Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{-1}{\sqrt{x}+\sqrt{x+\delta x}}$$

Taking $\lim\limits_{\delta x \to 0}$ on both sides:

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{-1}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x}+\sqrt{x+0}} = \frac{-1}{\sqrt{x}+\sqrt{x}} = \frac{-1}{2\sqrt{x}}$$

(iii) $\frac{1}{\sqrt{x}}$

Solution:

Let $y = \frac{1}{\sqrt{x}}$ ...(1)

$y + \delta y = \frac{1}{\sqrt{x+\delta x}}$ ...(2)

Eq. (2) – Eq. (1):

$$y + \delta y - y = \frac{1}{\sqrt{x+\delta x}} - \frac{1}{\sqrt{x}}$$

$$= \frac{\sqrt{x}-\sqrt{x+\delta x}}{\sqrt{x}\sqrt{x+\delta x}}$$

By rationalizing the denominator

$$= \frac{\sqrt{x}-\sqrt{x+\delta x}}{\sqrt{x}\sqrt{x+\delta x}} \times \frac{\sqrt{x}+\sqrt{x+\delta x}}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$= \frac{(\sqrt{x})^2-(\sqrt{x+\delta x})^2}{\sqrt{x}\sqrt{x+\delta x}[\sqrt{x}+\sqrt{x+\delta x}]}$$

$$\delta y = \frac{x-x-\delta x}{\sqrt{x}\sqrt{x+\delta x}[\sqrt{x}+\sqrt{x+\delta x}]}$$

$$= \frac{-\delta x}{\sqrt{x}\sqrt{x+\delta x}[\sqrt{x}+\sqrt{x+\delta x}]}$$

Dividing by δx both sides

$$\frac{\delta y}{\delta x} = \frac{-1}{\sqrt{x}\sqrt{x+\delta x}[\sqrt{x}+\sqrt{x+\delta x}]}$$

Taking $\lim\limits_{\delta x \to 0}$ on both sides

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{-1}{\sqrt{x}\sqrt{x+\delta x}[\sqrt{x}+\sqrt{x+\delta x}]}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x}\sqrt{x}[\sqrt{x}+\sqrt{x}]} = \frac{-1}{x[2\sqrt{x}]} = \frac{-1}{2x^{\frac{3}{2}}}$$

(iv) $x(x-3)$

Solution:

Let $y = x(x-3)$

$y = x^2 - 3x$ ...(1)

$y + \delta y = (x+\delta x)^2 - 3(x+\delta x)$ ...(2)

Eq. (2) – Eq. (1):

$$y + \delta y - y = ((x+\delta x)^2 - 3(x+\delta x)) - (x^2 - 3x)$$

$$\delta y = x^2 + (\delta x)^2 + 2x\,\delta x - 3x - 3\delta x - x^2 + 3x$$

$$\delta y = (\delta x)^2 + 2x\,\delta x - 3\delta x = \delta x(\delta x + 2x - 3)$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x} = \frac{\delta x(\delta x+2x-3)}{\delta x}$$

Taking $\lim\limits_{\delta x \to 0}$ on both sides:

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(\delta x + 2x - 3)$$

$$\frac{dy}{dx} = 0 + 2x - 3 = 2x - 3$$

2. Find $\frac{dy}{dx}$ from first principle and find gradient of the curve at the given point:

(i) $\sqrt{x+2}$ at $x = 6$

Solution:

Let $y = \sqrt{x+2}$ ... (1)

$y + \delta y = \sqrt{x+\delta x+2}$ ... (2)

Eq. (2) – Eq. (1):

$$y + \delta y - y = \sqrt{x+\delta x+2} - \sqrt{x+2}$$

$$\delta y = \sqrt{x+\delta x+2} - \sqrt{x+2} \times \frac{\sqrt{x+\delta x+2}+\sqrt{x+2}}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

By rationalizing the denominator

$$= \frac{(\sqrt{x+\delta x+2})^2 - (\sqrt{x+2})^2}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\delta y = \frac{x+\delta x+2-x-2}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$= \frac{\delta x}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

Taking $\lim\limits_{\delta x \to 0}$ on both sides:

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{1}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+2}+\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$$

At $x = 6$

$$\left.\frac{dy}{dx}\right|_{x=6} = \frac{1}{2\sqrt{6+2}}$$

$$= \frac{1}{2\sqrt{8}}$$

$$= \frac{1}{4\sqrt{2}}$$

Hence, the gradient of the curve $y = \sqrt{x+2}$ at $x = 6$ is $\frac{1}{4\sqrt{2}}$

(ii) $\frac{1}{\sqrt{x+a}}$ at $x = a$

Solution:

Let $y = \frac{1}{\sqrt{x+a}}$ ... (1)

$y + \delta y = \frac{1}{\sqrt{x+\delta x+a}}$ ...(2)

Eq. (2) – Eq. (1)

$$y + \delta y - y = \frac{1}{\sqrt{x+\delta x+a}} - \frac{1}{\sqrt{x+a}}$$

$$\delta y = \frac{\sqrt{x+a}-\sqrt{x+\delta x+a}}{\sqrt{x+\delta x+a}\sqrt{x+a}}$$

$$\delta y = \frac{\sqrt{x+a}-\sqrt{x+\delta x+a}}{\sqrt{x+\delta x+a}\sqrt{x+a}} \cdot \frac{\sqrt{x+a}+\sqrt{x+\delta x+a}}{\sqrt{x+a}\sqrt{x+\delta x+a}}$$

By rationalizing the denominator

$$\delta y = \frac{(\sqrt{x+a})^2 - (\sqrt{x+\delta x+a})^2}{\sqrt{x+\delta x+a}\sqrt{x+a}[\sqrt{x+a}+\sqrt{x+\delta x+a}]}$$

$$= \frac{x+a-x-\delta x-a}{\sqrt{x+\delta x+a}\sqrt{x+a}[\sqrt{x+a}+\sqrt{x+\delta x+a}]}$$

$$= \frac{-\delta x}{\sqrt{x+\delta x+a}\sqrt{x+a}[\sqrt{x+a}+\sqrt{x+\delta x+a}]}$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x} = \frac{-1}{\sqrt{x+\delta x+a}\sqrt{x+a}[\sqrt{x+a}+\sqrt{x+\delta x+a}]}$$

Taking $\lim\limits_{\delta x \to 0}$ on both sides:

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{-1}{\sqrt{x+\delta x+a}\sqrt{x+a}[\sqrt{x+a}+\sqrt{x+\delta x+a}]}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x+a}\sqrt{x+a}[\sqrt{x+a}+\sqrt{x+a}]}$$

$$\frac{dy}{dx} = \frac{-1}{(x+a)[2\sqrt{x+a}]} = \frac{-1}{2(x+a)^{\frac{3}{2}}}$$

At $x = a$

$$\left.\frac{dy}{dx}\right|_{x=a} = \frac{-1}{2(a+a)^{\frac{3}{2}}}$$

$$= \frac{-1}{2(2a)^{\frac{3}{2}}}$$

$$= \frac{-1}{2 \cdot 2^{\frac{3}{2}} \cdot a^{\frac{3}{2}}}$$

$$= \frac{-1}{2 \cdot 2\sqrt{2} \cdot a^{\frac{3}{2}}} \qquad \because 2^{\frac{3}{2}} = 2\sqrt{2}$$

$$= \frac{-1}{4\sqrt{2}a^{\frac{3}{2}}}$$

Hence, the gradient of the curve $y = \frac{1}{\sqrt{x+a}}$ at $x = a$ is $\frac{-1}{4\sqrt{2}a^{\frac{3}{2}}}$

3. (i) Find the derivative of $x^{\frac{2}{3}}$ at $x = 8$ from the first principle.

Solution:

Let $y = (x)^{\frac{2}{3}}$ ... (i)

$y + \delta y = (x+\delta x)^{\frac{2}{3}}$ ...(ii)

Eq. (ii) – Eq. (i)

$$\delta y = (x+\delta x)^{\frac{2}{3}} - (x)^{\frac{2}{3}}$$

$$= x^{\frac{2}{3}}\left(1+\frac{\delta x}{x}\right)^{\frac{2}{3}} - x^{\frac{2}{3}} = x^{\frac{2}{3}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{2}{3}} - 1\right]$$

Using binomial theorem:

$$\delta y = x^{\frac{2}{3}}\left[\left\{1 + \frac{2}{3}\left(\frac{\delta x}{x}\right) + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}\left(\frac{\delta x}{x}\right)^3 + \ldots\right\} - 1\right]$$

$$= x^{\frac{2}{3}}\left[\frac{2}{3}\left(\frac{\delta x}{x}\right) + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}\left(\frac{\delta x}{x}\right)^3 + \ldots\right]$$

Taking common $\frac{2}{3}\left(\frac{\delta x}{x}\right)$

$$\delta y = x^{\frac{2}{3}}\frac{2}{3}\left(\frac{\delta x}{x}\right)\left[1 + \frac{\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right) + \frac{\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}\left(\frac{\delta x}{x}\right)^2 + \ldots\right]$$

Dividing both sides by 'δx':

$$\frac{\delta y}{\delta x} = x^{\frac{2}{3}}\frac{2}{3}\left(\frac{1}{x}\right)\left[1 + \frac{\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right) + \frac{\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}\left(\frac{\delta x}{x}\right)^2 + \ldots\right]$$

Taking limit $\lim\limits_{\delta x\to 0}$ on both sides:

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{\frac{2}{3}}\frac{2}{3}\left(\frac{1}{x}\right)\left[1+\frac{\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)+\frac{\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}\left(\frac{\delta x}{x}\right)^2+\dots\right]$$

$$\frac{dy}{dx}=x^{\frac{2}{3}}\frac{2}{3}\left(\frac{1}{x}\right)[1+0+0+0+\dots]=\frac{2}{3}x^{\frac{2}{3}-1}=\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3x^{\frac{1}{3}}}$$

Now putting $x=8$

$$\left(\frac{dy}{dx}\right)_{x=8}=\frac{2}{3(8)^{\frac{1}{3}}}=\frac{2}{3(2^3)^{\frac{1}{3}}}=\frac{2}{3(2)}=\frac{1}{3}$$

---

3. (ii) Find the derivative of x^2+2x+3 by definition.

Solution:

Let $\quad y = x^2+2x+3 \quad$ ...(i)

$y+\delta y = (x+\delta x)^2+2(x+\delta x)+3 \quad$ ...(ii)

Eq (ii) – Eq (i)

$$\delta y=\left[(x+\delta x)^2+2(x+\delta x)+3\right]-(x^2+2x+3)$$

$$\delta y = x^2+(\delta x)^2+2x\delta x+2x+2\delta x+3-x^2-2x-3$$

$$\delta y = (\delta x)^2+\delta x(2x+2)$$

$$\delta y = \delta x(\delta x+2x+2)$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = \delta x+2x+2$$

Taking $\lim\limits_{\delta x\to 0}$ on both sides:

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(\delta x+2x+2)$$

$$\frac{dy}{dx} = 0+2x+2$$

$$\frac{dy}{dx} = 2x+2$$

---

4. Find from first principle, the derivatives of the following expressions w.r.t. their respective independent variables:

(i) $(3x-2)^{-2}$

Solution:

Let $\quad y=(3x-2)^{-2}=\dfrac{1}{(3x-2)^2} \quad$ ...(1)

$$y+\delta y = \frac{1}{(3(x+\delta x)-2)^2} \quad \text{...(2)}$$

Eq. (2) – Eq. (1)

$$y+\delta y-y = \frac{1}{(3x+3\delta x-2)^2}-\frac{1}{(3x-2)^2}$$

$$\delta y = \frac{(3x-2)^2-[(3x-2)+3\delta x]^2}{(3x-2+3\delta x)^2(3x-2)^2}$$

$$\delta y = \frac{(3x-2)^2-(3x-2)^2-(3\delta x)^2-6\delta x(3x-2)}{(3x-2+3\delta x)^2(3x-2)^2}$$

$$\delta y = \frac{-3\delta x\left[3\delta x+2(3x-2)\right]}{(3x-2+3\delta x)^2(3x-2)^2}$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x} = \frac{-3\left[3\delta x+2(3x-2)\right]}{(3x-2+3\delta x)^2(3x-2)^2}$$

Taking $\lim\limits_{\delta x\to 0}$ on both sides:

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{-3\left[3\delta x+2(3x-2)\right]}{(3x-2+3\delta x)^2(3x-2)^2}$$

$$\frac{dy}{dx} = \frac{-3\left[0+2(3x-2)\right]}{(3x-2+0)^2(3x-2)^2}$$

$$\frac{dy}{dx} = \frac{-6(3x-2)}{(3x-2)^2(3x-2)^2}=\frac{-6}{(3x-2)^3}$$

(ii) $(2t+3)^5$

Solution:

Let $\quad y = (2t+3)^5 \quad$ ...(1)

$y+\delta y = (2(t+\delta t)+3)^5 \quad$ ...(2)

Eq. (2) – Eq. (1):

$$y+\delta y-y = ((2t+3)+2\delta t)^5-(2t+3)^5$$

By using binomial theorem

$$\delta y = (2t+3)^5+\binom{5}{1}(2t+3)^4.(2\delta t)+\binom{5}{2}(2t+3)^3(2\delta t)^2+\dots+(2\delta t)^5-(2t+3)^5$$

$$\delta y = 5(2t+3)^4(2\delta t)+\binom{5}{2}(2t+3)^2(2\delta t)^2+\dots+(2\delta t)^5$$

$$\delta y = (2\delta t)\left\{(5)(2t+3)^4+\binom{5}{2}(2t+3)^3(2\delta t)+\dots+(2\delta t)^4\right\}$$

Dividing both sides by δt:

$$\frac{\delta y}{\delta t} = 2\left\{5(2t+3)^4+\binom{5}{2}(2t+3)^3(2\delta t)+\dots+(2\delta t)^4\right\}$$

Taking $\lim\limits_{\delta t\to 0}$ on both sides:

$$\lim_{x\to 0}\frac{\delta y}{\delta t}=2\lim_{\delta t\to 0}\left\{5(2t+3)^4+\binom{5}{2}(2t+3)^3(2\delta t)+\dots+(2\delta t)^4\right\}$$

$$\frac{dy}{dt}=2\{5(2t+3)^4+0+\dots+0\}=10(2t+3)^4$$

(iii) $(aw+b)^7$

Solution:

Let $\quad y = (aw+b)^7 \quad$ ...(1)

$y+\delta y = (a(w+\delta w)+b)^7 \quad$ ...(2)

Eq. (2) – Eq. (1)

$$y+\delta y-y = (aw+a\delta w+b)^7-(aw+b)^7$$

$$\delta y = ((aw+b)+a\delta w)^7-(aw+b)^7$$

By using binomial theorem

$$\delta y=(aw+b)^7+\binom{7}{1}(aw+b)^6(a\delta w)+\binom{7}{2}(aw+b)^5(a\delta w)^2+\dots+(a\delta w)^7-(aw+b)^7$$

$$\delta y=7(aw+b)^6(a\delta w)+\binom{7}{2}(aw+b)^5(a\delta w)^2+\dots+(a\delta w)^7$$

$$\delta y=(a\delta w)\left\{7(aw+b)^6+\binom{7}{2}(aw+b)^5(a\delta w)+\dots+(a\delta w)^6\right\}$$

Dividing both sides by δw:

$$\frac{\delta y}{\delta w}=a\left\{7(aw+b)^6+\binom{7}{2}(aw+b)^5(a\delta w)+\dots+(a\delta w)^6\right\}$$

Taking $\lim\limits_{\delta w\to 0}$ on both sides

$$\lim_{\delta w\to 0}\frac{\delta y}{\delta w}=a\lim_{\delta w\to 0}\left\{7(aw+b)^6+\binom{7}{2}(aw+b)^5(a\delta w)+\dots+(a\delta w)^6\right\}$$

$$\frac{dy}{dw}=a\left\{7(aw+b)^6+0+\dots+0\right\}=7a(aw+b)^6$$

---

5. Find the gradient and equation of the tangent line to $y=3x^2-4x+1$ at $x=2$.

Solution:

Function: $y = 3x^2-4x+1 \quad$ ...(1)

Differentiating (1) w.r.t. 'x'.

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(3x^2-4x+1\right)$$

$$\frac{dy}{dx} = 3\frac{d}{dx}(x^2)-4\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

$$\frac{dy}{dx} = 3(2x)-4(1)+0$$

$$\frac{dy}{dx} = 6x-4$$

Slope of tangent at $(x=2)=\left.\dfrac{dy}{dx}\right|_{x=2}$

$\Rightarrow \quad m = 6(2)-4 = 12-4 = 8$

For y-coordinate of the point of tangency, put $x=2$ in (1).

$$y = 3(2)^2-4(2)+1 = 12-8+1 = 5$$

Point of tangency = $P(2, 5)$

Equation of tangent line is

$$y-y_1 = m(x-x_1)$$

$$y-5 = 8(x-2)$$

$$y-5 = 8x-16$$

$$\boxed{y = 8x-11}$$

---

6. For the function $f(x)=2x^3+x$, calculate the equation of the tangent line at $x=-1$.

Solution:

Function: $f(x)=2x^3+x \quad$ ...(1)

Differentiating (1) w.r.t. 'x'.

$$\frac{d}{dx}f(x) = \frac{d}{dx}(2x^3+x)$$

$$f'(x) = 2(3x^2)+1$$

$$f'(x) = 6x^2+1$$

Slope of tangent at $(x=-1)=f'(-1)$

$$\Rightarrow m = 6(-1)^2+1 = 6+1 = 7$$

For y-coordinate of the point of tangency,
put $x=-1$ in (1)

$$y = f(-1) = 2(-1)^3-1 = -2-1 = -3$$

Point of tangency = $(-1, -3)$

Equation of tangent line is

$$y-y_1 = m(x-x_1)$$

$$y+3 = 7(x+1)$$

$$y = 7x+7-3$$

$$\boxed{y = 7x+4}$$

---

7. Find the coordinates of the point of tangency and the equation of the tangent line for $f(x)=x^3-2x+1$ at $x=1$.

Solution:

Function: $f(x) = x^3-2x+1 \cdots$ (1)

Differentiating (1) w.r.t. 'x'.

$$\frac{d}{dx}f(x) = \frac{d}{dx}(x^3-2x+1)$$

$$f'(x) = 3x^2-2(1)+0$$

$$f'(x) = 3x^2-2$$

Slope of tangent at $(x=1)=f'(1)$

$$\Rightarrow m = 3(1)^2-2 = 3-2 = 1$$

For y-coordinate of the point of tangency, put $x=1$ in (1)

$$y = f(1) = (1)^3-2(1)+1 = 1-2+1 = 0$$

Point of tangency = (1, 0)

Equation of tangent line is

$$y - y_1 = m(x - x_1)$$
$$y - 0 = 1(x - 1)$$
$$\boxed{y = x - 1}$$

8. Find the gradient of the curve $f(x)=3x^2+2x$ at $x=1$.

Solution:

Curve: $f(x)=3x^2+2x$ ...(1)

Differentiating (1) w.r.t. 'x'

$$\frac{d}{dx}f(x) = \frac{d}{dx}(3x^2+2x)$$
$$f'(x) = 3(2x) + 2(1)$$
$$f'(x) = 6x + 2$$

Gradient of the curve at $x = 1$ is

$$m = f'(1)$$
$$m = 6(1) + 2$$
$$\boxed{m = 8}$$

9. Find the gradient and an equation of tangent line to the graph of $f(x)=\sqrt{x}$ at $x=9$

Solution:

Function: $f(x) = \sqrt{x}$ ...(1)

Differentiating (1) w.r.t. 'x'

$$\frac{d}{dx}f(x) = \frac{d}{dx}(x^{\frac{1}{2}})$$
$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1}.1$$
$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$f'(x) = \frac{1}{2\sqrt{x}}$$

Gradient of the tangent line at $x = 9$ is

$$m = f'(9)$$
$$m = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$$

For y-coordinate of the point of tangency, put $x = 9$ in (1).

$$y = f(9) = \sqrt{9} = 3$$

Point of tangency = (9, 3)

Equation of tangent line is

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{1}{6}(x-9)$$

Multiply both sides by '6'

$$6y - 18 = 1(x-9)$$
$$6y = x - 9 + 18$$
$$6y = x + 9$$

10. The position of a car after t hours is given by: $s(t)=2t^3-3t^2+t$ (in kilometres)

(i) Find the average velocity over the interval [1, 4]

(ii) Find the instantaneous velocity at $t=2$

Solution:

Position function: $s(t) = 2t^3 - 3t^2 + t$, where $1 \le t \le 4$

(i) The average velocity over the interval [1, 4] is

$$v_{avg} = \frac{s(4)-s(1)}{4-1}$$
$$= \frac{\left(2(4)^3-3(4)^2+4\right)-\left(2(1)^3-3(1)^2+1\right)}{3}$$
$$= \frac{1}{3}(128-48+4-2+3-1)$$
$$v_{avg} = \frac{1}{3}(135-51) = \frac{1}{3}(84) = 28$$

Thus, average velocity over the interval [1, 4] is 28 km/hour.

(ii) The instantaneous velocity at $t = 2$ is

$$f'(2) = \lim_{\delta t\to 0}\frac{s(2+\delta t)-s(2)}{\delta t}$$
$$s(2+\delta t) = 2(2+\delta t)^3 - 3(2+\delta t)^2 + (2+\delta t)$$
$$= 2\left(8+(\delta t)^3+6\delta t(2+\delta t)\right) - 3\left(4+(\delta t)^2+4\delta t\right)+2+\delta t$$
$$= 16+2(\delta t)^3+24\delta t+12(\delta t)^2-12-3(\delta t)^2-12\delta t+2+\delta t$$
$$= 2(\delta t)^3+9(\delta t)^2+13\delta t+6$$
$$s(2) = 2(2)^3-3(2)^2+2$$
$$= 16-12+2 = 6$$

Now,

$$f'(2) = \lim_{\delta t\to 0}\frac{2(\delta t)^3+9(\delta t)^2+13\delta t+6-6}{\delta t}$$
$$= \lim_{\delta t\to 0}\frac{\delta t\left(2(\delta t)^2+9\delta t+13\right)}{\delta t}$$
$$= 2(0)^2 + 9(0) + 13 = 13$$

Thus, instantaneous velocity at $t = 2$ is 13 km/hour

11. A stone is thrown upwards and its height after t seconds is given by: $s(t)=-16t^2+32t+10$ (in feet). Find the instantaneous velocity at $t = 1$

Solution:

Height Function: $s(t) = -16t^2+32t+10$

The instantaneous velocity at $t = 1$ is

$$f'(1) = \lim_{\delta t\to 0}\frac{s(1+\delta t)-s(1)}{\delta t}$$
$$s(1+\delta t) = -16(1+\delta t)^2+32(1+\delta t)+10$$
$$= -16\left(1+(\delta t)^2+2\delta t\right)+32+32\delta t+10$$
$$= -16-16(\delta t)^2-32\delta t+32+32\delta t+10$$
$$= -16(\delta t)^2+26$$

$$s(1) = -16(1)^2+32(1)+10 = 26$$

Now,

$$f'(1) = \lim_{\delta t\to 0}\frac{s(1+\delta t)-s(1)}{\delta t}$$
$$= \lim_{\delta t\to 0}\frac{-16(\delta t)^2+26-26}{\delta t}$$
$$= \lim_{\delta t\to 0} -16\delta t = -16(0) = 0$$

Thus, the instantaneous velocity of stone at $t = 1$ is 0 .

12. The outdoor temperature (in °C) over time is modeled by: $T(t) = -t^2 + 12t + 10$, where t is the time in hours. Find the instantaneous rate of change at $t = 2$.

Solution:

Temperature Function: $T(t) = -t^2 + 12t + 10$.

The instantaneous rate of change at $t = 2$ is

$$T'(2) = \lim_{\delta t\to 0}\frac{T(2+\delta t)-T(2)}{\delta t}$$
$$T(2+\delta t) = -(2+\delta t)^2+12(2+\delta t)+10$$
$$= -4-(\delta t)^2-4\delta t+24+12\delta t+10$$
$$= -(\delta t)^2+8\delta t+30$$
$$T(2) = -(2)^2+12(2)+10$$
$$= -4+24+10 = 30$$

Now,

$$T'(2) = \lim_{\delta t\to 0}\frac{T(2+\delta t)-T(2)}{\delta t}$$
$$= \lim_{\delta t\to 0}\frac{-(\delta t)^2+8\delta t+30-30}{\delta t}$$
$$= \lim_{\delta t\to 0}\frac{\delta t(-\delta t+8)}{\delta t}$$
$$= \lim_{\delta t\to 0}(-\delta t+8) = 0+8 = 8$$

Thus, the instantaneous rate of change of the temperature at $t = 2$ is 8°C/hour.

Theorems on Differentiation:

We have, so far, proved the following two formulas:

1. $\frac{d}{dx}(c)=0$ i.e., the derivative of a constant function is zero.
2. $\frac{d}{dx}(x^n)=nx^{n-1}$, power formula (or rule) when n is any real number.

Now we will prove other important formulas (or rules) which are used to determine derivatives of different functions efficiently. Henceforth, in all subsequent discussion, f, g, h etc, all denote functions differentiable at x, unless stated otherwise.

Theorem 2: (The Power Rule)

Prove that $\frac{d}{dx}(x^n)=nx^{n-1}$, where n is any real number.

Case-I (when $n \in N$)

Let $y = x^n$ ...(i)

Then $y+\delta y = (x+\delta x)^n$ ...(ii)

Equation(ii) – Equation(i)

$$\delta y = (x+\delta x)^n - x^n$$

Using the binomial theorem, we have

$$\delta y = \left[x^n+\binom{n}{1}x^{n-1}\cdot\delta x+\binom{n}{2}x^{n-2}(\delta x)^2+\cdots+(\delta x)^n\right]-x^n$$
$$\delta y = nx^{n-1}\cdot\delta x+\frac{n(n-1)}{2!}x^{n-2}(\delta x)^2+\cdots+(\delta x)^n$$
$$\delta y = \delta x\left[nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}.\delta x+\ldots+(\delta x)^{n-1}\right]$$

Dividing both sides by δx, gives

$$\frac{\delta y}{\delta x} = nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}\cdot\delta x+\cdots+(\delta x)^{n-1}$$

Taking $\lim\limits_{\delta x\to 0}$ on both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left\{nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}\cdot\delta x+\cdots+(\delta x)^{n-1}\right\}$$

$$\frac{dy}{dx}=nx^{n-1}+0+0+\ldots+0=nx^{n-1}$$

As $y=x^n$, so $\frac{d}{dx}(x^n)=n\cdot x^{n-1}$

Case-II (When $n\in R-N$).

Let $y=x^n$ ...(i)

Then $y+\delta y=(x+\delta x)^n$ ...(ii)

Equation(ii) – Equation(i)

$$\delta y=(x+\delta x)^n-x^n$$

$$\delta y=x^n.\left(1+\frac{\delta x}{x}\right)^n-x^n$$

$$\delta y=x^n\left[\left(1+\frac{\delta x}{x}\right)^n-1\right]$$

Using the binomial theorem, we have

$$\delta y=x^n\left[1+n\left(\frac{\delta x}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots-1\right]$$

$$\delta y=x^n.\delta x\left[n.\frac{1}{x}+\frac{n(n-1)}{2!}.\frac{\delta x}{x^2}+\cdots\right]$$

Dividing both sides by δx, gives

$$\frac{\delta y}{\delta x}=x^n\left[\frac{n}{x}+\frac{n(n-1)}{2!}.\frac{\delta x}{x^2}+\cdots\right]$$

Taking $\lim\limits_{\delta x\to 0}$ on both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^n\left[\frac{n}{x}+\frac{n(n-1)}{2!}.\frac{\delta x}{x^2}+\cdots\right]$$

$$\frac{dy}{dx}=x^n\left[\frac{n}{x}+0+0+\cdots\right]=nx^{n-1}$$

Theorem 3: Derivative of $y=cf(x)$

Prove that $\frac{d}{dx}[c.f(x)]=c.f'(x)$

Proof:

Let $y=c.f(x)$ ...(i)

$y+\delta y=c.f(x+\delta x)$ ...(ii)

Equation (ii) – Equation (i)

$$y+\delta y-y=c.f(x+\delta x)-c.f(x)$$

$$\delta y=c.[f(x+\delta x)-f(x)]$$

Dividing both sides by 'δx':

$$\frac{\delta y}{\delta x}=c.\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

Taking $\lim\limits_{\delta x\to 0}$ on both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=c.\lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

$$\frac{dy}{dx}=c.f'(x)$$

Thus, $\frac{dy}{dx}=cf'(x)$, that is $[cf(x)]'=cf'(x)$ or $\frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)]$

Example 6: Calculate $\frac{d}{dx}\left(3x^{\frac{4}{3}}\right)$

Solution:

$$\frac{d}{dx}\left(3x^{\frac{4}{3}}\right)=3\frac{d}{dx}\left(x^{\frac{4}{3}}\right) \qquad \because \frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)]$$

$$=3\left(\frac{4}{3}x^{\frac{4}{3}-1}\right)=4x^{\frac{1}{3}} \qquad \text{(Using power rule)}$$

Theorem 4: Derivative of a sum of functions (Sum Rule)

If f and g are differentiable at x, then $f+g$ are also differentiable at x and $[f(x)+g(x)]'=f'(x)+g'(x)$, i.e.,

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$$

Proof:

Let $y=f(x)+g(x)$ ...(i)

$y+\delta y=f(x+\delta x)+g(x+\delta x)$ ...(ii)

Equation (ii) – Equation (i)

$$y+\delta y-y=f(x+\delta x)+g(x+\delta x)-f(x)-g(x)$$

$$\delta y=[f(x+\delta x)-f(x)]+[g(x+\delta x)-g(x)]$$

Dividing both sides by 'δx':

$$\frac{\delta x}{\delta y}=\frac{[f(x+\delta x)-f(x)]+[g(x+\delta x)-g(x)]}{\delta x}$$

Taking $\lim\limits_{\delta x\to 0}$ on both sides:

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]+\lim_{\delta x\to 0}\left[\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$\frac{dy}{dx}=f'(x)+g'(x)$$

or $\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$

Theorem 5: Derivative of a difference of functions (Difference Rule)

If f and g are differentiable at x, then $f-g$ are also differentiable at x and $[f(x)-g(x)]'=f'(x)-g'(x)$, i.e.,

$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]$$

Proof:

Let

$y=f(x)-g(x)$ ...(i)

$y+\delta y=f(x+\delta x)-g(x+\delta x)$ ...(ii)

Equation (ii) – Equation (i)

$$y+\delta y-y = f(x+\delta x)-g(x+\delta x)-f(x)+g(x)$$
$$\delta y = [f(x+\delta x)-f(x)]-[g(x+\delta x)-g(x)]$$

Dividing both sides by 'δx':

$$\frac{\delta x}{\delta y} = \frac{[f(x+\delta x)-f(x)]+[g(x+\delta x)-g(x)]}{\delta x}$$

Taking $\lim_{\delta x\to 0}$ on both sides:

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]-\lim_{\delta x\to 0}\left[\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$
$$\frac{dy}{dx} = f'(x)-g'(x)$$

or $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]$

> **Note:**
> Sum or difference formula can be extended to find derivative of more than two functions.

Example 7: Find the derivative of $y=\frac{3}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+2x+5$ w.r.t. x.

Solution:

$$y = \frac{3}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+2x+5$$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{3}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+2x+5\right]=\frac{d}{dx}\left(\frac{3}{4}x^4\right)+\frac{d}{dx}\left(\frac{2}{3}x^3\right)+\frac{d}{dx}\left(\frac{1}{2}x^2\right)+\frac{d}{dx}(2x)+\frac{d}{dx}(5)$$
$$= \frac{3}{4}\frac{d}{dx}(x^4)+\frac{2}{3}\frac{d}{dx}(x^3)+\frac{1}{2}\frac{d}{dx}(x^2)+2\frac{d}{dx}(x)+0$$
$$= \frac{3}{4}(4x^{4-1})+\frac{2}{3}(3x^{3-1})+\frac{1}{2}(2x^{2-1})+2(1\cdot x^{1-1}) \quad \text{(By power formula)}$$
$$= 3x^3+2x^2+x+2$$

Example 8: Find the derivative of $y=(x^2+5)(x^3+7)$ with respect to x.

Solution: $y = (x^2+5)(x^3+7)=x^5+5x^3+7x^2+35$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^5+5x^3+7x^2+35\right]$$
$$= \frac{d}{dx}(x^5)+5\frac{d}{dx}(x^3)+7\frac{d}{dx}(x^2)+\frac{d}{dx}(35)$$
$$= 5x^{5-1}+5\times 3x^{3-1}+7\times 2x^{2-1}+0$$
$$= 5x^4+15x^2+14x$$

Example 9: Find the derivative of $y=(2\sqrt{x}+2)(x-\sqrt{x})$

Solution: $y=(2\sqrt{x}+2)(x-\sqrt{x})$

$$= 2(\sqrt{x}+1)\cdot\sqrt{x}(\sqrt{x}-1)=2\sqrt{x}(\sqrt{x}+1)(\sqrt{x}-1)$$

$$= 2\sqrt{x}(x-1)=2\left(x^{\frac{3}{2}}-x^{\frac{1}{2}}\right) \qquad \because (a+b)(a-b)=a^2-b^2$$

Differentiating with respect to x we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[2\left(x^{\frac{3}{2}}-x^{\frac{1}{2}}\right)\right]$$
$$= 2\left[\frac{d}{dx}x^{\frac{3}{2}}-\frac{d}{dx}x^{\frac{1}{2}}\right]=2\left[\frac{3}{2}x^{\frac{3}{2}-1}-\frac{1}{2}x^{\frac{1}{2}-1}\right]$$
$$= 3x^{\frac{1}{2}}-x^{-\frac{1}{2}}=3\sqrt{x}-\frac{1}{\sqrt{x}}=\frac{3x-1}{\sqrt{x}}$$

Theorem 6: Derivative of a Product (The Product Rule)

If f and g are differentiable at x, then $f.g$ is also differentiable at x and $[f(x)g(x)]' = f'(x)g(x)+f(x)g'(x)$,

that is $\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}[f(x)]\right]g(x)+f(x)\left[\frac{d}{dx}[g(x)]\right]$.

Proof:

Let
$$\phi(x) = f(x)\,g(x) \qquad \text{...(i)}$$
$$\phi(x+\delta x) = f(x+\delta x)\,g(x+\delta x) \qquad \text{...(ii)}$$

Equation (ii) – Equation (i)

$$\phi(x+\delta x)-\phi(x) = f(x+\delta x)\,g(x+\delta x)-f(x)\,g(x)$$

Subtracting and adding $f(x)\,g(x+\delta x)$

$$\phi(x+\delta x)-\phi(x) = f(x+\delta x)g(x+\delta x)-f(x)g(x+\delta x)-f(x)g(x)+f(x)g(x+\delta x)$$
$$= g(x+\delta x)[f(x+\delta x)-f(x)]+f(x)[g(x+\delta x)-g(x)]$$

Dividing both sides by 'δx':

$$\frac{\phi(x+\delta x)-\phi(x)}{\delta x} = \left[g(x+\delta x).\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)+f(x)\left(\frac{g(x+\delta x)-g(x)}{\delta x}\right)\right]$$

Taking $\lim_{\delta x\to 0}$ on both sides

$$\lim_{\delta x\to 0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x} = \lim_{\delta x\to 0}\left[g(x+\delta x).\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)+f(x)\left(\frac{g(x+\delta x)-g(x)}{\delta x}\right)\right]$$
$$\phi'(x) = f'(x)g(x)+f(x)g'(x)$$
$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x)+f(x)g'(x)$$

or $\frac{d}{dx}[f(x)\cdot g(x)]=\frac{d}{dx}[f(x)]\cdot g(x)+f(x)\left[\frac{d}{dx}g(x)\right]$

Example 10: Find derivative of $y=(2\sqrt{x}+2)(x-\sqrt{x})$ with respect to x.

Solution:
$$y = (2\sqrt{x}+2)(x-\sqrt{x})$$
$$=2(\sqrt{x}+1)(x-\sqrt{x})$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2\frac{d}{dx}\left[(\sqrt{x}+1)(x-\sqrt{x})\right] \qquad \text{By using product rule}$$
$$= 2\left[\left(\frac{d}{dx}(\sqrt{x}+1)\right)(x-\sqrt{x})+(\sqrt{x}+1)\frac{d}{dx}(x-\sqrt{x})\right]$$

$$= 2\left[\left(\frac{1}{2}x^{\frac{1}{2}-1}+0\right)(x-\sqrt{x})+(\sqrt{x}+1)\times\left(1-\frac{1}{2}x^{\frac{1}{2}-1}\right)\right]$$

$$= 2\left[\frac{1}{2\sqrt{x}}(x-\sqrt{x})+(\sqrt{x}+1)\times\left(1-\frac{1}{2\sqrt{x}}\right)\right] = 2\left[\frac{x-\sqrt{x}}{2\sqrt{x}}+(\sqrt{x}+1)\left(\frac{2\sqrt{x}-1}{2\sqrt{x}}\right)\right]$$

$$\frac{dy}{dx} = \frac{2}{2\sqrt{x}}\left[x-\sqrt{x}+2x-\sqrt{x}+2\sqrt{x}-1)\right] = \frac{3x-1}{\sqrt{x}}$$

Theorem 7: Derivative of a Quotient (The Quotient Rule)

If f and g are differentiable at x and $g(x) \neq 0$, for any $x \in D(g)$ then $\frac{f}{g}$ is differentiable at x and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}, \text{ that is } \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}[f(x)]g(x)-f(x)\left[\frac{d}{dx}[g(x)]\right]\right]}{[g(x)]^2}$$

Proof:

Let $\quad \phi(x) = \dfrac{f(x)}{g(x)} \quad \ldots(i)$

$\phi(x+\delta x) = \dfrac{f(x+\delta x)}{g(x+\delta x)} \quad \ldots(ii)$

Equation (ii) – Equation (*i*)

$$\Rightarrow \quad \phi(x+\delta x)-\phi(x) = \frac{f(x+\delta x)}{g(x+\delta x)}-\frac{f(x)}{g(x)}$$

$$= \frac{g(x).f(x+\delta x)-f(x).g(x+\delta x)}{g(x+\delta x)g(x)}$$

$$= \frac{1}{g(x+\delta x)g(x)}[g(x).f(x+\delta x)-f(x).g(x+\delta x)]$$

Subtracting and adding $f(x).g(x)$

$$= \frac{1}{g(x+\delta x)g(x)}[g(x).f(x+\delta x)-f(x)g(x)-f(x)g(x+\delta x)+f(x)g(x)]$$

$$= \frac{1}{g(x+\delta x)g(x)}[g(x).(f(x+\delta x)-f(x))-f(x)(g(x+\delta x)-g(x))]$$

Dividing both sides by 'δx':

$$\frac{\phi(x+\delta x)-\phi(x)}{\delta x} = \frac{1}{g(x+\delta x)g(x)}\left[g(x).\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)-f(x)\left(\frac{g(x+\delta x)-g(x)}{\delta x}\right)\right]$$

Taking $\lim\limits_{\delta x\to 0}$ on both sides:

$$\lim_{\delta x\to 0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x} = \lim_{\delta x\to 0}\frac{1}{g(x+\delta x)g(x)}\left[g(x).\left(\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}\right)-f(x)\left(\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x}\right)\right]$$

$$\phi'(x) = \frac{1}{g(x).g(x)}[g(x).f'(x)-f(x).g'(x)]$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$$

Thus $\quad \left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

or $\quad \dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\left[\frac{d}{dx}(f(x))\right]g(x)-f(x)\left[\frac{d}{dx}(g(x))\right]}{[g(x)]^2}$

Example 11: Differentiate $\dfrac{2x^3-3x^2+5}{x^2+1}$ with respect to x.

Solution: $\quad y = \dfrac{2x^3-3x^2+5}{x^2+1}$

Differentiating with respect to x, we get

$$\frac{d}{dx}\left[\frac{2x^3-3x^2+5}{x^2+1}\right] = \frac{(x^2+1).\frac{d}{dx}(2x^3-3x^2+5)-(2x^3-3x^2+5).\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \quad \text{By using product rule}$$

$$= \frac{(x^2+1).(2(3x^2)-3(2x)+0)-(2x^3-3x^2+5).((2x)+0)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(6x^2-6x)-(2x^3-3x^2+5)(2x)}{(x^2+1)^2}$$

$$= \frac{6x^4-6x^3+6x^2-6x-(4x^4-6x^3+10x)}{(x^2+1)^2}$$

$$= \frac{6x^4-6x^3+6x^2-6x-4x^4+6x^3-10x}{(x^2+1)^2}$$

$$= \frac{2x^4+6x^2-16x}{(x^2+1)^2}$$

Exercise 13.2

1. Differentiate w.r.t 'x'.

(i) $x^4+2x^3+x^2$

Solution:

Let $\quad y = x^4+2x^3+x^2$

Differentiating w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^4+2x^3+x^2)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4)+2\frac{d}{dx}(x^3)+\frac{d}{dx}(x^2)$$

$$= 4x^{4-1}.1+2.3x^{3-1}.1+2x^{2-1}.1$$

$$\frac{dy}{dx} = 4x^3+6x^2+2x$$

(ii) $x^{-3}+2x-\frac{3}{2}+3$

Solution:

Let $\quad y = x^{-3}+2x^{-\frac{3}{2}}+3$

Differentiating w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^{-3}+2x^{-\frac{3}{2}}+3)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-3})+2\frac{d}{dx}(x^{-\frac{3}{2}})+\frac{d}{dx}(3)$$

$$= -3x^{-3-1}.1+2.\left(\frac{-3}{2}\right)x^{-\frac{3}{2}-1}.1+0$$

$$\frac{dy}{dx} = -3x^{-4}-3x^{-\frac{5}{2}}+0 = -3\left[\frac{1}{x^4}+\frac{1}{x^{\frac{5}{2}}}\right]$$

(iii) $\dfrac{2x-3}{2x+1}$

Solution:

Let $y=\dfrac{2x-3}{2x+1}$

Differentiating w.r.t. 'x':

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x-3}{2x+1}\right)$$

$$=\frac{(2x+1)\frac{d}{dx}(2x-3)-(2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$=\frac{(2x+1)(2.1-0)-(2x-3)(2.1+0)}{(2x+1)^2}$$

$$=\frac{(2x+1)(2)-(2x-3)(2)}{(2x+1)^2}$$

$$=\frac{4x+2-4x+6}{(2x+1)^2}=\frac{8}{(2x+1)^2}$$

(iv) $\dfrac{(1+\sqrt{x})\left(x-x^{\frac{3}{2}}\right)}{\sqrt{x}}$

Solution:

Let $y=\dfrac{(1+\sqrt{x})(x-x^{\frac{3}{2}})}{\sqrt{x}}$

$$y=\frac{x-x\sqrt{x}+x\sqrt{x}-x^2}{\sqrt{x}}$$

$$y=\frac{x}{\sqrt{x}}-\frac{x^2}{\sqrt{x}}$$

$$y=x^{\frac{1}{2}}-x^{\frac{3}{2}}$$

Differentiating w.r.t. 'x':

$$\frac{dy}{dx}=\frac{d}{dx}(x^{\frac{1}{2}}-x^{\frac{3}{2}})$$

$$=\frac{d}{dx}(x^{\frac{1}{2}})-\frac{d}{dx}(x^{\frac{3}{2}})$$

$$=\frac{1}{2}x^{\frac{1}{2}-1}.1-\frac{3}{2}x^{\frac{3}{2}-1}.1$$

$$=\frac{1}{2}x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}=\frac{1}{2}\left(\frac{1}{\sqrt{x}}-3\sqrt{x}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1-3x}{\sqrt{x}}\right)=\frac{1-3x}{2\sqrt{x}}$$

(v) $\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2$

Solution:

Let $y=\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2$

$$y=(\sqrt{x})^2+\left(\frac{1}{\sqrt{x}}\right)^2-2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)$$

$$y=x+\frac{1}{x}-2$$

$$y=x+x^{-1}-2$$

Differentiating w.r.t. 'x'

$$\frac{d}{dx}(y)=\frac{d}{dx}(x+x^{-1}-2)$$

$$\frac{dy}{dx}=\frac{d}{dx}(x)-\frac{d}{dx}(x^{-1})-\frac{d}{dx}(2)$$

$$\frac{dy}{dx}=1-(-1x^{-1-1}\cdot 1)-0$$

$$\frac{dy}{dx}=1-(-x^{-2})$$

$$\frac{dy}{dx}=1+x^{-2}=1+\frac{1}{x^2}=\frac{x^2+1}{x^2}$$

(vi) $(x-5)(3-x)$

Solution:

Let $y=(x-5)(3-x)$

$\Rightarrow$ $y=3x-x^2-15+5x$

$\Rightarrow$ $y=-x^2+8x-15$

Differentiating w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}(-x^2+8x-15)$$

$$\frac{dy}{dx}=-1\frac{d}{dx}(x^2)+8\frac{d}{dx}(x)-0$$

$$\frac{dy}{dx}=-2x+8$$

(vii) $\dfrac{(x^2+1)^2}{x^2-1}$

Solution:

Let $y=\dfrac{(x^2+1)^2}{x^2-1}$

Differentiating w.r.t 'x'.

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{(x^2+1)^2}{x^2-1}\right)$$

$$=\frac{(x^2-1)\frac{d}{dx}(x^2+1)^2-(x^2+1)^2\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$=\frac{(x^2-1).2(x^2+1)^{2-1}(2x+0)-(x^2+1)^2(2x-0)}{(x^2-1)^2}$$

$$=\frac{4x(x^2-1)(x^2+1)-2x(x^2+1)^2}{(x^2-1)^2}$$

$$=\frac{2x(x^2+1).[2(x^2-1)-(x^2+1)]}{(x^2-1)^2}$$

$$=\frac{2x(x^2+1)[2x^2-2-x^2-1]}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{2x(x^2+1)(x^2-3)}{(x^2-1)^2}$$

(viii) $\dfrac{x^2+1}{x^2-3}$

Solution:

Let $y=\dfrac{x^2+1}{x^2-3}$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

$$=\frac{(x^2-3)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-3)}{(x^2-3)^2}$$

$$=\frac{(x^2-3)(2x+0)-(x^2+1)(2x-0)}{(x^2-3)^2}$$

$$\frac{dy}{dx}=\frac{2x^3-6x-2x^3-2x}{(x^2-3)^2}=\frac{-8x}{(x^2-3)^2}$$

(ix) $\dfrac{2x-1}{\sqrt{x^2}+1}$

Solution:

Let $y=\dfrac{2x-1}{(x^2+1)^{\frac{1}{2}}}$

Differentiating w.r.t. 'x':

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x-1}{(x^2+1)^{\frac{1}{2}}}\right)$$

$$=\frac{(x^2+1)^{\frac{1}{2}}\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x^2+1)^{\frac{1}{2}}}{(\sqrt{x^2+1})^2}$$

$$=\frac{\sqrt{x^2+1}(2-0)-(2x-1).\frac{1}{2}(x^2+1)^{\frac{1}{2}-1}(2x+0)}{(x^2+1)}$$

$$=\frac{2\sqrt{x^2+1}-(2x-1).2x.\frac{1}{2}(x^2+1)^{-\frac{1}{2}}}{(x^2+1)}$$

$$=\frac{2\sqrt{x^2+1}-\frac{x(2x-1)}{\sqrt{x^2+1}}}{(x^2+1)}$$

$$=\frac{\frac{2(\sqrt{x^2+1})^2-(2x^2-x)}{\sqrt{x^2+1}}}{(x^2+1)}$$

$$\frac{dy}{dx}=\frac{2x^2+2-2x^2+x}{\sqrt{x^2+1}}\times\frac{1}{(x^2+1)^1}$$

$$=\frac{2+x}{(x^2+1)^{\frac{1}{2}+1}}=\frac{2+x}{(x^2+1)^{\frac{3}{2}}}$$

(x) $\sqrt{\dfrac{a-x}{a+x}}$

Solution:

Let $y=\left(\dfrac{a-x}{a+x}\right)^{\frac{1}{2}}$

Differentiating w.r.t. 'x':

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}$$

$$=\frac{1}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}-1}.\frac{d}{dx}\left(\frac{a-x}{a+x}\right)$$

$$=\frac{1}{2}.\left(\frac{a-x}{a+x}\right)^{-\frac{1}{2}}.\frac{(a+x)\frac{d}{dx}(a-x)-(a-x)\frac{d}{dx}(a+x)}{(a+x)^2}$$

$$=\frac{1}{2}.\frac{(a-x)^{-\frac{1}{2}}}{(a+x)^{-\frac{1}{2}}}.\frac{(a+x)(0-1)-(a-x)(0+1)}{(a+x)^2}$$

$$=\frac{-a-x-a+x}{2(a-x)^{\frac{1}{2}}.(a+x)^{2-\frac{1}{2}}}=\frac{-2a}{2(a-x)^{\frac{1}{2}}.(a+x)^{\frac{3}{2}}}$$

$$\frac{dy}{dx}=\frac{-a}{\sqrt{a-x}(a+x)^{\frac{3}{2}}}$$

(xi) $\dfrac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

Solution:

Let $y=\left(\dfrac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$

Differentiating w.r.t. 'x':

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-\frac{1}{2}}.\frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2+1)^{-\frac{1}{2}}}{2(x^2-1)^{-\frac{1}{2}}}.\frac{(x^2-1)(2x+0)-(x^2+1)(2x-0)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x)-(x^2+1)(2x)}{2(x^2+1)^{\frac{1}{2}}(x^2-1)^{2-\frac{1}{2}}}$$

$$= \frac{2x^3-2x-2x^3-2x}{2(x^2+1)^{\frac{1}{2}}(x^2-1)^{\frac{3}{2}}}$$

$$= \frac{-4x}{2\sqrt{x^2+1}(x^2-1)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{-2x}{\sqrt{x^2+1}(x^2-1)^{\frac{3}{2}}}$$

2. Find $\frac{dy}{dx}$ if $y=\frac{(\sqrt{x}+1)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{1}{2}}-1}, (x\neq 1)$

Solution:

Given that: $y=\frac{(\sqrt{x}+1)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{1}{2}}-1}, (x\neq 1)$

$$y = \frac{x^2-\sqrt{x}+x^{\frac{3}{2}}-1}{\sqrt{x}-1}$$

Differentiating w.r.t 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{x^2-\sqrt{x}+x^{\frac{3}{2}}-1}{\sqrt{x}-1}\right)$$

$$\frac{dy}{dx}=\frac{(\sqrt{x}-1)\frac{d}{dx}(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)-(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)\frac{d}{dx}(\sqrt{x}-1)}{(\sqrt{x}-1)^2}$$

$$= \frac{(\sqrt{x}-1)\left(2x-\frac{1}{2\sqrt{x}}+\frac{3}{2}x^{\frac{1}{2}}\right)-(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x}-1)^2}$$

$$= \frac{(\sqrt{x}-1)\left(\frac{4x\sqrt{x}-1+3x}{2\sqrt{x}}\right)-\frac{1}{2\sqrt{x}}(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)}{(\sqrt{x}-1)^2}$$

$$= \frac{\frac{1}{2\sqrt{x}}((\sqrt{x}-1)(4x^{\frac{3}{2}}-1+3x)-(x^2-\sqrt{x}+x^{\frac{3}{2}}-1))}{(\sqrt{x}-1)^2}$$

$$= \frac{4x^2\cancel{-\sqrt{x}}+3x^{\frac{3}{2}}-4x^{\frac{3}{2}}+1-3x-x^2\cancel{+\sqrt{x}}-x^{\frac{3}{2}}+1}{2\sqrt{x}(\sqrt{x}-1)^2}$$

$$\frac{dy}{dx}=\frac{3x^2-2x^{\frac{3}{2}}-3x+2}{2\sqrt{x}(\sqrt{x}-1)^2}$$

3. Differentiate $\frac{(\sqrt{x}+1)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$ with respect to x.

Solution:

Given that: $y=\frac{(\sqrt{x}+1)(x^{\frac{3}{2}}-1)}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$

$$y = \frac{x^2-\sqrt{x}+x^{\frac{3}{2}}-1}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$$

Differentiating w.r.t. 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{x^2-\sqrt{x}+x^{\frac{3}{2}}-1}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}\right)$$

$$\frac{dy}{dx}=\frac{(x^{\frac{3}{2}}-x^{\frac{1}{2}})\frac{d}{dx}(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)-(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)\frac{d}{dx}(x^{\frac{3}{2}}-x^{\frac{1}{2}})}{(x^{\frac{3}{2}}-x^{\frac{1}{2}})^2}$$

$$=\frac{(x^{\frac{3}{2}}-x^{\frac{1}{2}})\left(2x-\frac{1}{2\sqrt{x}}+\frac{3}{2}x^{\frac{1}{2}}\right)-(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)\left(\frac{3}{2}x^{\frac{1}{2}}-\frac{1}{2\sqrt{x}}\right)}{(x^{\frac{3}{2}}-x^{\frac{1}{2}})^2}$$

$$=\frac{(x^{\frac{3}{2}}-x^{\frac{1}{2}})\left(\frac{4x^{\frac{3}{2}}-1+3x}{2\sqrt{x}}\right)-(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)\left(\frac{3x-1}{2\sqrt{x}}\right)}{(x^{\frac{3}{2}}-x^{\frac{1}{2}})^2}$$

$$=\frac{\frac{1}{2\sqrt{x}}((x^{\frac{3}{2}}-x^{\frac{1}{2}})(4x^{\frac{3}{2}}-1+3x)-(x^2-\sqrt{x}+x^{\frac{3}{2}}-1)(3x-1))}{(x^{\frac{3}{2}}-x^{\frac{1}{2}})^2}$$

$$=\frac{4x^3\cancel{-x^{\frac{3}{2}}}\cancel{+3x^{\frac{5}{2}}}-4x^2\cancel{+\sqrt{x}}\cancel{-3x^{\frac{3}{2}}}-3x^3+x^2\cancel{+3x^{\frac{3}{2}}}\cancel{-\sqrt{x}}\cancel{-3x^{\frac{5}{2}}}\cancel{+x^{\frac{3}{2}}}+3x-1}{2\sqrt{x}.(x^{\frac{3}{2}}-x^{\frac{1}{2}})^2}$$

$$\frac{dy}{dx}=\frac{x^3-3x^2+3x-1}{2\sqrt{x}(x^{\frac{3}{2}}-x^{\frac{1}{2}})^2}$$

4. If $y=\sqrt{x}-\frac{1}{\sqrt{x}}$, show that $2x\frac{dy}{dx}+y=2\sqrt{x}$

Solution:

$$y=\sqrt{x}-\frac{1}{\sqrt{x}}$$

$\sqrt{x}.y = x-1$ Multiplying both sides by $\sqrt{x}$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(\sqrt{x}.y) = \frac{d}{dx}(x-1)$$

$$\sqrt{x}\frac{dy}{dx}+y.\frac{d}{dx}(x^{\frac{1}{2}}) = 1-0$$

$$\sqrt{x}\frac{dy}{dx}+y\left(\frac{1}{2}x^{\frac{1}{2}-1}.1\right) = 1$$

$$\sqrt{x}\frac{dy}{dx}+y\left(\frac{1}{2\sqrt{x}}\right) = 1$$

Multiplying both sides by '$2\sqrt{x}$'

$$2\sqrt{x}.\left[\sqrt{x}\frac{dy}{dx}+\frac{y}{2\sqrt{x}}\right] = 2\sqrt{x}[1]$$

$2x\frac{dy}{dx}+y = 2\sqrt{x}$ as required.

5. If $y=x^4+2x^2+2$, prove that $\frac{dy}{dx}=4x\sqrt{y-1}$.

Solution:

$y = x^4+2x^2+2$ ...(1)

Differentiating w.r.t. 'x':

$$\frac{dy}{dx} = 4x^3+4x+0$$

$$= 4x(x^2+1)$$

$$= 4x\sqrt{(x^2+1)^2} = 4x\sqrt{x^4+1+2x^2}$$

$= 4x\sqrt{x^4+1+2x^2+1-1}$ (Add and subtract '1')

$$= 4x\sqrt{x^4+2x^2+2-1}$$

$\frac{dy}{dx} = 4x\sqrt{y-1}$ (Proved) using eq. (1)

Application of Differentiation:

We will apply concept of differentiation to real-world problems such as (profits on diminishing returns, environmental factors, financial investments, population growth, spread of diseases, movement of particles, time-speed in transportation, structural stress, material required that is changes in construction).

Profits on Diminishing Returns:

Example 12: A company's profit function is given by $P(x)=100x-5x^2$, where x is the number of units produced. Determine the marginal profit when $x = 8$ units.

Solution:

Profit function: $P(x)=100x-5x^2$

The marginal profit is the derivative of the profit function with respect to x.

$$P'(x)=\frac{dP}{dx}=\frac{d}{dx}(100x-5x^2)$$

$$=10(1)-5(2x)=100-10x$$

Marginal profit when $x = 8$ is

$$P'(8)=100-10(8)=20$$

So, the marginal profit is 20 when 8 units are produced (in the given currency).

Movement of Particles

Example 13: A particle moves along a line according to the position function $s(t)=4t^3-3t^2+2t$, where $s(t)$ is the position in metres and t is the time in seconds. Find the velocity and acceleration at $t = 2$ seconds.

Solution:

Position function: $s(t)=4t^3-3t^2+2t$