

**Divergent Sequences:** A sequence is divergent if it is not convergent. Divergence can occur in the following ways:

- The sequence may increase or decrease without bound (e.g.,  $a_n = n^2$  diverges to infinity).
- The sequence may oscillate between different values and not settle near any one value (e.g.,  $a_n = (-1)^n$  oscillates between -1 and 1, so it does not converge).

**Methods for Evaluating the Limits at Infinity:**

In this case we first divide each term of both the numerator and the denominator by the highest power of  $x$  that appears in the denominator and then use the theorems on limit.

**Example 3:** Evaluate  $\lim_{x \rightarrow \infty} \frac{5x^4 - 10x^2 + 1}{-3x^3 + 10x^2 + 50}$

**Solution:**

Dividing numerator and denominator by  $x^3$ , we get

$$\lim_{x \rightarrow \infty} \frac{5x^4 - 10x^2 + 1}{-3x^3 + 10x^2 + 50} = \lim_{x \rightarrow \infty} \frac{5x - \frac{10}{x} + \frac{1}{x^3}}{-3 + \frac{10}{x} + \frac{50}{x^3}} = \frac{\infty - 0 + 0}{-3 + 0 + 0} = -\infty \quad \therefore \lim_{x \rightarrow \infty} \frac{a}{x^p} = 0$$

**Example 4:** Evaluate  $\lim_{x \rightarrow \infty} \frac{4x^4 - 5x^3}{-3x^5 + 2x^2 + 1}$

**Solution:**

Dividing numerator and denominator by  $x^5$ , we get

$$\lim_{x \rightarrow \infty} \frac{4x^4 - 5x^3}{-3x^5 + 2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{\frac{4}{x} - \frac{5}{x^2}}{-3 + \frac{2}{x^3} + \frac{1}{x^5}} = \frac{0 - 0}{-3 + 0 + 0} = 0 \quad \therefore \lim_{x \rightarrow \infty} \frac{a}{x^p} = 0$$

**Example 5:** Evaluate: (i)  $\lim_{x \rightarrow \infty} \frac{2-3x}{\sqrt{3+4x^2}}$  (ii)  $\lim_{x \rightarrow \infty} \frac{2-3x}{\sqrt{3+4x^2}}$

**Solution:**

(i)  $\lim_{x \rightarrow \infty} \frac{2-3x}{\sqrt{3+4x^2}}$  Here  $\sqrt{x^2} = |x| = -x$  as  $x < 0$

$\therefore$  Dividing numerator and denominator by  $-x$ , we get

$$\lim_{x \rightarrow \infty} \frac{2-3x}{\sqrt{3+4x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{-x} + 3}{\sqrt{\frac{3}{x^2} + 4}} = \frac{0+3}{0+4} = \frac{3}{2} \quad \therefore \lim_{x \rightarrow \infty} \frac{a}{x^p} = 0$$

(ii) Here  $\sqrt{x^2} = |x| = x$  as  $x > 0$

$\therefore$  Dividing numerator and denominator by  $x$ , we get

$$\lim_{x \rightarrow \infty} \frac{2-3x}{\sqrt{3+4x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x} - 3}{\sqrt{\frac{3}{x^2} + 4}} = \frac{0-3}{\sqrt{0+4}} = \frac{-3}{2} \quad \therefore \lim_{x \rightarrow \infty} \frac{a}{x^p} = 0$$

**Theorem:** Show that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ .

**Proof:**

By the binomial theorem, we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \dots$$

$$= 1 + 1 + \frac{1}{2!} \frac{n(n-1)}{n^2} + \frac{1}{3!} \frac{n(n-1)(n-2)}{n^3} + \dots$$

$$= 2 + \frac{1}{2!} \frac{n}{n} \frac{n-1}{n} + \frac{1}{3!} \frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} + \dots$$

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots$$

When  $n \rightarrow +\infty$ ,  $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots$  all tends to zero, therefore

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}(1-0) + \frac{1}{3!}(1-0)(1-0) + \dots$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$= 1 + 1 + 0.5 + 0.166667 + 0.0416667 + \dots = 2.718281 \dots$$

As approximate value of  $e$  is 2.718281.

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

**Deduction:**  $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$

We know that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$  ... (i)

Put  $n = \frac{1}{x}$  in (i) then  $x = \frac{1}{n}$

When  $n \rightarrow \infty$ ,  $x \rightarrow 0$  so,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$$

$$e = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \quad \therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Hence  $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$

**Theorem:** Show that  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$ .

**Proof:**

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$$

$$\text{L.H.S} = \lim_{x \rightarrow 0} \frac{a^x - 1}{x} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$\text{Put } a^x - 1 = y \quad \dots (i)$$

$$\text{then } a^x = 1 + y$$

$$x = \log_a(1+y) \quad (\text{In logarithmic form})$$

From (i) when  $x \rightarrow 0$ ,  $y \rightarrow 0$

$$\text{L.H.S} = \lim_{x \rightarrow 0} \frac{a^x - 1}{x}$$

**Note:**

We can also show that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{y}{\log_e(1+y)} = \lim_{y \rightarrow 0} \frac{1}{\frac{1}{y} \log_e(1+y)} \\
 &= \lim_{y \rightarrow 0} \frac{1}{\log_e(1+y)^{\frac{1}{y}}} \quad \because n \cdot \log(m) = \log(m)^n \\
 &= \frac{1}{\log_e e} \quad \because \lim_{y \rightarrow 0} (1+y)^{\frac{1}{y}} = e \\
 &= \log_e a \quad \because \frac{1}{\log_e e} = \log_e a \\
 &= \text{R. H. S (Proved)}
 \end{aligned}$$

Deduction:  $\lim_{x \rightarrow 0} \left( \frac{e^x - 1}{x} \right) = \log_e e = 1$

We know that  $\lim_{x \rightarrow 0} \left( \frac{a^x - 1}{x} \right) = \log_e a \quad \dots (ii)$

Put  $a = e$  in (ii), we have

$$\lim_{x \rightarrow 0} \left( \frac{e^x - 1}{x} \right) = \log_e e = 1 \text{ (Proved)}$$

**Important Results to Remember**

- (i)  $\lim_{x \rightarrow \infty} e^x = \infty$   
 (ii)  $\lim_{x \rightarrow \infty} e^x = \lim_{x \rightarrow \infty} \left( \frac{1}{e^x} \right) = 0$

**Example 6:** Express each limit in terms of  $e$ : (i)  $\lim_{n \rightarrow \infty} \left( 1 + \frac{3}{n} \right)^{2n}$  (ii)  $\lim_{n \rightarrow \infty} (1 + 2n)^{\frac{1}{n}}$

**Solution:**

(i)

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left( 1 + \frac{3}{n} \right)^{2n} &= \lim_{n \rightarrow \infty} \left[ \left( 1 + \frac{3}{n} \right)^n \right]^2 \\
 &= \lim_{n \rightarrow \infty} \left[ \left( 1 + \frac{3}{n} \right)^{\frac{n}{3} \cdot 3} \right]^2 \\
 &= \lim_{n \rightarrow \infty} \left[ \left( 1 + \frac{3}{n} \right)^{\frac{n}{3}} \right]^6 \\
 &= e^6 \quad \because \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e
 \end{aligned}$$

(ii)

$$\begin{aligned}
 \lim_{n \rightarrow \infty} (1 + 2n)^{\frac{1}{n}} &= \lim_{n \rightarrow \infty} (1 + 2n)^{\frac{1}{2n} \cdot 2} \\
 &= \lim_{n \rightarrow \infty} \left[ (1 + 2n)^{\frac{1}{2n}} \right]^2 \\
 &\text{Using Formula: } \lim_{x \rightarrow \infty} (1 + x)^{\frac{1}{x}} = e \\
 \lim_{n \rightarrow \infty} (1 + 2n)^{\frac{1}{n}} &= e^2
 \end{aligned}$$

**The Sandwich Theorem:**

Let  $f, g$  and  $h$  be functions such that  $f(x) \leq g(x) \leq h(x)$  for all numbers  $x$  in some open interval containing " $c$ ", except possibly at  $c$  itself.

If  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} h(x) = L$ , then  $\lim_{x \rightarrow c} g(x) = L$

> Many limit problems arise that cannot be directly evaluated by algebraic techniques. They require geometric arguments, so we evaluate an important theorem.

**Theorem:** If  $\theta$  is measured in radian, then prove that  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ .

**Proof:**

Let  $\theta$  be the positive acute central angle of a sector  $OAB$  of a circle with radius  $r = 1$ .

Join  $A$  and  $B$  and extend  $OB$  to  $D$  such that  $\overline{OA} \perp \overline{AD}$ .

Also draw  $\overline{BC} \perp \overline{OC}$  on  $\overline{OA}$ .

In the right  $\triangle OCB$

$$\sin \theta = \frac{|BC|}{|OB|}$$

$$\sin \theta = \frac{|BC|}{1} \quad \because |OB| = r = 1$$

$$\sin \theta = |BC| \quad \dots (i)$$

In the right  $\triangle OAD$

$$\tan \theta = \frac{|AD|}{|OA|}$$

$$\tan \theta = \frac{|AD|}{1} \quad \because |OA| = r = 1$$

$$\tan \theta = |AD| \quad \dots (ii)$$

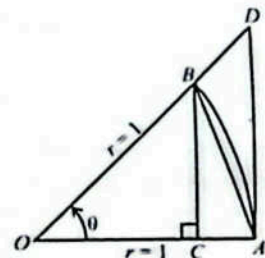


Figure: 1

$$\text{Area of } \triangle OAB = \frac{1}{2} (\text{Base})(\text{Height}) = \frac{1}{2} |\overline{OA}| |\overline{BC}| = \frac{1}{2} (1)(\sin \theta) = \frac{1}{2} \sin \theta \quad \text{Using eq.(i)}$$

$$\text{Area of sector } OAB = \frac{1}{2} r^2 \theta = \frac{1}{2} (1)^2 (\theta) = \frac{1}{2} \theta \quad \because r = 1$$

$$\text{Area of } \triangle OAD = \frac{1}{2} (\text{Base})(\text{Height}) = \frac{1}{2} |\overline{OA}| |\overline{AD}| = \frac{1}{2} (1)(\tan \theta) = \frac{1}{2} \tan \theta \quad \text{Using eq.(ii)}$$

From the figure(1) we see that

Area of  $\triangle OAB$  < Area of sector  $OAB$  < Area of  $\triangle OAD$

$$\begin{aligned}
 \Rightarrow \frac{1}{2} \sin \theta &< \frac{\theta}{2} < \frac{1}{2} \tan \theta \\
 \sin \theta &< \theta < \tan \theta \quad (\text{Multiply by 2})
 \end{aligned}$$

As  $\sin \theta$  is positive, so on division by  $\sin \theta$ , we get

$$1 < \frac{\theta}{\sin \theta} < \frac{\tan \theta}{\sin \theta}$$

$$1 < \frac{\theta}{\sin \theta} < \frac{\sin \theta / \cos \theta}{\sin \theta}$$

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta} \quad \left( 0 < \theta < \frac{\pi}{2} \right)$$

Taking reciprocal of each term

$$1 > \frac{\sin \theta}{\theta} > \cos \theta$$

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

Taking Lim of each term

$$\lim_{\theta \rightarrow 0} \cos \theta < \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} < \lim_{\theta \rightarrow 0} 1$$

$$1 < \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} < 1$$

By the sandwich theorem, we have

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ (Proved)}$$

**Example 7:** Evaluate  $\lim_{\theta \rightarrow 0} \frac{\sin 7\theta}{\theta}$

**Solution:**

$$\lim_{\theta \rightarrow 0} \frac{\sin 7\theta}{\theta} \quad \left(\frac{0}{0}\right) \text{ form}$$

Multiply and divide by '7'

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\sin 7\theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin 7\theta}{\theta} \times \frac{7}{7} \\ &= 7 \cdot \lim_{\theta \rightarrow 0} \frac{\sin 7\theta}{7\theta} \\ &= 7 \cdot (1) \quad \because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \\ &= 7 \end{aligned}$$

**Example 8:** Evaluate  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta}$

**Solution:**  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta}$   $\left(\frac{0}{0}\right)$  form

Multiply and divide by  $(1 + \cos \theta)$

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} = \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \sin \theta \cdot \frac{\sin \theta}{\theta} \cdot \frac{1}{1 + \cos \theta} \\ &= \lim_{\theta \rightarrow 0} \sin \theta \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{1 + \cos \theta} \\ &= (0)(1) \left(\frac{1}{1+1}\right) = 0 \end{aligned}$$

### Exercise 12.1

1. Find the limit of the following sequences if exists:

(i)  $a_n = \frac{2n+3}{n+1}$

**Solution:**

$$a_n = \frac{2n+3}{n+1}$$

Consider:  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{2n+3}{n+1}\right)$

**Note:**

The same result holds for

$$\frac{-\pi}{2} < \theta < 0$$

$$= \frac{2+0}{1+0}$$

$$= 2 \in R$$

$$\therefore \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$= \frac{\infty}{2}$$

$$= \infty \notin R$$

So, the given sequence is convergent.

(ii)  $b_n = \frac{2n+3}{n^2+1}$

**Solution:**

$$b_n = \frac{2n+3}{n^2+1}$$

Consider:

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \left(\frac{2n+3}{n^2+1}\right)$$

Dividing numerator and denominator by  $n^2$ , we get

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \left(\frac{\frac{2n}{n^2} + \frac{3}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}\right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{\frac{2}{n} + \frac{3}{n^2}}{1 + \frac{1}{n^2}}\right)$$

$$= \frac{0+0}{1+0}$$

$$= \frac{0}{1}$$

$$= 0 \in R$$

$$\therefore \lim_{x \rightarrow \infty} \frac{a}{x^p} = 0$$

So, the given sequence is convergent.

(iii)  $c_n = \frac{5n^2}{2n+3}$

**Solution:**

$$c_n = \frac{5n^2}{2n+3}$$

Consider:

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} \left(\frac{5n^2}{2n+3}\right)$$

Dividing numerator and denominator by  $n$ , we get

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} \left(\frac{\frac{5n^2}{n}}{\frac{2n}{n} + \frac{3}{n}}\right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{5n}{2 + \frac{3}{n}}\right)$$

$$= \frac{\infty}{2+0}$$

$$\therefore \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

So, the given sequence is divergent.

(iv)  $d_n = \frac{n^2 - 3n + 1}{2n^2 + n + 4}$

**Solution:**

$$d_n = \frac{n^2 - 3n + 1}{2n^2 + n + 4}$$

Consider:  $\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} \left(\frac{n^2 - 3n + 1}{2n^2 + n + 4}\right)$

Dividing numerator and denominator by  $n^2$ , we get

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} \left(\frac{\frac{n^2}{n^2} - \frac{3n}{n^2} + \frac{1}{n^2}}{\frac{2n^2}{n^2} + \frac{n}{n^2} + \frac{4}{n^2}}\right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1 - \frac{3}{n} + \frac{1}{n^2}}{2 + \frac{1}{n} + \frac{4}{n^2}}\right)$$

$$= \frac{1-0+0}{2+0+0}$$

$$= \frac{1}{2} \in R$$

$$\therefore \lim_{x \rightarrow \infty} \frac{a}{x^p} = 0$$

So, the given sequence is convergent.

2. Evaluate each limit by using theorems of limits:

(i)  $\lim_{x \rightarrow 3} (2x+4)$

**Solution:**

$$\lim_{x \rightarrow 3} (2x+4)$$

$$= \lim_{x \rightarrow 3} (2x) + \lim_{x \rightarrow 3} (4)$$

$$= 2 \lim_{x \rightarrow 3} (x) + 4 = 2(3) + 4$$

$$= 6 + 4 = 10$$

(ii)  $\lim_{x \rightarrow 1} (3x^2 - 2x + 4)$

**Solution:**

$$\lim_{x \rightarrow 1} (3x^2 - 2x + 4)$$

$$= \lim_{x \rightarrow 1} (3x^2) - \lim_{x \rightarrow 1} (2x) + \lim_{x \rightarrow 1} (4)$$

$$= 3 \lim_{x \rightarrow 1} (x)^2 - 2 \lim_{x \rightarrow 1} (x) + 4$$

$$= 3(1^2) - 2(1) + 4 = 3 - 2 + 4 = 5$$

(iii)  $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4}$

Solution:

$$\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4}$$

$$= [\lim_{x \rightarrow 3} (x^2 + x + 4)]^{\frac{1}{2}}$$

$$= [\lim_{x \rightarrow 3} (x^2) + \lim_{x \rightarrow 3} (x) + \lim_{x \rightarrow 3} (4)]^{\frac{1}{2}}$$

$$= (3^2 + 3 + 4)^{\frac{1}{2}} = (16)^{\frac{1}{2}} = 4$$

(iv)  $\lim_{x \rightarrow 2} \sqrt{x^2 + 4}$

Solution:

$$\lim_{x \rightarrow 2} \sqrt{x^2 + 4} = \lim_{x \rightarrow 2} [x^2 + 4]^{\frac{1}{2}}$$

$$= [\lim_{x \rightarrow 2} (x^2) + \lim_{x \rightarrow 2} (4)]^{\frac{1}{2}}$$

$$= [(2)^2 + 4]^{\frac{1}{2}}$$

$$= \sqrt{4+4} = \sqrt{8}$$

(v)  $\lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5})$

Solution:

$$\lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5})$$

$$= \lim_{x \rightarrow 2} \sqrt{x^3 + 1} - \lim_{x \rightarrow 2} \sqrt{x^2 + 5}$$

$$= (\lim_{x \rightarrow 2} (x^3) + \lim_{x \rightarrow 2} 1)^{\frac{1}{2}} - (\lim_{x \rightarrow 2} (x^2) + \lim_{x \rightarrow 2} 5)^{\frac{1}{2}}$$

$$= (2^3 + 1)^{\frac{1}{2}} - (2^2 + 5)^{\frac{1}{2}}$$

$$= \sqrt{9} - \sqrt{9} = 0$$

(vi)  $\lim_{x \rightarrow -2} \frac{2x^3 + 5x}{3x - 2}$

Solution:

$$\lim_{x \rightarrow -2} \frac{2x^3 + 5x}{3x - 2}$$

$$= \frac{\lim_{x \rightarrow -2} (2x^3 + 5x)}{\lim_{x \rightarrow -2} (3x - 2)}$$

$$= \frac{2 \lim_{x \rightarrow -2} (x^3) + 5 \lim_{x \rightarrow -2} (x)}{3 \lim_{x \rightarrow -2} (x) - \lim_{x \rightarrow -2} (2)}$$

$$= \frac{2(-2)^3 + 5(-2)}{3(-2) - 2} = \frac{-16 - 10}{-6 - 2} = \frac{-26}{-8} = \frac{13}{4}$$

3. Evaluate each limit by using algebraic techniques:

(i)  $\lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1}$

Solution:

$$\lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow -1} \frac{x(x^2 - 1)}{x + 1}$$

Using  $a^2 - b^2 = (a+b)(a-b)$ 

$$= \lim_{x \rightarrow -1} \frac{x(x+1)(x-1)}{(x+1)}$$

$$= \lim_{x \rightarrow -1} x(x-1) = (-1)(-1-1)$$

$$= -1(-2) = 2$$

(ii)  $\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 2x - 3}$

Solution:

$$\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 2x - 3} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 3} \frac{(x^2 - 3x - 2x + 6)}{(x^2 - 3x + x - 3)}$$

$$= \lim_{x \rightarrow 3} \frac{x(x-3) - 2(x-3)}{(x-3)(x+1)}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x-2)}{(x+1)(x-3)}$$

$$= \lim_{x \rightarrow 3} \frac{x-2}{x+1} = \frac{3-2}{3+1} = \frac{1}{4}$$

(iii)  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 5x + 6}$

Solution:

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 5x + 6} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 2} \frac{x^3 - 2^3}{x^2 - 3x - 2x + 6}$$

Using  $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ 

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{x(x-3) - 2(x-3)}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-3)(x-2)}$$

$$= \lim_{x \rightarrow 2} \frac{x^2 + 2x + 4}{x-3}$$

$$= \frac{2^2 + 2(2) + 4}{2-3}$$

$$= \frac{4+4+4}{-1}$$

$$= -12$$

(iv)  $\lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x}$

Solution:

$$\lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 1} \frac{x^3 - 1^3 - 3x(x-1)}{x(x^2 - 1)}$$

Using  $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$ 

$$= \lim_{x \rightarrow 1} \frac{(x-1)^3}{x(x+1)(x-1)}$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)^2}{x(x+1)} = \frac{(1-1)^2}{1(1+1)}$$

$$= \frac{0}{2}$$

$$= 0$$

(v)  $\lim_{x \rightarrow 2} \frac{x^3 - 6x^2 + 12x - 8}{x^3 - 4x}$

Solution:

$$\lim_{x \rightarrow 2} \frac{x^3 - 6x^2 + 12x - 8}{x^3 - 4x} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 2} \frac{x^3 - 2^3 - 6x(x-2)}{x(x^2 - 4)}$$

Using formula:  $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$ 

$$= \lim_{x \rightarrow 2} \frac{(x-2)^3}{x(x^2 - 2^2)}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x-2)^2}{x(x-2)(x+2)}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)^2}{x(x+2)}$$

$$= \frac{(2-2)^2}{2(2+2)}$$

$$= \frac{(0)^2}{2(4)} = \frac{0}{8} = 0$$

(vi)  $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x^2 - 3x + 2}$

Solution:

$$\lim_{x \rightarrow 1} \frac{x^4 - 1}{x^2 - 3x + 2} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 1} \frac{(x^2)^2 - 1^2}{x^2 - 2x - x + 2}$$

$$= \lim_{x \rightarrow 1} \frac{(x^2 + 1)(x^2 - 1)}{(x(x-2) - 1(x-2))}$$

$$= \lim_{x \rightarrow 1} \frac{(x^2 + 1)(x+1)(x-1)}{(x-2)(x-1)}$$

$$= \lim_{x \rightarrow 1} \frac{(x^2 + 1)(x+1)}{x-2}$$

$$= \frac{(1^2 + 1)(1+1)}{1-2}$$

$$= \frac{(2)(2)}{-1} = -4$$

(vii)  $\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x+2} - \sqrt{6-x}}$

Solution:

$$\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x+2} - \sqrt{6-x}} \quad \left(\frac{0}{0}\right) \text{ form}$$

Rationalizing the denominator

$$= \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x+2} - \sqrt{6-x}} \times \frac{\sqrt{x+2} + \sqrt{6-x}}{\sqrt{x+2} + \sqrt{6-x}}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x+2} + \sqrt{6-x})}{(\sqrt{x+2})^2 - (\sqrt{6-x})^2}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x+2} + \sqrt{6-x})}{x+2-6+x}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x+2} + \sqrt{6-x})}{2x-4}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x+2} + \sqrt{6-x})}{2(x-2)}$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{x+2} + \sqrt{6-x}}{2}$$

$$= \frac{\sqrt{2+2} + \sqrt{6-2}}{2}$$

$$= \frac{\sqrt{4} + \sqrt{4}}{2} = \frac{2+2}{2} = \frac{4}{2} = 2$$

(viii)  $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

Solution:

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \left(\frac{0}{0}\right) \text{ form}$$

By rationalization

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{x \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{x \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

(ix)  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m}$

Solution:

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m} \left( \frac{0}{0} \right) \text{ form}$$

Dividing up and down by  $(x-a)$ 

$$= \lim_{x \rightarrow a} \frac{\frac{x^n - a^n}{x-a}}{\frac{x^m - a^m}{x-a}} = \frac{\lim_{x \rightarrow a} \left( \frac{x^n - a^n}{x-a} \right)}{\lim_{x \rightarrow a} \left( \frac{x^m - a^m}{x-a} \right)}$$

Using  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x-a} = na^{n-1}$  and  $\lim_{x \rightarrow a} \frac{x^m - a^m}{x-a} = ma^{m-1}$ 

$$= \frac{na^{n-1}}{ma^{m-1}} = \frac{n}{m} a^{n-1-m} = \frac{n}{m} a^{n-m}$$

4. Evaluate the following limits:

(i)  $\lim_{x \rightarrow 0} \frac{\sin 5x}{x}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{x} \left( \frac{0}{0} \right) \text{ form}$$

Multiply and divide by '5'

$$= \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot 5$$

$$= 5 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x}$$

$$= 5(1)$$

$$= 5$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

(ii)  $\lim_{x \rightarrow 0} \frac{\sin x^{\circ}}{x}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin x^{\circ}}{x} \left( \frac{0}{0} \right) \text{ form}$$

As  $1^{\circ} = \frac{\pi}{180} \text{ rad} \Rightarrow x^{\circ} = \frac{x\pi}{180} \text{ rad}$ , so

$$= \lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{180}}{x}$$

Multiplying and dividing by  $\frac{\pi}{180}$ 

$$= \lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{180}}{x \cdot \frac{\pi}{180}} \cdot \frac{\pi}{180} = \frac{\pi}{180} \cdot \lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{180}}{\frac{x\pi}{180}}$$

$$= \frac{\pi}{180} (1) = \frac{\pi}{180}$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

(iii)  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta}$

Solution:

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} \left( \frac{0}{0} \right) \text{ form}$$

$$= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta}$$

$$= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin \theta (1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin \theta (1 + \cos \theta)}$$

$$\therefore \sin^2 \theta = 1 - \cos^2 \theta$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{1 + \cos \theta} = \frac{\sin 0}{1 + \cos 0} = \frac{0}{1+1} = \frac{0}{2} = 0$$

(iv)  $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

Solution:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} \left( \frac{0}{0} \right) \text{ form}$$

Multiply and divide by  $\sqrt{2}$ 

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\sqrt{2}}{\sqrt{2}}(\sin x - \cos x)}{x - \frac{\pi}{4}}$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} \left( \sin x \cdot \frac{1}{\sqrt{2}} - \cos x \cdot \frac{1}{\sqrt{2}} \right)}{x - \frac{\pi}{4}}$$

$$= \sqrt{2} \cdot \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x \cdot \cos \frac{\pi}{4} - \cos x \cdot \sin \frac{\pi}{4}}{x - \frac{\pi}{4}} \quad \therefore \sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

## SCHOLAR MATHEMATICS - 11

Using formula:  $\sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta)$ 

$$= \sqrt{2} \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin \left( x - \frac{\pi}{4} \right)}{x - \frac{\pi}{4}}$$

Put  $x - \frac{\pi}{4} = \theta$

As  $x \rightarrow \frac{\pi}{4}$ , so  $\theta \rightarrow 0$

$$= \sqrt{2} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$$

$$= \sqrt{2}(1) = \sqrt{2}$$

(v)  $\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2} \left( \frac{0}{0} \right) \text{ form}$$

Using formula:  $\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$ 

$$= \lim_{x \rightarrow 0} \frac{-2 \sin \left( \frac{ax+bx}{2} \right) \sin \left( \frac{ax-bx}{2} \right)}{x \cdot x}$$

$$= -2 \lim_{x \rightarrow 0} \frac{\sin \left( \frac{a+b}{2} \right) x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin \left( \frac{a-b}{2} \right) x}{x}$$

$$= -2 \lim_{x \rightarrow 0} \frac{\sin \left( \frac{a+b}{2} \right) x}{\left( \frac{a+b}{2} \right) x} \cdot \left( \frac{a+b}{2} \right) \cdot \lim_{x \rightarrow 0} \frac{\sin \left( \frac{a-b}{2} \right) x}{\left( \frac{a-b}{2} \right) x} \cdot \left( \frac{a-b}{2} \right)$$

$$= -2 \left( \frac{a+b}{2} \right) \left( \frac{a-b}{2} \right) \cdot \lim_{x \rightarrow 0} \frac{\sin \left( \frac{a+b}{2} \right) x}{\left( \frac{a+b}{2} \right) x} \cdot \lim_{x \rightarrow 0} \frac{\sin \left( \frac{a-b}{2} \right) x}{\left( \frac{a-b}{2} \right) x}$$

$$= -2 \left( \frac{a^2 - b^2}{4} \right) (1)(1)$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$= \frac{b^2 - a^2}{2}$$

(vi)  $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$

Solution:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} \left( \frac{0}{0} \right) \text{ form}$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\sin x}{\cos x} - 1}{x - \frac{\pi}{4}}$$

$$\frac{\sin x - \cos x}{\cos x}$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\sqrt{2}}{\sqrt{2}}(\sin x - \cos x)}{\cos x \left( x - \frac{\pi}{4} \right)}$$

$$= \frac{\sqrt{2} \cdot \lim_{x \rightarrow \frac{\pi}{4}} \left( \sin x \cdot \frac{1}{\sqrt{2}} - \cos x \cdot \frac{1}{\sqrt{2}} \right)}{\cos x \left( x - \frac{\pi}{4} \right)}$$

$$= \sqrt{2} \cdot \lim_{x \rightarrow \frac{\pi}{4}} \frac{1}{\cos x} \cdot \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x \cdot \cos \frac{\pi}{4} - \cos x \cdot \sin \frac{\pi}{4}}{x - \frac{\pi}{4}}$$

Using formula:  $\sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta)$ 

$$= \sqrt{2} \cdot \frac{1}{\cos \frac{\pi}{4}} \cdot \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin \left( x - \frac{\pi}{4} \right)}{x - \frac{\pi}{4}}$$

Put  $x - \frac{\pi}{4} = \theta$

As  $x \rightarrow \frac{\pi}{4}$ , so  $\theta \rightarrow 0$

$$= \sqrt{2} \cdot \frac{1}{\frac{1}{\sqrt{2}}} \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$$

$$= (\sqrt{2})^2 \cdot (1)$$

$$= 2$$

(vii)  $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$

Solution:

$$\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} \left( \frac{0}{0} \right) \text{ form}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \quad \therefore \begin{cases} \cos 2x = 1 - 2 \sin^2 x \\ 1 - \cos 2x = 2 \sin^2 x \end{cases}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}$$

$$= 2 \left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2 = 2(1)^2 = 2$$

$$(viii) \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx}$$

Solution:

$$\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$\text{Using formula: } \cos P - \cos Q = -2 \sin \frac{P+Q}{2} \cdot \sin \frac{P-Q}{2}$$

$$= \lim_{x \rightarrow 0} \frac{-2 \sin \left(\frac{ax+bx}{2}\right) \cdot \sin \left(\frac{ax-bx}{2}\right)}{-2 \sin \left(\frac{cx+dx}{2}\right) \cdot \sin \left(\frac{cx-dx}{2}\right)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin \left(\frac{a+b}{2}\right)x \cdot \sin \left(\frac{a-b}{2}\right)x}{\sin \left(\frac{c+d}{2}\right)x \cdot \sin \left(\frac{c-d}{2}\right)x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin \left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x} \cdot \left(\frac{a+b}{2}\right)x \cdot \frac{\sin \left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x} \cdot \left(\frac{a-b}{2}\right)x}{\frac{\sin \left(\frac{c+d}{2}\right)x}{\left(\frac{c+d}{2}\right)x} \cdot \left(\frac{c+d}{2}\right)x \cdot \frac{\sin \left(\frac{c-d}{2}\right)x}{\left(\frac{c-d}{2}\right)x} \cdot \left(\frac{c-d}{2}\right)x}$$

$$= \lim_{x \rightarrow 0} \frac{\left(\frac{a+b}{2}\right) \left(\frac{a-b}{2}\right) \lim_{x \rightarrow 0} \frac{\sin \left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x} \cdot \lim_{x \rightarrow 0} \frac{\sin \left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x}}{\left(\frac{c+d}{2}\right) \left(\frac{c-d}{2}\right) \lim_{x \rightarrow 0} \frac{\sin \left(\frac{c+d}{2}\right)x}{\left(\frac{c+d}{2}\right)x} \cdot \lim_{x \rightarrow 0} \frac{\sin \left(\frac{c-d}{2}\right)x}{\left(\frac{c-d}{2}\right)x}}$$

$$= \frac{\frac{a^2-b^2}{4} (1)(1)}{\frac{c^2-d^2}{4} (1)(1)}$$

$$= \frac{a^2-b^2}{c^2-d^2}$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$(ix) \lim_{x \rightarrow 1} \frac{x^3-1}{x^2-1}$$

Solution:

$$\lim_{x \rightarrow 1} \frac{x^3-1}{x^2-1} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 1} \frac{x^3-1^3}{x^2-1^2}$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} \quad \therefore \begin{cases} a^3-b^3=(a-b)(a^2+ab+b^2) \\ a^2-b^2=(a-b)(a+b) \end{cases}$$

$$= \lim_{x \rightarrow 1} \frac{x^2+x+1}{x+1}$$

$$= \frac{1^2+1+1}{1+1} = \frac{3}{2}$$

$$(x) \lim_{x \rightarrow 3} \frac{x^2 - x \log x + 3 \log x - 9}{x-3}$$

Solution:

$$\lim_{x \rightarrow 3} \frac{x^2 - x \log x + 3 \log x - 9}{x-3} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 3} \frac{(x^2-3^2) - \log x(x-3)}{x-3}$$

$$= \lim_{x \rightarrow 3} \frac{(x+3)(x-3) - (x-3) \cdot \log x}{x-3}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+3 - \log x)}{x-3}$$

$$= \lim_{x \rightarrow 3} (x+3 - \log x) = 3+3 - \log 3 = 6 - \log 3$$

$$(xi) \lim_{x \rightarrow 0} \frac{x(2^x-1)}{1-\cos x}$$

Solution:

$$\lim_{x \rightarrow 0} \frac{x(2^x-1)}{1-\cos x} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 0} \frac{x \cdot x(2^x-1)}{x \cdot 1-\cos x}$$

$$= \lim_{x \rightarrow 0} x^2 \cdot \frac{2^x-1}{x} \cdot \frac{1}{2 \sin^2 \frac{x}{2}} \quad \begin{cases} \cos 2x = 1 - 2 \sin^2 x \\ 2 \sin^2 x = 1 - \cos 2x \\ 1 - \cos x = 2 \sin^2 \frac{x}{2} \end{cases}$$

$$= \lim_{x \rightarrow 0} \frac{x^2}{2} \cdot \log_e 2 \cdot \frac{1}{\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2} \quad \therefore \lim_{x \rightarrow 0} \frac{a^x-1}{x} = \log_e a$$

$$= \frac{1}{2} \cdot \log_e 2 \cdot \lim_{x \rightarrow 0} x^2 \cdot \frac{1}{\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2} \cdot \frac{x^2}{4}$$

$$= \frac{1}{2} \log_e 2 \cdot \lim_{x \rightarrow 0} x^2 \cdot \frac{4}{x^2} \cdot \frac{1}{(1)^2} \quad \therefore \lim_{x \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$= \frac{4}{2} \log_e 2 \cdot \lim_{x \rightarrow 0} (1)$$

$$= 2 \log_e 2$$

$$= 2 \log 2 \quad \therefore \log_e 2 = \ln 2 = \log 2$$

5. Express each limit in terms of e.

$$(i) \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{2n}$$

Solution:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{2n}$$

$$= \left[ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right]^2$$

$$= e^2$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$(ii) \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{\frac{n}{2}}$$

Solution:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{\frac{n}{2}}$$

$$= \left[ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right]^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}} = \sqrt{e}$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$(iii) \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n$$

Solution:

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n$$

$$= \left[ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{-n}\right)^{-n} \right]^{-1}$$

$$= e^{-1} = \frac{1}{e}$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$(iv) \lim_{n \rightarrow \infty} \left(1 + \frac{1}{3n}\right)^n$$

Solution:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{3n}\right)^n$$

$$= \left[ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{3n}\right)^{3n} \right]^{\frac{1}{3}}$$

$$= e^{\frac{1}{3}}$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$(v) \lim_{n \rightarrow \infty} \left(1 + \frac{4}{n}\right)^n$$

Solution:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{4}{n}\right)^n$$

$$= \left[ \lim_{n \rightarrow \infty} \left(1 + \frac{4}{n}\right)^{\frac{n}{4}} \right]^4$$

$$= e^4$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$(vi) \lim_{x \rightarrow 0} (1+3x)^{\frac{2}{x}}$$

Solution:

$$\lim_{x \rightarrow 0} (1+3x)^{\frac{2}{x}}$$

$$= \left[ \lim_{x \rightarrow 0} (1+3x)^x \right]^2$$

$$= \left[ \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x}} \right]^{2 \cdot 3}$$

$$= e^6$$

$$\therefore \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$$

$$(vii) \lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{x^2}}$$

Solution:

$$\lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{x^2}}$$

$$= \left[ \lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{2x^2}} \right]^2$$

$$= e^2$$

$$\therefore \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$$

$$(viii) \lim_{x \rightarrow 0} \frac{e^{ax} - e^{bx}}{abx}$$

Solution:

$$= \lim_{x \rightarrow 0} \frac{e^{ax} - e^{bx}}{abx} \quad \left(\frac{0}{0}\right) \text{ form}$$

Add and Subtract '1' in numerator.

$$= \lim_{x \rightarrow 0} \frac{e^{ax} - 1 - e^{bx} + 1}{abx}$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{e^{ax} - 1}{abx} - \frac{e^{bx} - 1}{abx} \right\}$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{1}{b} \frac{e^{ax} - 1}{ax} - \frac{1}{a} \frac{e^{bx} - 1}{bx} \right\}$$

$$= \frac{1}{b} \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{ax} - \frac{1}{a} \lim_{x \rightarrow 0} \frac{e^{bx} - 1}{bx}$$

Using formula:  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \ln e = 1$

$$= \frac{1}{b}(1) - \frac{1}{a}(1)$$

$$= \frac{a-b}{ab}$$

(ix)  $\lim_{x \rightarrow 0} \left( \frac{x}{1+x} \right)^x$

Solution:

$$\lim_{x \rightarrow \infty} \left( \frac{x}{1+x} \right)^x$$

$$= \lim_{x \rightarrow \infty} \left( \frac{1+x}{x} \right)^{-x}$$

$$= \lim_{x \rightarrow \infty} \left( 1 + \frac{1}{x} \right)^{-x}$$

$$= \left[ \lim_{x \rightarrow \infty} \left( 1 + \frac{1}{x} \right)^x \right]^{-1}$$

$$= e^{-1}$$

$$= \frac{1}{e}$$

(x)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{e^x + 1}, x < 0$

Solution:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{e^x + 1}; x < 0$$

As  $x < 0$ , so  $\lim_{x \rightarrow 0} \frac{1}{x} = -\infty$

$$= \frac{\lim_{x \rightarrow 0} (e^x - 1)}{\lim_{x \rightarrow 0} (e^x + 1)}$$

$$= \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = \frac{0-1}{0+1}$$

$$= \frac{-1}{1} = -1$$

$\therefore e^{-\infty} = 0$

(xi)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{\frac{1}{e^x + 1}}, x > 0$

Solution:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\frac{1}{e^x + 1}}, x > 0 \quad \left( \frac{\infty}{\infty} \right) \text{ form}$$

As  $x > 0$ , so  $\lim_{x \rightarrow 0} \frac{1}{x} = +\infty$

$$= \lim_{x \rightarrow 0} \frac{e^x \left[ 1 - \frac{1}{e^x} \right]}{\frac{1}{e^x \left[ 1 + \frac{1}{e^x} \right]}} = \lim_{x \rightarrow 0} \frac{1 - e^{-\frac{1}{x}}}{1 + e^{-\frac{1}{x}}} = \frac{1 - e^{-\frac{1}{0}}}{1 + e^{-\frac{1}{0}}}$$

$$= \frac{1 - e^{-\infty}}{1 + e^{-\infty}} = \frac{1 - 0}{1 + 0} = \frac{1}{1} = 1$$

$\therefore e^{-\infty} = 0$

(xii)  $\lim_{x \rightarrow 2} \frac{e^x - e^2}{x - 2}$

Solution:

$$\lim_{x \rightarrow 2} \frac{e^x - e^2}{x - 2} \quad \left( \frac{0}{0} \right) \text{ form}$$

Put  $x - 2 = y \Rightarrow x = y + 2$

As  $x \rightarrow 2$ , so  $y \rightarrow 0$

$$\lim_{x \rightarrow 2} \frac{e^x - e^2}{x - 2} = \lim_{y \rightarrow 0} \frac{e^{y+2} - e^2}{y}$$

$$= \lim_{y \rightarrow 0} \frac{e^y \cdot e^2 - e^2}{y}$$

$$= \lim_{y \rightarrow 0} \frac{e^2 (e^y - 1)}{y}$$

$$= e^2 \cdot \lim_{y \rightarrow 0} \frac{e^y - 1}{y}$$

$$= e^2 \cdot (1)$$

$$= e^2$$

$\therefore \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

### Continuity and Discontinuity of Functions:

#### One-Sided Limits:

In defining  $\lim_{x \rightarrow c} f(x)$ , we restricted  $x$  in an open interval containing  $c$  i.e., we studied the behaviour of  $f$  on both sides of  $c$ . However, in some cases it is necessary to investigate one sided limits that is, the left hand limit and the right hand limit.

(i) **The Left Hand Limit:**

$\lim_{x \rightarrow c^-} f(x) = L$  is read as the limit of  $f(x)$  is equal to  $L$  as  $x$  approaches  $c$  from the left i.e., for all  $x$  sufficiently close to  $c$ , but less than  $c$ , the value of  $f(x)$  can be made as close as we please to  $L$ .

(ii) **The Right Hand Limit:**

$\lim_{x \rightarrow c^+} f(x) = M$  is read as the limit of  $f(x)$  is equal to  $M$  as  $x$  approaches  $c$  from the right i.e., for all  $x$  sufficiently close to  $c$ , but greater than  $c$ , the value of  $f(x)$  can be made as close as we please to  $M$ .

**Note:**  
The rules for calculating the left hand and the right hand limits are the same as we studied to calculate limits in the preceding section.

#### Criterion for Existence of Limit of a Function:

$\lim_{x \rightarrow c} f(x) = L$  if and only if  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$

**Example 9:** Determine whether  $\lim_{x \rightarrow 2} f(x)$  and  $\lim_{x \rightarrow 4} f(x)$  exist, when  $f(x) = \begin{cases} 2x+1 & \text{if } 0 \leq x \leq 2 \\ 7-x & \text{if } 2 < x < 4 \\ x & \text{if } 4 \leq x \leq 6 \end{cases}$

Solution:

(i)

Left Hand Limit =  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x+1) = 2(2)+1 = 5$

Right Hand Limit =  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (7-x) = 7-2 = 5$

Since  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 5$

$\Rightarrow \lim_{x \rightarrow 2} f(x)$  exists and is equal to 5.

(ii)

Left Hand Limit =  $\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} (7-x) = 7-4 = 3$

Right Hand Limit =  $\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} (x) = 4$

Since  $\lim_{x \rightarrow 4^-} f(x) \neq \lim_{x \rightarrow 4^+} f(x)$

Therefore,  $\lim_{x \rightarrow 4} f(x)$  does not exist.

#### Continuity of a Function at a Point:

(a) **Continuous Function:**

A function  $f$  is said to be continuous at a number " $c$ " if and only if the following three conditions are satisfied.

(i)  $f(c)$  is defined

(ii)  $\lim_{x \rightarrow c} f(x)$  exists i.e.,  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$

(iii)  $\lim_{x \rightarrow c} f(x) = f(c)$

(b) **Discontinuous Function:**

If one or more of these three conditions fail to hold at " $c$ ", then the function  $f$  is said to be discontinuous at " $c$ ".

**Remember!**

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = -\infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = +\infty$$

**Example 10:** Consider the function  $f(x) = \frac{x^2 - 1}{x - 1}$ , discuss the continuity of  $f$  at  $x = 1$ .

**Solution:**

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Here  $f(1)$  is not defined.

$\Rightarrow f(x)$  is discontinuous at 1.

**Example 11:** For  $f(x) = 3x^2 - 5x + 4$ , discuss continuity of  $f$  at  $x = 1$ .

**Solution:**

$$f(x) = 3x^2 - 5x + 4$$

**Condition: (i)**

$$f(1) = 3(1)^2 - 5(1) + 4 = 3 - 5 + 4 = 2 \text{ (defined)}$$

**Condition: (ii)**

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (3x^2 - 5x + 4) = 3 - 5 + 4 = 2$$

**Condition: (iii)**

$$\lim_{x \rightarrow 1} f(x) = f(1) = 2$$

From conditions (i), (ii) and (iii), it is clear that  $f(x)$  is continuous at  $x = 1$ .

**Example 12:** Discuss the continuity of the functions  $f(x)$  and  $g(x)$  at  $x = 3$

$$(a) f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases} \quad (b) g(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \end{cases}$$

**Solution:**

$$(a) f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

**Condition: (i)**

$$f(3) = 6 \text{ (defined)}$$

**Condition: (ii)**

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \rightarrow 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x \rightarrow 3} (x+3) = 3+3 = 6$$

**Condition: (iii)**

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

From conditions (i), (ii) and (iii), it is clear that  $f(x)$  is continuous at  $x = 3$ .

$$(b) g(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \end{cases}$$

As  $g(x)$  is not defined at  $x = 3$

$\therefore g(x)$  is discontinuous at  $x = 3$ .

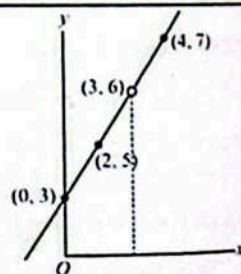


Figure 12.5

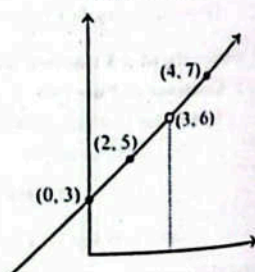


Figure 12.6

**Example 13:** Discuss continuity of  $f(x)$  at  $x = 3$ , when:

$$f(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 2x + 1 & \text{if } x \geq 3 \end{cases}$$

**Solution:**

$$f(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 2x + 1 & \text{if } x \geq 3 \end{cases}$$

**Condition: (i)**

$$f(3) = 2(3) - 1 = 7 \text{ (defined)}$$

**Condition: (ii)**

$$\text{L.H.L} = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x - 1) = 3 - 1 = 2$$

$$\text{R.H.L} = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (2x + 1) = 2(3) + 1 = 7$$

As  $\lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 3^+} f(x)$ , so  $\lim_{x \rightarrow 3} f(x)$  does not exist.

From condition (ii), it is clear that  $f(x)$  is discontinuous at  $x = 3$ .

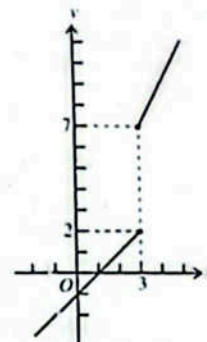


Figure 12.7

## Exercise 12.2

1. Determine the left hand limit and the right hand limit and then, find limit of the following functions when  $x \rightarrow c$ .

(i)  $f(x) = 2x^2 + x - 5, c = 1$

**Solution:**

$$f(x) = 2x^2 + x - 5, c = 1$$

**Left Hand Limit**

$$\begin{aligned} \text{L.H.L} &= \lim_{x \rightarrow 1^-} f(x) \\ &= \lim_{x \rightarrow 1^-} (2x^2 + x - 5) \\ &= 2(1)^2 + 1 - 5 \\ &= 2 + 1 - 5 = 3 - 5 = -2 \end{aligned}$$

**Right Hand Limit:**

$$\begin{aligned} \text{R.H.L} &= \lim_{x \rightarrow 1^+} f(x) \\ &= \lim_{x \rightarrow 1^+} (2x^2 + x - 5) \\ &= 2(1)^2 + 1 - 5 \\ &= 2 + 1 - 5 = 3 - 5 = -2 \end{aligned}$$

As Left hand limit = Right hand limit = -2

So,  $\lim_{x \rightarrow 1} f(x) = -2$

(ii)  $f(x) = \frac{x^2 - 9}{x - 3}, c = -3$

**Solution:**

$$f(x) = \frac{x^2 - 9}{x - 3}, c = -3$$

**Left Hand Limit:**

$$\begin{aligned} \text{L.H.L} &= \lim_{x \rightarrow -3^-} f(x) \\ &= \lim_{x \rightarrow -3^-} \frac{x^2 - 9}{x - 3} \\ &= \frac{(-3)^2 - 9}{-3 - 3} = \frac{9 - 9}{-6} = \frac{0}{-6} = 0 \end{aligned}$$

**Right Hand Limit:**

$$\begin{aligned} \text{R.H.L} &= \lim_{x \rightarrow -3^+} f(x) \\ &= \lim_{x \rightarrow -3^+} \frac{x^2 - 9}{x - 3} \\ &= \frac{(-3)^2 - 9}{-3 - 3} = \frac{9 - 9}{-6} = \frac{0}{-6} = 0 \end{aligned}$$

As Left hand limit = Right hand limit = 0

So,  $\lim_{x \rightarrow -3} f(x) = 0$

(iii)  $f(x) = |x - 5|, c = 5$

**Solution:**

$$f(x) = |x - 5|, c = 5$$

$$f(x) = \begin{cases} (x - 5) & \text{if } x - 5 \geq 0 \text{ or } x \geq 5 \\ -(x - 5) & \text{if } x - 5 < 0 \text{ or } x < 5 \end{cases}$$

**Left Hand Limit:**

$$\begin{aligned} \text{L.H.L} &= \lim_{x \rightarrow 5^-} f(x) \\ &= \lim_{x \rightarrow 5^-} [-(x - 5)] \\ &= (5 - 5) = 0 \end{aligned}$$

Right Hand Limit:

R.H.L. =  $\lim_{x \rightarrow 5^+} f(x)$

=  $\lim_{x \rightarrow 5^+} (x-5)$

=  $5-5=0$

As Left hand limit = Right hand limit = 0

So,  $\lim_{x \rightarrow 5} f(x) = 0$

2. Discuss the continuity of  $f(x)$  at  $x = c$ 

(i)  $f(x) = \begin{cases} 2x+5 & \text{if } x \leq 2 \\ 4x+1 & \text{if } x > 2 \end{cases}, c=2$

Solution:

$$f(x) = \begin{cases} 2x+5 & \text{if } x \leq 2 \\ 4x+1 & \text{if } x > 2 \end{cases}, \text{ at } c=2$$

Condition: (i)

$$f(2) = 2(2) + 5 = 9 \text{ (defined)}$$

Condition: (ii)

L.H.L. =  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x+5) = 2(2)+5=9$

R.H.L. =  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (4x+1) = 4(2)+1=9$

As  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$ , so  $\lim_{x \rightarrow 2} f(x) = 9$

Condition: (iii)

$$\lim_{x \rightarrow 2} f(x) = f(2)$$

From conditions (i), (ii) and (iii), it is clear that  $f(x)$  is continuous at  $x = 2$ 

(ii)  $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1, c=1 \\ 2x & \text{if } x > 1 \end{cases}$

Solution:

$$f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1, c=1 \\ 2x & \text{if } x > 1 \end{cases}$$

Condition: (i)

$$f(1) = 4 \text{ (defined)}$$

Condition: (ii)

L.H.L. =  $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x-1) = 3(1)-1=2$

R.H.L. =  $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x) = 2(1)=2$

As  $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$ , so  $\lim_{x \rightarrow 1} f(x) = 2$

Condition: (iii)

$$\lim_{x \rightarrow 1} f(x) \neq f(1)$$

From condition (iii), it is clear that  $f(x)$  is discontinuous at  $x = 1$ .

3. If  $f(x) = \begin{cases} 3x & \text{if } x \leq -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$

Discuss continuity at  $x = 2$  and  $x = -2$ .

Solution:

$$f(x) = \begin{cases} 3x & \text{if } x \leq -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases} \text{ when } x = 2$$

Condition: (i)

$$f(2) = 3 \text{ (defined)}$$

Condition: (ii)

L.H.L. =  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 - 1) = (2)^2 - 1 = 3$

R.H.L. =  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3) = 3$

As  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 3$ , so  $\lim_{x \rightarrow 2} f(x) = 3$

Condition: (iii)

$$\lim_{x \rightarrow 2} f(x) = f(2)$$

From conditions (i), (ii) and (iii), it is clear that  $f(x)$  is continuous at  $x = 2$ when  $x = -2$ 

Condition: (i)

$$f(-2) = 3(-2) = -6 \text{ (defined)}$$

Condition: (ii)

L.H.L. =  $\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (3x) = 3(-2) = -6$

R.H.L. =  $\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (x^2 - 1) = (-2)^2 - 1 = 4 - 1 = 3$

As  $\lim_{x \rightarrow -2^-} f(x) \neq \lim_{x \rightarrow -2^+} f(x)$ , so  $\lim_{x \rightarrow -2} f(x)$  does not exist.

From condition (ii), it is clear that  $f(x)$  is discontinuous at  $x = -2$ 

4. If  $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ c+2 & \text{if } x > -1 \end{cases}$

find "c" so that  $\lim_{x \rightarrow -1} f(x)$  exists.

Solution:

$$f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ c+2 & \text{if } x > -1 \end{cases}$$

L.H.L. =  $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (x+2) = -1+2=1$

R.H.L. =  $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (c+2) = c+2$

Since  $\lim_{x \rightarrow -1} f(x)$  exists, so

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$$

$$\Rightarrow 1 = c + 2$$

$$\Rightarrow c = -1$$

5. Find the values of  $m$  and  $n$ , so that given function  $f$  is continuous at  $x = 3$ 

$$f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x+9 & \text{if } x > 3 \end{cases}$$

Solution:

$$f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x+9 & \text{if } x > 3 \end{cases}$$

$$f(3) = n$$

L.H.L. =  $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (mx) = 3m$

R.H.L. =  $\lim_{x \rightarrow 3^+} f(x)$

$$= \lim_{x \rightarrow 3^+} (-2x+9) = -2(3)+9=3$$

Since  $f(x)$  is continuous at  $x = 3$ , therefore

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

$$\Rightarrow \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

$$3m = 3 = n$$

$$\Rightarrow 3m = 3 \text{ and } 3 = n$$

$$\boxed{m=1} \text{ and } \boxed{n=3}$$

(ii)  $f(x) = \begin{cases} mx & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$

Solution:

$$f(x) = \begin{cases} mx & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$$

$$f(3) = (3)^2 = 9 \text{ (defined)}$$

L.H.L. =  $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (mx) = 3$

R.H.L. =  $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x^2) = (3)^2 = 9$

Since  $f(x)$  is continuous at  $x = 3$ , therefore  $\lim_{x \rightarrow 3} f(x)$  exists

$$\Rightarrow \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x)$$

$$3m = 9$$

$$\boxed{m=3}$$

6.  $f(x) = \begin{cases} \sqrt{2x+5} - \sqrt{x+7} & , x \neq 2 \\ k & , x = 2 \end{cases}$

Find value of  $k$  so that  $f$  is continuous at  $x = 2$ .

Solution:

Since  $f(x)$  is continuous at  $x = 2$ , therefore

$$\lim_{x \rightarrow 2} f(x) = f(2)$$

$$\lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} = k \quad \left(\frac{0}{0}\right) \text{ form}$$

Rationalize the numerator

$$\lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \times \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} = k$$

Using  $(a+b)(a-b) = a^2 - b^2$ 

$$\lim_{x \rightarrow 2} \frac{(\sqrt{2x+5})^2 - (\sqrt{x+7})^2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = k$$

$$\lim_{x \rightarrow 2} \frac{2x+5 - x-7}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = k$$

$$\lim_{x \rightarrow 2} \frac{(x-2)}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = k$$

$$\lim_{x \rightarrow 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}} = k$$

$$\frac{1}{\sqrt{2(2)+5} + \sqrt{2+7}} = k$$

$$\frac{1}{\sqrt{9} + \sqrt{9}} = k \Rightarrow \frac{1}{3+3} = k \Rightarrow k = \frac{1}{6}$$

7. Given the function  $f(x) = \begin{cases} 2x+3, & x \leq 1 \\ -x+4, & x > 1 \end{cases}$

Discuss the limit and continuity at  $x = 1$ .

Solution:

$$f(x) = \begin{cases} 2x+3, & x \leq 1 \\ -x+4, & x > 1 \end{cases} \text{ at } x=1$$

Condition: (i)

$$f(1) = 2(1)+3 = 5 \text{ (defined)}$$

Condition: (ii)

L.H.L. =  $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x+3) = 2(1)+3=5$

R.H.L. =  $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (-x+4) = -1+4=3$

As  $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$ , so  $\lim_{x \rightarrow 1} f(x)$  does not exist.From condition (ii), it is clear that  $f(x)$  is discontinuous at  $x = 1$ .

### Application of Transcendental Functions to Limits and Continuity on Real World problems:

Limit and continuity of transcendental functions are fundamental concepts in calculus with numerous real-world applications. These concepts help us model, analyze and solve problems in various fields such as growth and decay, finance, economics, surveying and predicting long-term stock prices.

#### Example 14: Growth and Decay (Radioactive Decay)

The radioactive decay of a substance is given by the function  $A(t) = A_0 e^{-kt}$ , where  $A_0$  is the initial amount of the substance,  $k$  is the decay constant, and  $t$  is the time in years. Find the limit of the amount of substance as  $t \rightarrow \infty$ .

**Solution:** We need to compute the limit:

$$\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} A_0 e^{-kt}$$

As  $t \rightarrow \infty$ ,  $e^{-kt} \rightarrow 0$ , so

$$\lim_{t \rightarrow \infty} A_0 e^{-kt} = A_0 \times 0 = 0$$

Thus, the amount of radioactive substance approaches 0 as time increases indefinitely.

#### Example 15: Finance (Compound Interest)

The value of an investment grows according to the formula for continuous compounding  $A(t) = P_0 e^{rt}$ , where  $P_0$  is the initial principal,  $r$  is the annual interest rate, and  $t$  is the time in years. What happens to the value of the investment as  $t \rightarrow \infty$ ?

**Solution:** We need to compute the limit:

$$\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} P_0 e^{rt}$$

Since  $e^{rt} \rightarrow \infty$  as  $t \rightarrow \infty$  for any positive  $r$ , therefore

$$\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} P_0 e^{rt} = \infty$$

Thus, the value of the investment increases indefinitely as time approaches infinity.

#### Example 16: Economics (Supply and Demand)

In economics, the demand function  $D(p)$  decreases as the price  $p$  increases. Suppose the demand function is given by  $D(p) = \frac{100}{p+1}$ , where  $p$  is the price in dollars. Find the limit of the demand as the price becomes very large, i.e.,

$$\lim_{p \rightarrow \infty} D(p)$$

**Solution:** We need to compute the limit:

$$\lim_{p \rightarrow \infty} D(p) = \lim_{p \rightarrow \infty} \frac{100}{p+1}$$

As  $p \rightarrow \infty$ , the denominator becomes very large, so  $\lim_{p \rightarrow \infty} \frac{100}{p+1} = 0$

$$\Rightarrow \lim_{p \rightarrow \infty} D(p) = \lim_{p \rightarrow \infty} \frac{100}{p+1} = 0$$

Thus, as the price becomes very large, the demand approaches 0.

#### Example 17: Astronomy

The apparent brightness  $B(d)$  of a star decreases as the distance from Earth increases following the inverse square law  $B(d) = \frac{L}{d^2}$ , where  $L$  is the star's luminosity. Find the limit of the brightness as  $d \rightarrow \infty$ .

**Solution:** We need to compute the limit:

$$\lim_{d \rightarrow \infty} B(d) = \lim_{d \rightarrow \infty} \frac{L}{d^2}$$

As  $d \rightarrow \infty$  the denominator becomes very large, so  $\lim_{d \rightarrow \infty} \frac{L}{d^2} = 0$

$$\Rightarrow \lim_{d \rightarrow \infty} B(d) = \lim_{d \rightarrow \infty} \frac{L}{d^2} = 0$$

Thus, as the distance increases indefinitely, the apparent brightness of the star approaches 0.

### Exercise 12.3

1. A substance decays exponentially following the formula  $A(t) = A_0 e^{-0.1t}$ , where  $A_0$  is the initial amount. Find the limit of  $A(t)$  as  $t \rightarrow \infty$ .

**Solution:**

We need to compute the limit:

$$\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} A_0 e^{-0.1t}$$

$$= A_0 \cdot \lim_{t \rightarrow \infty} e^{-0.1t}$$

$$= A_0 \cdot e^{-\infty}$$

$$= A_0 \cdot (0)$$

$$= 0$$

$\therefore e^{-\infty} = 0$

2. A town's population is modeled by  $P(t) = \frac{100,000}{1+9e^{-0.5t}}$ . What is the long-term population as  $t \rightarrow \infty$ .

**Solution:**

$$\text{Population Model: } P(t) = \frac{100,000}{1+9e^{-0.5t}}$$

Long term population as  $t \rightarrow \infty$  is

$$\lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} \frac{100000}{1+9e^{-0.5t}}$$

$$= \frac{100000}{1+9e^{-\infty}}$$

$$= \frac{100000}{1+9(0)}$$

$$= \frac{100000}{1-9(0)}$$

$$= \frac{100000}{1-0} = 100000$$

$\therefore e^{-\infty} = 0$

3. A company's weekly sales (in thousands) follow the function  $S(t) = \frac{500t}{t+10}$ . What is the limit of  $S(t)$  as  $t \rightarrow \infty$  and what does it represent?

**Solution:**

$$\text{Weekly Sales Function: } S(t) = \frac{500t}{t+10}$$

$$\lim_{t \rightarrow \infty} S(t) = \lim_{t \rightarrow \infty} \frac{500t}{t+10} \quad \left( \frac{0}{0} \right) \text{ form}$$

Dividing numerator and denominator by 't', we get

$$\frac{500t}{t+10}$$

$$= \lim_{t \rightarrow \infty} \frac{500t}{t+10}$$

$$= \lim_{t \rightarrow \infty} \frac{500}{1+\frac{10}{t}}$$

$$= \frac{500}{1+0}$$

$$= \frac{500}{1+0}$$

$$= 500$$

$\therefore \lim_{t \rightarrow \infty} \frac{a}{x^p} = 0$

Thus, it represents the company's weekly sales increases as time approaches infinity.

4. Signal strength  $S(d)$  at a distance  $d$  from a tower is modeled as  $S(d) = \frac{1000}{d^2}$ .

(i) What is the signal strength at  $d = 10$ ?

(ii) What happens to signal strength as  $d \rightarrow \infty$ ?

**Solution:**

$$\text{Signal strength Formula: } S(d) = \frac{1000}{d^2}$$

(i) Signal strength at  $d = 10$  is:

$$S(10) = \frac{1000}{10^2}$$

$$= \frac{1000}{100} = 10$$

(ii) Signal strength as  $d \rightarrow \infty$  is:

$$\lim_{d \rightarrow \infty} S(d) = \lim_{d \rightarrow \infty} \frac{1000}{d^2}$$

$$= 0$$

$\therefore \lim_{x \rightarrow \infty} \frac{a}{x^p} = 0$

Thus, as the distance 'd' increases indefinitely, the signal strength from tower approaches 0.

5. A stock price grows according to the function  $P(t) = 50e^{0.05t}$ .

(i) Find the limit of  $P(t)$  as  $t \rightarrow \infty$ .

**Solution:**

$$\text{Stock price function: } P(t) = 50e^{0.05t}$$

$$\lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} (50e^{0.05t})$$

$$= 50 \lim_{t \rightarrow \infty} e^{0.05t}$$

$$= 50 \cdot e^{\infty}$$

$$= 50(\infty)$$

$$= \infty$$

(ii) Calculate the price after 10 years.

Solution:

$$\text{Price after 10 years} = P(10)$$

$$= 50e^{0.05(10)}$$

$$= 50e^{0.5}$$

$$= 50(1.6487)$$

$$= 82.44$$

6. The factory's cost function is given as:

$$C(x) = \begin{cases} 10x + 500 & \text{if } x \leq 100 \\ 12x + 300 & \text{if } x > 100 \end{cases}$$

Is the cost function continuous at  $x = 100$ ?

Solution:

$$\text{Factory cost function: } C(x) = \begin{cases} 10x + 500 & \text{if } x \leq 100 \\ 12x + 300 & \text{if } x > 100 \end{cases}$$

Condition (i):

$$C(100) = 10(100) + 500 = 1500 \text{ (defined)}$$

Condition (ii):

$$\text{L.H.L} = \lim_{x \rightarrow 100^-} C(x) = \lim_{x \rightarrow 100^-} (10x + 500)$$

$$= 10(100) + 500 = 1500$$

$$\text{R.H.L} = \lim_{x \rightarrow 100^+} C(x) = \lim_{x \rightarrow 100^+} (12x + 300)$$

$$= 12(100) + 300 = 1500$$

As  $\lim_{x \rightarrow 100^-} C(x) = \lim_{x \rightarrow 100^+} C(x) = 1500$ , so  $\lim_{x \rightarrow 100} C(x) = 1500$

Condition (iii):

$$\lim_{x \rightarrow 100} C(x) = C(100)$$

From conditions (i), (ii) and (iii), it is clear that if  $C(x)$  is continuous at  $x = 100$ .

7. Inflation is modeled by  $I(t) = I_0 e^{0.03t}$ , where  $I_0$  is the initial price index and  $t$  is the number of years

(i) Find the inflation rate after 5 years if  $I_0 = 100$ .

Solution:

$$\text{Inflation Model: } I(t) = I_0 e^{0.03t} \quad \dots(1)$$

$$\text{Inflation rate} = ? , t = 5, I_0 = 100$$

$$I(5) = 100 \cdot e^{0.03(5)} \quad \text{Using (1)}$$

$$= 100 \cdot e^{0.15}$$

$$= 100(1.1618)$$

$$= 116.18$$

As we know

$$\text{Inflation rate} = \frac{I(5) - I_0}{I_0} \times 100\%$$

$$= \frac{116.18 - 100}{100} \times 100\%$$

$$= 16.18\%$$

(ii) What is the expected price index after 10 years?

Solution:

$$\text{Price index after 10 years} = I(10)$$

$$= 100 \cdot e^{0.03(10)}$$

$$= 100 \cdot e^{0.3}$$

$$= 100(1.3499)$$

$$= 134.99$$

8. The cost to produce  $x$  units is:

$$C(x) = \begin{cases} 5x + 20 & \text{if } x \leq 10 \\ 6x + 10 & \text{if } x > 10 \end{cases}$$

Is the cost function continuous at  $x = 10$ ?

Solution:

$$\text{Cost function: } C(x) = \begin{cases} 5x + 20 & \text{if } x \leq 10 \\ 6x + 10 & \text{if } x > 10 \end{cases}$$

Condition (i):

$$C(10) = 5(10) + 20 = 50 + 20 = 70 \text{ (defined)}$$

Condition (ii):

$$\text{L.H.L} = \lim_{x \rightarrow 10^-} C(x) = \lim_{x \rightarrow 10^-} (5x + 20)$$

$$= 5(10) + 20 = 70$$

$$\text{R.H.L} = \lim_{x \rightarrow 10^+} C(x) = \lim_{x \rightarrow 10^+} (6x + 10)$$

$$= 6(10) + 10 = 70$$

As  $\lim_{x \rightarrow 10^-} C(x) = \lim_{x \rightarrow 10^+} C(x) = 70$ , so  $\lim_{x \rightarrow 10} C(x) = 70$

Condition (iii):

$$\lim_{x \rightarrow 10} C(x) = C(10)$$

From conditions (i), (ii) and (iii), it is clear that  $C(x)$  is continuous at  $x = 10$ .

## Formula Sheet

### Theorems on Limits of Functions:

|   |  |
|---|--|
| 1. $\lim_{x \rightarrow a} x^p = a^p$ , where $p > 0$ and $p \in \mathbb{R}$  | 2. $\lim_{x \rightarrow a} c = c$ , where $c$ is any real number                                       |
| 3. $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$   | 4. $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$  |
| 5. $\lim_{x \rightarrow a} [kf(x)] = k \lim_{x \rightarrow a} f(x)$ , $k$ is any real number  | 6. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$ |
| 7. $\lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  | 8. $\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n$                    |
| 1. $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$ , where $n$ is a non-zero integer and $a > 0$ .  |  |
| 2. Let $p$ be a positive rational number. If $x^p$ is defined, then<br>$\lim_{x \rightarrow +\infty} \frac{a}{x^p} = 0 \text{ and } \lim_{x \rightarrow -\infty} \frac{a}{x^p} = 0$ , where $a$ is any real number. |  |
| 3. $\lim_{n \rightarrow +\infty} \left( 1 + \frac{1}{n} \right)^n = e$  | 4. $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$  |
| 5. $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a = \ln a$  | 6. $\lim_{x \rightarrow 0} \left( \frac{e^x - 1}{x} \right) = \log_e e = 1$                            |
| 7. $\lim_{x \rightarrow +\infty} e^x = \infty$  | 8. $\lim_{x \rightarrow -\infty} e^x = \lim_{x \rightarrow +\infty} \left( \frac{1}{e^x} \right) = 0$  |
| 9. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$   |  |

## Multiple Choice Questions (MCQs)

### Exercise 12.1

1.  $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4} = \dots$   
 (A) 4 (B) -4 (C) 6 (D) 0
2.  $\lim_{x \rightarrow 0} \frac{x^2 - 1}{x^2 - x} = \dots$   
 (A) 2 (B) 1 (C) 4 (D) 5
3.  $\lim_{x \rightarrow 2} \frac{x^3 - 32}{x - 2}$  equals  $\dots$   
 (A) 40 (B) 60 (C) 80 (D) 120
4. If  $\lim_{x \rightarrow 0} \frac{f(x) - 1}{x} = \ln a$ ;  $a > 0$ , then  $\dots$   
 (A)  $a^x$  (B)  $a^x$  (C)  $e^{-x}$  (D)  $e^x$
5. If  $f(x) \leq g(x) \leq h(x)$  and  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = 2$ , then  $\lim_{x \rightarrow c} g(x) = \dots$   
 (A)  $< 2$  (B)  $> 2$  (C) 2 (D) not defined
6.  $\lim_{x \rightarrow 0} \frac{x^2}{\sin 7x \cdot \sin 5x} = \dots$   
 (A)  $\frac{7}{5}$  (B)  $\frac{5}{7}$  (C) 2 (D)  $\frac{1}{35}$

7.  $\lim_{n \rightarrow \infty} \left(\frac{5n+1}{5n}\right)^n = \dots$

- (A)
- $e^5$
- (B)
- $e^5$
- (C)
- $e^{-5}$
- (D)
- $e^{-5}$

8.  $\lim_{x \rightarrow \infty} \left(\frac{1}{2}\right)^x = \dots$

- (A) 0 (B) 1 (C)
- $\frac{1}{2}$
- (D)
- $\infty$

**Exercise 12.2**9. When  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$ , then -----

- (A)
- $f(x)$
- is continuous at
- $x = c$
- (B)
- $\lim_{x \rightarrow c} f(x)$
- exists
- 
- (C)
- $\lim_{x \rightarrow c} f(x)$
- does not exist (D) none of these

10. If  $f(x)$  is continuous at  $x = c$ , then:

- (A)
- $\lim_{x \rightarrow c} f(x) \neq 0$
- (B)
- $\lim_{x \rightarrow c} f(x) = 1$
- (C)
- $\lim_{x \rightarrow c} f(x) = f(c)$
- (D)
- $\lim_{x \rightarrow c} f(x) \neq f(c)$

11. The value of "k" for which the function  $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$  is continuous at  $x = 0$ :

- (A)
- $k = -1$
- (B)
- $k = 1$
- (C)
- $k = 2$
- (D)
- $k = 0$

12. The function  $f(x) = \frac{x}{x^2 - 4}$  is discontinuous at:

- (A) 0 (B)
- $\pm 2$
- (C) 1 (D)
- $\pm 1$

**Exercise 12.3**

13.  $\lim_{t \rightarrow \infty} A_0 e^{-kt} = \dots$

- (A) 0 (B) 1 (C) -1 (D)
- $\infty$

14.  $\lim_{t \rightarrow \infty} P_0 e^{rt} = \dots$

- (A) 0 (B) 1 (C) -1 (D)
- $\infty$

15.  $\lim_{p \rightarrow \infty} \frac{100}{p+1} = \dots$

- (A) 0 (B) 1 (C) -1 (D)
- $\infty$

**ANSWER KEY**

|     |   |     |   |     |   |     |   |     |   |    |   |    |   |    |   |    |   |     |   |
|-----|---|-----|---|-----|---|-----|---|-----|---|----|---|----|---|----|---|----|---|-----|---|
| 1.  | A | 2.  | B | 3.  | C | 4.  | B | 5.  | C | 6. | D | 7. | A | 8. | A | 9. | B | 10. | C |
| 11. | B | 12. | B | 13. | A | 14. | D | 15. | A |    |   |    |   |    |   |    |   |     |   |

**Differentiation****Introduction**

The ancient Greeks knew the concepts of area, volume, centroids etc. which are related to integral calculus. Later on, in the seventeenth century, Sir Isaac Newton, an English mathematician (1642 - 1727) and Gottfried Wilhelm G. Leibniz, a German mathematician, (1646-1716) considered the problem of instantaneous rates of change. They reached independently to the invention of differential calculus. After the development of calculus, mathematics became a powerful tool for dealing with rates of change and describing the physical universe.

**Tangent to a Curve at a Point**Curve:  $y = f(x)$ Let  $P(x, f(x))$  and  $Q(x + \delta x, f(x + \delta x))$  be two points on arc  $AB$  of graph of the curve  $f$ . (See Figure 13.1)Where  $\delta x$  is the increment in the value of  $x$  (read as delta  $x$ )The line  $PQ$  is secant of the curve and slope of secant line passing through  $P(x, f(x))$  and  $Q(x + \delta x, f(x + \delta x))$  is:

$$m_{sec} = \frac{RQ}{PR} = \frac{f(x + \delta x) - f(x)}{\delta x} \quad \dots(1)$$

where  $m_{sec}$  is slope of the secant line.Revolving the secant line  $PQ$  towards  $P$ , some of its successive positions  $PQ_1, PQ_2, PQ_3, \dots$  are shown in the Figure 13.2.Points  $Q_i (i = 1, 2, 3, \dots)$  are getting closer and closer to the point  $P$  and  $PR$  are approaching zero.In other words, as  $\delta x \rightarrow 0$ , the point  $Q$  approaches  $P$ , and the secant line becomes the tangent line. The revolving secant line becomes the tangent line  $PT$  at  $P$  while  $\delta x$  approaches zero.

That is,

$$m_{tan} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad \dots(2)$$

where  $m_{tan}$  denote the slope of tangent line.We see that  $m_{tan}$  is the limit of  $m_{sec}$  as  $Q$  approaches  $P$  along the curve  $y = f(x)$ .**Example 1:** Find the gradient and an equation of tangent line to the graph of  $f(x) = x^2 - 2$  at the point  $P(-1, -1)$ .**Solution:**Given that:  $f(x) = x^2 - 2$  at the point  $P(-1, -1)$ As we know:  $m_{tan} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ For gradient of the tangent line at point  $(-1, -1)$ , put  $x = -1$ , we get

$$m_{tan} = \lim_{\delta x \rightarrow 0} \frac{f(-1 + \delta x) - f(-1)}{\delta x}$$

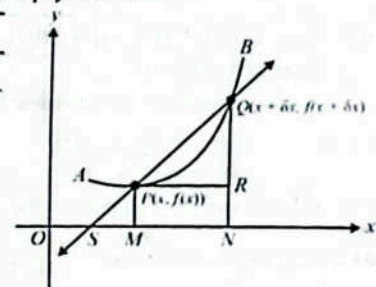


Figure 13.1

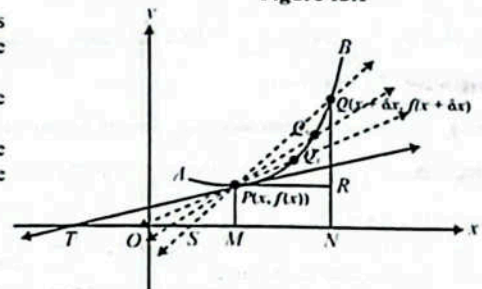
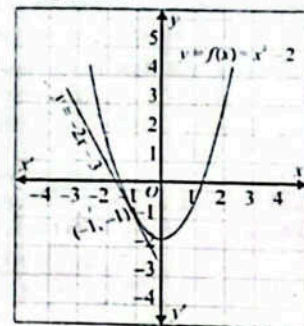


Figure 13.2



$$\begin{aligned}
 m_{\text{tan}} &= \lim_{\delta x \rightarrow 0} \frac{(-1 + \delta x)^2 - 2 - ((-1)^2 - 2)}{\delta x} \\
 &= \lim_{\delta x \rightarrow 0} \frac{1 - 2\delta x + \delta x^2 - 2 - (1 - 2)}{\delta x} \\
 &= \lim_{\delta x \rightarrow 0} \frac{1 - 2\delta x + \delta x^2 - 2 + 1}{\delta x} \\
 m_{\text{tan}} &= \lim_{\delta x \rightarrow 0} \frac{\delta x(-2 + \delta x)}{\delta x} = \lim_{\delta x \rightarrow 0} (-2 + \delta x) = -2
 \end{aligned}$$

Equation of tangent line with slope = -2 and point (-1, -1) is

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = -2(x - (-1)) \Rightarrow y + 1 = -2x - 2 \text{ or } y = -2x - 3, \text{ which is the required equation of}$$

tangent line.

The graph of  $f$  and tangent line are shown in the above figure.

### Derivative as the Limit of a Difference Quotient:

#### Average Rate of Change:

Let  $f$  be a real valued function continuous in the interval  $(x, x_1) \subseteq D_f$  (domain of  $f$ ), then difference quotient

$$\frac{f(x_1) - f(x)}{x_1 - x}$$

represents the average rate of change in the value of  $f$  with respect to the change  $x_1 - x$  in the value of independent variable  $x$ .

#### Instantaneous Rate of Change:

Let  $f$  be a real valued function continuous in the interval  $(x, x_1) \subseteq D_f$  (domain of  $f$ ), then  $\lim_{x_1 \rightarrow x} \frac{f(x_1) - f(x)}{x_1 - x}$

provided this limit exists, is called the instantaneous rate of change of  $f$  with respect to  $x$  and is written as  $f'(x)$ .

#### Derivative of a Function:

Let  $f$  be a real valued function continuous in the interval  $(x, x + \delta x)$ , then  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$  provided the limit exist, is defined to be the derivative of  $f$  with respect to  $x$  and is denoted by  $f'(x)$ .

#### Remember That:

- >  $f'(x)$  also called the differential coefficient of  $f$ .
- >  $f'(x)$  is read as "f-prime of x".
- > The process of finding  $f'(x)$  is called differentiation.
- > The domain of  $f'$  consists of all  $x$  for which the limit exists.
- > If  $x \in D_f$  and  $f'(x)$  exists, then  $f$  is said to be differentiable at  $x$ .

#### Derivative as the Rate of Change of Velocity:

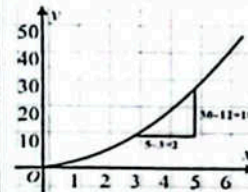
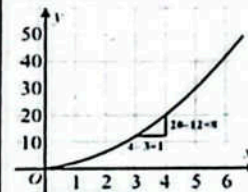
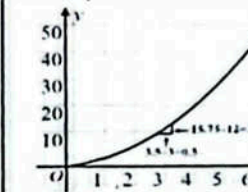
The rate of change is a fundamental concept in describing the motion of an object moving in a straight line. In physics, this is typically analyzed using position, velocity, and acceleration, which are all related through derivatives (rates of change).

The position versus time graph provides a simple interpretation of the average velocity over a given time interval. Suppose a particle moves in a straight line and its position at time  $t$  is given by the function  $s(t)$ . The average velocity over the interval from  $t$  to  $t_1$  denoted by  $v_{\text{avg}}$  is defined as:

$$v_{\text{avg}} = \frac{s(t_1) - s(t)}{t_1 - t} \quad \dots(i)$$

Equation (i) also represent the slope of secant line passing through the points  $(t, s(t))$  and  $(t_1, s(t_1))$ . If the interval  $t_1 - t$  is not small, this average velocity does not accurately represent the rate of change at time  $t$ .

To illustrate this, consider a particle whose position at time  $t$  (in seconds) is given by a function  $s(t) = t^2 + t$  in metres. The average rate of change over various time intervals starting at  $t = 3$  seconds is shown in the table below:

| Interval         | $t = 3$ secs to $t = 5$ secs  | $t = 3$ secs to $t = 4$ secs  | $t = 3$ secs to $t = 3.5$ secs  |
|------------------|---|---|---|
| Average velocity | $\frac{s(5) - s(3)}{5 - 3} = \frac{30 - 12}{2} = 9$                                 | $\frac{s(4) - s(3)}{4 - 3} = \frac{20 - 12}{1} = 8$                                 | $\frac{s(3.5) - s(3)}{3.5 - 3} = \frac{6.3 - 12}{0.5} = 7.5$                        |
|                  |  |  |  |

We observe that these values are not closely approximate the particle's velocity at exactly 3 seconds. To obtain a better approximation of velocity at  $x = 3$ , we use smaller intervals:

| Interval                         | Average velocity  |
|----------------------------------|---|
| $t = 3$ secs to $t = 3.1$ secs   | $\frac{\{(3.1)^2 + 3.1\} - 12}{3.1 - 3} = \frac{0.71}{0.1} = 7.1$               |
| $t = 3$ secs to $t = 3.01$ secs  | $\frac{\{(3.01)^2 + 3.01\} - 12}{3.01 - 3} = \frac{0.0701}{0.01} = 7.01$        |
| $t = 3$ secs to $t = 3.001$ secs | $\frac{\{(3.001)^2 + 3.001\} - 12}{3.001 - 3} = \frac{0.007001}{0.001} = 7.001$ |

We see as the length of the time interval decreases, the average velocity becomes instantaneous velocity at  $t = 3$ . Based on the trend, we estimate the instantaneous velocity to be approximately 7 m/sec.

Thus, over a sufficiently small interval, the velocity changes negligibly. If  $t_1$  is very close to  $t$ , the average velocity over  $t_1 - t$  approximates the instantaneous velocity at  $t$ . As  $t_1$  approaches  $t$ , the average velocity is called the instantaneous velocity.

This is similar to approximating the slope of a tangent line by calculating the slope of a secant line. Mathematically, the instantaneous velocity denoted by  $v_{\text{inst}}$  is given by the following limit:

$$v_{\text{inst}} = \lim_{t_1 \rightarrow t} \frac{s(t_1) - s(t)}{t_1 - t} \quad (\text{Provide the limit exist})$$

For convenient, if  $t_1 = t + \delta t$ , then as  $t_1 \rightarrow t \Rightarrow \delta t \rightarrow 0$ , thus above equation becomes:

$$v_{\text{inst}} = \lim_{\delta t \rightarrow 0} \frac{s(t + \delta t) - s(t)}{\delta t} \quad \dots(ii)$$

In other words, the instantaneous velocity is the derivative of the position function  $s(t)$  with respect to time.

**Example 2:** A particle moves along a line such that its position after  $t$  hours is given by:  $s(t) = 4t^2 + 2t + 1$  (in miles)

(a) Find the average velocity over the interval  $[2, 5]$  (b) Find the instantaneous velocity at  $t = 3$

**Solution:**

(a) Given position function  $s(t) = 4t^2 + 2t + 1$ , where  $2 \leq t \leq 5$

The average velocity over the interval  $2 \leq t \leq 5$  is:

$$\text{Average velocity} = v_{\text{avg}} = \frac{s(5) - s(2)}{5 - 2} = \frac{[4(5)^2 + 2(5) + 1] - [4(2)^2 + 2(2) + 1]}{3}$$

$$= \frac{111-21}{3} = \frac{90}{3} = 30 \text{ miles/hours}$$

(b) Instantaneous velocity can be found using the formula:

$$\begin{aligned} \text{Instantaneous velocity} = v_{\text{inst}} &= \lim_{\delta t \rightarrow 0} \frac{s(t+\delta t) - s(t)}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{[4(3+\delta t)^2 + 2(3+\delta t) + 1] - [4(3)^2 + 2(3) + 1]}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{4(9+6\delta t + \delta t^2) + 6 + 2\delta t + 1 - 43}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{36 + 24\delta t + 4\delta t^2 + 6 + 2\delta t + 1 - 43}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{43 + 26\delta t + 4\delta t^2 - 43}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{26\delta t + 4\delta t^2}{\delta t} \\ \lim_{\delta t \rightarrow 0} \frac{\delta t(26 + 4\delta t)}{\delta t} &= \lim_{\delta t \rightarrow 0} (26 + 4\delta t) = 26 \end{aligned}$$

Thus, instantaneous velocity at  $t = 3$  is 26 miles/hour

### Process of Finding Derivative $f'(x)$ by Definition:

#### Notation of Derivative:

Several notations are used for derivatives. We have used the functional symbol  $f'(x)$ , for the derivative of  $f$  at  $x$ .

For the function  $y = f(x)$ ,

$$y + \delta y = f(x + \delta x)$$

$$\delta y = f(x + \delta x) - y$$

$$\delta y = f(x + \delta x) - f(x)$$

Dividing both the sides by  $\delta x$ , we get

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit of both the sides as  $\delta x \rightarrow 0$ , we have

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad \dots(i)$$

$$= \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} \text{ is denoted by } \frac{dy}{dx}, \text{ so (i) is written as } \frac{dy}{dx} = f'(x)$$

**Note** The symbol  $\frac{dy}{dx}$  is used for the derivative of  $y$  with respect to  $x$  and here it is not a quotient of  $dy$  and  $dx$ .

$\frac{dy}{dx}$  is also denoted by  $y'$ .

Now we write, in a table the notations for derivative of  $y = f(x)$  used by different mathematicians:

| Name of mathematician        | Leibniz                            | Newton          | Lagrange | Euler   |
|------------------------------|------------------------------------|-----------------|----------|---------|
| Notation used for derivative | $\frac{dy}{dx}$ or $\frac{df}{dx}$ | $f'(x)$ or $y'$ | $f'(x)$  | $Df(x)$ |

**Derivative or gradient of  $f(x)$  at  $x = a$ :**

If we replace  $x + \delta x$  by  $x$  and  $x$  by  $a$ , then the expression  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$  is written as  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a)$

Here  $f'(a)$  is called the derivative or gradient of  $f$  at  $x = a$ .

### Finding $f'(x)$ by Definition of Derivative:

Given a function  $f$ , then  $f'(x)$  if it exists, can be found by the following four steps:

Step I: Find  $f(x + \delta x)$

Step II: Simplify  $f(x + \delta x) - f(x)$

Step III: Divide  $f(x + \delta x) - f(x)$  by  $\delta x$  to get  $\frac{f(x + \delta x) - f(x)}{\delta x}$  and simplify it.

Step IV: Find  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$

The method of finding derivatives by this process is called differentiation by definition or by ab-initio or from first principle.

**Example 3:** Find the derivative of the following functions by definition:

(a)  $f(x) = c$

(b)  $f(x) = x^2$

**Solution: (a)**

$$f(x) = c$$

Let  $y = c$

$$y + \delta y = c$$

Equation (ii) - Equation (i)

$$y + \delta y - y = c - c$$

$$\delta y = 0$$

Dividing both sides by ' $\delta x$ ':

$$\frac{\delta y}{\delta x} = \frac{0}{\delta x}$$

Taking  $\lim_{\delta x \rightarrow 0}$  on both sides:

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (0)$$

$$\frac{dy}{dx} = 0$$

**Solution: (b)**

$$f(x) = x^2$$

Let  $y = x^2$  ... (i)

$\Rightarrow y + \delta y = (x + \delta x)^2$  ... (ii)

Eq. (ii) - Eq. (i)

$$y + \delta y - y = (x + \delta x)^2 - x^2$$

$$\delta y = x^2 + (\delta x)^2 + 2x \cdot \delta x - x^2$$

$$\delta y = \delta x(\delta x + 2x)$$

Dividing both sides by  $\delta x$ :

$$\frac{\delta y}{\delta x} = \delta x + 2x$$

Taking  $\lim_{\delta x \rightarrow 0}$  on both sides:

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (\delta x + 2x)$$

$$\frac{dy}{dx} = 0 + 2x$$

$$\frac{dy}{dx} = 2x$$

**Example 4:** Find the derivative of  $\sqrt{x}$  at  $x = a$  from first principle.

**Solution:**

$$\text{Let } y = \sqrt{x} \quad \dots(i)$$

$$\Rightarrow y + \delta y = \sqrt{x + \delta x} \quad \dots(ii)$$

Eq. (ii) - Eq. (i)

$$y + \delta y - y = \sqrt{x + \delta x} - \sqrt{x}$$

$$\delta y = \sqrt{x + \delta x} - \sqrt{x}$$

$$\delta y = (\sqrt{x + \delta x} - \sqrt{x}) \times \frac{\sqrt{x + \delta x} + \sqrt{x}}{\sqrt{x + \delta x} + \sqrt{x}} \quad \text{by rationalizing}$$

$$\delta y = \frac{(\sqrt{x + \delta x})^2 - (\sqrt{x})^2}{\sqrt{x + \delta x} + \sqrt{x}} = \frac{x + \delta x - x}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$\delta y = \frac{\delta x}{\sqrt{x + \delta x} + \sqrt{x}}$$

Dividing both sides by ' $\delta x$ ':

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

Taking  $\lim_{\delta x \rightarrow 0}$  on both sides:

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Now putting  $x = a$

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{2\sqrt{a}}$$

which is the gradient of  $f$  at  $x = a$ .

**Example 5:** If  $y = \frac{1}{x^2}$ , then find  $\frac{dy}{dx}$  at  $x = -1$  by ab-initio method.

**Solution:**

$$\text{Let } y = \frac{1}{x^2} \quad \dots(i)$$

$$\Rightarrow y + \delta y = \frac{1}{(x + \delta x)^2} \quad \dots(ii)$$

Eq. (ii) - Eq. (i)

$$\Rightarrow y + \delta y - y = \frac{1}{(x + \delta x)^2} - \frac{1}{x^2}$$

$$\delta y = \frac{x^2 - (x + \delta x)^2}{x^2(x + \delta x)^2} = \frac{x^2 - (x^2 + 2x\delta x + (\delta x)^2)}{x^2(x + \delta x)^2}$$

$$\delta y = \frac{x^2 - x^2 - 2x\delta x - (\delta x)^2}{x^2(x + \delta x)^2}$$

$$\delta y = \frac{-2x\delta x - (\delta x)^2}{x^2(x + \delta x)^2} = \frac{\delta x(-2x - \delta x)}{x^2(x + \delta x)^2}$$

Dividing both sides by ' $\delta x$ '

$$\frac{\delta y}{\delta x} = \frac{-2x - \delta x}{x^2(x + \delta x)^2}$$

Taking  $\lim_{\delta x \rightarrow 0}$  on both sides

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-2x - \delta x}{x^2(x + \delta x)^2}$$

$$\frac{dy}{dx} = \frac{-2x - 0}{x^2(x + 0)^2} = \frac{-2x}{x^2 \cdot x^2} = -\frac{2}{x^3}$$

Now putting  $x = -1$

$$\left(\frac{dy}{dx}\right)_{x=-1} = \frac{-2}{(-1)^3} = \frac{-2}{-1} = 2$$

The gradient of  $f$  at  $x = -1$  is 2.

**Theorem:** Find the derivation of  $x^n$  where  $n \in \mathbb{Z}$  Or Prove that  $\frac{d}{dx}(x^n) = nx^{n-1}$ , where  $n \in \mathbb{Z}$

**Case-I** when  $n$  is positive integer.

$$\text{Let } y = x^n \quad \dots(i)$$

$$\text{Then } y + \delta y = (x + \delta x)^n \quad \dots(ii)$$

Equation(ii) - Equation(i)

$$\delta y = (x + \delta x)^n - x^n$$

Using the binomial theorem, we have

$$\delta y = \left[ x^n + \binom{n}{1} x^{n-1} \cdot \delta x + \binom{n}{2} x^{n-2} (\delta x)^2 + \dots + (\delta x)^n \right] - x^n$$

$$\delta y = nx^{n-1} \cdot \delta x + \frac{n(n-1)}{2!} x^{n-2} (\delta x)^2 + \dots + (\delta x)^n$$

$$\delta y = \delta x \left[ nx^{n-1} + \frac{n(n-1)}{2!} x^{n-2} \cdot \delta x + \dots + (\delta x)^{n-1} \right]$$

Dividing both sides by  $\delta x$ , gives

$$\frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{2!} x^{n-2} \cdot \delta x + \dots + (\delta x)^{n-1}$$

Taking  $\lim_{\delta x \rightarrow 0}$  on both sides

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left\{ nx^{n-1} + \frac{n(n-1)}{2!} x^{n-2} \cdot \delta x + \dots + (\delta x)^{n-1} \right\}$$

$$\frac{dy}{dx} = nx^{n-1} + 0 + 0 + \dots + 0 = nx^{n-1}$$

As  $y = x^n$ , so  $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

**Case-II** When  $n$  is negative integer.

Let  $n = -m$  ( $m$  is a positive integer).

$$\text{Let } y = x^n$$

$$\text{Then } y = x^{-m} = \frac{1}{x^m} \quad \dots(i)$$

$$y + \delta y = \frac{1}{(x + \delta x)^m} \quad \dots(ii)$$

Equation (ii) - Equation (i)

$$\delta y = \frac{1}{(x + \delta x)^m} - \frac{1}{x^m} = \frac{x^m - (x + \delta x)^m}{x^m(x + \delta x)^m}$$

$$= \frac{x^m - \left[ x^m + \binom{m}{1} x^{m-1} \delta x + \frac{m(m-1)}{2!} x^{m-2} (\delta x)^2 + \dots + (\delta x)^m \right]}{x^m(x + \delta x)^m}$$

Using the binomial theorem, we have

$$\delta y = \frac{\delta x \left( mx^{m-1} + \frac{m(m-1)}{2!} x^{m-2} \delta x + \dots + (\delta x)^{m-1} \right)}{x^m(x + \delta x)^m}$$

Dividing both sides by  $\delta x$ , gives

$$\frac{\delta y}{\delta x} = \frac{-1}{x^m(x + \delta x)^m} \left( mx^{m-1} + \frac{m(m-1)}{2!} x^{m-2} \delta x + \dots + (\delta x)^{m-1} \right)$$

Taking  $\lim_{\delta x \rightarrow 0}$  on both sides

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-1}{x^m(x + \delta x)^m} \left( mx^{m-1} + \frac{m(m-1)}{2!} x^{m-2} \delta x + \dots + (\delta x)^{m-1} \right)$$

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{-1}{x^m(x + 0)^m} (mx^{m-1} + 0 + \dots + 0)$$

$$\frac{dy}{dx} = \frac{-1}{x^m \cdot x^m} \cdot mx^{m-1}$$

$$= -mx^{m-1} \cdot x^{-2m}$$

$$= -mx^{(-m)-1}$$

$$\frac{dy}{dx} = nx^{n-1} \quad [\because -m = n]$$

$$\frac{d(x)^n}{dx} = nx^{n-1}$$

So, we have proved that  $\frac{d}{dx}(x^n) = nx^{n-1}$ , if  $n \in \mathbb{Z}$ **Note:** The above rule also holds if  $n \in \mathbb{Q} - \mathbb{Z}$ , i.e. for rational powers.The proof of  $\frac{d}{dx}(x^n) = nx^{n-1}$  when  $n \in \mathbb{Q} - \mathbb{Z}$  is left as an exercise.

$$\text{For example, } \frac{d}{dx} \left( x^{\frac{2}{3}} \right) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3x^{\frac{1}{3}}}$$

**Connection Between Derivatives and Continuity:**

Calculus is a powerful branch of mathematics that allows us to study change and motion. Two of its foundational concepts of continuity and derivatives are deeply connected. While each concept has its own definition and application, understanding how they relate to each other is essential for solving real-world problems in mathematics. As discussed in previous units, a function is continuous at a point if its graph has no breaks, jumps, or holes at that point. On the other hand, the derivative of a function at a point measures the instantaneous rate of change or equivalently, the slope of the tangent line at that point. However, this definition depends on the function being well-behaved around the point. This leads to a well-known result:

**Remember That:**

- > If a function is differentiable at a point, it must also be continuous there.
- > This means that differentiability implies continuity, but the reverse is not necessarily true.

For example, consider the function  $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Clearly this function is continuous at  $x = 0$  (see Figure 13.3).Now, we check the differentiability of  $f(x) = |x|$  at  $x = 0$ .

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

By definition of derivative of  $f(x)$ 

$$f'(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$f'(0) = \lim_{\delta x \rightarrow 0} \frac{f(0 + \delta x) - f(0)}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{f(\delta x) - f(0)}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{|\delta x| - |0|}{\delta x}$$

$$\text{Thus } f'(0) = \lim_{\delta x \rightarrow 0} \frac{|\delta x|}{\delta x}$$

Because  $|\delta x| = \delta x$  when  $\delta x > 0$  and  $|\delta x| = -\delta x$  when  $\delta x < 0$ , so we consider one-sided limits

$$\text{R.H.L.} = \lim_{\delta x \rightarrow 0^+} \frac{|\delta x|}{\delta x} = \lim_{\delta x \rightarrow 0^+} \frac{\delta x}{\delta x} = 1 \quad \text{and} \quad \text{L.H.L.} = \lim_{\delta x \rightarrow 0^-} \frac{|\delta x|}{\delta x} = \lim_{\delta x \rightarrow 0^-} \frac{-\delta x}{\delta x} = -1$$

As L.H.L.  $\neq$  R.H.L., therefore, the  $\lim_{\delta x \rightarrow 0} \frac{|\delta x|}{\delta x}$  does not exist.

This implies that derivative of  $f$  at  $x = 0$  does not exist, and thus, there is no tangent line to the graph of  $f$  at this point (see Figure 13.3). However, the derivative exists at all other points of  $f$  i.e., it is 1 on the right side and -1 on the left side. A function can be continuous at a point but not necessarily differentiable there.

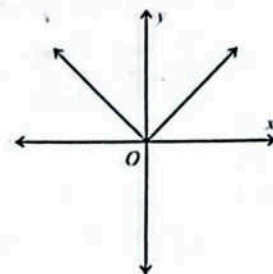


Figure 13.3

**Exercise 13.1**

1. Find by definition, the derivatives w.r.t. 'x' of the following functions defined as:

(i)  $2x^2 + 1$

Solution:

Let  $y = 2x^2 + 1$  ... (1)

$y + \delta y = 2(x + \delta x)^2 + 1$  ... (2)

Eq. (2) - Eq. (1):

$y + \delta y - y = 2(x + \delta x)^2 + 1 - (2x^2 + 1)$

$\delta y = 2(x^2 + (\delta x)^2 + 2x\delta x) + 1 - 2x^2 - 1$

$\delta y = 2x^2 + 2(\delta x)^2 + 4x\delta x - 2x^2 = 2(\delta x)^2 + 4x\delta x$

$\delta y = 2\delta x(\delta x + 2x)$

Dividing both sides by  $\delta x$ :

$\frac{\delta y}{\delta x} = \frac{2\delta x(\delta x + 2x)}{\delta x} = 2(\delta x + 2x)$

Taking  $\lim_{\delta x \rightarrow 0}$  both sides:

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 2 \lim_{\delta x \rightarrow 0} (\delta x + 2x)$

$\frac{dy}{dx} = 2(0 + 2x) = 4x$

(ii)  $2 - \sqrt{x}$

Solution:

Let  $y = 2 - \sqrt{x}$  ... (1)

$y + \delta y = 2 - \sqrt{x + \delta x}$  ... (2)

Eq. (2) - Eq. (1):

$y + \delta y - y = 2 - \sqrt{x + \delta x} - 2 + \sqrt{x}$

$\delta y = -\sqrt{x + \delta x} + \sqrt{x} = \sqrt{x} - \sqrt{x + \delta x}$

$\delta y = \sqrt{x} - \sqrt{x + \delta x} \times \frac{\sqrt{x} + \sqrt{x + \delta x}}{\sqrt{x} + \sqrt{x + \delta x}}$

By rationalizing the denominator

$= \frac{(\sqrt{x})^2 - (\sqrt{x + \delta x})^2}{\sqrt{x} + \sqrt{x + \delta x}} = \frac{x - x - \delta x}{\sqrt{x} + \sqrt{x + \delta x}}$

$= \frac{-\delta x}{\sqrt{x} + \sqrt{x + \delta x}}$