\cdot(\underline{w}\times\underline{u})$$

Since $\underline{u}\cdot(\underline{v}\times\underline{w}) = \underline{v}\cdot(\underline{w}\times\underline{u}) = \underline{w}\cdot(\underline{u}\times\underline{v})$, therefore

$$= \underline{u}\cdot(\underline{v}\times\underline{w}) + \underline{u}\cdot(\underline{v}\times\underline{w})$$
$$= 2\underline{u}\cdot(\underline{v}\times\underline{w})$$
$$= 2[\underline{u}\ \underline{v}\ \underline{w}] = \text{R.H.S (Proved)}$$

10. Consider a parallelepiped determined by the vector $\underline{u} = 2\underline{i} + 4\underline{j} - 3\underline{k}$, $\underline{v} = 5\underline{i} - 3\underline{j} + 6\underline{k}$ and $\underline{w} = 4\underline{i} - 7\underline{j} - 2\underline{k}$. If the base of the parallelepiped is define by the vectors $\underline{u}$ and $\underline{v}$ then find the height of the parallelepiped.

Solution:



Given that:

$\underline{u} = 2\underline{i} + 4\underline{j} - 3\underline{k}, \underline{v} = 5\underline{i} - 3\underline{j} + 6\underline{k}, \underline{w} = 4\underline{i} - 7\underline{j} - 2\underline{k}$

and base of || piped is defined by $\underline{u}$ and $\underline{v}$

Area of base $= |\underline{u}\times\underline{v}|$

As we know

Volume of || piped = (Area of base) (Height)

$$\Rightarrow \quad \text{Height} = \frac{\text{volume of llpiped}}{\text{Area of base}}$$

$$H = \frac{\underline{u}\cdot(\underline{v}\times\underline{w})}{|\underline{u}\times\underline{v}|} \qquad \ldots(1)$$

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = \begin{vmatrix} 2 & 4 & -3 \\ 5 & -3 & 6 \\ 4 & -7 & -2 \end{vmatrix}$$
$$= 2(6+42) - 4(-10-24) - 3(-35+12)$$
$$= 2(48) - 4(-34) - 3(-23)$$
$$= 96 + 136 + 69 = 301$$

$$\underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 4 & -3 \\ 5 & -3 & 6 \end{vmatrix}$$
$$= \underline{i}(24-9) - \underline{j}(12+15) + \underline{k}(-6-20)$$
$$= 15\underline{i} - 27\underline{j} - 26\underline{k}$$

$$|\underline{u}\times\underline{v}| = \sqrt{(15)^2 + (-27)^2 + (-26)^2}$$
$$= \sqrt{225 + 729 + 676} = \sqrt{1630}$$

Putting values in (1), we have

$$H = \frac{301}{\sqrt{1630}} \text{ (As required)}$$

11. A mechanic applies a force of 50 pounds along the positive x-axis on a wrench connected to a bolt. The pivot point of the wrench is at the origin $(0,0,0)$, and the force is applied at the point $(0\text{ft}, 2\text{ft}, 3\text{ft})$. Determine the torque produced by this force about the pivot point.

Solution:

Since a force of 50 pounds is applied along tire x-axis, Therefore

$$\vec{F} = 50\underline{i} + 0\underline{j} + 0\underline{k}$$

Pivot point $= O(0,0,0)$

Since the force is applied at $A(0,2,3)$, therefore

$$\vec{r} = \overrightarrow{OA}$$
$$= (0-0)\underline{i} + (2-0)\underline{j} + (3-0)\underline{k}$$
$$= 0\underline{i} + 2\underline{j} + 3\underline{k}$$

Torque produced by $\vec{F}$ about O is

$$\underline{\tau} = \vec{r}\times\vec{F}$$
$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 2 & 3 \\ 50 & 0 & 0 \end{vmatrix}$$
$$= \underline{i}(0-0) - \underline{j}(0-150) + \underline{k}(0-100)$$
$$= 0\underline{i} + 150\underline{j} - 100\underline{k} \text{ (Pound feet)}$$

12. A drone flies from point $(1,2,5)$ to point $(4,6,9)$, with each unit representing a meter. What is the magnitude of the displacement the drone experienced during this flight?

Solution:

Given that: $A(1,2,5), B(4,6,9)$

Displacement: $\vec{d} = \overrightarrow{AB}$

$$\vec{d} = (4-1)\underline{i} + (6-2)\underline{j} + (9-5)\underline{k}$$
$$= 3\underline{i} + 4\underline{j} + 4\underline{k}$$
$$\Rightarrow \quad |\vec{d}| = \sqrt{x^2 + y^2 + z^2}$$
$$= \sqrt{(3)^2 + (4)^2 + (4)^2}$$
$$= \sqrt{9 + 16 + 16}$$
$$= \sqrt{41} \text{ meters.}$$

13. The vector $\underline{u} = 50\underline{i} + 75\underline{j} + 65\underline{k}$ shows how many belts, pants, and shirts were sold at a store. The vector $\underline{w} = 1500\underline{i} + 3500\underline{j} + 3000\underline{k}$ shows the price (in rupees) of each item. Find $\underline{u}\cdot\underline{w}$ and explain what the result tells us in real life.

Solution:

Given vectors are

$\underline{u}=50\underline{i}+75\underline{j}+65\underline{k}, \underline{v}=1500\underline{i}+3500\underline{j}+3000\underline{k}$

Dot product of $\underline{u}$ and $\underline{v}$ = $\underline{u}\,.\,\underline{v}$

$$
\begin{aligned}
&= (50\underline{i}+75\underline{j}+65\underline{k})\cdot(1500\underline{i}+3500\underline{j}+3000\underline{k})\\
&= (50)(1500)+(75)(3500)+(65)(3000)\\
&= 75000+262500+195000\\
&= 532{,}500
\end{aligned}
$$

The result $\underline{u}\,.\,\underline{v}$ = 532,500 tells us that total revenue generated from selling all the three product is Rs. 532,500.

14. A force $\underline{F} = (20,-10,30)$N is applied at a point $P(2,-1,4)$ in 3D space. The pivot point is at $M(1,2,-3)$. Calculate the torque produced by this force about the pivot point M.

Solution:

Solution:

$$
\begin{aligned}
\text{Force: } \bar{F} &= (20,-10,30)\text{N}\\
&= 20i-10i+30k\\
\text{Pivot point} &= M(1,2,-3)
\end{aligned}
$$

Since the force is applied at $P(2,-1,4)$, so

$$
\begin{aligned}
\bar{r} &= \overline{MP}\\
&= (2-1)\underline{i}+(-1-2)\underline{j}+(4+3)\underline{k}\\
&= \underline{i}-3j+7\underline{k}
\end{aligned}
$$

Torque produced by $\bar{F}$ about M is

$$
\begin{aligned}
\underline{\tau} &= \bar{r}\times\bar{F}\\
&= \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1. & -3 & 7\\ 20 & -10 & 30\end{vmatrix}\\
&= \underline{i}(-90+70)-\underline{j}(30-140)+\underline{k}(-10+60)\\
&= -20\underline{i}+110\underline{j}+50\underline{k}(\text{Nm})
\end{aligned}
$$

15. An electric shop sells three types of appliances: Fans, Heaters, and Ovens. The monthly sales quantities are 500 units of Fans, 300 units of Heaters and 200 units of Ovens. The profit per unit for each appliance is Rs 500 for Fans, Rs 400 for Heaters, and Rs 2,000 for Ovens.

(a) Represent the monthly sales quantities and the profit per unit as vectors.

Solution:

Given that:

Monthly bales quantities: 500 fans, 300 Healers, 2000 ven profit per unit: Rs 500 for Fans, Rs 400 for Heaters, Rs 2000 for Ovens.

Representation of the sales quantities as a vector

$$\underline{u}=500\underline{i}+300\underline{j}+200\underline{k}=[500,300,200]$$

Representation of the profit per unit as a vector

$$\underline{v}=500\underline{i}+400\underline{i}+2000\underline{k}=[500,400,2000]$$

(b) Calculate the total monthly profit using vector operations.

Solution:

The total monthly profit is the dot product of $\underline{u}$ and $\underline{v}$.

$$
\begin{aligned}
\underline{u}\cdot\underline{v} &= (500\underline{i}+300\underline{j}+200\underline{k})\cdot(500\underline{i}+400\underline{j}+2000\underline{k})\\
&= (500)(500)+(300)(400)+(200)(2000)\\
&= 250000+120000+400000\\
&= 770000
\end{aligned}
$$

The total monthly profit is Rs 770,000.

Formula Sheet

1. $d=\lvert\overline{P_1P_2}\rvert=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$, where $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.		
2. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, where $\cos\alpha=\frac{x}{r},\cos\beta=\frac{y}{r}$ and $\cos\gamma=\frac{z}{r}$		
3. For $\underline{u}=[x,y,z]$, Magnitude of $\underline{u}=\lvert\underline{u}\rvert=\sqrt{x^2+y^2+z^2}$		
Dot or Scalar Product		
1. $\underline{u}\cdot\underline{v}=\lvert\underline{u}\rvert\lvert\underline{v}\rvert\cos\theta$, $0\le\theta\le\pi$ and $\cos\theta=\frac{\underline{u}\cdot\underline{v}}{\lvert\underline{u}\rvert\lvert\underline{v}\rvert}$		
2. If $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$, then $\underline{u}\cdot\underline{v}=a_1a_2+b_1b_2+c_1c_2$		
3. $\underline{i}\cdot\underline{i}=\underline{j}\cdot\underline{j}=\underline{k}\cdot\underline{k}=1$	4. $\underline{i}\cdot\underline{j}=\underline{j}\cdot\underline{k}=\underline{k}.\underline{i}=0$	5. Work done = $\underline{F}\cdot\underline{d}$
6. Projection of $\underline{v}$ along $\underline{u}=\frac{\underline{u}\cdot\underline{v}}{\lvert\underline{u}\rvert}=\underline{v}.\hat{\underline{u}}$	7. Projection of $\underline{u}$ along $\underline{v}=\frac{\underline{u}\cdot\underline{v}}{\lvert\underline{v}\rvert}=\underline{u}.\hat{\underline{v}}$	
Cross Product or Vector Product		
1. $\underline{u}\times\underline{v}=(\lvert\underline{u}\rvert\lvert\underline{v}\rvert\sin\theta)\underline{n}$, $0\le\theta\le\pi$ and $\sin\theta=\frac{\lvert\underline{a}\times\underline{b}\rvert}{\lvert\underline{a}\rvert\lvert\underline{b}\rvert}$		
2. If $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$, then $\underline{u}\times\underline{v}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\a_1&b_1&c_1\\a_2&b_2&c_2\end{vmatrix}$		
3. $\underline{i}\times\underline{i}=\underline{j}\times\underline{j}=\underline{k}\times\underline{k}=\underline{0}$	4. $\underline{i}\times\underline{j}=\underline{k}$, $\underline{j}\times\underline{k}=\underline{i}$, $\underline{k}\times\underline{i}=\underline{j}$	
5. In any triangle ABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ (Law of Sines)		
6. Area of parallelogram $=\lvert\underline{u}\times\underline{v}\rvert$	7. Area of triangle $=\frac{1}{2}\lvert\underline{u}\times\underline{v}\rvert$	8. Moment of Force = $\underline{r}\times\underline{F}$
Scalar Triple Product		
1. If $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}, \underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$, then $\underline{u}\cdot(\underline{v}\times\underline{w})=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}$		
2. Volume of the parallelepiped $=(\underline{u}\times\underline{v})\cdot\underline{w}$	3. Volume of tetrahedron $=\frac{1}{6}(\underline{u}\times\underline{v})\cdot\underline{w}=\frac{1}{6}[\underline{u}\ \underline{v}\ \underline{w}]$	

Multiple Choice Questions (MCQs)

Exercise 14.1

1. If $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, then $\overrightarrow{AB}$ = -----

(A) $\vec{a}-\vec{b}$ (B) $\vec{a}+\vec{b}$ (C) $\vec{b}-\vec{a}$ (D) $\vec{a}\,.\,\vec{b}$

2. If $\underline{u}=[1,2,-1]$; $\underline{v}=[3,-2,1]$, then $\lvert\underline{u}+\underline{v}\rvert$ equals -----

(A) 1 (B) 2 (C) 3 (D) 4

3. The magnitude of vector is also called its -----

(A) parameter (B) variable (C) point (D) norm

4. For what value of α, vectors $5\underline{i}+\underline{i}+\underline{k}$ and $\alpha\underline{i}+3\underline{i}+3\underline{k}$ are parallel to each other?

(A) −3 (B) −15 (C) 15 (D) 3

5. If $\bar{v}$ is any vector then vector of magnitude 5 opposite to $\bar{v}$ is -----

(A) $5\bar{v}$ (B) $-5\bar{v}$ (C) $5\frac{\bar{v}}{\lvert v\rvert}$ (D) $-5\frac{\bar{v}}{\lvert v\rvert}$

Exercise 14.2

6. Two vectors $\underline{a}$ and $\underline{b}$ are perpendicular if -----

(A) $\underline{a}\times\underline{b}=0$ (B) $\underline{a}\,.\,\underline{b}=0$ (C) $\underline{a}=\underline{b}$ (D) $\underline{a}=-\underline{b}$

7. If $\underline{u}=2\alpha\underline{i}+\underline{i}-\underline{k}$ and $\underline{v}=\underline{i}+\alpha\underline{i}+4\underline{k}$ are perpendicular, then α = -----

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) 3

8. **Projection of $\underline{v}$ along $\underline{u}$ is - - - - -**

(A) $\underline{u} \cdot \hat{v}$ (B) $\underline{v} \cdot \hat{u}$ (C) $\frac{\underline{u} \cdot \underline{v}}{|\underline{v}|}$ (D) $\frac{\underline{u} \times \underline{v}}{|\underline{v}|}$

9. **$a = b \cos C + ?$**

(A) $c \cos B$ (B) $b \cos A$ (C) $c \sin B$ (D) $b \sin C$

10. **The work done by a force $\underline{F}$ during displacement $\underline{d}$ is equal to - - - - -**

(A) $\underline{d} \times \underline{F}$ (B) $\underline{F} \times \underline{d}$ (C) $-\underline{F} . \underline{d}$ (D) $\underline{F} . \underline{d}$

Exercise 14.3

11. **The non-zero vectors $\underline{a}$ and $\underline{b}$ are parallel if $\underline{a} \times \underline{b} =$ - - - - -**

(A) 1 (B) −1 (C) 0 (D) ab

12. **If $\underline{u} = 2\underline{i} + 7\underline{i} + 9\underline{k}$ then $u \times u$ - - - - -**

(A) $19\underline{j}$ (B) 0 (C) $3\underline{i} + 5\underline{j} + 19\underline{k}$ (D) $4\underline{j}$

13. **$2\hat{\underline{i}} \times 2\hat{\underline{i}} =$ - - - - -**

(A) $2\hat{k}$ (B) $-2\hat{k}$ (C) $4\hat{k}$ (D) 0

14. **$|\vec{a} \times \vec{b}| =$ - - - - -**

(A) area of triangle (B) area of rhombus (C) area of trapezium (D) area of parallelogram

15. **Moment of force $\underline{F}$ about a point with position vector $\underline{r}$ will be equal to - - - - -**

(A) $\underline{F} \times \underline{r}$ (B) $\underline{r} \times \underline{F}$ (C) $\underline{d} \times \underline{F}$ (D) $\underline{F} \times \underline{d}$

Exercise 14.4

16. **If any two vectors of scalar triple product are equal, then its value is - - - - -**

(A) 1 (B) −1 (C) 2 (D) 0

17. **$2\underline{i} \times 3\underline{i} . \underline{k} =$ - - - - -**

(A) −4 (B) 4 (C) 1 (D) 6

18. **The volume of parallelepiped with $\underline{u}$, $\underline{v}$, $\underline{w}$ as its coterminous edges is - - - - -**

(A) $\underline{u} \times \underline{v}$ (B) $(\underline{u} \times \underline{v}) \cdot \underline{w}$ (C) $\underline{u} \times (\underline{v} \times \underline{w})$ (D) $\underline{u} \times (\underline{u} \times \underline{v})$

19. **What is the volume of a parallelepiped determined by $\underline{i} + 2\underline{i} - \underline{k}$, $\underline{i} - 2\underline{i} + 3\underline{k}$ and $\underline{i} - 7\underline{i} - 4\underline{k}$?**

(A) 20 (B) 48 (C) 8 (D) 38

20. **Scalar triple product of coplanar vectors is - - - - -**

(A) 1 (B) 0 (C) 2 (D) −1

ANSWER KEY

1.	C	2.	D	3.	D	4.	C	5.	D	6.	B	7.	C	8.	B	9.	A	10.	D
11.	C	12.	B	13.	D	14.	D	15.	B	16.	D	17.	D	18.	B	19.	B	20.	B

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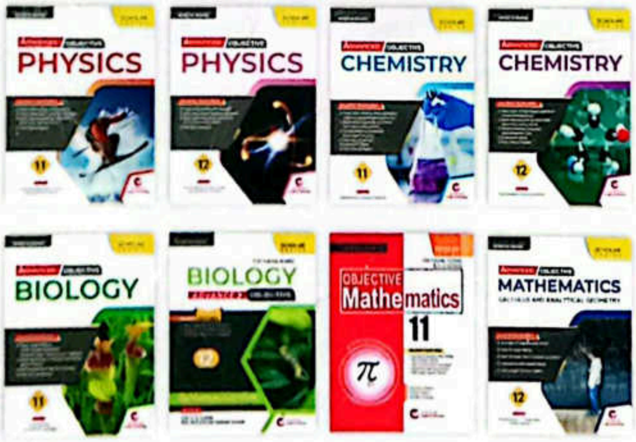

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