Domains and Ranges of Sine and Cosine Functions:

Let $P(x,y)$ be any point on unit circle with centre at the origin O such that $m\angle XOP = \theta$ in standard position, then

$$\cos\theta = x \text{ and } \sin\theta = y$$

⇒ for any real number θ there is one and only one value of each x and y i.e., of each $\cos\theta$ and $\sin\theta$.

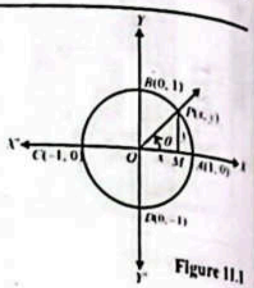


Figure 11.1

Hence $\sin\theta$ and $\cos\theta$ are the functions of θ and their domain is R, the set of real numbers. i.e.,

Domain of $\sin\theta$: R ; **Domain of $\cos\theta$: R**

Domain of $\sin\theta = R$ and Domain of $\cos\theta = R$

Since (x,y) is a point on the unit circle with centre at the origin O, therefore

$-1 \le x \le 1$ and $-1 \le y \le 1$

⇒ $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

Range of $\sin\theta$: $[-1,1]$; **Range of $\cos\theta$: $[-1,1]$**

Domains and Ranges of Tangent and Cotangent Functions:

(i) From the Figure 11.1

$$\tan\theta = \frac{y}{x}, x \ne 0$$

⇒ terminal side $\overline{OP}$ should not coincide with OY or OY' (the Y-axis)

⇒ $\theta \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

⇒ $\theta \ne (2n+1)\frac{\pi}{2}$, where $n \in Z$

Domain of tangent function $= R - \left\{x \mid x = (2n+1)\frac{\pi}{2}, n \in Z\right\}$

If $y = \frac{1}{2}, \tan\theta = \frac{1}{2x}$ as $x \to 0, \frac{1}{2x} \to \pm\infty$ therefore the

Range of tangent function $= R =$ Set of real numbers

(ii) From Figure 11.1

$$\cot\theta = \frac{x}{y}, y \ne 0$$

⇒ terminal side $\overline{OP}$ should not coincide with OX or OX' (the X-axis)

⇒ $\theta \ne 0, \pm\pi, \pm 2\pi, \ldots$

⇒ $\theta \ne n\pi$, where $n \in Z$

Domain of cotangent function $= R - \{x \mid x = n\pi, n \in Z\}$

If $x = \frac{1}{2}, \cot\theta = \frac{1}{2y}$ as $y \to 0, \frac{1}{2y} \to \pm\infty$ therefore the

Range of cotangent function $= R =$ Set of real numbers

Domain and Range of Secant Function:

From the Figure 11.1

$$\sec\theta = \frac{1}{x}, x \ne 0$$

⇒ terminal side $\overline{OP}$ should not coincide with OY or OY' (the Y-axis)

⇒ $\theta \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

⇒ $\theta \ne (2n+1)\frac{\pi}{2}$, where $n \in Z$

⇒ $\theta \ne (2n+1)\frac{\pi}{2}$, where $n \in Z$

Domain of secant function $= R - \left\{x \mid x = (2n+1)\frac{\pi}{2}, n \in Z\right\}$

As $0 \le x \le 1$ so, $\frac{1}{x} \ge 1, \sec\theta \ge 1$ and $-1 \le x \le 0$ so, $\frac{1}{x} \le -1, \sec\theta \le -1$

As $\sec\theta$ attains all real values except those between -1 and 1

Range of secant function $= R - \{x \mid -1 < x < 1\}$

Domain and Range of Cosecant Function:

From the Figure 11.1

$$\csc\theta = \frac{1}{y}, y \ne 0$$

⇒ terminal side $\overline{OP}$ should not coincide with OX or OX' (the X-axis)

⇒ $\theta \ne 0, \pm\pi, \pm 2\pi, \ldots$

⇒ $\theta \ne n\pi$, where $n \in Z$

Domain of cosecant function $= R - \{x \mid x = n\pi, n \in Z\}$

As $\csc\theta$ attains all values except those between -1 and 1

Range of cosecant function $= R - \{x \mid -1 < x < 1\}$

The following table summarizes the domains and ranges of the trigonometric functions:

Function	Domain	Range
$y = \sin x$	$(-\infty,\infty) = R$	$[-1, 1]$
$y = \cos x$	$(-\infty,\infty) = R$	$[-1, 1]$
$y = \tan x$	$R = (-\infty,\infty),\ x \ne (2n+1)\frac{\pi}{2},\ n \in Z$	$(-\infty,\infty) = R$
$y = \cot x$	$R = (-\infty,\infty),\ x \ne n\pi,\ n \in Z$	$(-\infty,\infty) = R$
$y = \sec x$	$(-\infty,\infty),\ x \ne (2n+1)\frac{\pi}{2},\ n \in Z$	$(-\infty,-1] \cup [1,\infty)$
$y = \operatorname{cosec} x$	$(-\infty,\infty),\ x \ne n\pi,\ n \in Z$	$(-\infty,-1] \cup [1,\infty)$

Even and Odd Functions:

Even Function: A function f is said to be even if $f(-x) = f(x)$, $\forall x \in$ Domain f

For example: $f(x) = x^2$ is even function of x.

Here $f(-x) = (-x)^2$
$= x^2$
$= f(x)$

> **Remember!**
> The graph of even function is always symmetric about y-axis.

Odd Function: A function f is said to be odd if $f(-x) = -f(x)$, $\forall x \in$ Domain f

For example: $f(x) = x^3$ is an odd function of x.

Here $f(-x) = (-x)^3$

$= -x^3$

$= -f(x)$

> **Remember!**
> The graph of odd function is always symmetric about the origin.

The function $f(\theta) = \cos\theta$ for all $\theta \in R$ is an even function (See figure 11.2).

Here $f(-\theta) = \cos(-\theta) = \cos\theta = f(\theta)$.

Thus, $f(\theta) = \cos\theta$ is an even function.

Similarly, the function $f(\theta) = \sin\theta$ for all $\theta \in R$ is an odd function.

Here $f(-\theta) = \sin(-\theta) = -\sin\theta = -f(\theta)$.

Thus, $f(\theta) = \sin\theta$ is an odd function.

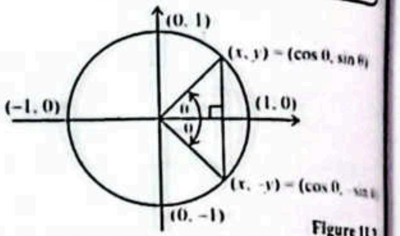


Figure 11.2

> **Note:**
> In both the cases, for each x in the domain of f, $-x$ must also be in the domain of f.

Period of Trigonometric Functions:

Periodicity: All the six trigonometric functions repeat their values for each increase or decrease of 2π in θ therefore the values of trigonometric functions for θ and $\theta \pm 2n\pi$, where $\theta \in R$ and $n \in Z$, are the same. This behaviour of trigonometric functions is called periodicity.

Period of a trigonometric function: Period of a trigonometric function is the smallest positive number which when added to the original circular measure of the angle, gives the same value of the function.

Or

A function is periodic, if $f(\theta + p) = f(\theta)$, for all θ in domain of function and the least positive value of p is called the period of the function.

Now, let us discover the periods of the trigonometric functions.

Theorem 11.1: Sine is a periodic function and its period is 2π.

Proof: Suppose p is the period of sine function such that

$\sin(\theta + p) = \sin\theta$ for all $\theta \in R$ ... (A)

Now put $\theta = 0$, we have

$\sin(0 + p) = \sin 0$

$\Rightarrow \quad \sin p = 0$

$\Rightarrow \quad p = +\pi, +2\pi, +3\pi, \ldots$ (Taking +ve values only)

Checking:

If $p = \pi$, then from (A)

$$\sin(\theta + \pi) = \sin\left(2.\frac{\pi}{2} + \theta\right) = \sin\theta \quad \text{(not true)}$$

Thus π is not the period of $\sin\theta$

If $p = 2\pi$, then from (A)

$$\sin(\theta + 2\pi) = \sin\left(4.\frac{\pi}{2} + \theta\right) = \sin\theta \quad \text{(true)} \qquad \because \sin(\theta + 2\pi) = \sin\theta$$

As 2π is the smallest positive real number for which $\sin(\theta + 2\pi) = \sin\theta$

$\therefore$ 2π is the period of $\sin\theta$.

Theorem 11.2: Cosecant is a periodic function and its period is 2π.

Suppose p is the period of cosecant function, such that

$\operatorname{cosec}(\theta + p) = \operatorname{cosec}\theta$ for all $\theta \in R$ ... (A)

$\Rightarrow \quad \dfrac{1}{\sin(\theta + p)} = \dfrac{1}{\sin\theta}$

$\Rightarrow \quad \sin(\theta + p) = \sin\theta$

Now put $\theta = 0$, we have

$\sin(0 + p) = \sin 0$

$\sin p = 0$

$\Rightarrow \quad p = \pi, 2\pi, 3\pi, \ldots$ (Taking +ve values only)

Checking:

If $p = \pi$, then from (A)

$\operatorname{cosec}(\theta + \pi) = \operatorname{cosec}\theta$

$\operatorname{cosec}(2 \times 90^\circ + \theta) = \operatorname{cosec}\theta$

$\Rightarrow \quad -\operatorname{cosec}\theta = \operatorname{cosec}\theta$ (False)

$\therefore \quad \pi$ is not the period of $\operatorname{cosec}\theta$.

If $p = 2\pi$, then from (A)

$\operatorname{cosec}(\theta + 2\pi) = \operatorname{cosec}\theta$

$\operatorname{cosec}(4 \times 90^\circ + \theta) = \operatorname{cosec}\theta$

$\Rightarrow \quad \operatorname{cosec}\theta = \operatorname{cosec}\theta$ (True)

As $\quad 2\pi$ is the smallest +ve real number for which $\operatorname{cosec}(\theta + 2\pi) = \operatorname{cosec}\theta$

$\therefore \quad 2\pi$ is the period of $\operatorname{cosec}\theta$

Theorem 11.3: Tangent is a periodic function and its period is π.

Proof: Suppose p is the period of tangent function such that

$\tan(\theta + p) = \tan\theta$ for all $\theta \in R$ ...(A)

$p = \pi, 2\pi, 3\pi, \ldots$

Checking:

If $p = \pi$, then from (A)

$\tan(\theta + \pi) = \tan\theta$, which is true

As π is the smallest positive number for which $\tan(\theta + \pi) = \tan\theta$

Therefore, π is the period of $\tan\theta$.

> **Note:**
> By adopting the procedure used in finding the periods of sine and tangent, we can prove that
> (i) 2π is the period of $\cos\theta$
> (ii) 2π is the period of $\csc\theta$
> (iii) 2π is the period of $\sec\theta$
> (iv) π is the period of $\cot\theta$.

Theorem 11.4: Cosine is a periodic function and its period is 2π.

Proof: Suppose p is the period of $\cos\theta$, such that

$\cos(\theta + p) = \cos\theta$ ... (1) $\quad \forall \theta \in R$

Now put $\theta = 0$, we have

$\cos(0 + p) = \cos 0$

$\cos p = 1$

$\Rightarrow \quad p = 2\pi, 4\pi, 6\pi, \ldots$ (Taking +ive values only)

Checking:

If $p = 2\pi$, then from (1)

$\cos(\theta + 2\pi) = \cos\theta$

$\cos(4 \times 90^\circ + \theta) = \cos\theta$

$\Rightarrow \quad \cos\theta = \cos\theta$ (True)

As $\quad 2\pi$ is the smallest +ive real number for which $\cos(\theta + 2\pi) = \cos\theta$

$\therefore \quad 2\pi$ is the period of $\cos\theta$

Theorem 11.5: Secant is a periodic function and its period is 2π.

Proof:

Suppose p is the period of $\sec\theta$, such that

$\sec(\theta+p) = \sec\theta \quad \ldots (1) \qquad \forall\, \theta \in R$

$\Rightarrow \dfrac{1}{\cos(\theta+p)} = \dfrac{1}{\cos\theta}$

$\Rightarrow \cos(\theta+p) = \cos\theta$

Now put $\theta = 0$, we have

$\cos(0+p) = \cos 0$

$\cos p = 1$

$\Rightarrow p = 2\pi, 4\pi, 6\pi, \ldots$ (Taking +ive values only)

Checking:

If $p = 2\pi$, then from (1)

$\sec(\theta+2\pi) = \sec\theta$

$\sec(4\times 90° + \theta) = \sec\theta$

$\Rightarrow \sec\theta = \sec\theta$ (True)

As 2π is the smallest +ive real number for which $\sec(\theta+2\pi) = \sec\theta$

$\therefore$ 2π is the period of $\sec\theta$

Theorem 11.6: Cotangent is a periodic function and its period is π.

Proof:

Suppose p is the period of $\cot\theta$, such that

$\cot(\theta+p) = \cot\theta \ldots (1) \qquad \forall\, \theta \in R$

$\Rightarrow \dfrac{1}{\tan(\theta+p)} = \dfrac{1}{\tan\theta}$

$\Rightarrow \tan(\theta+p) = \tan\theta$

Now put $\theta = 0$, we have

$\tan(0+p) = \tan 0$

$\tan p = 0$

$\Rightarrow p = \pi, 2\pi, 3\pi, \ldots$ (Taking +ive values only)

Checking

If $p = \pi$, then from (1)

$\cot(\theta+\pi) = \cot\theta$

$\cot(2\times 90° + \theta) = \cot\theta$

$\Rightarrow \cot\theta = \cot\theta$ (True)

As π is the smallest +ive real number for which $\cot(\theta+\pi) = \cot\theta$

$\therefore$ π is the period of $\cot\theta$

Example 1: Find the periods of: (i) $\sin 2x$ (ii) $3+\tan\dfrac{x}{3}$

Solution:

(i) Since period of $\sin x$ is 2π, therefore

$\sin 2x = \sin(2x+2\pi)$

$\Rightarrow \sin 2x = \sin 2(x+\pi)$

It means that the value of $\sin 2x$ repeats when x is increased by π.

Hence π is the period of $\sin 2x$.

(ii) To find the period of $3+\tan\dfrac{x}{3}$, consider only $\tan\dfrac{x}{3}$.

Since period of $\tan x$ is π, therefore

$\tan\dfrac{x}{3} = \tan\left(\dfrac{x}{3}+\pi\right)$

$\Rightarrow \tan\dfrac{x}{3} = \tan\dfrac{1}{3}(x+3\pi)$

It means that the value of $\tan\dfrac{x}{3}$ repeats when x is increased by 3π.

Hence the period of $3+\tan\dfrac{x}{3}$ is 3π. The addition of constant number 3 to the tangent function does not affect the period.

Exercise 11.1

1. Determine whether the following functions are even, odd or neither odd nor even.

(i) $\sin^2 x$

Solution:

Let $f(x) = \sin^2 x$

Replace x by $-x$

$f(-x) = [\sin(-x)]^2$

$f(-x) = [-\sin x]^2 \qquad \because \sin(-\theta) = -\sin\theta$

$f(-x) = \sin^2 x$

$f(-x) = f(x)$

Thus, $f(x) = \sin^2 x$ is an even function.

(ii) $\sin x + \cos x$

Solution:

Let $f(x) = \sin x + \cos x$

Replace x by $-x$

$f(-x) = \sin(-x) + \cos(-x)$

$f(-x) = -\sin x + \cos x \qquad \because \begin{cases}\sin(-\theta) = -\sin\theta\\ \cos(-\theta) = \cos\theta\end{cases}$

As $f(-x) \neq \pm f(x)$

So, $f(x) = \sin x + \cos x$ is neither even nor odd function.

(iii) $\sin^4 x + \cos^4 x$

Solution:

Let $f(x) = \sin^4 x + \cos^4 x$

Replace x by $-x$

$f(-x) = [\sin(-x)]^4 + [\cos(-x)]^4$

$f(-x) = \left[-\sin x^4\right] + [\cos x]^4 \qquad \because \begin{cases}\sin(-\theta) = -\sin\theta\\ \cos(-\theta) = \cos\theta\end{cases}$

$f(-x) = \sin^4 x + \cos^4 x$

$f(-x) = f(x)$

Thus, $f(x) = \sin^4 x + \cos^4 x$ is an even function.

(iv) $\tan x + \sec x$

Solution:

Let $f(x) = \tan x + \sec x$

Replace x by $-x$

$f(-x) = \tan(-x) + \sec(-x)$

$f(-x) = -\tan x + \sec x \qquad \because \begin{cases}\tan(-\theta) = -\tan\theta\\ \sec(-\theta) = \sec\theta\end{cases}$

As $f(-x) \neq \pm f(x)$

So, $f(x) = \tan x + \sec x$ is neither even nor odd function.

(v) $\dfrac{1}{\operatorname{cosec}^3 x}$

Solution:

Let $f(x) = \dfrac{1}{\operatorname{cosec}^3 x}$

Replace x by $-x$

$f(-x) = \dfrac{1}{[\operatorname{cosec}(-x)]^3}$

$f(-x) = \dfrac{1}{[-\operatorname{cosec} x]^3} \qquad \because \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

$f(-x) = -\dfrac{1}{\operatorname{cosec}^3 x}$

$f(-x) = -f(x)$

Thus, $f(x) = \dfrac{1}{\operatorname{cosec}^3 x}$ is an odd function.

(vi) $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x}$

Solution:

Let $f(x) = \dfrac{\sin x + \sin 3x}{\cos x + \cos 3x}$

Replace x by $-x$

$$f(-x) = \frac{\sin(-x)+\sin3(-x)}{\cos(-x)+\cos3(-x)}$$

$$f(-x) = \frac{-\sin x-\sin3x}{\cos x+\cos3x} \quad \because \begin{cases}\sin(-\theta)=-\sin\theta\\ \cos(-\theta)=\cos\theta\end{cases}$$

$$f(-x) = -\frac{\sin x+\sin3x}{\cos x+\cos3x}$$

$$f(-x) = -f(x)$$

Thus, $f(x)=\dfrac{\sin x+\sin3x}{\cos x+\cos3x}$ is an odd function.

(vii) $\dfrac{1}{\sec x+\sec^3x}$

Solution:

Let $f(x)=\dfrac{1}{\sec x+\sec^3x}$

Replace x by $-x$

$$f(-x) = \frac{1}{\sec(-x)+[\sec(-x)]^3}$$

$$f(-x) = \frac{1}{\sec x+[\sec x]^3} \quad \because \sec(-\theta)=\sec\theta$$

$$f(-x) = \frac{1}{\sec x+\sec^3x}$$

$$f(-x) = f(x)$$

Thus, $f(x)=\dfrac{1}{\sec x+\sec^3x}$ is an even function.

(viii) $\dfrac{1}{\sec x+\cot^2x}$

Solution:

Let $f(x)=\dfrac{1}{\sec x+\cot^2x}$

Replace x by $-x$

$$f(-x)= \frac{1}{\sec(-x)+[\cot(-x)]^2}$$

$$f(-x)= \frac{1}{\sec x+[-\cot x]^2} \quad \because \begin{cases}\sec(-\theta)=\sec\theta\\ \cot(-\theta)=-\cot\theta\end{cases}$$

$$f(-x)= \frac{1}{\sec x+\cot^2x}$$

$$f(-x)= f(x)$$

Thus, $f(x)=\dfrac{1}{\sec x+\cot^2x}$ is an even function.

2. Find the periods of the following functions:

(i) $\sin 5x$

Solution:

Since period of $\sin x$ is 2π, therefore

$$\sin5x = \sin(5x+2\pi)$$

$$= \sin5\left(x+\frac{2\pi}{5}\right)$$

Hence period of $\sin5x$ is $\dfrac{2\pi}{5}$.

(ii) $\cos 7x$

Solution:

Since period of $\cos x$ is 2π, therefore

$$\cos7x = \cos(7x+2\pi)$$

$$= \cos7\left(x+\frac{2\pi}{7}\right)$$

Hence period of $\cos7x$ is $\dfrac{2\pi}{7}$.

(iii) $\tan 3x$

Solution:

Since period of $\tan x$ is π, therefore

$$\tan3x = \tan(3x+\pi)$$

$$= \tan3\left(x+\frac{\pi}{3}\right)$$

Hence period of $\tan3x$ is $\dfrac{\pi}{3}$.

(iv) $\cot\dfrac{x}{2}$

Solution:

Since period of $\cot x$ is π, therefore

$$\cot\frac{x}{2} = \cot\left(\frac{x}{2}+\pi\right)$$

$$= \cot\left(\frac{x+2\pi}{2}\right)$$

$$= \cot\frac{1}{2}(x+2\pi)$$

Hence period of $\cot\dfrac{x}{2}$ is 2π.

(v) $19\sin\left(\dfrac{\pi}{20}x\right)$

Solution:

Since period of $\sin x$ is 2π, therefore

$$19\sin\left(\frac{\pi}{20}x\right) = 19\sin\left(\frac{\pi}{20}x+2\pi\right)$$

$$= 19\sin\left(\frac{\pi x+40\pi}{20}\right)$$

$$= 19\sin\frac{\pi}{20}(x+40)$$

Hence period of $19\sin\left(\dfrac{\pi}{20}x\right)$ is 40.

(vi) $\text{cosec}\left(\dfrac{2x}{5}\right)$

Solution:

Since period of $\text{cosec}\, x$ is 2π, therefore

$$\text{cosec}\left(\frac{2x}{5}\right) = \text{cosec}\left(\frac{2x}{5}+2\pi\right)$$

$$= \text{cosec}\left(\frac{2x+10\pi}{5}\right)$$

$$= \text{cosec}\frac{2}{5}(x+5\pi)$$

Hence period of $\text{cosec}\left(\dfrac{2x}{5}\right)$ is 5π.

(vii) $\dfrac{1}{2}\sin\left(\dfrac{3x}{2}-\dfrac{\pi}{2}\right)$

Solution:

$$\frac{1}{2}\sin\left(\frac{3x}{2}-\frac{\pi}{2}\right)$$

Consider $\dfrac{1}{2}\sin\left(\dfrac{3x}{2}\right)$

Since period of $\sin x$ is 2π, therefore

$$\frac{1}{2}\sin\left(\frac{3x}{2}\right) = \frac{1}{2}\sin\left(\frac{3x}{2}+2\pi\right)$$

$$= \frac{1}{2}\sin\left(\frac{3x+4\pi}{2}\right)$$

$$= \frac{1}{2}\sin\frac{1}{2}(3x+4\pi)$$

$$= \frac{1}{2}\sin\frac{3}{2}\left(x+\frac{4\pi}{3}\right)$$

$\Rightarrow$ period of $\dfrac{1}{2}\sin\left(\dfrac{3x}{2}\right)$ is $\dfrac{4\pi}{3}$

Hence period of $\dfrac{1}{2}\sin\left(\dfrac{3x}{2}-\dfrac{\pi}{2}\right)$ is also $\dfrac{4\pi}{3}$

(viii) $-5-3\sec\left(7\pi x+\dfrac{\pi}{4}\right)$

Solution:

$$-5-3\sec\left(7\pi x+\frac{\pi}{4}\right)$$

Consider: $-3\sec(7\pi x)$

Since period of $\sec x$ is 2π, therefore

$$-3\sec(7\pi x) = -3\sec(7\pi x+2\pi)$$

$$= -3\sec7\pi\left(x+\frac{2\pi}{7\pi}\right)$$

$$= -3\sec7\pi\left(x+\frac{2}{7}\right)$$

$\Rightarrow$ Period of $-3\sec(7\pi x)$ is $\dfrac{2}{7}$.

Hence pried of $-5-3\sec\left(7\pi x+\dfrac{\pi}{4}\right)$ is also $\dfrac{2}{7}$.

(ix) $12+10\tan\left(\dfrac{\pi}{30}x\right)$

Solution:

$$12+10\tan\left(\frac{\pi}{30}x\right)$$

Consider: $10\tan\left(\dfrac{\pi}{30}x\right)$

Since period of $\tan x$ is π, therefore

$$10\tan\left(\frac{\pi}{30}x\right) = 10\tan\left(\frac{\pi}{30}x+\pi\right)$$

$$= 10\tan\left(\frac{\pi x+30\pi}{30}\right)$$

$$= 10\tan\frac{\pi}{30}(x+30)$$

$\Rightarrow$ period of $10\tan\left(\dfrac{\pi}{30}x\right)$ is 30

Hence period of $12+10\tan\left(\dfrac{\pi}{30}\right)$ is also 30.

(x) $6-4\cot\left(\dfrac{7x}{4}+\dfrac{\pi}{4}\right)$

Solution:

$$6-4\cot\left(\frac{7x}{4}+\frac{\pi}{4}\right)$$

consider: $-4\cot\left(\dfrac{7x}{4}\right)$

since period of $\cot x$ is π, therefore

$$-4\cot\left(\frac{7x}{4}\right) = -4\cot\left(\frac{7x}{4}+\pi\right)$$

$$= -4\cot\left(\frac{7x+4\pi}{4}\right)$$

$$= -4\cot\frac{1}{4}(7x+4\pi)$$

$$= -4\cot\frac{7}{4}\left(x+\frac{4\pi}{7}\right)$$

$\Rightarrow$ Period of $-4\cot\left(\dfrac{7x}{4}\right)$ is $\dfrac{4\pi}{7}$

Hence pried of $6-4\cot\left(\dfrac{7x}{4}+\dfrac{\pi}{4}\right)$ is also $\dfrac{4\pi}{7}$.

(xi) $9+30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$

Solution:

$9+30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$

Consider: $30\sec\left(\frac{x}{15}\right)$

Since period of $\sec x$ is 2π, therefore

$30\sec\left(\frac{x}{15}\right)=30\sec\left(\frac{x}{15}+2\pi\right)$

$=30\sec\left(\frac{x+30\pi}{15}\right)$

$=30\sec\frac{1}{15}(x+30\pi)$

$\Rightarrow$ Period of $30\sec\left(\frac{x}{15}\right)$ is 30π

Hence period of $9+30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$ is also 30π.

Values of Trigonometric Functions:

We know the values of trigonometric functions for angles of measure 0°, 30°, 45°, 60°, and 90°. We have also established the following identities:

$\sin(-\theta)=-\sin\theta$	$\cos(-\theta)=\cos\theta$	$\tan(-\theta)=-\tan\theta$
$\sin(\pi-\theta)=\sin\theta$	$\cos(\pi-\theta)=-\cos\theta$	$\tan(\pi-\theta)=-\tan\theta$
$\sin(\pi+\theta)=-\sin\theta$	$\cos(\pi+\theta)=-\cos\theta$	$\tan(\pi+\theta)=\tan\theta$
$\sin(2\pi-\theta)=-\sin\theta$	$\cos(2\pi-\theta)=\cos\theta$	$\tan(2\pi-\theta)=-\tan\theta$

By using the above identities, we can easily find the values of trigonometric functions of the angles of the following measures:

−30°, −45°, −60°, −90° ±120°, ±135°, ±150°, ±180°

±210°, ±225°, ±240°, ±270° ±300°, ±315°, ±330°, ±360°

Graphs of Trigonometric Functions:

To plot the graph we shall follow these steps:

(i) Table of ordered pairs (x, y) is constructed, when x is the measure of the angle and y is the value of the trigonometric function for the angle of measure x.

(ii) The measures of the angles are taken along the X-axis.

(iii) The values of the trigonometric functions are taken along the Y-axis.

(iv) The points corresponding to the ordered pairs are plotted on the graph paper.

(v) These points are joined with the help of smooth curves.

Graph of $y=\sin x$ from -2π to 2π:

We know that the period of sine function is 2π so, we will first draw the graph for the interval from 0° to 360° (from 0 to 2π).

To graph the sine function, first, recall that $-1\le\sin x\le 1$ for all $x\in R$.

We know the range of the sine function is $[-1,1]$, so the graph will be between the horizontal lines $y=+1$ and $y=-1$

The table of the ordered pairs satisfying $y=\sin x$ is as follows:

x	0 0°	$\frac{\pi}{6}$ 30°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°	$\frac{2\pi}{3}$ 120°	$\frac{5\pi}{6}$ 150°	π 180°	$\frac{7\pi}{6}$ 210°	$\frac{4\pi}{3}$ 240°	$\frac{3\pi}{2}$ 270°	$\frac{5\pi}{3}$ 300°	$\frac{11\pi}{6}$ 330°	2π 360°
$\sin x$	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0

To draw the graph:

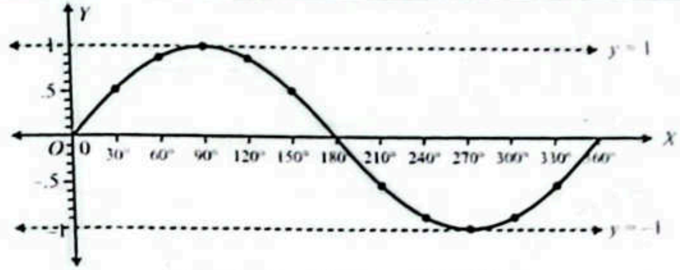
(i) Take a convenient scale $\begin{cases}\text{1 side of small square on the } x\text{-axis} = 10°\\ \text{1 side of big square on the } y\text{-axis} = 1 \text{ unit}\end{cases}$

(ii) Draw the coordinate axes.

(iii) Plot the points corresponding to the ordered pairs in the table above i.e., (0, 0), (30°, 0.5), (60°, 0.87) and so on.

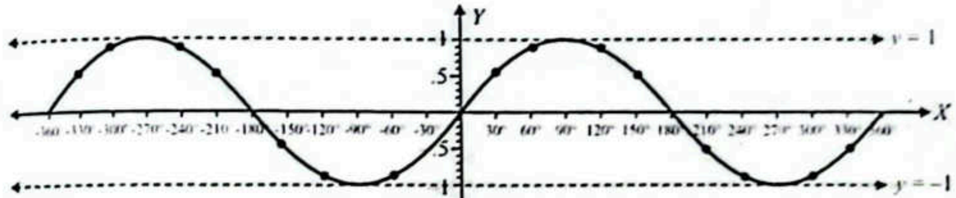
(iv) Join the points with the help of a smooth curve as shown. So, we get the graph of $y=\sin x$ from 0 to 360° i.e., from 0 to 2π.

Note:
As we see that the graphs of trigonometric functions are smooth curves and none of them is line segment or has sharp corners or breaks within their domain. This behaviour of the curve is called continuity. It means that the trigonometric functions are continuous, wherever they are defined. Moreover, as the trigonometric functions are periodic so their curves repeat after fixed intervals.



Graph of $y=\sin x$ from 0° to 360°

In the similar way, we can draw the graph for the interval from 0° to −360°. This will complete the graph of $y=\sin x$ from −360° to 360° (from -2π to 2π), which is given below:



Graph of $y=\sin x$ from −360° to 360°

The graph in the interval $[0,2\pi]$ is called a cycle and the maximum height of the wave from its mid line is called amplitude. Since the period of sine function is 2π, so the sine graph can be extended on both sides of x-axis through every interval of 2π.

Properties of graph of sine function ($y=\sin x$)

(i) The domain is the set of real numbers ($-\infty<x<\infty$).

(ii) The range includes all real numbers from −1 to 1, inclusive, $[-1,1]$.

(iii) The graph of sine function is continuous for all real numbers.

(iv) The period of sine function is 2π. Mathematically, we can express it as $\sin(\theta+2\pi)=\sin\theta$.

(v) The sine function is an odd function. As the graph of sine function is symmetric about the origin. Mathematically, it can be written as $\sin(-\theta)=-\sin\theta$.

(vi) The maximum value of $y=\sin x$ is 1 when $x=\frac{\pi}{2}+2\pi n$, where $n\in Z$.

(vii) The minimum value of $y=\sin x$ is −1 when $x=\frac{3\pi}{2}+2\pi n$, where $n\in Z$.

(viii) The x-intercepts of the sine function occurs at $x=\pi n$, where $n\in Z$.

(ix) The y-intercept of the sine function is 0.

(x) The amplitude of sine function is 1.

(xi) In unit circle $\sin\theta$ is equal to the y-coordinate of the given point.

Graph of $y = \cos x$ from -2π to 2π:

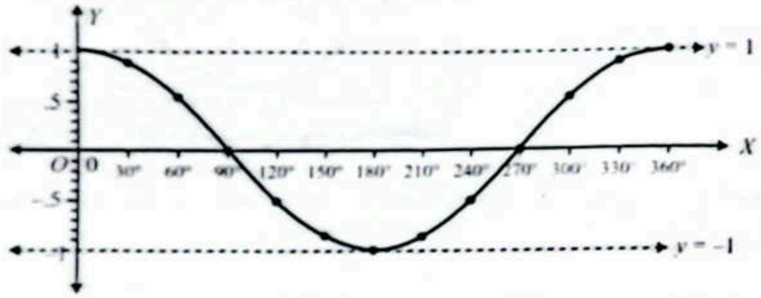
We know that the period of cosine function is 2π so, we will first draw the graph for the interval from 0° to 360° (from 0 to 2π).

We know the range of the cosine function is $[-1, 1]$, so the graph will be between the horizontal lines $y = +1$ and $y = -1$.

The table of the ordered pairs satisfying $y = \cos x$ is as follows:

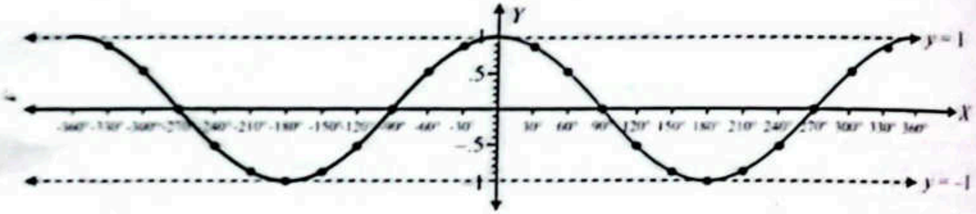
x	0 0°	$\frac{\pi}{6}$ 30°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°	$\frac{2\pi}{3}$ 120°	$\frac{5\pi}{6}$ 150°	π 180°	$\frac{7\pi}{6}$ 210°	$\frac{4\pi}{3}$ 240°	$\frac{3\pi}{2}$ 270°	$\frac{5\pi}{3}$ 300°	$\frac{11\pi}{6}$ 330°	2π 360°
sin x	1	0.87	0.5	0	− 0.5	−0.87	− 1	−0.87	− 0.5	0	0.5	0.87	1

The graph of $y = \cos x$ from 0° to 360° is given below:



Graph of $y = \cos x$ from 0° to 360°

In the similar way, we can draw the graph for the interval from 0° to −360°. This will complete the graph of $y = \cos x$ from −360° to 360° i.e. from -2π to 2π, which is given below:



Graph of $y = \cos x$ from −360° to 360°

As in the case of sine graph, the cosine graph is also extended on both sides of x-axis through an interval of 2π.

Properties of graph of cosine function ($y = \cos x$)

(i) The domain is the set of real numbers $(-\infty < x < \infty)$.
(ii) The range includes all real numbers from -1 to 1 , inclusive, $[-1, 1]$.
(iii) The graph of cosine function is continuous for all real numbers.
(iv) The period of cosine function is 2π. Mathematically, we can express it as $\cos(\theta + 2\pi) = \cos\theta$.
(v) The cosine function is an even function, as the graph of cosine function is symmetric about the y-axis. Mathematically, it can be written as $\cos(-\theta) = \cos\theta$.
(vi) The maximum value of $y = \cos x$ is 1 when $x = \pi n$, where n is an even integer.
(vii) The minimum value of $y = \cos x$ is −1 when $x = \pi n$, where n is an odd integer.
(viii) The x-intercepts of the cosine function occurs at $x = \frac{\pi}{2} + \pi n$, where $n \in Z$.
(ix) The y-intercept of the cosine function is 1 .
(x) The amplitude of cosine function is 1 .
(xi) In unit circle $\cos\theta$ is equal to the x-coordinate of the given point.

Graph of $y = \tan x$ from $-\pi$ to π:

We know that $\tan(-x) = -\tan x$ and $\tan(\pi - x) = -\tan x$, so the values of $\tan x$ for $x = 0°, 30°, 60°, 90°$ can help us in making the table.

Also, we know that $\tan x$ is undefined at $x = \pm 90°$, when

(i) x approaches $\frac{\pi}{2}$ from left $x \to \left(\frac{\pi}{2}\right)^-$, $\tan x$ decreases indefinitely in Quard I.

(ii) x approaches $\frac{\pi}{2}$ from right i.e., $x \to \left(\frac{\pi}{2}\right)^+$, $\tan x$ decreases indefinitely in Quard IV.

(iii) x approaches $-\frac{\pi}{2}$ from left i.e., $x \to \left(-\frac{\pi}{2}\right)^-$, $\tan x$ increases indefinitely in Quard II.

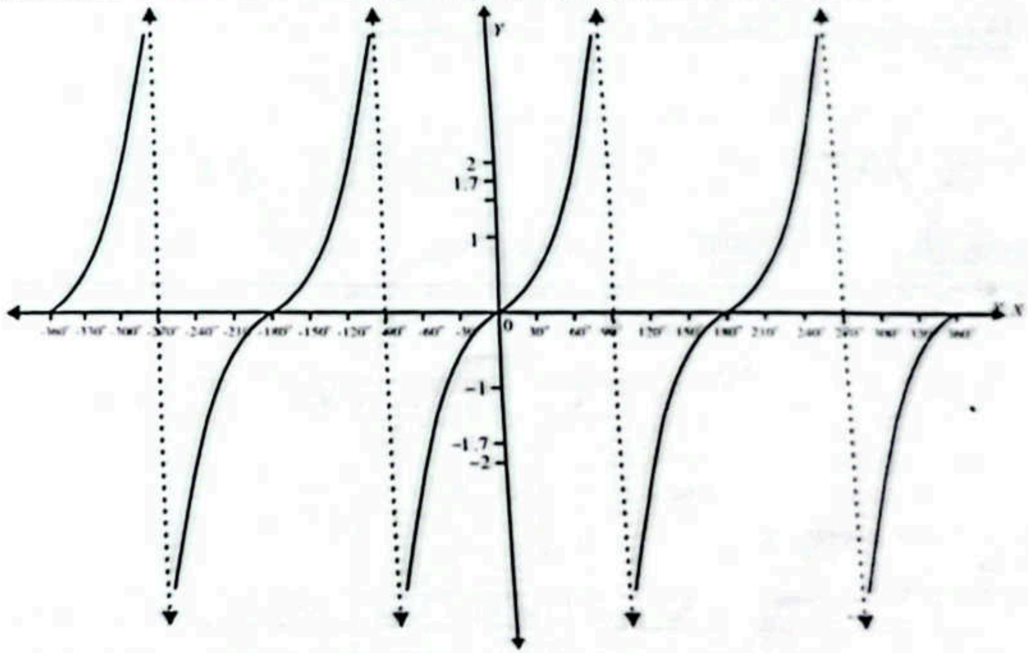
(iv) x approaches $-\frac{\pi}{2}$ from right i.e., $x \to \left(-\frac{\pi}{2}\right)^+$, $\tan x$ decreases indefinitely in Quard III.

We know that the period of tangent is π, so we shall first draw the graph for the interval from 0 to π (from 0° to 180°).

The table of ordered pairs satisfying $y = \tan x$ is given below:

x	0 0	$\frac{\pi}{6}$ 30°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2} - 0$ 90° − 0	$\frac{\pi}{2} + 0$ 90° + 0	$\frac{2\pi}{3}$ 120°	$\frac{5\pi}{6}$ 150°	π 180°
sin x	0	0.58	1.73	$+\infty$	$-\infty$	− 1.73	− 0.58	0

Since the period of $\tan x$ is π, so we have the following graph of $y = \tan x$ from −360° to 360°.



Graph of $y = \tan x$ from −360° to 360°

Properties of graph of tangent function ($y = \tan x$)

(i) The domain is the set of real numbers except the values where function is undefined domain of $\tan x = (-\infty, \infty), x \neq (2n+1)\frac{\pi}{2}$, where $n \in Z$

(ii) The range includes all real numbers $(-\infty, \infty)$

(iii) The graph of tan x is not continuous for all real numbers. It breaks at $x = (2n+1)\frac{\pi}{2}$, where $n \in Z$

(iv) The period of tan function is π. Mathematically, we can express it as $\tan(\theta + \pi) = \tan\theta$

(v) The tan function is an odd function, as the graph of tan function is symmetric about the origin. Mathematically, it can be written as $\tan(-\theta) = -\tan\theta$

(vi) The x-intercepts of the tangent function occurs at $x = \pi n$, where $n \in Z$.

(vii) The y-intercept of the tangent function is 0

(viii) The amplitude of tangent function is undefined because it has no maximum or minimum values.

Exercise 11.2

1. Draw the graph of each of the following function for the intervals mentioned against each:

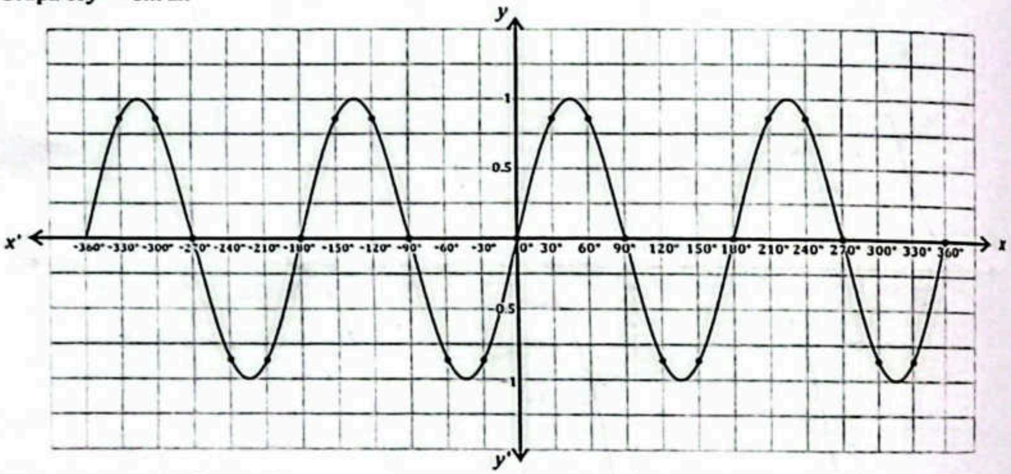
(i) $y = -\sin 2x, x \in [-2\pi, 2\pi]$

Solution:

Table for $y = -\sin 2x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = -\sin 2x$	0	0.87	0.87	0	-0.87	-0.87	0	0.87	0.87	0	-0.87	-0.87	0
x		-30°	-60°	-90°	-120°	-150°	-180°	-210°	-240°	-270°	-300°	-330°	-360°
$y = -\sin 2x$		-0.87	-0.87	0	0.87	0.87	0	-0.87	-0.87	0	0.87	0.87	0

Graph of $y = -\sin 2x$



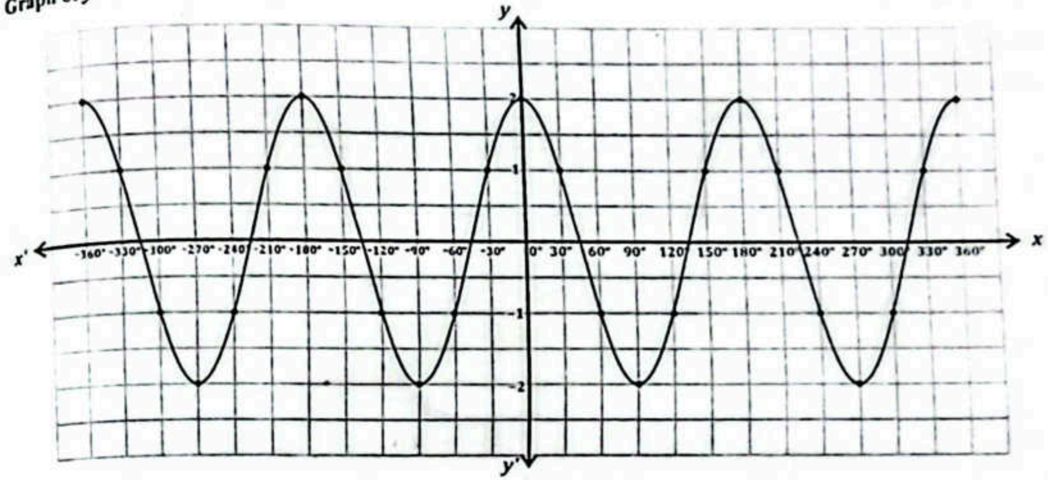
(ii) $y = 2\cos 2x, x \in [-2\pi, 2\pi]$

Solution:

Table for $y = 2\cos 2x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = 2\cos 2x$	2	1	-1	-2	-1	1	2	1	-1	-2	-1	1	2
x		-30°	-60°	-90°	-120°	-150°	-180°	-210°	-240°	-270°	-300°	-330°	-360°
$y = 2\cos 2x$		1	-1	-2	-1	1	2	1	-1	-2	-1	1	2

Graph of $y = 2\cos 2x$



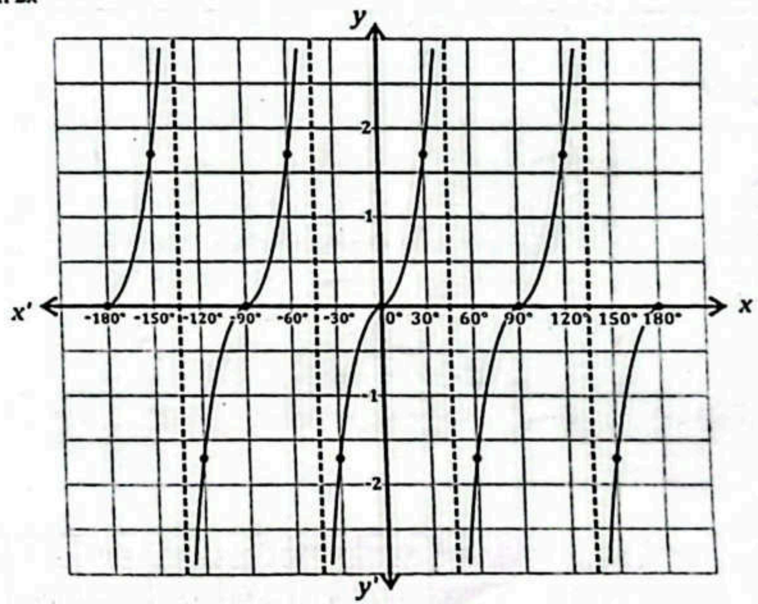
(iii) $y = \tan 2x, x \in [-\pi, \pi]$

Solution:

Table for $y = \tan 2x$

x	0°	30°	60°	90°	120°	150°	180°
$y = \tan 2x$	0	1.73	-1.73	0	1.73	-1.73	0
x		-30°	-60°	-90°	-120°	-150°	-180°
$y = \tan 2x$		-1.73	1.73	0	-1.73	1.73	0

Graph of $y = \tan 2x$



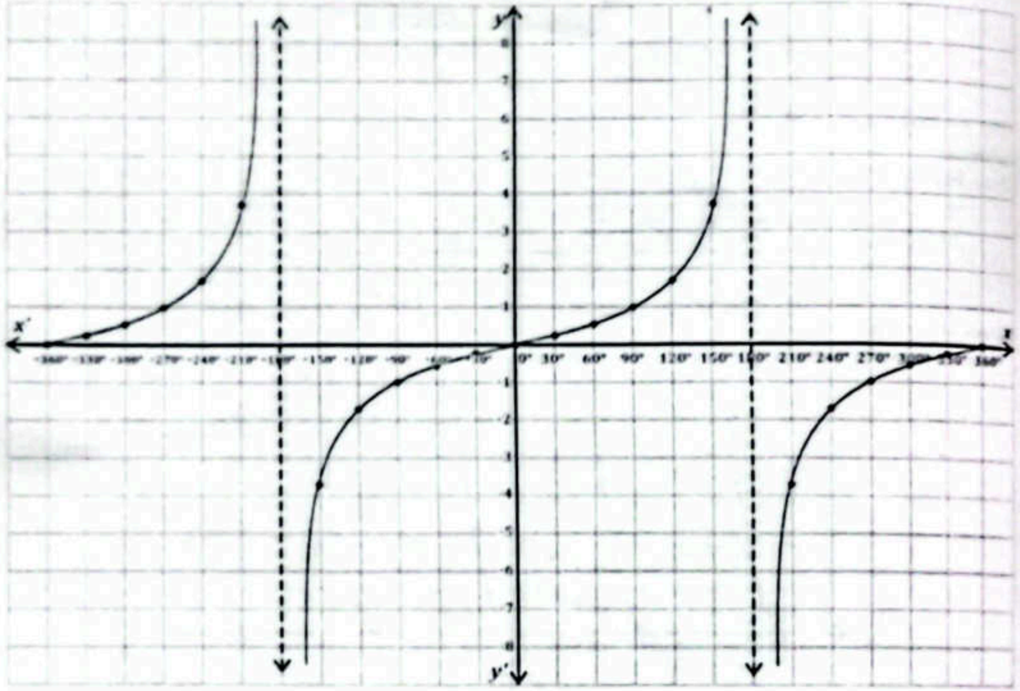
(iv) $y = \tan\frac{x}{2}, x \in [-2\pi, 2\pi]$

Solution:

Table for $y = \tan\frac{x}{2}$

x	0°	30°	60°	90°	120°	150°	180°-0	180°+0	210°	240°	270°	300°	330°	360°
$y = \tan\frac{x}{2}$	0	0.27	0.58	1	1.73	3.73	+∞	−∞	-3.73	-1.73	-1	-0.58	-0.27	0
x		-30°	-60°	-90°	-120°	-150°	-180°+0	-180°-0	-210°	-240°	-270°	-300°	-330°	-360°
$y = \tan\frac{x}{2}$		-0.27	-0.58	-1	-1.73	-3.73	−∞	∞	3.73	1.73	1	0.58	0.27	0

Graph of $y = \tan\frac{x}{2}$



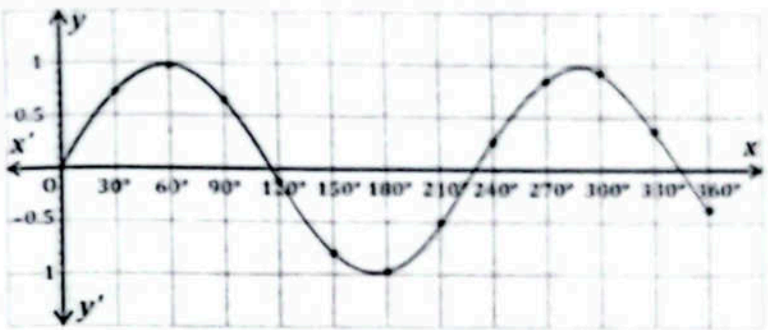
(v) $y = \sin\frac{\pi}{2}x, x \in [0, 2\pi]$

Solution:

Table for $y = \sin\frac{\pi}{2}x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin\frac{\pi}{2}x$	0	0.73	1	0.62	-0.15	-0.83	-0.96	-0.50	0.29	0.90	0.93	0.37	-0.43

Graph of $y = \sin\frac{\pi}{2}x$



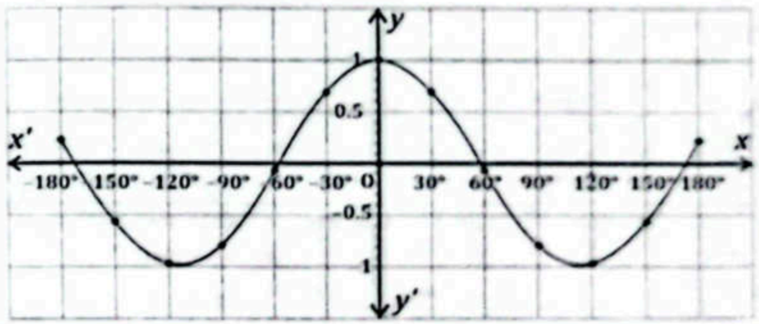
(vi) $y = \cos\frac{\pi}{2}x, x \in [-\pi, \pi]$

Solution:

Table for $y = \cos\frac{\pi}{2}x$

x	0°	30°	60°	90°	120°	150°	180°
$y = \cos\frac{\pi}{2}x$	1	0.68	-0.07	-0.78	-0.99	-0.56	0.22
x		-30°	-60°	-90°	-120°	-150°	-180°
$y = \cos\frac{\pi}{2}x$		0.68	-0.07	-0.78	-0.99	-0.56	0.22

Graph of $y = \cos\frac{\pi}{2}x$



2. On the same axes and to the same scale, draw the graphs of the following functions for their complete period:

(i) $y = \sin x$ and $y = \sin 2x$

Solution:

$y = \sin x$

$y = \sin 2x$

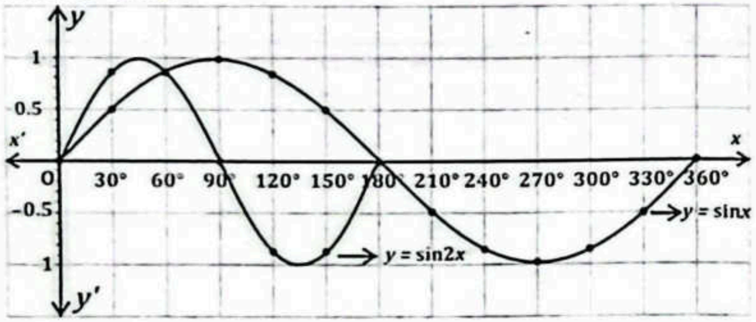
As period of $\sin x = 2\pi$, so $x \in [0, 2\pi]$

and period of $\sin 2x = \pi$, so $x \in [0, \pi]$

Table of $y = \sin x$ and $y = \sin 2x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin x$	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0
$y = \sin 2x$	0	0.87	0.87	0	-0.87	-0.87	0						

Graph for $y = \sin x$ and $y = \sin 2x$



(ii) $y = \cos x$ and $y = \cos 2x$

Solution:

$y = \cos x$

$y = \cos 2x$

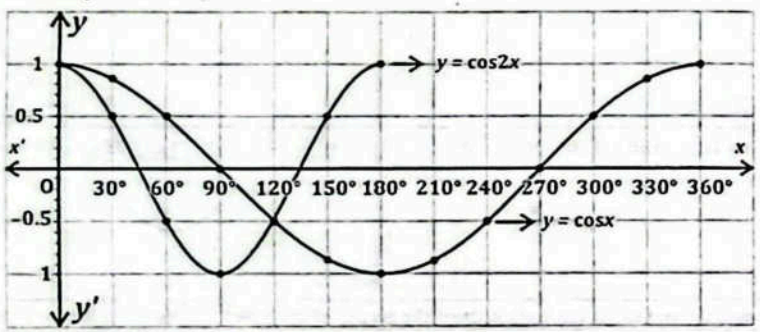
As period of $\cos x = 2\pi$, so $x \in [0, 2\pi]$

and period of $\cos 2x = \pi$, so $x \in [0, \pi]$

Table for $y = \cos x$ and $y = \cos 2x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \cos x$	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1
$y = \cos 2x$	1	0.5	-0.5	-1	-0.5	0.5	1						

Graph for $y = \sin x$ and $y = \cos 2x$



3. Solve graphically:

(i) $\sin x = \cos x, x \in [0, \pi]$

Solution:

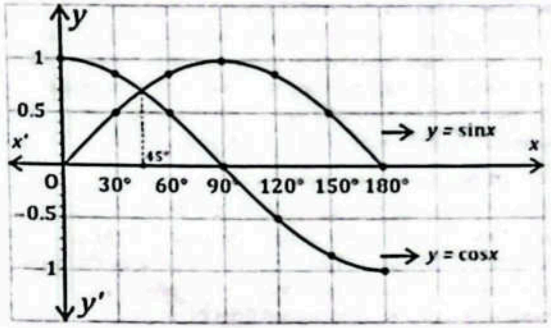
We draw the graph of two functions:

$y = \sin x$ and $y = \cos x$

Table for $y = \sin x$ and $y = \cos x$

x	0°	30°	60°	90°	120°	150°	180°
$y = \sin x$	0	0.5	0.87	1	0.87	0.5	0
$y = \cos x$	1	0.87	0.5	0	-0.5	-0.87	-1

Graph for $y = \sin x$ and $y = \cos x$



The two curves intersect each other at a point where $x = 45°$

The solution of the equation $\sin x = \cos x$ is $x = 45°$

(ii) $\sin x = x, x \in [0, \pi]$

Solution:

We draw the graph of two functions:

$y = \sin x$, $y = x$

Table for $y = \sin x$, $y = x$

x	0°	30°	60°	90°	120°	150°	180°
$y = \sin x$	0	0.5	0.87	1	0.87	0.5	0
$y = x$	0	0.52	1.05	1.57	2.09	2.62	3.14

$30° = 30 \times \frac{\pi}{180}$

$= 0.52$ rad

$60° = 60 \times \frac{\pi}{180}$

$= 1.05$ rad

$90° = 90 \times \frac{\pi}{180}$

$= 1.57$ rad

$120° = 120 \times \frac{\pi}{180}$

$= 2.09$ rad

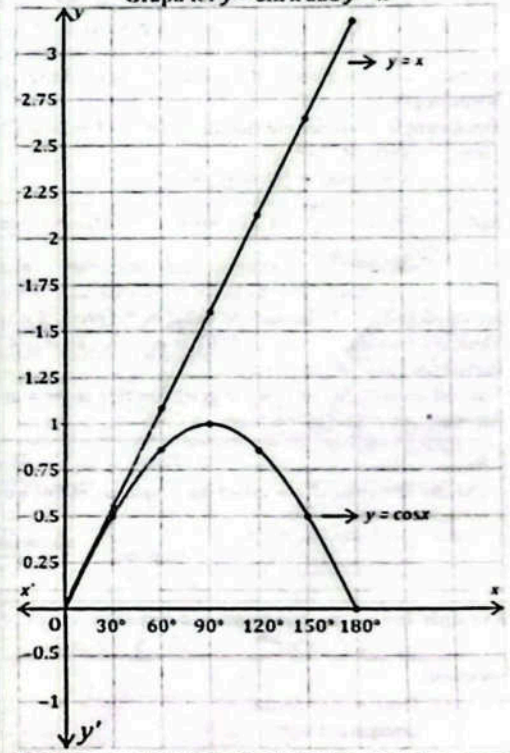
$150° = 150 \times \frac{\pi}{180}$

$= 2.62$ rad

$180° = 180 \times \frac{\pi}{180}$

$= 3.14$ rad

Graph for $y = \sin x$ and $y = x$



The two cures intersect each other at a point where $x = 0$

The solution of the equation $\sin x = x$ is $x = 0$

Maximum and Minimum Values of Given Functions of the Type:

- $a + b\sin\theta$
- $a + b\cos\theta$
- $a + b\sin(c\theta + d)$
- $a + b\cos(c\theta + d)$
- **The reciprocals of the above, where a, b, c and d are real numbers.**

The trigonometric functions like sine and cosine are periodic function because the values of these function repeat over regular intervals. These functions are fundamental in mathematics because of the repetition of their values at definite cycles and are used to model various real-life situations, such as radio waves, light wave, and alternating current in electricity and are also known as a specific case of sinusoidal functions.

Sinusoidal Functions: The functions of the form $f(\theta) = a + b\sin\theta$, $g(\theta) = a + b\cos\theta$, $f_1(\theta) = a + b\sin(c\theta + d)$ and $g_1(\theta) = a + b\cos(c\theta + d)$ are the types of sinusoidal functions.

Now consider the general form of sinusoidal function $f_1(\theta) = a + b\sin(c\theta + d)$ ...(i)

Here 'a' represent the **vertical shift** refers to the vertical translation of the graph of the function, achieved by shifting the entire graph upward or downward. This shift, also known as the vertical displacement, moves the function's position along the y-axis without altering its shape or period.

Amplitude $|b|$ is the maximum height of a wave measured from its midline.

The period of (i) is equal to $\frac{2\pi}{c}$.

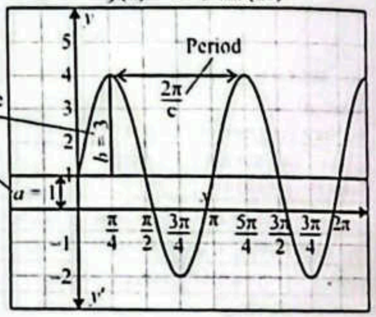
Phase shift 'd' indicates the horizontal translation of the graph of the function, determining how far the wave is shifted left or right along the x-axis. A positive d shifts the graph to the left, while a negative d shifts it to the right, altering the starting point of the wave without changing its shape or period.

For Example, consider the function $f(\theta) = 1 + 3\sin(2\theta)$.

Here Vertical Shift: $a = 1$

Amplitude $= |b| = |3| = 3$

and Period $= \frac{2\pi}{2} = \pi$ as shown in the adjacent figure.



Now, finding the maximum and minimum values of the functions $f(\theta) = a + b\sin(c\theta + d)$ and $g(\theta) = a + b\cos(c\theta + d)$ is not a difficult task. We know that the maximum absolute values of sine and cosine are equal to 1, so the maximum value of the product $b\sin\theta$ is $|b|$.

Thus, the maximum value of $f(\theta)$ or $g(\theta)$ is : $M = a + |b|$, whenever $\sin\theta = 1$ or $\cos\theta = 1$ where M denotes the maximum value of the function.

The minimum value of $f(\theta)$ or $g(\theta)$ function is: $m = a - |b|$, whenever $\sin\theta = -1$ or $\cos\theta = -1$ and m denotes the minimum value of the function.

> **Note:**
> The absolute value of b is called the Amplitude of $f(\theta) = a + b\sin\theta$. The value of the amplitude can also be determined using the formula
>
> $$\text{Amplitude} = \frac{\text{Maximum value} - \text{Minimum value}}{2}$$

Example 2: Find the maximum and minimum values of the following functions:

(i) $2 + 3\sin x$ (ii) $5 - 2\cos 3x$ (iii) reciprocal of (ii)

Solution:

(i) Let $f(x) = 2 + 3\sin x$

Compare it with

$f(x) = a + b\sin cx$

Here $a = 2$ and $b = 3$

Maximum value of $f(x)$: $M = a + |b| = 2 + 3 = 5$

Minimum value of $f(x)$: $m = a - |b| = 2 - 3 = -1$

Thus, maximum value of the function is 5 and the minimum value is -1

(ii) Let $f(x) = 5 - 2\cos 3x$

Compare it with

$f(x) = a + b\cos cx$

Here $a = 5$ and $b = -2$

Maximum value of the function: $M = a + |b| = 5 + |-2| = 5 + 2 = 7$.

Minimum value of the function: $m = a - |b| = 5 - |-2| = 5 - 2 = 3$.

Thus, maximum value of the function is 7 and the minimum value is 3.

(iii) reciprocal of part (ii)

The reciprocal of $5 - 2\cos 3x$ is $\frac{1}{5 - 2\cos 3x}$

Let $g(x) = \frac{1}{5 - 2\cos 3x}$

First, we will find the maximum and minimum values of $5 - 2\cos 3x$.

Let $f(x) = 5 - 2\cos 3x$

Compare it with

$f(x) = a + b\cos cx$

Here $a = 5$ and $b = -2$

Maximum value of the function: $M = a + |b| = 5 + |-2| = 5 + 2 = 7$.

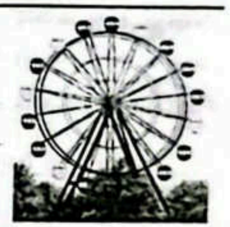
Minimum value of the function: $m = a - |b| = 5 - |-2| = 5 - 2 = 3$.

Now, Maximum value of $g(x) = \frac{1}{m} = \frac{1}{3} = 0.33$

and Minimum value of $g(x) = \frac{1}{M} = \frac{1}{7} = 0.14$

Real World Applications:

Ferris Wheel Problems



The first Ferris wheel was invented by George W. Ferris. He built the first one for 1893 World's Fair. A Ferris wheel is an important example of periodic motion that can be described using trigonometric functions, specifically sinusoidal functions. When we model the height of a rider on a Ferris wheel over time, we can use these functions to capture the periodic nature of the motion. The motion of Ferris wheel can be modeled by $f(t) = a + b\sin(ct + d)$ or $f(t) = a + b\cos(ct + d)$.

Example 3: A Ferris wheel with a radius of 45 feet has its lowest point located 5 feet above the ground. It completes one full revolution every 60 seconds in counter clock wise direction. Model an equation that describes the height of a rider on the Ferris wheel as a function of time t. How high is the rider from the ground after 40 seconds?. Also graph the model equation.

Solution:

Since it takes 60 seconds for the Ferris wheel to complete one full revolution (one cycle), which is the period of the Ferris wheel, that is

period $= 60$

$\frac{2\pi}{c} = 60 \Rightarrow c = \frac{2\pi}{60} \Rightarrow c = \frac{\pi}{30}$

Amplitude: b = radius of a Ferris wheel $b = 45$

Vertical Shift: a = height of the center of the Ferris wheel above the ground

$a = 5 + b = 5 + 45 = 50$ ∵ The lowest point is 5 feet above the ground

Since the rider starts at the lowest point and goes up, so

$h(t) = -b\cos(ct) + a$, where t is time and h is height.

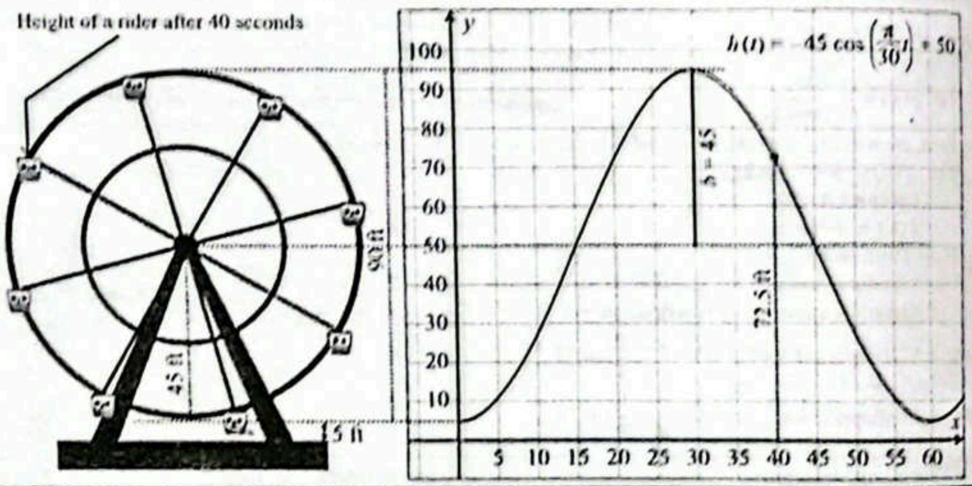
Now substituting the above values, we get

$h(t) = -45\cos\left(\frac{\pi}{30}t\right) + 50$, which is the required equation of Ferris wheel.

Next, we find the height of the rider at $t = 40$ seconds.

$$h(40) = -45\cos\left(\frac{\pi}{30}\cdot 40\right) + 50 = 72.5 \text{ feet}$$

Thus, height of rider after 40 second is 72.5 feet.
The graph of the model equation is shown below.



Example 4: The water level L (in feet) of a tidal river varies throughout the day. Suppose the level of the tidal river can be modeled by the equation: $L(t) = 8 + 4\sin\left(\frac{\pi}{6}t\right)$, where t denotes the time (in hours). The water level oscillates 4 feet above and below an average level of 8 feet.

(a) Find the water level at $t = 3$ hours?
(b) What is the minimum water level?

Solution:

(a) Given equation of water level: $L(t) = 8 + 4\sin\left(\frac{\pi}{6}t\right)$

To find the water level, substitute $t = 3$ into the equation

$$L(3) = 8 + 4\sin\left(\frac{\pi}{6}\cdot 3\right) = 8 + 4\sin\left(\frac{\pi}{2}\right)$$

$$L(3) = 8 + 4(1) = 12$$

Thus, water level at $t = 3$ hours is 12 feet.

(b) Now, to find the minimum water level, we need to determine when the sine function attains its minimum value.

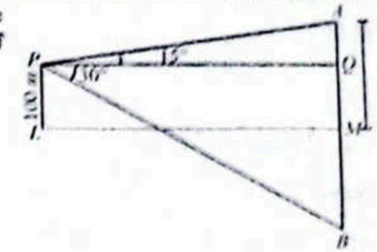
We know that the minimum value of $\sin t = -1$, substitute the $\sin\left(\frac{\pi}{6}t\right) = -1$ into the equation

$$L(t) = 8 + 4\sin\left(\frac{\pi}{6}t\right) = 8 + 4(-1) = 8 - 4 = 4$$

Thus, minimum water level of the tidal river is 4 feet.

Example 5: From a point 100 m above the surface of a lake, the angle of elevation of a peak of a cliff is found to be 15° and the angle of depression of the image of the peak is 30°. Find the height of the peak.

Solution: Let A be the top of the peak $\overline{AM}$ and $\overline{MB}$ be its image. Let P be the point of observation and L be the point just below P (on the surface of the lake).



From P, draw $\overline{PQ} \perp \overline{AM}$.

Let $m\overline{PQ} = y$ metres and $m\overline{AM} = h$ metres.

$\therefore\ m\overline{AQ} = h - m\overline{QM} = h - m\overline{PL} = h - 100$

From the figure,

$$\tan 15^\circ = \frac{AQ}{PQ} = \frac{h-100}{y} \quad \text{and} \quad \tan 30^\circ = \frac{BQ}{PQ} = \frac{100+h}{y}$$

By division, we get

$$\frac{\tan 15^\circ}{\tan 30^\circ} = \frac{h-100}{h+100}$$

By Componendo and Dividendo, we have

$$\frac{\tan 15^\circ + \tan 30^\circ}{\tan 15^\circ - \tan 30^\circ} = \frac{h-100+h+100}{h-100-h-100}$$

$$\frac{\tan 15^\circ + \tan 30^\circ}{\tan 15^\circ - \tan 30^\circ} = \frac{2h}{-200}$$

$$\frac{\tan 15^\circ + \tan 30^\circ}{\tan 30^\circ - \tan 15^\circ} = \frac{h}{100}$$

$$\therefore\ h = \frac{\tan 30^\circ + \tan 15^\circ}{\tan 30^\circ - \tan 15^\circ} \times 100 = \left[\frac{0.5774 + 0.2679}{0.5774 - 0.2679}\right] \times 100$$

$\Rightarrow\ h = 273.1179.$

Hence height of the peak = 273 m. (approximately)

Exercise 11.3

1. Find the maximum and minimum values of the following functions:

(i) $3 - \sin 3x$

Solution:

Let $f(x) = 3 - \sin 3x$

compare it with

$f(x) = a + b\sin(cx)$

Here $a = 3$, $b = -1$

Maximum value of $f(x): M = a + |b|$

$M = 3 + |-1| = 3 + 1 = 4$

Minimum value of $f(x): m = a - |b|$

$m = 3 - |-1| = 3 - 1 = 2$

Thus, $M = 4$ and $m = 2$

(ii) $3 + \sin 2x$

Solution:

Let $f(x) = 3 + \sin 2x$

Compare it with

$f(x) = a + b\sin(cx)$

Here $a = 3, b = 1$

Maximum value of $f(x): M = a + |b|$

$M = 3 + |1| = 3 + 1 = 4$

Minimum value of $f(x): m = a - |b|$

$m = 3 - |1| = 3 - 1 = 2$

Thus, $M = 4$ and $m = 2$

(iii) $\frac{1}{2} + \sin(5x + \pi)$

Solution:

Let $f(x) = \frac{1}{2} + \sin(5x + \pi)$

compare it with

$f(x) = a + b\sin(cx + d)$

Here, $a = \frac{1}{2}, b = 1$

Maximum value of $f(x): M = a + |b|$

$M = \frac{1}{2} + |1| = \frac{1}{2} + 1 = \frac{3}{2}$

Minimum value of $f(x): m = a - |b|$

$$m = \frac{1}{2} - |1| = \frac{1}{2} - 1 = -\frac{1}{2}$$

Thus, $M = \frac{3}{2}$ and $m = -\frac{1}{2}$

(iv) $\frac{3}{2} + \cos\left(x - \frac{\pi}{4}\right)$

Solution:

Let $f(x) = \frac{3}{2} + \cos\left(x - \frac{\pi}{4}\right)$

compare it with

$$f(x) = a + b\cos(cx + d)$$

Here, $a = \frac{3}{2}, b = 1$

Maximum value of $f(x): M = a + |b|$

$$M = \frac{3}{2} + |1| = \frac{3}{2} + 1 = \frac{5}{2}$$

Minimum value of $f(x): m = a - |b|$

$$m = \frac{3}{2} - |1| = \frac{3}{2} - 1 = \frac{1}{2}$$

Thus, $M = \frac{5}{2}$ and $m = \frac{1}{2}$

(v) $1 - 3\cos 2x$

Solution:

Let $f(x) = 1 - 3\cos 2x$

compare it with

$$f(x) = a + b\sin(cx)$$

Here $a = 1, b = -3$

Maximum value of $f(x): M = a + |b|$

$$M = 1 + |-3| = 1 + 3 = 4$$

Minimum value of $f(x): m = a - |b|$

$$m = 1 - |-3| = 1 - 3 = -2$$

Thus, $M = 4$ and $m = -2$

(vi) $1 + 2\sin\left(x + \frac{\pi}{6}\right)$

Solution:

Let $f(x) = 1 + 2\sin\left(x + \frac{\pi}{6}\right)$

compare it with

$$f(x) = a + b\sin(cx + d)$$

Here, $a = 1, b = 2$

Maximum value of $f(x): M = a + |b|$

$$M = 1 + |2| = 1 + 2 = 3$$

Minimum value of $f(x): m = a - |b|$

$$m = 1 - |2| = 1 - 2 = -1$$

Thus, $M = 3$ and $m = -1$

(vii) $\frac{1}{10 - 2\sin 3x}$

Solution:

Let $f(x) = \frac{1}{10 - 2\sin 3x}$

First, we will find the maximum and minimum values of $10 - 2\sin 3x$

Let, $g(x) = 10 - 2\sin 3x$

Here, $a = 10, b = -2$

Max. value of $g(x): M = a + |b|$

$$M = 10 + |-2| = 10 + 2 = 12$$

Min. value of $g(x): m = a - |b|$

$$m = 10 - |-2| = 10 - 2 = 8$$

Hence, Max. value of $f(x) = \frac{1}{m} = \frac{1}{8}$ and

Min. value of $f(x) = \frac{1}{M} = \frac{1}{12}$

(viii) $\frac{1}{7 + 3\cos(-2x)}$

Solution:

Let $f(x) = \frac{1}{7 + 3\cos(-2x)}$

First, we will find the maximum and minimum values of $7 + 3\cos(-2x)$.

Let $g(x) = 7 + 3\cos(-2x)$

Here, $a = 7, b = 3$

Max. value of $g(x): M = a + |b|$

$$M = 7 + |3| = 7 + 3 = 10$$

Min. value of $g(x): m = a - |b|$

$$m = 7 - |3| = 7 - 3 = 4$$

Hence, Max. value of $f(x) = \frac{1}{m} = \frac{1}{4}$ and

Min. value of $f(x) = \frac{1}{M} = \frac{1}{10}$

(ix) $\frac{1}{5 - 3\cos(3x - 1)}$

Solution:

Let $f(x) = \frac{1}{5 - 3\cos(3x - 1)}$

First, we will find the maximum and minimum values of $5 - 3\cos(3x - 1)$

Let $g(x) = 5 - 3\cos(3x - 1)$

Here, $a = 5, b = -3$

Max. value of $g(x): M = a + |b|$

$$m = 5 + |-3| = 5 + 3 = 8$$

Min. value of $g(x): m = a - |b|$

$$m = 5 - |-3| = 5 - 3 = 2$$

Hence, Max. value of $f(x) = \frac{1}{m} = \frac{1}{2}$ and

Min. value of $f(x) = \frac{1}{M} = \frac{1}{8}$.

2. The temperature T in a certain city varies throughout the day according to the equation $T(t) = \frac{13}{2}\sin\left(\frac{\pi}{6}t - \frac{\pi}{9}\right) + 15$, where t is the time in hours, with $t = 0$ corresponding to midnight.

(a) Find the maximum and minimum temperature during the day.

(b) Find the temperature at $t = 9$ hours (9:00 a.m.).

Solution:

Temperature function: $T(t) = 15 + \frac{13}{2}\sin\left(\frac{\pi}{6}t - \frac{\pi}{9}\right)$

compare it with

$$T(t) = a + b\sin(ct + d)$$

Here, $a = 15, b = \frac{13}{2}$

(a)

Max. value of $T(t): M = a + |b|$

$$M = 15 + \left|\frac{13}{2}\right| = 15 + \frac{13}{2} = \frac{43}{2} = 21.5°C$$

Min. value of $T(t): m = a - |b|$

$$m = 15 - \left|\frac{13}{2}\right| = 15 - \frac{13}{2} = \frac{17}{2} = 8.5°C$$

(b)

$T = ?$, $t = 9$ hours (9:00 am)

put $t = 9$ in temperature function

$$T(9) = 15 + \frac{13}{2}\sin\left(\frac{\pi}{6}(9) - \frac{\pi}{9}\right)$$

$$= 15 + \frac{13}{2}\sin\left(\frac{3\pi}{2} - \frac{\pi}{9}\right)$$

$$= 15 - \frac{13}{2}\cos\frac{\pi}{9} \quad \because \frac{3\pi}{2} - \frac{\pi}{9} = 250° \text{ lies in III quad}$$

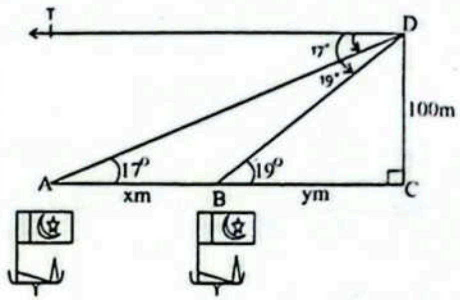
$$= 15 - \frac{13}{2}(0.9397)$$

$$= 15 - 6.108$$

$$T(9) = 8.892°C$$

3. A man on the top of a 100 m high light-house is in line with two ships on the same side of it, whose angles of depression from the man are 17° and 19° respectively. Find the distance between the ships.

Solution:



Let distance between ships $= AB = x$ m

Clearly $m\angle CAD = m\angle ADT = 17°$

and $m\angle CBD = m\angle BDT = 19°$

In right ΔBCD

$$\tan 19° = \frac{CD}{BC}$$

$$0.3443 = \frac{100}{y}$$

$$y = \frac{100}{0.3443} = 290.42\text{m}$$

Now, in right ΔACD

$$\tan 17° = \frac{CD}{AC}$$

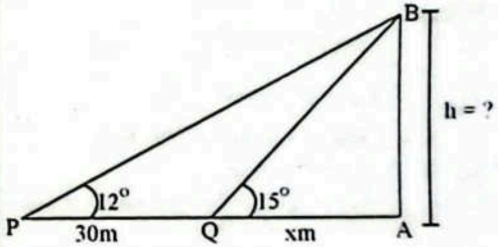
$$0.3057 = \frac{100}{x + y} \Rightarrow x + y = \frac{100}{0.3057}$$

$$x = 327.12 - 290.42 \quad \because y = 290.42\text{m}$$

$$x = 36.70\text{m}$$

4. P and Q are two points in line with a tree. If the distance between P and Q be 30 m and the angles of elevation of the top of the tree at P and Q are 12° and 15° respectively, find the height of the tree.

Solution:



Let the height of tree $= AB = h$ m

In right ΔQAB

$$\tan 15° = \frac{AB}{AQ}$$

$$\tan 15° = \frac{h}{x}$$

$$x = \frac{h}{\tan 15°} \quad \ldots(i)$$

Now, in right ΔPAB

$$\tan 12° = \frac{AB}{PA}$$

$$0.2126 = \frac{h}{30+x} \quad \Rightarrow h = (30+x)(0.2126)$$

$$\Rightarrow h = 6.378 + 0.2126x$$

$$h = 6.378 + 0.2126\left(\frac{h}{\tan 15°}\right) \quad \text{using (i)}$$

$$h = 6.378 + 0.7934h \quad \Rightarrow h - 0.7934h = 6.378$$

$$\Rightarrow 0.2066h = 6.378 \quad \Rightarrow h = 30.87\text{m}$$

5. A giant Ferris wheel has a diameter of 60 feet. The lowest point of the wheel is located 6 feet above the ground. The wheel completes one full revolution every 80 seconds.

(a) Model an equation that represent the height $h(t)$ of a rider on the Ferris wheel at any given time t.

(b) Find the maximum height of the rider.

(c) Find the height of the rider from the ground after 35 seconds.

Solution:

(a) Since the Ferris wheel takes 80 seconds to complete one full revolution (one cycle), so

Period = 80

$$\frac{2\pi}{c} = 80 \quad \Rightarrow \quad c = \frac{\pi}{40}$$

Amplitude: b = radius of wheel

$$b = \frac{1}{2}(\text{diameter}) = \frac{1}{2}(60) = 30$$

Vertical shift: $a = 6 + b = 6 + 30 = 36$

Since the rider starts at the lowest point and goes up, so The equation for the height $h(t)$ is

$$h(t) = -b\cos(ct) + a$$

$$h(t) = -30\cos\left(\frac{\pi}{40}t\right) + 36 \quad \ldots(1)$$

(b)
$$h(t) = 36 - 30\cos\left(\frac{\pi}{40}t\right)$$

Max. height: $M = a + |b| = 36 + 30 = 66$ ft.

(c) $h = ?$, $t = 30$ seconds

$$h(35) = -30\cos\left(\frac{\pi}{40}(35)\right) + 36 \quad \text{Using (1)}$$

$$= -30\cos\left(\frac{7\pi}{8}\right) + 36$$

$$= -30(-0.9239) + 36$$

$$= 27.72 + 36$$

$$h(35) = 63.72 \text{ ft}$$

6. A child is playing on a swing in a playground. The height $h(t)$ of the swing seat above the ground (in metres) at time t (in seconds) is modeled by the function: $h(t) = 1.5 + 1.2\sin(3\pi t)$

(a) What is the maximum height reached by the swing seat?

(b) What is the minimum height reached by the swing seat?

(c) How long does it take for the swing to complete one full back-and-forth motion (period)?

(d) At what time(s) does the swing seat first reach a height of 2.12 metres?

Solution:

(a) Swing height function is

$$h(t) = 1.5 + 1.2\sin(3\pi t)$$

Compare it with

$$h(t) = a + b\sin(ct)$$

Here $a = 1.5, b = 1.2, c = 3\pi$

(a) Max. Height:

$$M = a + |b|$$

$$M = 1.5 + 1.2 = 2.7 \text{ metres}$$

(b) Min. Height:

$$m = a - |b|$$

$$m = 1.5 - 1.2 = 0.3 \text{ metres}$$

(c) The swing will take time to complete one full back-and-forth motion

$$t = \text{Period}$$

$$t = \frac{2\pi}{3\pi} = \frac{2}{3} \text{ seconds}$$

(d)
$$h(t) = 2.12 \text{ metres}, \quad t = ?$$

$$h(t) = 1.5 + 1.2\sin(3\pi t)$$

$$2.12 = 1.5 + 1.2\sin(3\pi t)$$

$$2.12 - 1.5 = 1.2\sin(3\pi t)$$

$$\frac{0.62}{1.2} = \sin(3\pi t)$$

$$\sin(3\pi t) = 0.5167$$

$$3\pi t = \sin^{-1}(0.5167)$$

$$t = \frac{1}{3\pi}\sin^{-1}(0.5167)$$

$$t = 0.058 \text{ seconds}$$

7. A carnival ride consists of a vertical wheel with a diameter of 40 feet. The centre of the wheel is 28 feet above the ground. The wheel rotates at a constant speed and takes 120 seconds to make one complete revolution. Model an equation that describes the height $h(t)$ of a rider on the wheel as a function of time t. How high is the rider from the ground after 90 seconds? At what times will the rider be 36 feet above the ground?

Solution:

Since the wheel takes 120 seconds to make one complete revolution, so

Period = 120

$$\frac{2\pi}{c} = 120 \quad \Rightarrow \quad c = \frac{\pi}{60}$$

Amplitude: b = radius of wheel

$$= \frac{1}{2}(\text{diameter}) = \frac{1}{2}(40) = 20$$

Vertical shift:

$$a = 28$$

Since the rider starts at the lowest point and goes up, so the equation for the $h(t)$ is

$$h(t) = -b\cos(ct) + a$$

$$\boxed{h(t) = -20\cos\left(\frac{\pi}{60}t\right) + 28}$$

Height of rider after 90 seconds = $h(90)$

$$= -20\cos\left(\frac{\pi}{60}(90)\right) + 28$$

$$= -20\cos\left(\frac{3\pi}{2}\right) + 28$$

$$= -20(0) + 28$$

$$= 28 \text{ feet}$$

Now, we will find the time, when $h(t) = 36$ ft

As
$$h(t) = -20\cos\left(\frac{\pi}{60}t\right) + 28$$

$$\Rightarrow \quad 36 = -20\cos\left(\frac{\pi}{60}t\right) + 28$$

$$36 - 28 = -20\cos\left(\frac{\pi}{60}t\right)$$

$$\frac{-8}{20} = \cos\left(\frac{\pi}{60}t\right)$$

$$\frac{\pi}{60}t = \cos^{-1}\left(\frac{-2}{5}\right)$$

Since cos function is negative in II and III-quad., so

II-quad	III-quad
$\frac{\pi}{60}t = \pi - \cos^{-1}\left(\frac{2}{5}\right)$	$\frac{\pi}{60}t = \pi + \cos^{-1}\left(\frac{2}{5}\right)$
$t = \frac{60}{\pi}\left(\pi - \cos^{-1}\frac{2}{5}\right)$	$t = \frac{60}{\pi}\left(\pi + \cos^{-1}\frac{2}{5}\right)$
$t = 60 - \frac{60}{\pi}\cos^{-1}\frac{2}{5}$	$t = 60 + \frac{60}{\pi}\cos^{-1}\frac{2}{5}$
$t = 37.86$ seconds	$t = 82.14$ seconds

8. Suppose the temperature T in degrees Fahrenheit of Lahore city in a month of December throughout the day can be modeled by the equation: $T = 64 + 8\sin\left(\frac{\pi}{12}(t-8)\right)$, where t is the time in hours. The temperature oscillates 8 degrees above and below an average temperature of 64 degrees.

(a) Find the temperature at $t = 9$ hours?

(b) At what time the temperature will be maximum?

(c) Calculate the maximum temperature.

Solution:

Temperature equation

$$T = 64 + 8\sin\left(\frac{\pi}{12}(t-8)\right) \quad \ldots(1)$$

(a) $T = ?$ $\quad t = 9$ hours

$$T = 64 + 8\sin\left(\frac{\pi}{12}(9-8)\right) \quad \text{using (1)}$$

$$= 64 + 8\sin\frac{\pi}{12}$$

$$= 64 + 8(0.2588)$$

$$= 64 + 2.0704$$

$$= 66.07°\text{F}$$

(b) Temperature will be maximum, if

$$\sin\left(\frac{\pi}{12}(t-8)\right) = 1$$

$$\frac{\pi}{12}(t-8) = \sin^{-1}(1)$$

$$t - 8 = \frac{12}{\pi}\left(\frac{\pi}{2}\right)$$

$$t = 6 + 8$$

$t = 14$ hours or 2 p.m.

(c)
$$T = 64 + 8\sin\left(\frac{\pi}{12}(t-8)\right)$$

Here $a = 64, b = 8$

Max. temperature: $M = a + |b| = 64 + 8 = 72°\text{F}$

9. Suppose the population of a coastal city follows a sinusoidal pattern due to seasonal migration. The population of the city over the course of a year can be modeled by the equation: $P(t) = 70000 + 10000\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right)$, $P(t)$ is the population at time t (t is the time in months, with $t = 0$ corresponding to January 1st). where t denoted the months in a year.

(a) Find the population of the city at $t = 7$ months.

(b) Find the maximum population.

Solution:

Population equation

$$P(t) = 70000 + 10000\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) \quad \ldots(1)$$

(a) Population at $t = 7$ months = $P(7)$

$$= 70000 + 10000\cos\left(\frac{7\pi}{6} - \frac{\pi}{2}\right)$$

$$= 70000 + 10000\cos\left(\frac{7\pi - 3\pi}{6}\right)$$

$= 70000 + 10000\cos\left(\frac{4\pi}{6}\right)$

$= 70000 + 10000\cos\left(\frac{2\pi}{3}\right)$

$= 70000 + 10000\left(\frac{-1}{2}\right)$

$= 70000 - 5000$

$P(7) = 65000$

(b) $P(t) = 70000 + 10000\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right)$

Here $a = 70000$, $b = 10000$

Max. population:

$M = a + |b|$

$M = 70000 + 10000 = 80000$

Formula Sheet

Sinusoidal Functions			
1. $f(\theta) = a + b\sin\theta$,	2. $g(\theta) = a + b\cos\theta$,	3. $f_1(\theta) = a + b\sin(c\theta + d)$	4. $g_1(\theta) = a + b\cos(c\theta + d)$
5. For $f(\theta) = a + b\sin(c\theta + d)$, where a = vertical shift, $\|b\|$ = amplitude, d = phase shift, $\frac{2\pi}{c}$ = period			
(i) Maximum value of $f(\theta) = M = a + \|b\|$		(ii) Minimum value of $f(\theta) = m = a - \|b\|$	
6. Amplitude = $\frac{\text{Maximum value} - \text{Minimum value}}{2}$			

Multiple Choice Questions (MCQs)

Exercise 11.1

1. **Domain of $y = \cos x$ is -----**
   (A) $-\infty < x < \infty$ (B) $-1 \le x \le 1$
   (C) $R - \{x \mid x = n\pi\}$ $n \in Z$ (D) $R - \{x \mid x = 2n\pi\}$ $n \in Z$
2. **Range of $\tan x$ is -----**
   (A) $-\infty < y < \infty$ (B) $-\infty < y < 0$ (C) $\infty \le y \le \infty$ (D) $0 < y < 2\pi$
3. **If $f(x + p) = f(x)$, where 'p' is the smallest positive number then 'p' is called -----**
   (A) domain of f(x) (B) codomain of f(x) (C) period of f(x) (D) range of f(x)
4. **Period of $3 + \tan\frac{x}{3}$ is:**
   (A) 3π (B) $\frac{3\pi}{2}$ (C) $3 + 3\pi$ (D) $3 - 3\pi$
5. **$f(x) = x^{2/3}$ is a/an -----**
   (A) even function (B) odd function (C) neither even nor odd (D) cubic function
6. **If $f(-x) = -f(x)$ $\forall x \in \text{dom}(f)$, then the function f(x) is called -----**
   (A) linear function (B) periodic function (C) odd function (D) even function
7. **$f(x) = x^3 + \sin x$ then f(x) is -----**
   (A) constant function (B) even function (C) odd function (D) neither even nor odd

Exercise 11.2

8. **In general form of sinusoidal function $f(\theta) = a + b\sin(c\theta + d)$, is called -----**
   (A) phase shift (B) vertical shift (C) amplitude (D) period
9. **Maximum value of the function $f(\theta) = a + b\cos(c\theta + d)$ is -----**
   (A) $a - |b|$ (B) $\frac{2\pi}{c}$ (C) $a + |b|$ (D) 1
10. **Maximum value of the function $5 - 2\cos 3x$ is -----**
   (A) 3 (B) 7 (C) 9 (D) 2

ANSWER KEY

1.	A	2.	A	3.	C	4.	A	5.	A	6.	C	7.	C	8.	B	9.	C	10.	B

Unit 12 Limit and Continuity

Introduction

In mathematics, the concept of limit and continuity is foundational in understanding the behaviour of functions and sequences, especially when applied to real-world scenarios. This unit will introduce and explore how to demonstrate and find the limit of a sequence and a function, understand continuous and discontinuous functions, and apply these concepts in various contexts such as economics, finance, and natural sciences

Limit of a Function

The concept of limit of a function is the basis on which the structure of calculus rests. Before the definition of the limit of a function, it is necessary to have a clear understanding of the following phrases.

Meaning of the Phrase "x approaches zero":

Suppose a sequence $x_n = \frac{1}{n^2}$ assumes a sequence of values as:

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots, \frac{1}{2^n}, \ldots$$

We can see that the sequence $x_n = \frac{1}{n^2}$ is becoming smaller and smaller as n increases and can be made as small as we please by taking "n" sufficiently large. In other words, $x_n = \frac{1}{n^2}$ becoming closer and closer to 0 as n becoming large. This unending decrease of x_n is denoted by $x_n \to 0$ and read as "x_n approaches zero" or "x_n tends to zero as $n \to \infty$." That is, the limit of the sequence x_n is 0.

Meaning of the Phrase "x approaches infinity"

Suppose a sequence $x_n = 10^n$ assumes values as $1, 10, 10^2, 10^3, \ldots, 10^n, \ldots$

It is clear that the sequence x_n is becoming larger and larger as n increases and can be made as large as we please by taking n sufficiently large. This unending increase of the sequence x_n is symbolically written as "$x_n \to \infty$" and is read as "x_n approaches infinity" or "x_n tends to infinity as $n \to \infty$." That is, the limit of the sequence x_n is ∞.

Meaning of the Phrase "x approaches a":

Symbolically it is written as "$x \to a$" which means that x is sufficiently close to a but different from the number a, from both the left and right sides of a that is $x - a$ becomes smaller and smaller as we please but $x - a \neq 0$.

> **Point to remember:**
> The symbol $x \to 0$ is quite different from $x = 0$.
> $x \to 0$ means that x is very close to zero but not actually zero.
> $x = 0$ means that x is actually zero.

Concept of Limit of a Function:

(i) By Finding the Area of Circumscribed Regular Polygon:

Consider a circle of unit radius which circumscribes a square (4-sided regular polygon) as shown in Figure 12.1.

The side of square is $\sqrt{2}$ and its area is 2 square units. It is clear that the area of inscribed 4-sided polygon is less than the area of the circumcircle $\pi = 3.142$ $\because$ (Area $= \pi r^2 = \pi(1)^2 = \pi = 3.142$).

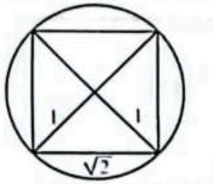


Figure 12.1: 4-sided polygon

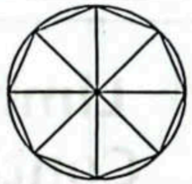


Figure 12.2: 8-sided polygon

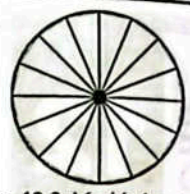


Figure 12.3: 16-sided polygon

Bisecting the arcs between the vertices of the square, we get an inscribed 8 -sided regular polygon as shown in Figure 12.2. Its area is $2\sqrt{2} = 2.828$ square units which is closer to the area of circum-circle. A further similar bisection of the arcs gives an inscribed 16-sided regular polygon as shown in Figure 12.3 with area 3.061 square units which is more closer to the area of circumcircle.

It follows that as "n", the number of sides of the inscribed polygon increases, the area of polygon increases and becoming near to 3.142 which is the area of circle of unit radius.

We express this situation by saying that the limiting value of the area of the inscribed polygon is the area of the circle as n approaches infinity, i.e.,

Area of inscribed polygon → Area of circle as $n \to \infty$

(ii) Numerical Approach

Consider the function: $f(x) = x^3$,

Domain $f = R$ = The set of all real numbers

Let us find the limit of $f(x) = x^3$ as x approaches 2.

The table of values of $f(x)$ for different values of x as x approaches 2 from left and right is as follows:

From left of 2 ⟶ 2 ⟶ from right of 2

x	1	1.5	1.8	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1	2.2	2.5	3
$f(x) = x^3$	1	3.375	5.832	6.859	7.8806	7.8806	7.9988	8.0012	8.012	8.1206	9.261	10.648	15.625	27

The table shows that, as x gets closer and closer to 2 (sufficiently close to 2), from both sides, $f(x)$ gets closer and closer to 8.

We say that 8 is the limit of $f(x)$ when x approaches 2 and is written as:

$$f(x) \to 8 \text{ as } x \to 2 \text{ or } \lim_{x\to 2}(x^3) = 8$$

Limit of a Function:

Let a function $f(x)$ be defined in an open interval near the number "a" (need not be at a). If, as x approaches "a" from both left and right side of "a" $f(x)$ approaches a specific number "L" then "L", is called the limit of $f(x)$ as x approaches a. Symbolically it is written as:

$$\lim_{x\to a} f(x) = L \text{ read as "limit of } f(x) \text{ as } x \to a, \text{ is } L\text{"}.$$

- It is neither desirable nor practicable to find the limit of a function by numerical approach. We must be able to evaluate a limit in some mechanical way. The theorems on limits will serve this purpose. Their proofs will be discussed in higher classes.

Theorems on Limits of Functions:

Let f and g be two functions for which $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

Theorem 1: (i) $\lim_{x\to a} x^p = a^p$, where $p > 0$ and $p \in R$

(ii) $\lim_{x\to a} c = c$

Theorem 2: (a) The limit of the sum of two functions is equal to the sum of their limits.

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

For example, $\lim_{x\to 1}(x+5) = \lim_{x\to 1} x + \lim_{x\to a} 5 = 1 + 5 = 6$

(b) The limit of the difference of two functions is equal to the difference of their limits.

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$$

For example, $\lim_{x\to 3}(x-5) = \lim_{x\to 3}(x) - \lim_{x\to 3} 5 = 3 - 5 = -2$

(c) If k is any real number, then

$$\lim_{x\to a}[kf(x)] = k\lim_{x\to a} f(x) = kL$$

For example, $\lim_{x\to 2}(3x) = 3\lim_{x\to 2}(x) = 3(2) = 6$

(d) The limit of the product of the functions is equal to the product of their limits.

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) = L.M$$

For example, $\lim_{x\to 1}(2x)(x+4) = \lim_{x\to 1}(2x)\cdot\lim_{x\to 1}(x+4) = (2)(5) = 10$

(e) The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of denominator is non-zero.

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \text{ provided } g(x) \neq 0 \text{ in a neighborhood of } a \text{ and } M \neq 0$$

For example: $\lim_{x\to 2}\left[\frac{3x+4}{x+3}\right] = \frac{\lim_{x\to 2}(3x+4)}{\lim_{x\to 2}(x+3)} = \frac{6+4}{2+3} = \frac{10}{5} = 2$

(f) Limit of $[f(x)]^n$, where n is an integer

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$$

For example, $\lim_{x\to 4}(2x-3)^3 = (\lim_{x\to 4}(2x-3))^3 = (5)^3 = 125$

We conclude from the theorems on limits that limits are evaluated by merely substituting the number that x approaches into the function.

Limits of Important Functions:

If by substituting the number that x approaches into the function, we get $\left(\frac{0}{0}\right)$, then one possible way to evaluate the limits is as follows:

We simplify the given function by using algebraic techniques of making factors if possible and cancel the common factors. The method explained in the following important limits.

Question 1: $\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$ **where n is a non-zero integer and $a > 0$.**

Case I: Suppose n is a positive integer.

$$\text{L.H.S} = \lim_{x\to a}\frac{x^n-a^n}{x-a} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-1})}{x-a} \quad \text{By Factorization of } (x^n - a^n)$$

$$= \lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-1})$$

$$= a^{n-1}+aa^{n-2}+a^2a^{n-3}+a^3a^{n-4}+\ldots+a^{n-1}$$

$$= a^{n-1}+a^{n-1}+a^{n-1}+a^{n-1}\ldots+a^{n-1} = na^{n-1} = \text{R.H.S}$$

Case II: Suppose n is a negative integer (say $= -m$) where m is a positive integer.

$$\text{L.H.S} = \lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$= \lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a} = \lim_{x\to a}\frac{\frac{1}{x^m}-\frac{1}{a^m}}{x-a} = \lim_{x\to a}\frac{\frac{a^m-x^m}{x^m a^m}}{x-a}$$

$$= \lim_{x\to a}\left(\frac{-1}{x^m a^m}\right)\left(\frac{x^m-a^m}{x-a}\right)$$

$$= \lim_{x\to a}\left(\frac{-1}{x^m a^m}\right).\lim_{x\to a}\left(\frac{x^m-a^m}{x-a}\right)$$

$$= \frac{-1}{a^m a^m}(ma^{m-1}) \qquad \text{(by case-I)}$$

$$= -ma^{m-1-m-m} = -ma^{-m-1}$$

$$= na^{n-1} = \text{R.H.S} \qquad \because n=-m$$

Hence proved L.H.S = R.H.S

Question 2: $\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x} = \frac{1}{2\sqrt{a}}$, **where n is an integer and $a>0$.**

$$\text{L.H.S} = \lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x} \qquad \left(\frac{0}{0}\right)\text{ form}$$

$$= \lim_{x\to 0}\left(\frac{\sqrt{x+a}-\sqrt{a}}{x}\right)\left(\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right) \qquad \text{By Rationalizing}$$

$$= \lim_{x\to 0}\frac{\left(\sqrt{x+a}\right)^2-\left(\sqrt{a}\right)^2}{x\left(\sqrt{x+a}+\sqrt{a}\right)} = \lim_{x\to 0}\frac{x+a-a}{x\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$= \lim_{x\to 0}\frac{x}{x\left(\sqrt{x+a}+\sqrt{a}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+a}+\sqrt{a}} = \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$$

Example 1: Evaluate: (i) $\lim_{x\to 1}\frac{x^2-1}{x^2-x}$ (ii) $\lim_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}}$

Solution:

(i) $\lim_{x\to 1}\frac{x^2-1}{x^2-x}$ $\left(\frac{0}{0}\right)$ form

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x(x-1)} \qquad \text{using } a^2-b^2=(a+b)(a-b)$$

$$= \lim_{x\to 1}\frac{x+1}{x} = \frac{1+1}{1} = \frac{2}{1} = 2$$

(ii) $\lim_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}}$ $\left(\frac{0}{0}\right)$ form

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}}\times\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} \qquad \text{By Rationalizing}$$

$$= \lim_{x\to 3}\frac{(x-3)(\sqrt{x}+\sqrt{3})}{(\sqrt{x})^2-(\sqrt{3})^2} = \lim_{x\to 3}\frac{(x-3)(\sqrt{x}+\sqrt{3})}{(x-3)}$$

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3}(\sqrt{x}+\sqrt{3}) = \sqrt{3}+\sqrt{3} = 2\sqrt{3}$$

Limit at Infinity:

We have studied the limits of the functions $f(x), f(x).g(x)$ and $\frac{f(x)}{g(x)}$, when $x\to c$ (a number)

Let us see what happens to the limit of the function $f(x)$ if c is $+\infty$ or $-\infty$ (limits at infinity) i.e., when $x\to+\infty$ or $x\to-\infty$.

(a) Limit as $x\to+\infty$

Let $f(x)=\frac{1}{x}$, when $x\neq 0$

This function has the property that the value of $f(x)$ can be made as close as we please to zero when the number x is sufficiently large.

We express this phenomenon by writing $\lim_{x\to\infty}\frac{1}{x}=0$

(b) Limit as $x\to-\infty$

This type of limits are handled in the same way as limits as $x\to+\infty$. i.e., $\lim_{x\to-\infty}\frac{1}{x}=0$, where $x\neq 0$.

The following theorem is useful for evaluating limit at infinity.

Theorem: Let p be a positive rational number. If x^p is defined, then

$\lim_{x\to+\infty}\frac{a}{x^p}=0$ and $\lim_{x\to-\infty}\frac{a}{x^p}=0$, where a is any real number.

For example, $\lim_{x\to\pm\infty}\frac{6}{x^3}=0$ and $\lim_{x\to-\infty}\frac{-5}{\sqrt[3]{x}}=0$

Limit of a Sequence:

Let $\{a_n\}$ be a sequence, the limit of a sequence $\{a_n\}$ is the value L that the terms of the sequence approach as $n\to\infty$, that is,

$$\lim_{n\to\infty}a_n = L$$

If such an L exists, the sequence is said to converge to L and $\{a_n\}$ is called convergent sequence. If no such L exists, the sequence is said to diverge.

For example, consider the sequence $\left\{a_n=\frac{1}{n}\right\}$: As $n\to\infty, \frac{1}{n}\to 0$

So, we write $\lim_{n\to\infty}a_n = \lim_{n\to\infty}\frac{1}{n}=0$.

Example 2: Find the limit of the sequence $a_n=\frac{2n+3}{n+1}$.

Solution:

Consider: $\lim_{n\to\infty}a_n = \lim_{n\to\infty}\frac{2n+3}{n+1}$

Dividing numerator and denominator by n, we get

$$\lim_{n\to\infty}a_n = \lim_{n\to\infty}\left(\frac{\frac{2n}{n}+\frac{3}{n}}{\frac{n}{n}+\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}}{1+\frac{1}{n}}\right) = \frac{2+0}{1+0} \qquad \because \lim_{n\to\infty}\frac{1}{n}=0$$

$$\lim_{n\to\infty}a_n = \frac{2}{1} = 2$$