Now, $\Delta AOB \cong \Delta COD$ [(SAS) theorem]

Therefore, $|\overline{AB}|=|\overline{CD}|$

$\Rightarrow$ $|\overline{AB}|^2=|\overline{CD}|^2$ Squaring both sides

Using the distance formula, we have:

$$\left[\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}\right]^2=\left[\sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2}\right]^2$$

$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2$$

$$\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta+\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta=\cos^2(\alpha-\beta)+1-2\cos(\alpha-\beta)+\sin^2(\alpha-\beta)$$

$$1-2\cos\alpha\cos\beta+1-2\sin\alpha\sin\beta=1+1-2\cos(\alpha-\beta) \quad \because \sin^2\theta+\cos^2\theta=1$$

$$2-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=2-2\cos(\alpha-\beta)$$

Hence, $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

> **Note:**
> Although we have proved this law for $\alpha>\beta>0$, it is true for all values of α and β.

Suppose we know the values of sin and cos of two angles α and β, we can find $\cos(\alpha-\beta)$ using this law as explained in the following example:

Example 2: Find the value of $\sin\frac{5\pi}{12}$.

Solution: As $\frac{5\pi}{12}=75°=45°+30°=\frac{\pi}{4}+\frac{\pi}{6}$

$$\therefore \quad \sin\frac{5\pi}{12}=\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

Deductions from Fundamental Law:

1. We know that:

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Putting $\alpha=\frac{\pi}{2}$ in it, we get

$$\cos\left(\frac{\pi}{2}-\beta\right)=\cos\frac{\pi}{2}\cos\beta+\sin\frac{\pi}{2}\sin\beta$$

$$\Rightarrow \cos\left(\frac{\pi}{2}-\beta\right)=0\cdot\cos\beta+1\cdot\sin\beta \quad \left(\because \cos\frac{\pi}{2}=0,\ \sin\frac{\pi}{2}=1\right)$$

$$\therefore \boxed{\cos\left(\frac{\pi}{2}-\beta\right)=\sin\beta} \quad ...(i)$$

2. We know that:

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Putting $\beta=-\frac{\pi}{2}$ in it, we get

$$\cos\left[\alpha-\left(-\frac{\pi}{2}\right)\right]=\cos\alpha\cdot\cos\left(-\frac{\pi}{2}\right)+\sin\alpha.\sin\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \cos\left(\alpha+\frac{\pi}{2}\right)=\cos\alpha\cdot 0+\sin\alpha(-1) \quad \because \sin\left(-\frac{\pi}{2}\right)=-1 \text{ and } \cos\left(-\frac{\pi}{2}\right)=0$$

$$\therefore \boxed{\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha} \quad ...(ii)$$

3. We know that:

$$\cos\left(\frac{\pi}{2}-\beta\right)=\sin\beta \quad \text{[(i) above]}$$

Putting $\beta=\frac{\pi}{2}+\alpha$ in it, we get

$$\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}+\alpha\right)\right]=\sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \cos(-\alpha)=\sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \cos\alpha=\sin\left(\frac{\pi}{2}+\alpha\right) \quad [\because \cos(-\alpha)=\cos\alpha]$$

$$\boxed{\sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha} \quad ...(iii)$$

4. We know that:

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Replacing β by $-\beta$ we get

$$\cos[\alpha-(-\beta)]=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta) \quad [\because \cos(-\beta)=\cos\beta,\ \sin(-\beta)=-\sin\beta]$$

$$\Rightarrow \boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta} \quad ...(iv)$$

5. We know that:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

Replacing α by $\frac{\pi}{2}+\alpha$, we get

$$\cos\left[\left(\frac{\pi}{2}+\alpha\right)+\beta\right]=\cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$$

$$\Rightarrow \cos\left[\frac{\pi}{2}+(\alpha+\beta)\right]=-\sin\alpha.\cos\beta-\cos\alpha.\sin\beta$$

$$\Rightarrow -\sin(\alpha+\beta)=-[\sin\alpha\cos\beta+\cos\alpha\sin\beta]$$

$$\therefore \boxed{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta} \quad ...(v)$$

6. We know that:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \quad \text{[from (v) above]}$$

Replacing β by $-\beta$, we get

$$\sin(\alpha-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta) \quad \because \sin(-\beta)=-\sin\beta \text{ and } \cos(-\beta)=\cos\beta$$

$$\therefore \boxed{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta} \quad ...(vi)$$

7. We know that:

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Let $\alpha=2\pi$ and $\beta=\theta$

$$\therefore \cos(2\pi-\theta)=\cos2\pi\cdot\cos\theta+\sin2\pi.\sin\theta$$

$$=1\cdot\cos\theta+0\cdot\sin\theta \quad \because \cos2\pi=1,\ \sin2\pi=0$$

$$\boxed{\cos(2\pi-\theta)=\cos\theta} \quad ...(vii)$$

8. We know that:

$$\sin(\alpha-\beta) = \sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta$$

Let $\alpha = 2\pi$ and $\beta = \theta$

$$\sin(2\pi-\theta) = \sin2\pi\cdot\cos\theta-\cos2\pi.\sin\theta$$

$$= 0\cdot\cos\theta-1\cdot\sin\theta \qquad \because \sin2\pi=0,\ \cos2\pi=1$$

$$\boxed{\sin(2\pi-\theta) = -\sin\theta} \qquad ...(viii)$$

9. $$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \qquad \text{Dividing up and down by } \cos\alpha\cos\beta$$

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}} \qquad ...(ix)$$

10. $$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

$$= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \qquad \text{Dividing up and down by } \cos\alpha\cos\beta$$

$$\boxed{\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}} \qquad ...(x)$$

Trigonometric Ratios of Allied Angles:

Two angles α and β are said to be allied, if $\alpha \pm \beta = n(90^\circ), n \in Z$

For example, $\pm\alpha$, $90^\circ \pm \alpha$, $180^\circ \pm \alpha$, $270^\circ \pm \alpha$ and $360^\circ \pm \alpha$ are some allied angles of α.

Using fundamental law of trigonometry, $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and its deductions, we derive the following identities:

$$\begin{cases}\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,\ \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta,\ \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta\\ \sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta,\ \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta,\ \tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta\end{cases}$$

$$\begin{cases}\sin(\pi-\theta)=\sin\theta, & \cos(\pi-\theta)=-\cos\theta, & \tan(\pi-\theta)=-\tan\theta\\ \sin(\pi+\theta)=-\sin\theta, & \cos(\pi+\theta)=-\cos\theta, & \tan(\pi+\theta)=\tan\theta\end{cases}$$

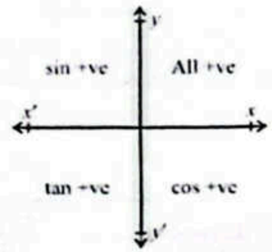
$$\begin{cases}\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta,\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta,\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta\\ \sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta,\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta,\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta\end{cases}$$

$$\begin{cases}\sin(2\pi-\theta)=-\sin\theta,\cos(2\pi-\theta)=\cos\theta,\tan(2\pi-\theta)=-\tan\theta\\ \sin(2\pi+\theta)=\sin\theta,\cos(2\pi+\theta)=\cos\theta,\tan(2\pi+\theta)=\tan\theta\end{cases}$$

Note:
The above results also apply to the reciprocals of sine, cosine and tangent. These results are to be applied frequently in the study of trigonometry and they can be remembered by using the following device:

1. If θ is added to or subtracted from **odd multiple** of right angle, the trigonometric ratios change into **co-ratios** and vice versa.
   i.e., $\sin \rightleftarrows \cos$, $\tan \rightleftarrows \cot$, $\sec \rightleftarrows \operatorname{cosec}$
   e.g., $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ and $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$
2. If θ is added to or subtracted from an even multiple of $\frac{\pi}{2}$, the trigonometric ratios shall remain the same.
3. So far as the sign of the results is concerned, it is determined by the quadrant in which the terminal arm of the angle lies.
   e.g., $\sin(\pi-\theta)=\sin\theta$, $\tan(\pi+\theta)=\tan\theta$, $\cos(2\pi-\theta)=\cos\theta$.

Measure of the angle	Quad.
$\frac{\pi}{2}-\theta$	I
$\frac{\pi}{2}+\theta$ or $\pi-\theta$	II
$\pi+\theta$ or $\frac{3\pi}{2}-\theta$	III
$\frac{3\pi}{2}+\theta$ or $2\pi-\theta$	IV



(a) In $\sin\left(\frac{\pi}{2}-\theta\right), \sin\left(\frac{\pi}{2}+\theta\right), \sin\left(\frac{3\pi}{2}-\theta\right)$ and $\sin\left(\frac{3\pi}{2}+\theta\right)$ odd multiplies of $\frac{\pi}{2}$ are involved.

Therefore, sin will change into cos.
Moreover, the angle of measure

(i) $\left(\frac{\pi}{2}-\theta\right)$ will have terminal side in Quad. I, so, $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$

(ii) $\left(\frac{\pi}{2}+\theta\right)$ will have terminal side in Quad. II, so, $\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$

(iii) $\left(\frac{3\pi}{2}-\theta\right)$ will have terminal side in Quad. III, so, $\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$

(iv) $\left(\frac{3\pi}{2}+\theta\right)$ will have terminal side in Quad. IV, so, $\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$

(b) In $\cos(\pi-\theta), \cos(\pi+\theta), \cos(2\pi-\theta)$ and $\cos(2\pi+\theta)$, even multiples of $\frac{\pi}{2}$ are involved.

Therefore, cos will remain as cos.
Moreover, the angle of measure

(i) $(\pi-\theta)$ will have terminal side in Quad. II, therefore $\cos(\pi-\theta)=-\cos\theta$

(ii) $(\pi+\theta)$ will have terminal side in Quad. III, so $\cos(\pi+\theta)=-\cos\theta$

(iii) $(2\pi-\theta)$ will have terminal side in Quad. IV, so $\cos(2\pi-\theta)=\cos\theta$

(iv) $(2\pi+\theta)$ will have terminal side in Quad. I, so $\cos(2\pi+\theta)=\cos\theta$

Example 3: Without using the tables, write down the values of:

(i) sin 225° (ii) tan 600° (iii) cot(−225°) (iv) cosec(−420°)

Solution:

(i) $\sin 225° = \sin(180+45)°$

$= \sin(2\times 90+45)° = -\sin 45° = -\dfrac{1}{\sqrt{2}}$ ∵ 225° lies in III - quad.

(ii) $\tan 600° = \tan(540+60)°$

$= \tan(6\times 90+60)° = \tan 60° = \sqrt{3}$ ∵ 600° lies in III - quad.

(iii) $\cot(-225°) = -\cot 225° = -\cot(180+45)°$ ∵ $\cot(-\theta) = -\cot\theta$

$= -\cot(2\times 90+45)°$

$= -\cot 45° = -\dfrac{1}{\tan 45°} = -\dfrac{1}{1} = -1$ ∵ 225° lies in III - quad.

(iv) $\text{cosec}(-420°) = -\text{cosec}\,420° = -\text{cosec}(360+60)°$ ∵ $\cot(-\theta) = -\cot\theta$

$= -\text{cosec}(4\times 90+60)°$

$= -\text{cosec}\,60°$ ∵ 420° lies in III - quad.

$= -\dfrac{1}{\sin 60°} = -\dfrac{1}{\frac{\sqrt{3}}{2}} = \dfrac{-2}{\sqrt{3}}$

Example 4: Simplify: $\dfrac{\sin(180°-\theta)\cos(360°-\theta)\tan(90°+\theta)}{\sin(90°-\theta)\cos(180°+\theta)\tan(270°-\theta)}$

Solution:

$$\frac{\sin(180°-\theta)\cos(360°-\theta)\tan(90°+\theta)}{\sin(90°-\theta)\cos(180°+\theta)\tan(270°-\theta)}$$

$$= \frac{\sin(2\times 90°-\theta)\cos(4\times 90°-\theta)\tan(1\times 90°+\theta)}{\sin(1\times 90°-\theta)\cos(2\times 90°+\theta)\tan(3\times 90°-\theta)}$$

$$= \frac{\sin(2\times 90°-\theta)\cos(4\times 90°-\theta)\tan(1\times 90°+\theta)}{\sin(1\times 90°-\theta)\cos(2\times 90°+\theta)\tan(3\times 90°-\theta)} = \frac{\sin\theta.\cos\theta.(-\cot\theta)}{\cos\theta.(-\cos\theta).\cot\theta}$$

$$= \frac{\sin\theta\cdot\cos\theta\cdot(-\cot\theta)}{\cos\theta\cdot(-\cos\theta)\cdot\cot\theta} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$

Exercise 10.1

1. Without using the tables, find the values of:

(i) cos(−1230°)

Solution:

$\cos(-1230°) = \cos(1230°)$ ∵ $\cos(-\theta) = \cos\theta$

$= \cos(13\times 90°+60°)$

$= -\sin 60°$ ∵ 1230° lies in II-quad

$= -\dfrac{\sqrt{3}}{2}$

(ii) tan (−1035°)

Solution:

$\tan(-1035°) = -\tan(1035°)$ ∵ $\tan(-\theta) = -\tan\theta$

$= -\tan(11\times 90°+45°)$

$= -(-\cot 45°)$ ∵ 1035° lies in II-quad

$= \cot 45°$

$= \dfrac{1}{\tan 45°}$

$= \dfrac{1}{1} = 1$

(iii) sec (1140°)

Solution:

$\sec(1140°) = -\sec(12\times 90°+60°)$

$= \sec 60°$ ∵ 1140° lies in I-quad

$= \dfrac{1}{\cos 60°}$

$= \dfrac{1}{\frac{1}{2}}$

$= 2$

(iv) cosec (−690°)

Solution:

$\text{cosec}(-690°) = -\text{cosec}(690°)$ ∵ $\text{cosec}(-\theta) = -\text{cosec}\,\theta$

$= -\text{cosec}(7\times 90° \times 60°)$

$= -(-\sec 60°)$ ∵ 690° lies in IV-quad

$= \dfrac{1}{\cos 60°}$

$= \dfrac{1}{\frac{1}{2}} = 2$

(v) cot (1320°)

Solution:

$\cot(1320°) = \cot(14\times 90°+60°)$

$= \cot 60°$ ∵ 1320° lies in III-quad

$= \dfrac{1}{\tan 60°}$

$= \dfrac{1}{\sqrt{3}}$

(vi) cos(−240°)

Solution:

$\cos(-240°) = \cos(240°)$ ∵ $\cos(-\theta) = \cos\theta$

$= \cos(2\times 90°+60°)$

$= -\cos 60°$ ∵ 240° lies in III-quad

$= -\dfrac{1}{2}$

2. Express each of the following as a trigonometric function of an angle of positive degree measure of less than 45°.

(i) cos 168°

Solution:

$\cos 168° = \cos(180°-12°)$

$= \cos(2\times 90°-12°)$

$= -\cos 12°$ ∵ 168° lies in II-quad

(ii) sin 192°

Solution:

$\sin 192° = \sin(180°+12°)$

$= \sin(2\times 90°+12°)$

$= -\sin 12°$ ∵ 192° lies in III-quad

(iii) cos 333°

Solution:

$\cos 333° = \cos(360°-27°)$

$= \cos(4\times 90°-27°)$

$= \cos 27°$ ∵ 333° lies in IV-quad

(iv) tan 213°

Solution:

$\tan 213° = \tan(180°+33°)$

$= \tan(2\times 90°+33°)$

$= \tan 33°$ ∵ 213° lies in III-quad

(v) cos (−435°)

Solution:

$\cos(-435°) = \cos 435°$ ∵ $\cos(-\theta) = \cos\theta$

$= \cos(450°-15°)$

$= \cos(5\times 90°-15°)$

$= \sin 15°$ ∵ 435° lies in I-quad

(vi) sin 219°

Solution:

$\sin 219° = \sin(180°+39°)$

$= \sin(2\times 90°+39°)$

$= -\sin 39°$ ∵ 219° lies in III-quad

(vii) tan (−597°)

Solution:

$\tan(-597°) = -\tan(597°)$ ∵ $\tan(-\theta) = -\tan\theta$

$= -\tan(630°-33°)$

$= -\tan(7\times 90°-33°)$

$= -\cot 33°$ ∵ 597° lies in III-quad

(viii) cos(−111°)

Solution:

$\cos(-111°) = \cos(111°)$ ∵ $\cos(-\theta) = \cos\theta$

$= \cos(90°+21°)$

$= \cos(1\times 90°+21°)$

$= -\sin 21°$ ∵ 111° lies in II-quad.

(ix) sin (−390°)

Solution:

$\sin(-390°) = -\sin 390°$ ∵ $\sin(-\theta) = -\sin\theta$

$= -\sin(360°+30°)$

$= -\sin(4\times 90° + 30°)$

$= -\sin 30°$ $\quad \because 390°$ lies in I-quad.

3. Prove the following:

(i) $\sin(180°+\alpha)\sin(90°-\alpha) = -\sin\alpha\cos\alpha$

Solution:

$$\begin{aligned} L.H.S &= \sin(180°+\alpha)\cdot\sin(90°-\alpha) \\ &= \sin(2\times 90°+\alpha)\cdot\sin(1\times 90°-\alpha) \\ &= (-\sin\alpha)(\cos\alpha) \\ &= -\sin\alpha\cos\alpha \\ &= R.H.S \text{ (Proved)} \end{aligned}$$

(ii) $\sin 810°\sin 630° + \cos 135°\sin 225° = -\frac{1}{2}$

Solution:

$$\begin{aligned} \sin 810° &= \sin(90°+2\times 360°) = \sin 90° = 1 \\ \sin 630° &= \sin(270°+1\times 360°) \\ &= \sin 270° = -1 \\ \cos 135° &= \cos(1\times 90°+45°) \\ &= -\sin 45° = -\frac{1}{\sqrt{2}} \\ \sin 225° &= \sin(2\times 90°+45°) = -\sin 45° = -\frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} L.H.S &= \sin 810°\cdot\sin 630° + \cos 135°\cdot\sin 225° \\ &= (1)(-1) + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) \\ &= -1 + \frac{1}{2} \\ &= \frac{-2+1}{2} \\ &= -\frac{1}{2} = R.H.S \text{ (Proved)} \end{aligned}$$

(iii) $\tan 150°\cot 330° - 2\sec 135°\,\mathrm{cosec}\,225° = -3$

Solution:

$$\begin{aligned} \tan 150° &= \tan(1\times 90°+60°) = -\cot 60° = -\frac{1}{\sqrt{3}} \\ \cot 330° &= \cot(3\times 90°+60°) = -\tan 60° = -\sqrt{3} \\ \sec 135° &= \sec(1\times 90°+45°) = -\mathrm{cosec}\,45° = -\sqrt{2} \\ \mathrm{cosec}\,225° &= \mathrm{cosec}(2\times 90°+45°) = -\cos 45° = -\sqrt{2} \end{aligned}$$

$$\begin{aligned} L.H.S &= \tan 150°\cdot\cot 330° - 2\sec 135°\cdot\mathrm{cosec}\,225° \\ &= \left(-\frac{1}{\sqrt{3}}\right)(-\sqrt{3}) - 2(-\sqrt{2})(-\sqrt{2}) \\ &= \frac{\sqrt{3}}{\sqrt{3}} - 2(\sqrt{2})^2 \\ &= 1 - 2(2) \\ &= -3 = R.H.S \text{ (Proved)} \end{aligned}$$

(iv) $\sin 210° + \cos 240° + \tan 225° + \cot 225° = 1$

Solution:

$$\begin{aligned} \sin 210° &= \sin(2\times 90°+30°) = -\sin 30° = -\frac{1}{2} \\ \cos 240° &= \cos(2\times 90°+60°) = -\cos 60° = -\frac{1}{2} \\ \tan 225° &= \tan(2\times 90°+45°) = \tan 45° = 1 \\ \cot 225° &= \cot(2\times 90°+45°) = \cot 45° = 1 \end{aligned}$$

$$\begin{aligned} L.H.S &= \sin 210° + \cos 240° + \tan 225° + \cot 225° \\ &= -\frac{1}{2} - \frac{1}{2} + 1 + 1 \\ &= \frac{-1-1+2+2}{2} \\ &= \frac{2}{2} \\ &= 1 = R.H.S \text{ (Proved)} \end{aligned}$$

4. Prove that:

(i) $\dfrac{\tan(180°+\alpha)\cot(90°-\alpha)}{\sin(360°-\alpha)\cos(270°+\alpha)} = -\sec^2\alpha$

Solution:

$$\begin{aligned} L.H.S &= \frac{\tan(180°+\alpha)\cot(90°-\alpha)}{\sin(360°-\alpha)\cos(270°+\alpha)} \\ &= \frac{\tan(2\times 90°+\alpha)\cot(1\times 90°-\alpha)}{\sin(4\times 90°-\alpha)\cos(3\times 90°+\alpha)} \\ &= \frac{(\tan\alpha)(\tan\alpha)}{(-\sin\alpha)(\sin\alpha)} \\ &= -\frac{\tan^2\alpha}{\sin^2\alpha} \\ &= -\frac{\sin^2\alpha}{\cos^2\alpha}\cdot\frac{1}{\sin^2\alpha} \\ &= -\frac{1}{\cos^2\alpha} \\ &= -\sec^2\alpha \\ &= R.H.S \text{ (proved)} \end{aligned}$$

(ii) $\dfrac{\sin^2(\pi+\theta)\tan\left(\frac{3\pi}{2}+\theta\right)}{\cot^2\left(\frac{3\pi}{2}-\theta\right)\cos^2(\pi-\theta)\,\mathrm{cosec}(2\pi-\theta)} = \cos\theta$

Solution:

$$\begin{aligned} L.H.S &= \frac{\sin^2(180°+\theta)\tan(270°+\theta)}{\cot^2(270°-\theta)\cos^2(180°-\theta)\,\mathrm{cosec}(360°-\theta)} \\ &= \frac{\sin^2(2\times 90°+\theta)\tan(3\times 90°+\theta)}{\cot^2(3\times 90°-\theta)\cos^2(2\times 90°-\theta)\,\mathrm{cosec}(4\times 90°-\theta)} \\ &= \frac{(-\sin\theta)^2(-\cot\theta)}{(\tan\theta)^2(-\cos\theta)^2(-\mathrm{cosec}\theta)} \\ &= \frac{-\sin^2\theta.\cot\theta}{-\tan^2\theta.\cos^2\theta.\mathrm{cosec}\theta} \\ &= \frac{\sin^2\theta.\frac{\cos\theta}{\sin\theta}}{\frac{\sin^2\theta}{\cos^2\theta}.\cos^2\theta.\frac{1}{\sin\theta}} = \frac{\sin\theta.\cos\theta}{\sin\theta} \\ &= \cos\theta = R.H.S \text{ (Proved)} \end{aligned}$$

(iii) $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)} = -1$

Solution:

$$\begin{aligned} L.H.S &= \frac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)} \\ &= \frac{\cos(1\times 90°+\theta)\sec(-\theta)\tan(2\times 90°-\theta)}{\sec(4\times 90°-\theta)\sin(2\times 90°+\theta)\cot(1\times 90°-\theta)} \\ &= \frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)} \qquad \because \sec(-\theta) = \sec\theta \\ &= -1 = R.H.S \text{ (Proved)} \end{aligned}$$

5. Show that:

$\sec\left(\frac{3\pi}{2}-\theta\right)\sec\left(\frac{5\pi}{2}-\theta\right) - \tan\left(\frac{3\pi}{2}-\theta\right)\tan\left(\frac{5\pi}{2}+\theta\right) = -1$

Solution:

$$\begin{aligned} L.H.S &= \sec\left(\frac{3\pi}{2}-\theta\right)\sec\left(\frac{5\pi}{2}-\theta\right) - \tan\left(\frac{3\pi}{2}-\theta\right)\tan\left(\frac{5\pi}{2}+\theta\right) \\ &= \sec(3\times 90°-\theta)\sec(5\times 90°-\theta) - \tan(3\times 90°-\theta)\tan(5\times 90°+\theta) \\ &= (-\mathrm{cosec}\theta)(\mathrm{cosec}\theta) - (+\cot\theta)(-\cot\theta) \\ &= -\mathrm{cosec}^2\theta + \cot^2\theta \\ &= \cot^2\theta - \mathrm{cosec}^2\theta \qquad \left\{\because 1+\cot^2\theta = \mathrm{cosec}^2\theta \Rightarrow \cot^2\theta - \mathrm{cosec}^2\theta = -1.\right. \\ &= -1 \\ &= R.H.S \text{ (Proved)} \end{aligned}$$

6. If α, β, γ are the angles of a triangle ABC, then prove that

(i) $\sin(\alpha+\beta) = \sin\gamma$

Solution:

Since α, β, γ are the angles of a triangle ABC, therefore

$\alpha+\beta+\gamma = 180° \Rightarrow \alpha+\beta = 180°-\gamma$ ...(i)

$$\begin{aligned} L.H.S &= \sin(\alpha+\beta) \\ &= \sin(180°-\gamma) && \text{using eq (i)} \\ &= \sin(2\times 90°-\gamma) \\ &= \sin\gamma = R.H.S \text{ (Proved)} \end{aligned}$$

(ii) $\sec\left(\frac{\alpha+\beta}{2}\right) = \csc\frac{\gamma}{2}$

Solution:

Since α, β, γ are the angles of a triangle ABC, therefore

$\alpha+\beta+\gamma = 180° \Rightarrow \alpha+\beta = 180°-\gamma$ ...(1)

$$\begin{aligned} L.H.S &= \sec\left(\frac{\alpha+\beta}{2}\right) \\ &= \sec\left(\frac{180°-\gamma}{2}\right) && \text{using eq. (1)} \\ &= \sec\left(\frac{180°}{2}-\frac{\gamma}{2}\right) \\ &= \sec\left(1\times 90°-\frac{\gamma}{2}\right) \\ &= \csc\frac{\gamma}{2} = R.H.S \text{ (Proved)} \end{aligned}$$

(iii) $\mathrm{cosec}\,\alpha = \dfrac{1}{\sin(\beta+\gamma)}$

Solution:

Since α, β, γ are the angles of a triangle ABC, therefore

$\alpha+\beta+\gamma = 180° \Rightarrow \beta+\gamma = 180°-\alpha$ ...(1)

$$\begin{aligned} R.H.S &= \frac{1}{\sin(\beta+\gamma)} \\ &= \mathrm{cosec}(\beta+\gamma) \\ &= \mathrm{cosec}(180°-\alpha) && \text{using eq (i)} \\ &= \mathrm{cosec}(2\times 90°-\alpha) \\ &= \mathrm{cosec}\,\alpha \\ &= L.H.S \text{ (Proved)} \end{aligned}$$

(iv) $\tan(\alpha+\beta)+\tan\gamma = 0$

Solution:

Since α, β, γ are the angles of a triangle ABC, therefore

$\alpha+\beta+\gamma = 180° \Rightarrow \alpha+\beta = 180°-\gamma p$ ...(i)

$$\begin{aligned} L.H.S &= \tan(\alpha+\beta)+\tan\gamma \\ &= \tan(180°-\gamma)+\tan\gamma && \text{using eq. (i)} \\ &= \tan(2\times 90°-\gamma)+\tan\gamma \\ &= -\tan\gamma+\tan\gamma \\ &= 0 = R.H.S \text{ (Proved)} \end{aligned}$$

Further Applications of Basic Identities:

Example 5: Prove that: $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$ ...(i)

$= \cos^2\beta - \cos^2\alpha$ ...(ii)

Solution:

$$\begin{aligned}
L.H.S &= \sin(\alpha+\beta)\sin(\alpha-\beta)\\
&= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\
&= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\
&= \sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta \qquad \because \sin^2\theta+\cos^2\theta=1\\
&= \sin^2\alpha-\sin^2\alpha\sin^2\beta-\sin^2\beta+\sin^2\alpha\sin^2\beta\\
&= \sin^2\alpha-\sin^2\beta \qquad \text{...(i)}\\
&= (1-\cos^2\alpha)-(1-\cos^2\beta) \qquad \because \sin^2\theta+\cos^2\theta=1\\
&= 1-\cos^2\alpha-1+\cos^2\beta\\
&= \cos^2\beta-\cos^2\alpha \qquad \text{...(ii)}
\end{aligned}$$

Example 6: Without using tables, find the values of all trigonometric functions of 105°

Solution: As $105° = 60° = 45°$

$$\sin 105° = \sin(60°+45°) = \sin 60°\cos 45° + \cos 60°\sin 45°$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos 105° = \cos(60°+45°) = \cos 60°\cos 45° - \sin 60°\sin 45°$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)=\frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$\tan 105° = \tan(60°+45°) = \frac{\tan 60°+\tan 45°}{1-\tan 60°\tan 45°} = \frac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$\cot 105° = \frac{1}{\tan 105°} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$

$$\text{cosec } 105° = \frac{1}{\sin 105°} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

and $$\sec 105° = \frac{1}{\cos 105°} = \frac{2\sqrt{2}}{1-\sqrt{3}}$$

Example 7: Prove that: $\dfrac{\cos 11°+\sin 11°}{\cos 11°-\sin 11°} = \tan 56°$

Solution:

$$\text{R.H.S.} = \tan 56° = \tan(45°+11°)$$

$$= \frac{\tan 45°+\tan 11°}{1-\tan 45°\tan 11°} = \frac{1+\tan 11°}{1-\tan 11°} \qquad \because \tan 45° = 1$$

$$= \frac{1+\frac{\sin 11°}{\cos 11°}}{1-\frac{\sin 11°}{\cos 11°}} = \frac{\frac{\cos 11°+\sin 11°}{\cos 11°}}{\frac{\cos 11°-\sin 11°}{\cos 11°}} = \frac{\cos 11°+\sin 11°}{\cos 11°-\sin 11°} = \text{L.H.S}$$

Hence $\dfrac{\cos 11°+\sin 11°}{\cos 11°-\sin 11°} = \tan 56°$

Example 8: If $\cos\alpha = -\dfrac{7}{25}$, $\tan\beta = \dfrac{12}{5}$, the terminal side of the angle of measure α is in the II quadrant and that of β is in the III quadrant, find the values of:

(i) $\sin(\alpha+\beta)$ (ii) $\cos(\alpha+\beta)$

In which quadrant does the terminal side of the angle of measure $(\alpha+\beta)$ lie?

Solution:

Given that: $\cos\alpha = -\dfrac{7}{25}, \tan\beta = \dfrac{12}{5}$

Therefore, $$\sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{1-\left(-\frac{7}{25}\right)^2} = \pm\sqrt{\frac{576}{625}} = \pm\frac{24}{25} \qquad \because \sin^2\alpha+\cos^2\alpha=1$$

As the terminal side of the angle of measure of α is in the II quadrant, where $\sin\alpha$ is positive.

So, $$\sin\alpha = \frac{24}{25}$$

Now, $$\sec\beta = \pm\sqrt{1+\tan^2\beta} = \pm\sqrt{1+\left(\frac{12}{5}\right)^2} = \pm\frac{13}{5} \qquad \because 1+\tan^2\alpha = \sec^2\alpha$$

As the terminal side of the angle of measure of β in the quadrant III, so $\sec\beta$ is negative

$$\sec\beta = -\frac{13}{5} \text{ and } \cos\beta = -\frac{5}{13}$$

$$\sin\beta = \pm\sqrt{1-\cos^2\beta} = \pm\sqrt{1-\left(-\frac{5}{13}\right)^2} = \pm\sqrt{\frac{144}{169}} = \pm\frac{12}{13} \qquad \because \sin^2\beta+\cos^2\beta=1$$

As the terminal arm of the angle of measure β is in the III quadrant, so $\sin\beta$ is negative

$$\sin\beta = -\frac{12}{13}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$= \left(\frac{24}{25}\right)\left(-\frac{5}{13}\right)+\left(-\frac{7}{25}\right)\left(-\frac{12}{13}\right)$$

$$= \frac{-120+84}{325} = -\frac{36}{325}$$

and $$\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$= \left(-\frac{7}{25}\right)\left(-\frac{5}{13}\right)-\left(\frac{24}{25}\right)\left(-\frac{12}{13}\right)$$

$$= \frac{35+288}{325} = \frac{323}{325}$$

As, $\sin(\alpha+\beta)$ is −ive and $\cos(\alpha+\beta)$ is +ive

Thus, the terminal arm of the angle of measure $(\alpha+\beta)$ is in the quadrant IV.

Example 9: If α, β, γ are the angles of ΔABC, prove that:

(i) $\tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\tan\beta\tan\gamma$ (ii) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$

Solution: As α, β, γ are the angles of ΔABC, therefore

$$\alpha+\beta+\gamma = 180°$$

$$\alpha+\beta = 180°-\gamma$$

(i) $\tan(\alpha+\beta) = \tan(2\times 90^\circ-\gamma)$

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = -\tan\gamma \qquad \because 180^\circ-\gamma \text{ lies in II-quad.}$$

$$\tan\alpha+\tan\beta = -\tan\gamma(1-\tan\alpha\tan\beta)$$

$$\tan\alpha+\tan\beta = -\tan\gamma+\tan\alpha\tan\beta\tan\gamma$$

$$\tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\tan\beta\tan\gamma \text{ (Proved)}$$

(ii) As $\alpha+\beta+\gamma = 180^\circ \Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=90^\circ$

So $$\frac{\alpha}{2}+\frac{\beta}{2} = 90^\circ-\frac{\gamma}{2}$$

$$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \tan\left(90^\circ-\frac{\gamma}{2}\right)$$

$$\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \cot\frac{\gamma}{2} \qquad \because 90^\circ-\frac{\gamma}{2} \text{ lies in I-quad.}$$

$$\tan\frac{\alpha}{2}+\tan\frac{\beta}{2} = \frac{1}{\tan\frac{\gamma}{2}}\left(1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

$$\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1 \text{ (Proved)}$$

Example 10: Express $3\sin\theta + 4\cos\theta$ in the form $r\sin(\theta+\phi)$, where the terminal side of the angle of measure ϕ is in quadrant I.

Solution:

Let $3\sin\theta + 4\cos\theta = r\sin(\theta+\phi)$ ... (A)

$3\sin\theta + 4\cos\theta = r(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$3\sin\theta + 4\cos\theta = r\sin\theta\cos\phi + r\cos\theta\sin\phi$

Comparing both sides, we have

$3 = r\cos\phi$ ... (i)

and $4 = r\sin\phi$ ... (ii)

Squaring then adding (i) and (ii)

$$3^2+4^2 = r^2\cos^2\phi+r^2\sin^2\phi$$

$$9+16 = r^2(\cos^2\phi+\sin^2\phi)$$

$$25 = r^2$$

$$5 = r$$

$$r = 5$$

Dividing (ii) by (i)

$$\frac{4}{3}=\frac{r\sin\phi}{r\cos\phi}$$

$$\frac{4}{3}=\tan\phi$$

$$\tan\phi=\frac{4}{3}$$

Putting values of r and ϕ in equation (A), we get

$$3\sin\theta + 4\cos\theta = 5\sin(\theta+\phi),$$

where $r=5$ and $\tan\phi=\frac{4}{3}$

Exercise 10.2

1. Without using table find the values of the following:

Hint: $15^\circ = (45^\circ-30^\circ)$ and $105^\circ = (60^\circ+45^\circ)$

(i) $\sin 15^\circ$

Solution:

$$\sin 15^\circ = \sin(45^\circ-30^\circ)$$

$$= \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) $\cos 15^\circ$

Solution:

$$\cos 15^\circ = \cos(45^\circ-30^\circ)$$

$$= \cos 45^\circ\cos 30^\circ + \sin 45^\circ\sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) $\tan 15^\circ$

Solution:

$$\tan 15^\circ = \tan(45^\circ-30^\circ)$$

$$= \frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}$$

$$= \frac{1-\frac{1}{\sqrt{3}}}{1+(1)\cdot\left(\frac{1}{\sqrt{3}}\right)} \qquad \because \tan 45^\circ = 1$$

$$= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Rationalize the denominator

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2-1^2}$$

$$= \frac{(\sqrt{3})^2+1^2-2\sqrt{3}}{3-1}$$

$$= \frac{3+1-2\sqrt{3}}{2}$$

$$= \frac{4-2\sqrt{3}}{2}$$

$$= 2-\sqrt{3}$$

(iv) $\sin 105^\circ$

Solution:

$$\sin 105^\circ = \sin(45^\circ+60^\circ)$$

$$= \sin 45^\circ\cos 60^\circ+\cos 45^\circ\sin 60^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(v) $\cos 105^\circ$

Solution:

$$\cos 105^\circ = \cos(45^\circ+60^\circ)$$

$$= \cos 45^\circ\cos 60^\circ-\sin 45^\circ\sin 60^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\sqrt{2}}-\frac{\sqrt{3}}{2\sqrt{2}}=\frac{1-\sqrt{3}}{2\sqrt{2}}$$

(vi) $\tan 105^\circ$

Solution:

$$\tan 105^\circ = \tan(45^\circ+60^\circ)$$

$$= \frac{\tan 45^\circ+\tan 60^\circ}{1-\tan 45^\circ\tan 60^\circ}$$

$$= \frac{1+\sqrt{3}}{1-(1)\cdot\sqrt{3}} \qquad \because \tan 45^\circ$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

Rationalize the denominator

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{(1+\sqrt{3})^2}{1^2-(\sqrt{3})^2}$$

$$= \frac{1+(\sqrt{3})^2+2\sqrt{3}}{1-3}$$

$$= \frac{1+3+2\sqrt{3}}{-2}$$

$$= \frac{4+2\sqrt{3}}{-2}$$
$$= \frac{-4}{2} - \frac{2\sqrt{3}}{2}$$
$$= -2-\sqrt{3}$$

2. Prove that:

(i) $\sin(45°+\alpha)=\frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$

Solution:

L.H.S $= \sin(45°+\alpha)$
$= \sin 45° \cos\alpha + \cos 45° \sin\alpha$
$= \left(\frac{1}{\sqrt{2}}\right).\cos\alpha + \left(\frac{1}{\sqrt{2}}\right).\sin\alpha$
$= \frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$
= R.H.S (Proved)

(ii) $\cos(\alpha+45°)=\frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)$

Solution:

L.H.S $= \cos(\alpha+45°)$
$= \cos\alpha\cos 45° - \sin\alpha\sin 45°$
$= \cos\alpha\left(\frac{1}{\sqrt{2}}\right) - \sin\alpha\left(\frac{1}{\sqrt{2}}\right)$
$= \frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)$
= R.H.S (Proved)

3. Prove that:

(i) $\tan(45°+A)\tan(45°-A)=1$

Solution:

L.H.S $= \tan(45°+A)\tan(45°-A)$
$$= \left[\frac{\tan 45°+\tan A}{1-\tan 45°\tan A}\right]\times\left[\frac{\tan 45°-\tan A}{1+\tan 45°\tan A}\right]$$
$$= \left[\frac{1+\tan A}{1-\tan A}\right]\times\left[\frac{1-\tan A}{1+\tan A}\right] \qquad \because \tan 45° = 1$$
= 1 = R.H.S (Proved)

(ii) $\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right)=0$

Solution:

L.H.S $= \tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right)$
$= \tan(45°-\theta)+\tan(135°+\theta)$
$$= \frac{\tan 45°-\tan\theta}{1+\tan 45°\tan\theta}+\frac{\tan 135°+\tan\theta}{1-\tan 135°\tan\theta}$$
$$= \frac{1-\tan\theta}{1+\tan\theta}+\frac{-1+\tan\theta}{1+\tan\theta} \qquad \because \begin{cases}\tan 45°=1\\ \tan 135°=-1\end{cases}$$
$$= \frac{1-\tan\theta-1+\tan\theta}{1+\tan\theta}$$
$$= \frac{0}{1+\tan\theta} = 0 = \text{R.H.S (Proved)}$$

(iii) $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right)=\cos\theta$

Solution:

L.H.S $= \sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right)$
$= \sin(\theta+30°)+\cos(\theta+60°)$
$= \sin\theta\cos 30°+\cos\theta\sin 30°+\cos\theta\cos 60°-\sin\theta\sin 60°$
$= \sin\theta.\left(\frac{\sqrt{3}}{2}\right)+\cos\theta.\left(\frac{1}{2}\right)+\cos\theta.\left(\frac{1}{2}\right)-\sin\theta.\left(\frac{\sqrt{3}}{2}\right)$
$= \frac{\cos\theta}{2}+\frac{\cos\theta}{2}=\frac{\cos\theta+\cos\theta}{2}=\frac{2\cos\theta}{2}$
$= \cos\theta$ = R.H.S (Proved)

(iv) $\frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}=\tan\frac{\theta}{2}$

Solution:

$$\text{L.H.S} = \frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}$$
$$= \frac{\sin\theta-\cos\theta.\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\cos\theta+\sin\theta.\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$
$$= \frac{\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\frac{\cos\theta\cos\frac{\theta}{2}-\sin\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$
$$= \frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2}+\sin\theta\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\theta-\frac{\theta}{2}\right)}{\cos\left(\theta-\frac{\theta}{2}\right)} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$
= R.H.S (Proved)

(v) $\frac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi}=\frac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$

Solution:

$$\text{L.H.S} = \frac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi}$$
$$= \frac{1-\frac{\sin\theta}{\cos\theta}.\frac{\sin\phi}{\cos\phi}}{1+\frac{\sin\theta}{\cos\theta}.\frac{\sin\phi}{\cos\phi}} = \frac{\frac{\cos\theta\cos\phi-\sin\theta\sin\phi}{\cos\theta\cos\phi}}{\frac{\cos\theta\cos\phi+\sin\theta\sin\phi}{\cos\theta\cos\phi}}$$
$$= \frac{\cos\theta\cos\phi-\sin\theta\sin\phi}{\cos\theta\cos\phi+\sin\theta\sin\phi}$$
$$= \frac{\cos(\theta+\phi)}{\cos(\theta-\phi)} = \text{R.H.S (Proved)}$$

4. Show that:

$\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha$

Solution:

L.H.S $= \cos(\alpha+\beta)\cos(\alpha-\beta)$
$= (\cos\alpha\cos\beta-\sin\alpha\sin\beta).(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$

By using $(a-b)(a+b)=a^2-b^2$
$= (\cos\alpha\cos\beta)^2-(\sin\alpha\sin\beta)^2$
$= \cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta$
$= \cos^2\alpha(1-\sin^2\beta)-(1-\cos^2\alpha)\sin^2\beta$
$= \cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta$
$= \cos^2\alpha-\sin^2\beta =$ Middle term

Now,
Middle term $= \cos^2\alpha-\sin^2\beta$
$= (1-\sin^2\alpha)-(1-\cos^2\beta)$
$= 1-\sin^2\alpha-1+\cos^2\beta$
$= \cos^2\beta-\sin^2\alpha =$ R.H.S

Hence proved.
$\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha$

5. Show that: $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}=\tan\alpha$

Solution:

$$\text{L.H.S} = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$
$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$
$$= \frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta}=\frac{\sin\alpha}{\cos\alpha}=\tan\alpha = \text{R.H.S (Proved)}$$

6. Show that:

(i) $\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

Solution:

L.H.S $= \cot(\alpha+\beta)$
$$= \frac{1}{\tan(\alpha+\beta)} = \frac{1}{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$
$$= \frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$
$$= \frac{1-\frac{1}{\cot\alpha.\cot\beta}}{\frac{1}{\cot\alpha}+\frac{1}{\cot\beta}}$$
$$= \frac{\frac{\cot\alpha.\cot\beta-1}{\cot\alpha.\cot\beta}}{\frac{\cot\beta+\cot\alpha}{\cot\alpha.\cot\beta}}$$
$$= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta} = \text{R.H.S (Proved)}$$

(ii) $\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

Solution:

L.H.S $= \cot(\alpha-\beta)$
$$= \frac{1}{\tan(\alpha-\beta)} = \frac{1}{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}$$
$$= \frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$$
$$= \frac{1+\frac{1}{\cot\alpha.\cot\beta}}{\frac{1}{\cot\alpha}-\frac{1}{\cot\beta}}$$
$$= \frac{\frac{\cot\alpha.\cot\beta+1}{\cot\alpha.\cot\beta}}{\frac{\cot\beta-\cot\alpha}{\cot\alpha.\cot\beta}}$$
$$= \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha} = \text{R.H.S (Proved)}$$

(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution:

$$\text{L.H.S} = \frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$$

$$= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}}$$

$$= \frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\ .\ \cos\beta}}{\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\ .\ \cos\beta}}$$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}=\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$$

= R.H.S (Proved)

7. Show that:

(i) $\cos(\alpha-\beta)=\dfrac{1+\tan\alpha\tan\beta}{\sec\alpha\sec\beta}$

Solution:

$$\text{R.H.S} = \frac{1+\tan\alpha\tan\beta}{\sec\alpha\sec\beta}$$

$$= \frac{1+\frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}{\frac{1}{\cos\alpha}\cdot\frac{1}{\cos\beta}}$$

$$= \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\times\frac{\cos\alpha\cos\beta}{1}$$

$$= \cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$= \cos(\alpha-\beta) = \text{L.H.S (proved)}$$

(ii) $\sin(\alpha+\beta)=\dfrac{1+\cot\alpha\tan\beta}{\operatorname{cosec}\alpha\sec\beta}$

Solution:

$$\text{R.H.S} = \frac{1+\cot\alpha\tan\beta}{\operatorname{cosec}\alpha\sec\beta}$$

$$= \frac{1+\frac{\cos\alpha}{\sin\alpha}\cdot\frac{\sin\beta}{\cos\beta}}{\frac{1}{\sin\alpha}\cdot\frac{1}{\cos\beta}}$$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta}\times\frac{\sin\alpha\cos\beta}{1}$$

$$= \sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$= \sin(\alpha+\beta) = \text{L.H.S (proved)}$$

Note: Parts (iii), (iv) and (v) of this question has been repeated in Q6. (i), (ii) and (iii).

(iii) $\cot(\alpha-\beta)=\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

(iv) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

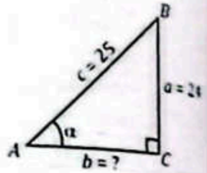
(v) $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

8. If $\sin\alpha=\dfrac{24}{25}$ and $\cos\beta=\dfrac{20}{29}$, where $0<\alpha<\dfrac{\pi}{2}$ and $0<\beta<\dfrac{\pi}{2}$. Show that $\sin(\alpha-\beta)=\dfrac{333}{725}$

Solution:

$$\sin\alpha=\frac{24}{25}$$

By Pythagoras theorem



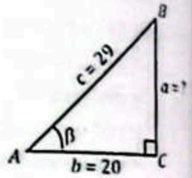
$$c^2 = a^2+b^2$$
$$(25)^2 = (24)^2+b^2$$
$$b^2 = 625-576$$
$$b^2 = 49$$
$$b = 7$$

Since α lies in I-quad, therefore

$$\cos\alpha=\frac{b}{c}=\frac{7}{25}$$

Now, $\cos\beta=\dfrac{20}{29}$

By Pythagoras theorem



$$c^2 = a^2+b^2$$
$$(29)^2 = a^2+(20)^2$$
$$a^2 = 841-400$$
$$a^2 = 441$$
$$a = 21$$

Since β lies in I-quad., therefore

$$\sin\beta=\frac{a}{c}=\frac{21}{29}$$

To show: $\sin(\alpha-\beta)=\dfrac{333}{725}$

$$\text{L.H.S} = \sin(\alpha-\beta)$$
$$= \sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$= \frac{24}{25}\cdot\frac{20}{29}-\frac{7}{25}\cdot\frac{21}{29}$$
$$= \frac{480}{725}-\frac{147}{725}=\frac{480-147}{725}$$
$$= \frac{333}{725} = \text{R.H.S (Proved)}$$

9. If $\sin\alpha=-\dfrac{8}{17}$ and $\cos\beta=-\dfrac{4}{5}$ where $\dfrac{3\pi}{2}<\alpha<2\pi$ and $\pi<\beta<\dfrac{3\pi}{2}$. Find:

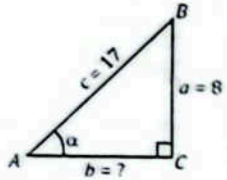
(i) $\sin(\alpha+\beta)$ (ii) $\cos(\alpha+\beta)$
(iii) $\tan(\alpha+\beta)$ (iv) $\sin(\alpha-\beta)$
(v) $\cos(\alpha-\beta)$ (vi) $\tan(\alpha-\beta)$

In which quadrants do the terminal sides of the angles of measures $(\alpha+\beta)$ and $(\alpha-\beta)$ lie?

Solution:

$$\sin\alpha=-\frac{8}{17}$$

By Pythagoras theorem



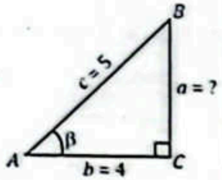
$$c^2 = a^2+b^2$$
$$(17)^2 = 8^2+b^2$$
$$b^2 = 289-64$$
$$b^2 = 225$$
$$b = 15$$

Since α lies in IV-quad., Therefore

$$\sin\alpha=-\frac{8}{17},\ \cos\alpha=\frac{15}{17},\ \tan\alpha=-\frac{8}{15}$$

Now, $\cos\beta=-\dfrac{4}{5}$

By Pythagoras theorem



$$c^2 = a^2+b^2$$
$$5^2 = a^2+4^2$$
$$a^2 = 25-16$$
$$a^2 = 9$$
$$a = 3$$

Since β lies in III-quad., therefore

$$\sin\beta=-\frac{3}{5},\ \cos\beta=-\frac{4}{5},\ \tan\beta=\frac{3}{4}$$

(i) $\sin(\alpha+\beta)=?$

Solution:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$= \left(-\frac{8}{17}\right)\left(-\frac{4}{5}\right)+\left(\frac{15}{17}\right)\left(-\frac{3}{5}\right)$$
$$= \frac{32}{85}-\frac{45}{85}$$
$$= \frac{32-45}{85}=-\frac{13}{85}$$

(ii) $\cos(\alpha+\beta)=?$

Solution:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$= \left(\frac{15}{17}\right)\left(-\frac{4}{5}\right)-\left(-\frac{8}{17}\right)\left(-\frac{3}{5}\right)$$
$$= -\frac{60}{85}-\frac{24}{85}$$
$$= \frac{-60-24}{85}=-\frac{84}{85}$$

(iii) $\tan(\alpha+\beta)=?$

Solution:

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$$
$$= \frac{-\frac{8}{15}+\frac{3}{4}}{1-\left(-\frac{8}{15}\right)\left(\frac{3}{4}\right)}$$
$$= \frac{\frac{-32+45}{60}}{\frac{60+24}{60}}$$
$$= \frac{13}{84}$$

(iv) $\sin(\alpha-\beta)=?$

Solution:

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$= \left(-\frac{8}{17}\right)\left(-\frac{4}{5}\right)-\left(\frac{15}{17}\right)\left(-\frac{3}{5}\right)$$
$$= \frac{32}{85}+\frac{45}{85}$$
$$= \frac{32+45}{85}=\frac{77}{85}$$

(v) $\cos(\alpha-\beta)=?$

Solution:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$= \left(\frac{15}{17}\right)\left(-\frac{4}{5}\right)+\left(-\frac{8}{17}\right)\left(-\frac{3}{5}\right)$$
$$= -\frac{60}{85}+\frac{24}{85}$$
$$= \frac{-60+24}{85}=-\frac{36}{85}$$

(vi) $\tan(\alpha-\beta)=?$

Solution:

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$$
$$= \frac{-\frac{8}{15}-\frac{3}{4}}{1+\left(-\frac{8}{15}\right)\left(\frac{3}{4}\right)}$$

$$= \frac{\frac{-32-45}{60}}{\frac{60-24}{60}} = -\frac{77}{36}$$

Since $\sin(\alpha+\beta) < 0$, $\cos(\alpha+\beta) < 0$ and $\tan(\alpha+\beta) > 0$, therefore $\alpha+\beta$ lies in III-quad.

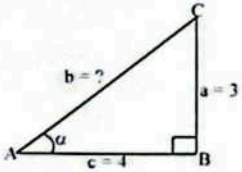
Since $\sin(\alpha-\beta) > 0$, $\cos(\alpha-\beta) < 0$ and $\tan(\alpha-\beta) < 0$, therefore $\alpha-\beta$ lies in II-quad.

10. Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

(i) $\tan\alpha = \frac{3}{4}$, $\cos\beta = \frac{5}{13}$ and neither the terminal side of the angle of measure α nor that of β is in the quadrant I.

Solution:

$$\tan\alpha = \frac{3}{4}$$



$b^2 = c^2 + a^2$

By Pythagoras theorem

$b^2 = (4)^2 + (3)^2$

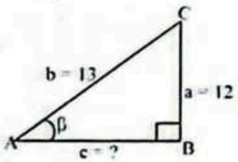
$= 16 + 9 = 25$

$b = 5$

As α lies in III-Quad, therefore

$$\sin\alpha = -\frac{3}{5} \quad , \quad \cos\alpha = -\frac{4}{5}$$

Now, $\cos\beta = \frac{5}{13}$



$b^2 = c^2 + a^2$

By Pythagoras theorem

$(13)^2 = c^2 + (12)^2$

$69 = c^2 + 144 \quad \Rightarrow \quad c^2 = 169 - 144$

$c^2 = 25 \quad \Rightarrow \quad c = 5$

As β lies in IV-Quad, therefore

$$\sin\beta = -\frac{12}{13}$$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$

$$= -\frac{15}{65} + \frac{48}{65}$$

$$= \frac{-15+48}{65} = \frac{33}{65}$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

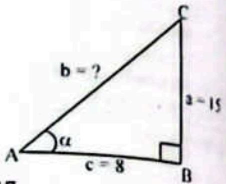
$$= \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$

$$= -\frac{20}{65} + \frac{-36}{65} = \frac{-20-36}{65} = -\frac{56}{65}$$

(ii) $\tan\alpha = -\frac{15}{8}$ and $\sin\beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the quadrant IV.

Solution:

$$\tan\alpha = -\frac{15}{8}$$



$b^2 = c^2 + a^2$

By Pythagoras theorem

$b^2 = (15)^2 + (8)^2$

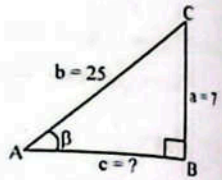
$= 225 + 64$

$b^2 = 289 \quad \Rightarrow b = 17$

As α lies in II-Quad, therefore

$$\sin\alpha = \frac{15}{17} \quad , \quad \cos\alpha = -\frac{8}{17}$$

Now, $\sin\beta = -\frac{7}{25}$



$b^2 = c^2 + a^2$

By Pythagoras theorem

$(25)^2 = c^2 + (7)^2$

$625 = c^2 + 49$

$\Rightarrow \quad c^2 = 625 - 49$

$c^2 = 576$

$\Rightarrow \quad c = 24$

As β lies in III-Quad, therefore

$$\cos\beta = -\frac{24}{25}$$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(\frac{15}{17}\right)\left(-\frac{24}{25}\right) + \left(-\frac{8}{17}\right)\left(-\frac{7}{25}\right) = -\frac{360}{425} + \frac{56}{425}$$

$$= \frac{-360+56}{425} = -\frac{304}{425}$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{8}{17}\right)\left(-\frac{24}{25}\right) - \left(\frac{15}{17}\right)\left(-\frac{7}{25}\right) = \frac{192}{425} + \frac{105}{425}$$

$$= \frac{192+105}{425} = \frac{297}{425}$$

11. Prove that: $\frac{\cos 19° + \sin 19°}{\cos 19° - \sin 19°} = \tan 64°$

Solution:

R.H.S $= \tan 64°$

$= \tan(45° + 19°)$

$$= \frac{\tan 45° + \tan 19°}{1 - \tan 45° \cdot \tan 19°} \quad \because \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{1 + \tan 19°}{1 - \tan 19°} \quad \because \tan 45° = 1$$

$$= \frac{1 + \frac{\sin 19°}{\cos 19°}}{1 - \frac{\sin 19°}{\cos 19°}}$$

$$= \frac{\frac{\cos 19° + \sin 19°}{\cos 19°}}{\frac{\cos 19° - \sin 19°}{\cos 19°}}$$

$$= \frac{\cos 19° + \sin 19°}{\cos 19° - \sin 19°} = \text{L.H.S (Proved)}$$

12. Prove that: $\cos(60° + \theta)\cos(60° - \theta) + \sin(60° + \theta)\sin(60° - \theta) = \cos 2\theta$

Solution:

L.H.S $= \cos(60° + \theta)\cos(60° - \theta) + \sin(60° + \theta)\sin(60° - \theta)$

Using formula: $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

L.H.S $= \cos((60° + \theta) - (60° - \theta))$

$= \cos(60° + \theta - 60° + \theta)$

$= \cos(2\theta)$

$= \cos 2\theta =$ R.H.S (Proved)

13. If α, β, γ are the angles of a triangle ABC, show that $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

Solution:

Since α, β, γ are the angles of triangle ABC, therefore

$\alpha + \beta + \gamma = 180°$

$\alpha + \beta = 180° - \gamma$

Dividing both sides by '2'

$$\frac{\alpha}{2} + \frac{\beta}{2} = 90° - \frac{\gamma}{2}$$

$$\Rightarrow \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(1 \times 90° - \frac{\gamma}{2}\right)$$

Using $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$\frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \cot\frac{\gamma}{2}$$

$$\frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \frac{1}{\tan\frac{\gamma}{2}}$$

$$\tan\frac{\gamma}{2} \cdot \left\{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}\right\} = 1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$\tan\frac{\gamma}{2} \cdot \tan\frac{\alpha}{2} + \tan\frac{\gamma}{2} \cdot \tan\frac{\beta}{2} + \tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} = 1$$

Dividing both sides by $\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} \cdot \tan\frac{\gamma}{2}$

$$\frac{1}{\tan\frac{\beta}{2}} + \frac{1}{\tan\frac{\alpha}{2}} + \frac{1}{\tan\frac{\gamma}{2}} = \frac{1}{\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} \cdot \tan\frac{\gamma}{2}}$$

$$\cot\frac{\beta}{2} + \cot\frac{\alpha}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2} \text{ (Proved)}$$

14. If $\alpha + \beta + \gamma = 180°$, show that: $\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma + \cot\alpha = 1$

Solution:

It is given that, $\alpha + \beta + \gamma = 180°$

$\alpha + \beta = 180° - \gamma$

$\Rightarrow \tan(\alpha + \beta) = \tan(2 \times 90° - \gamma)$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma$$

$\tan\alpha + \tan\beta = -\tan\gamma \cdot (1 - \tan\alpha\tan\beta)$

$\tan\alpha + \tan\beta = -\tan\gamma + \tan\alpha \cdot \tan\beta \cdot \tan\gamma$

$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \cdot \tan\beta \cdot \tan\gamma$

Dividing both sides by $\tan\alpha \cdot \tan\beta \cdot \tan\gamma$

$$\frac{1}{\tan\beta \cdot \tan\gamma} + \frac{1}{\tan\alpha \cdot \tan\gamma} + \frac{1}{\tan\alpha \cdot \tan\beta} = \frac{\tan\alpha \cdot \tan\beta \cdot \tan\gamma}{\tan\alpha \cdot \tan\beta \cdot \tan\gamma}$$

$\cot\beta \cdot \cot\gamma + \cot\alpha \cdot \cot\gamma + \cot\alpha \cdot \cot\beta = 1$

$\cot\alpha \cdot \cot\beta + \cot\beta \cdot \cot\gamma + \cot\gamma \cdot \cot\alpha = 1$

Hence proved.

15. Express the following in the form $r\sin(\theta + \phi)$ or $r\sin(\theta - \phi)$ where terminal sides of the angles of measures θ and ϕ are in the first quadrant:

(i) $24\sin\theta + 7\cos\theta$

Solution:

Let $24\sin\theta + 7\cos\theta = r\sin(\theta + \phi)$ ...(1)

$24\sin\theta + 7\cos\theta = r(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$24\sin\theta + 7\cos\theta = r\sin\theta\cos\phi + r\cos\theta\sin\phi$

Comparing both sides

$r\cos\phi = 24$ ...(2)

$r\sin\phi = 7$ ...(3)

Squaring equations (2) and (3), then adding

$r^2\cos^2\phi + r^2\sin^2\phi = (24)^2 + (7)^2$

$r^2(\cos^2\phi + \sin^2\phi) = 576 + 49$

$r^2 = 625 \quad \Rightarrow r = 25$

Dividing equation (3) by equation (2)

$$\frac{r\sin\phi}{} = \frac{7}{}$$

$$\tan\phi = \frac{7}{24} \Rightarrow \phi = \tan^{-1}\left(\frac{7}{24}\right)$$

Putting values of r and ϕ in equation (1).

$$24\sin\theta + 7\cos\theta = 25\sin(\theta+\phi), \text{ where } \phi = \tan^{-1}\left(\frac{7}{24}\right)$$

(ii) $12\sin\theta - 5\cos\theta$

Solution:

Let
$$12\sin\theta - 5\cos\theta = r\sin(\theta-\phi) \quad ...(1)$$
$$12\sin\theta - 5\cos\theta = r(\sin\theta\cos\phi - \cos\theta\sin\phi)$$
$$12\sin\theta - 5\cos\theta = r\sin\theta\cos\phi - r\cos\theta\sin\phi$$

Comparing both sides
$$r\cos\phi = 12 \quad ...(2)$$
$$r\sin\phi = 5 \quad ...(3)$$

Squaring equations (2) and (3), then adding
$$r^2\cos^2\phi + r^2\sin^2\phi = (12)^2 + (5)^2$$
$$r^2(\cos^2\phi + \sin^2\phi) = 144 + 25$$
$$r^2 = 169 \quad \Rightarrow r = 13$$

Dividing equation (3) by equation (2)
$$\frac{r\sin\phi}{r\cos\phi} = \frac{5}{12}$$
$$\tan\phi = \frac{5}{12} \quad \Rightarrow \phi = \tan^{-1}\left(\frac{5}{12}\right)$$

Putting values of r and ϕ in equation (1)
$$12\sin\theta - 5\cos\theta = 13\sin(\theta-\phi), \text{ where } \phi = \tan^{-1}\left(\frac{5}{12}\right)$$

(iii) $\sin\theta - \cos\theta$

Solution:

Let
$$\sin\theta - \cos\theta = r\sin(\theta-\phi) \quad ...(1)$$
$$\sin\theta - \cos\theta = r(\sin\theta\cos\phi - \cos\theta\sin\phi)$$
$$\sin\theta - \cos\theta = r\sin\theta\cos\phi - r\cos\theta\sin\phi$$

Comparing both sides
$$r\cos\phi = 1 \quad ...(2)$$
$$r\sin\phi = 1 \quad ...(3)$$

Squaring equations (2) and (3), then adding
$$r\cos^2\phi + r^2\sin^2\phi = 1^2 + 1^2$$
$$r^2(\cos^2\phi + \sin^2\phi) = 2$$
$$r^2 = 2 \quad \Rightarrow r = \sqrt{2}$$

Dividing equation (3) by equation (2)
$$\frac{r\sin\phi}{r\cos\phi} = \frac{1}{1}$$
$$\tan\phi = 1 \quad \Rightarrow \phi = \tan^{-1}(1)$$

Putting values of r and ϕ in equation (1)
$$\sin\theta - \cos\theta = \sqrt{2}\sin(\theta-\phi), \text{ where } \phi = \tan^{-1}(1)$$

(iv) $8\sin\theta - 6\cos\theta$

Solution:

Let
$$8\sin\theta - 6\cos\theta = r\sin(\theta-\phi) \quad ...(1)$$
$$8\sin\theta - 6\cos\theta = r(\sin\theta\cos\phi - \cos\theta\sin\phi)$$
$$8\sin\theta - 6\cos\theta = r\sin\theta\cos\phi - r\cos\theta\sin\phi$$

Comparing both sides
$$r\cos\phi = 8 \quad ...(2)$$
$$r\sin\phi = 6 \quad ...(3)$$

Squaring equations (2) and (3), then adding
$$r^2\cos^2\phi + r^2\sin^2\phi = 8^2 + 6^2$$
$$r^2(\cos^2\phi + \sin^2\phi) = 64 + 36$$
$$r^2 = 100 \quad \Rightarrow r = 10$$

Dividing equation (3) by equation (2)
$$\frac{r\sin\phi}{r\cos\phi} = \frac{6}{8}$$
$$\tan\phi = \frac{3}{4} \quad \Rightarrow \phi = \tan^{-1}\left(\frac{3}{4}\right)$$

Putting values of r and ϕ in equation (1)
$$8\sin\theta - 6\cos\theta = 10\sin(\theta-\phi), \text{ where } \phi = \tan^{-1}\left(\frac{3}{4}\right)$$

(v) $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta$

Solution:

Let
$$\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = r\sin(\theta+\phi) \quad ...(1)$$
$$\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = r(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = r\sin\theta\cos\phi + r\cos\theta\sin\theta$$

Comparing both side
$$r\cos\phi = \frac{1}{2} \quad ...(2)$$
$$r\sin\phi = \frac{\sqrt{3}}{2} \quad ...(3)$$

Squaring equations (2) and (3), then adding
$$r^2\cos^2\phi + r^2\sin^2\phi = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$
$$r^2(\cos^2\phi + \sin^2\phi) = \frac{1}{4} + \frac{3}{4}$$
$$r^2 = \frac{4}{4} \quad \Rightarrow r^2 = 1 \Rightarrow r = 1$$

Dividing equation (3) by equation (2)
$$\frac{r\sin\phi}{r\cos\phi} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$\tan\phi = \sqrt{3} \quad \Rightarrow \phi = \tan^{-1}(\sqrt{3})$$

Putting values of r and ϕ in equation (1)
$$\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = 1\cdot\sin(\theta+\phi), \text{ where } \phi = \tan^{-1}(\sqrt{3})$$

(vi) $13\sin\theta - 84\cos\theta$

Solution:

Let
$$13\sin\theta - 84\cos\theta = r\sin(\theta-\phi) \quad ...(1)$$
$$13\sin\theta - 84\cos\theta = r(\sin\theta\cos\phi - \cos\theta\sin\phi)$$
$$13\sin\theta - 84\cos\theta = r\sin\theta\cos\phi - r\cos\theta\sin\phi$$

Comparing both sides
$$r\cos\phi = 13 \quad ...(2)$$
$$r\sin\phi = 84 \quad ...(3)$$

Squaring equations (2) and (3), Then adding
$$r^2\cos^2\phi + r^2\sin^2\phi = (13)^2 + (84)^2$$
$$r^2(\cos^2\phi + \sin^2\phi) = 169 + 7056$$
$$r^2 = 7225 \quad \Rightarrow r = 85$$

Dividing equation (3) by equation (2)
$$\frac{r\sin\phi}{r\cos\phi} = \frac{84}{13}$$
$$\tan\phi = \frac{84}{13}$$
$$\phi = \tan^{-1}\left(\frac{84}{13}\right)$$

Putting values of r and ϕ in equation (1)
$$13\sin\theta - 84\cos\theta = 85\sin(\theta-\phi), \text{ where } \phi = \tan^{-1}\left(\frac{84}{13}\right)$$

Double Angle Identities:

We have discussed the following results:
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ and } \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

We can use them to obtain the double angle identities as follows:

i) Show that: $\sin 2\alpha = 2\sin\alpha\cos\alpha$

Proof
$$\begin{aligned}\text{L.H.S} &= \sin 2\alpha \\ &= \sin(\alpha+\alpha) \\ &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\ &= \sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ &= 2\sin\alpha\cos\alpha \\ &= \text{R.H.S (Proved)}\end{aligned}$$

ii) Show that: $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

Proof
$$\begin{aligned}\text{L.H.S} &= \cos 2\alpha \\ &= \cos(\alpha+\alpha) \\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\ &= \cos 2\alpha - \sin 2\alpha \\ &= \text{R.H.S (Proved)}\end{aligned}$$

iii) Show that: $\cos 2\alpha = 2\cos^2\alpha - 1$

Proof
$$\begin{aligned}\text{L.H.S} &= \cos 2\alpha \\ &= \cos(\alpha+\alpha) \\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\ &= \cos^2\alpha - \sin^2\alpha \\ &= \cos^2\alpha - (1-\cos^2\alpha) \\ &= \cos^2\alpha - 1 + \cos^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= \text{R.H.S (Proved)}\end{aligned}$$

iv) Show that: $\cos 2\alpha = 1 - 2\sin^2\alpha$

Proof
$$\begin{aligned}\text{L.H.S} &= \cos 2\alpha \\ &= \cos(\alpha+\alpha) \\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\ &= \cos^2\alpha - \sin^2\alpha \\ &= (1-\sin^2\alpha) - \sin^2\alpha \\ &= 1 - \sin^2\alpha - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha \\ &= \text{R.H.S (Proved)}\end{aligned}$$

v) Show that: $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

Proof
$$\begin{aligned}\text{L.H.S} &= \tan 2\alpha \\ &= \tan(\alpha+\alpha) \\ &= \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha} = \text{R.H.S (Proved)}\end{aligned}$$

Half Angle Identities:

The formulas proved above can also be written in the form of half angle identities, in the following way:

(i) $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \quad \Rightarrow \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \quad \Rightarrow \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

(ii) $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \quad \Rightarrow \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \quad \Rightarrow \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

(iii) $\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \pm\frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}} \quad \Rightarrow \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$

Triple Angle Identities:

(i) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ (ii) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

(iii) $\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

Proof: (i)

$\sin 3\alpha = \sin(2\alpha + \alpha)$

$= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha$

$= (2\sin\alpha\cos\alpha)\cos\alpha + (1 - 2\sin^2\alpha)\sin\alpha \qquad \because \begin{bmatrix}\sin 2\alpha = 2\sin\alpha\cos\alpha \\ \cos 2\alpha = 1 - 2\sin^2\alpha\end{bmatrix}$

$= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha$

$= 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha \qquad \because \cos^2\alpha = 1 - \sin^2\alpha$

$= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha$

$\boxed{\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha}$

(ii)

$\cos 3\alpha = \cos(2\alpha + \alpha)$

$= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha$

$= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha \qquad \because \cos 2\alpha = 2\cos^2\alpha - 1$

$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$

$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha \qquad \because \sin^2\alpha = 1 - \cos^2\alpha$

$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$

$\boxed{\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha}$

(iii)

$\tan 3\alpha = \tan(2\alpha + \alpha)$

$= \frac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha\tan\alpha}$

$= \frac{\frac{2\tan\alpha}{1-\tan^2\alpha} + \tan\alpha}{1 - \frac{2\tan\alpha}{1-\tan^2\alpha}\cdot\tan\alpha} = \frac{\frac{2\tan\alpha + \tan\alpha(1-\tan^2\alpha)}{1-\tan^2\alpha}}{\frac{1-\tan^2\alpha - 2\tan^2\alpha}{1-\tan^2\alpha}}$

$= \frac{2\tan\alpha + \tan\alpha - \tan^3\alpha}{1 - \tan^2\alpha - 2\tan^2\alpha}$

$\boxed{\tan^3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}}$

Example 11: Prove that: $\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$

Solution:

$\text{L.H.S.} = \frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta}$

$= \frac{\sin\theta + 2\sin\theta\cos\theta}{1 + \cos\theta + 2\cos^2\theta - 1} \qquad \because \cos 2\theta = 2\cos^2\theta - 1$

$= \frac{\sin\theta(1 + 2\cos\theta)}{\cos\theta(1 + 2\cos\theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{R.H.S}$

Hence $\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$.

Example 12: Show that:

(i) $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$ (ii) $\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

i) $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$

Solution

$\text{R.H.S} = \frac{2\tan\theta}{1 + \tan^2\theta}$

$= \frac{\frac{2\sin\theta}{\cos\theta}}{\sec^2\theta} \qquad \because 1 + \tan^2\theta = \sec^2\theta$

$= \frac{2\sin\theta}{\cos\theta}\cdot\frac{1}{\sec^2\theta} = \frac{2\sin\theta}{\cos\theta}\cdot\cos^2\theta$

$= 2\sin\theta\cdot\cos\theta = \sin 2\theta = \text{L.H.S (Proved)}$

ii) $\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

Solution:

$\text{R.H.S} = \frac{1 - \tan^2\theta}{\sec^2\theta} \qquad \because 1 + \tan^2\theta = \sec^2\theta$

$= \left(1 - \frac{\sin^2\theta}{\cos^2\theta}\right)\cdot\frac{1}{\sec^2\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta$

$= \cos^2\theta - \sin^2\theta$

$= \cos 2\theta = \text{L.H.S (Proved)}$

Example 13: Reduce $\cos^4\theta$ to an expression involving only function of multiples of θ, raised to the first power.

Solution:

Consider $\cos^4\theta = (\cos^2\theta)^2$

$= \left[\frac{1 + \cos 2\theta}{2}\right]^2 \qquad \because 2\cos^2\theta = 1 + \cos 2\theta \Rightarrow \cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$= \frac{1 + 2\cos 2\theta + \cos^2 2\theta}{4} = \frac{1}{4}\left[1 + 2\cos 2\theta + \cos^2 2\theta\right]$

$= \frac{1}{4}\left[1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right] \qquad \because \cos^2\theta = \frac{1 + \cos 2\theta}{2} \Rightarrow \cos^2 2\theta = \frac{1 + \cos 4\theta}{2}$

$= \frac{1}{4}\left[\frac{2 + 4\cos 2\theta + 1 + \cos 4\theta}{2}\right] = \frac{1}{8}[3 + 4\cos 2\theta + \cos 4\theta]$ (As Required)

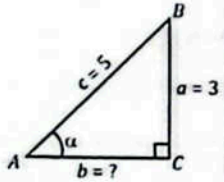
Exercise 10.3

1. Find the values of sin2α, cos2α and tan2α, when:

(i) $\sin\alpha = \frac{3}{5}$

Solution:

$$\sin\alpha = \frac{3}{5}$$



By Pythagoras theorem

$$c^2 = a^2 + b^2$$
$$5^2 = 3^2 + b^2$$
$$b^2 = 25 - 9$$
$$b^2 = 16$$
$$b = 4$$

Since α lies in I-quad., therefore

$\cos\alpha = \frac{4}{5}$, $\tan\alpha = \frac{3}{4}$

Now, $\sin2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$$

$$\cos2\alpha = 2\cos^2\alpha - 1$$
$$= 2\left(\frac{4}{5}\right)^2 - 1 = 2\left(\frac{16}{25}\right) - 1$$
$$= \frac{32-25}{25} = \frac{7}{25}$$

$$\tan2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$
$$= \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{1-\frac{9}{16}}$$
$$= \frac{\frac{3}{2}}{\frac{16-9}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

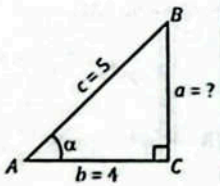
(ii) $\cos\alpha = \frac{4}{5}$, where $0 < \alpha < \frac{\pi}{2}$

Solution:

$$\cos\alpha = \frac{4}{5}$$

By Pythagoras theorem

$$c^2 = a^2 + b^2$$
$$5^2 = a^2 + 4^2$$
$$25 - 16 = a^2$$
$$a^2 = 9$$
$$a = 3$$



Since α lies in I-quad., therefore.

$\sin\alpha = \frac{3}{5}$, $\tan\alpha = \frac{3}{4}$

Now, $\sin2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$$

$$\cos2\alpha = 2\cos^2\alpha - 1$$
$$= 2\left(\frac{4}{5}\right)^2 - 1 = 2\left(\frac{16}{25}\right) - 1$$
$$= \frac{32-25}{25} = \frac{7}{25}$$

$$\tan2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$
$$= \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{1-\frac{9}{16}}$$
$$= \frac{\frac{3}{2}}{\frac{16-9}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

---

2. Prove the following identities:

(i) $\cot\alpha - \tan\alpha = 2\cot2\alpha$

Solution:

L.H.S = $\cot\alpha - \tan\alpha$

$$= \frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha} = \frac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha \,.\, \cos\alpha}$$

$$= \frac{\cos2\alpha}{\sin\alpha \,.\, \cos\alpha}$$ using $\cos2\alpha = \cos^2\alpha - \sin^2\alpha$

$$= \frac{2\cos2\alpha}{2\sin\alpha \cdot \cos\alpha}$$ Multiplying up and down by '2'

$$= \frac{2\cos2\alpha}{\sin2\alpha} = 2\cot2\alpha = \text{R.H.S (Proved)}$$

(ii) $\frac{\sin2\alpha}{1+\cos2\alpha} = \tan\alpha$

Solution:

L.H.S = $\frac{\sin2\alpha}{1+\cos2\alpha}$

$$= \frac{\sin2\alpha}{1+(2\cos^2\alpha - 1)}$$ using $\cos2\alpha = 2\cos^2\alpha - 1$

$$= \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \text{R.H.S (Proved)}$$

(iii) $\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$

Solution:

L.H.S = $\frac{1-\cos\alpha}{\sin\alpha}$

$$= \frac{1-\left(1-2\sin^2\frac{\alpha}{2}\right)}{\sin\alpha}$$ using $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$

$$= \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}}$$ using $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} = \text{R.H.S (Proved)}$$

(iv) $\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \sec2\alpha - \tan2\alpha$

Solution:

L.H.S = $\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}$

$$= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} \times \frac{\cos\alpha - \sin\alpha}{\cos\alpha - \sin\alpha} = \frac{(\cos\alpha - \sin\alpha)^2}{\cos^2\alpha - \sin^2\alpha}$$

using $\cos2\alpha = \cos^2\alpha - \sin^2\alpha$

$$= \frac{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha}{\cos2\alpha}$$

$$= \frac{1-\sin2\alpha}{\cos2\alpha}$$ $\because \begin{cases}\sin^2\alpha + \cos^2\alpha = 1 \\ 2\sin\alpha\cos\alpha = \sin2\alpha\end{cases}$

$$= \frac{1}{\cos2\alpha} - \frac{\sin2\alpha}{\cos2\alpha} = \sec2\alpha - \tan2\alpha = \text{R.H.S (Proved)}$$

(v) $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}}$

Solution:

L.H.S = $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$

Using $\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1$ and $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$= \sqrt{\frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$

$$= \sqrt{\frac{\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)^2}{\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)^2}} = \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}}$$

= R.H.S (Proved)

(vi) $\frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta} = \cot\frac{\theta}{2}$

Solution:

L.H.S = $\frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta}$

$$= \frac{\frac{1}{\sin\theta} + \frac{2}{\sin2\theta}}{\frac{1}{\cos\theta}}$$

$$= \left[\frac{1}{\sin\theta} + \frac{2}{2\sin\theta\cos\theta}\right] \times \cos\theta$$ $\because \sin2\theta = 2\sin\theta\cos\theta$

$$= \left[\frac{\cos\theta + 1}{\sin\theta\cos\theta}\right] \times \cos\theta = \frac{\cos\theta + 1}{\sin\theta}$$

$$= \frac{\left(2\cos^2\frac{\theta}{2} - 1\right) + 1}{\sin\theta}$$ $\because \cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$ $\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2} = \text{R.H.S (Proved)}$$

(vii) $1 + \tan\alpha\tan2\alpha = \sec2\alpha$

Solution:

L.H.S = $1 + \tan\alpha\tan2\alpha$

$$= 1 + \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\sin2\alpha}{\cos2\alpha}$$

$$= \frac{\cos2\alpha.\cos\alpha + \sin2\alpha.\sin\alpha}{\cos2\alpha \,.\, \cos\alpha}$$

Using $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

$$= \frac{\cos(2\alpha - \alpha)}{\cos2\alpha \,.\, \cos\alpha}$$

$$= \frac{\cos\alpha}{\cos2\alpha \,.\, \cos\alpha}$$

$$= \frac{1}{\cos2\alpha} = \sec2\alpha = \text{R.H.S (Proved)}$$

(viii) $\dfrac{2\sin\theta\sin2\theta}{\cos\theta+\cos3\theta}=\tan2\theta\tan\theta$

Solution:

$$\text{L.H.S} = \frac{2\sin\theta\,.\,\sin2\theta}{\cos\theta+\cos3\theta}$$

$$= \frac{2\sin\theta.\sin2\theta}{\cos\theta+4\cos^3\theta-3\cos\theta} \quad \because \cos3\theta=4\cos^3\theta-3\cos\theta$$

$$= \frac{2\sin\theta.\sin2\theta}{4\cos^3\theta-2\cos\theta}$$

$$= \frac{2\sin\theta.\sin2\theta}{2\cos\theta(2\cos^2\theta-1)}$$

$$= \frac{\sin\theta.\sin2\theta}{\cos\theta.\cos2\theta} \quad \because \cos2\theta=2\cos^2\theta-1$$

$= \tan\theta\,.\,\tan2\theta =$ R.H.S (Proved)

(ix) $\dfrac{\sin3\theta}{\sin\theta}-\dfrac{\cos3\theta}{\cos\theta}=2$

Solution:

$$\text{L.H.S} = \frac{\sin3\theta}{\sin\theta}-\frac{\cos3\theta}{\cos\theta}$$

$$= \frac{\sin3\theta\cos\theta-\cos3\theta\sin\theta}{\sin\theta\cos\theta}$$

Using $\sin\alpha.\cos\beta-\cos\alpha.\sin\beta=\sin(\alpha-\beta)$

$$= \frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$$

$$= \frac{2\sin2\theta}{2\sin\theta\cos\theta} \quad \text{Multiplying up and down by '2'}$$

$$= \frac{2\sin2\theta}{\sin2\theta} \quad \because 2\sin\theta\cos\theta=\sin2\theta$$

$= 2 =$ R.H.S (Proved)

(x) $\dfrac{\cos3\theta}{\cos\theta}+\dfrac{\sin3\theta}{\sin\theta}=4\cos2\theta$

Solution:

$$\text{L.H.S} = \frac{\cos3\theta}{\cos\theta}+\frac{\sin3\theta}{\sin\theta}$$

$$= \frac{\cos3\theta\sin\theta+\sin3\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin3\theta\cos\theta+\cos3\theta\sin\theta}{\sin\theta\cos\theta}$$

Using $\sin\alpha\,.\cos\beta+\cos\alpha\,.\sin\beta=\sin(\alpha+\beta)$

$$= \frac{\sin(3\theta+\theta)}{\sin\theta\cos\theta}$$

$$= \frac{2\sin4\theta}{2\sin\theta\cos\theta} \quad \text{Multiplying up and down by '2'}$$

$$= \frac{2\sin2(2\theta)}{\sin2\theta}$$

$$= \frac{2.2\sin2\theta\cos2\theta}{\sin2\theta} \quad \because \sin2\alpha=2\sin\alpha.\cos\alpha$$

$= 4\cos2\theta =$ R.H.S (Proved)

(xi) $\dfrac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}=\sec\theta$

Solution:

$$\text{L.H.S} = \frac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}$$

$$= \frac{\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}+\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}}{\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}-\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}} = \frac{\dfrac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}{\dfrac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}$$

$$= \frac{1}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}} \quad \because \sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}=1$$

$$= \frac{1}{\cos\theta} \quad \because \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}=\cos\theta$$

$= \sec\theta =$ R.H.S (Proved)

(xii) $\dfrac{\sin3\theta}{\cos\theta}+\dfrac{\cos3\theta}{\sin\theta}=2\cot2\theta$

Solution:

$$\text{L.H.S} = \frac{\sin3\theta}{\cos\theta}+\frac{\cos3\theta}{\sin\theta}$$

$$= \frac{\sin3\theta\sin\theta+\cos3\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos3\theta\cos\theta+\sin3\theta\sin\theta}{\sin\theta\cos\theta}$$

Using $\cos\alpha.\cos\beta+\sin\alpha.\sin\beta=\cos(\alpha-\beta)$

$$= \frac{\cos(3\theta-\theta)}{\sin\theta\cos\theta} = \frac{\cos2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\cos2\theta}{2\sin\theta\cos\theta} \quad \text{Multiplying up and down by '2'}$$

$$= \frac{2\cos2\theta}{\sin2\theta} \quad \because \sin2\theta=2\sin\theta\cos\theta$$

$= 2\cot2\theta =$ R.H.S (Proved)

(xiii) $\dfrac{3+\cos4\theta}{1-\cos4\theta}=\dfrac{1}{2}(\tan^2\theta+\cot^2\theta)$

Solution:

$$\text{L.H.S} = \frac{3+\cos4\theta}{1-\cos4\theta}$$

As $\cos2\theta=2\cos^2\theta-1 \Rightarrow \cos4\theta=2\cos^2 2\theta-1$

and $\cos2\theta=1-2\sin^2\theta \Rightarrow \cos4\theta=1-2\sin^2 2\theta$

$$\text{L.H.S} = \frac{3+(2\cos^2 2\theta-1)}{1-(1-2\sin^2 2\theta)}$$

$$= \frac{2+2\cos^2 2\theta}{2\sin^2 2\theta} = \frac{2}{2\sin^2 2\theta}+\frac{2\cos^2 2\theta}{2\sin^2 2\theta}$$

$$= \frac{1}{\sin^2 2\theta}+\cot^2 2\theta$$

$= \operatorname{cosec}^2 2\theta+(\operatorname{cosec}^2 2\theta-1) \quad \because 1+\cot^2 2\theta=\operatorname{cosec}^2 2\theta$

$= 2\operatorname{cosec}^2 2\theta-1$

$$\text{R.H.S} = \frac{1}{2}(\tan^2\theta+\cot^2\theta)$$

$$= \frac{1}{2}\left(\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= \frac{1}{2}\left(\frac{(\sin^2\theta)^2+(\cos^2\theta)^2}{\sin^2\theta\cos^2\theta}\right)$$

$$= \frac{((\sin^2\theta)^2+(\cos^2\theta)^2+2\sin^2\theta\cos^2\theta)-2\sin^2\theta\cos^2\theta}{2\sin^2\theta\cos^2\theta}$$

$$= \frac{(\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta}{2\sin^2\theta\cos^2\theta}$$

$$= \frac{(1)^2-2\sin^2\theta\cos^2\theta}{2\sin^2\theta\cos^2\theta}$$

Muliply up and down by '2'

$$= \frac{2-4\sin^2\theta\cos^2\theta}{4\sin^2\theta\cos^2\theta} = \frac{2-(2\sin\theta\cos\theta)^2}{(2\sin\theta\cos\theta)^2}$$

$$= \frac{2-(\sin2\theta)^2}{(\sin2\theta)} = \frac{2}{\sin^2 2\theta}-\frac{\sin^2 2\theta}{\sin^2 2\theta}$$

$= 2\operatorname{cosec}^2 2\theta-1$

Hence proved L.H.S = R.H.S

(xiv) $\dfrac{1+\sin2\theta}{1-\sin2\theta}=\tan^2\left(\dfrac{\pi}{4}+\theta\right)$

Solution:

$$\text{L.H.S} = \frac{1+\sin2\theta}{1-\sin2\theta}$$

$$= \frac{\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta} \quad \because \cos^2\theta+\sin^2\theta=1$$

$$= \frac{(\cos\theta+\sin\theta)^2}{(\cos\theta-\sin\theta)^2}$$

$$= \frac{\cos^2\theta\left(1+\dfrac{\sin\theta}{\cos\theta}\right)^2}{\cos^2\theta\left(1-\dfrac{\sin\theta}{\cos\theta}\right)^2}$$

$$= \frac{(1+\tan\theta)^2}{(1-1\cdot\tan\theta)^2}$$

$$= \left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\cdot\tan\theta}\right)^2 \quad \because \tan\frac{\pi}{4}=1$$

$$= \left(\tan\left(\frac{\pi}{4}+\theta\right)\right)^2 \quad \because \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\tan(\alpha+\beta)$$

$= \tan^2\left(\dfrac{\pi}{4}+\theta\right) =$ R.H.S (Proved)

(xv) $\cos^2\dfrac{\pi}{8}+\cos^2\dfrac{3\pi}{8}+\cos^2\dfrac{5\pi}{8}+\cos^2\dfrac{7\pi}{8}=2$

Solution:

$$\text{L.H.S} = \cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{5\pi}{8}+\cos^2\frac{7\pi}{8}$$

$$= \cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\left(\pi-\frac{3\pi}{8}\right)+\cos^2\left(\pi-\frac{\pi}{8}\right)$$

$$= \cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\left(2\times\frac{\pi}{2}-\frac{3\pi}{8}\right)+\cos^2\left(2\times\frac{\pi}{2}-\frac{\pi}{8}\right)$$

$$= \cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\left(-\cos\frac{3\pi}{8}\right)^2+\left(-\cos\frac{\pi}{8}\right)^2$$

$$= \cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{\pi}{8}$$

$$= 2\cos^2\frac{\pi}{8}+2\cos^2\frac{3\pi}{8}$$

$$= 2\cos^2\left(\frac{\pi}{2}-\frac{3\pi}{8}\right)+2\cos^2\frac{3\pi}{8}$$

$$= 2\sin^2\frac{3\pi}{8}+2\cos^2\frac{3\pi}{8} \quad \because \cos\left(\frac{\pi}{2}-\frac{3\pi}{8}\right)=\sin\frac{3\pi}{8}$$

$$= 2\left(\sin^2\frac{3\pi}{8}+\cos^2\frac{3\pi}{8}\right)$$

$= 2(1) \quad \because \sin^2\theta+\cos^2\theta=1$

$= 2 =$ R.H.S (Proved)

3. Show that: $2\cos\theta=\sqrt{2+\sqrt{2+2\cos4\theta}}$

Solution:

$$\text{R.H.S} = \sqrt{2+\sqrt{2+2\cos4\theta}}$$

$$= \sqrt{2+\sqrt{2(1+\cos4\theta)}}$$

As we know

$$\cos 2\theta = 2\cos^2\theta - 1 \quad \text{...(1)}$$
$$\cos 4\theta = 2\cos^2 2\theta - 1$$
$$1 + \cos 4\theta = 2\cos^2 2\theta \quad \text{...(2)}$$

$$\text{R.H.S} = \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} \quad \text{using (2)}$$
$$= \sqrt{2+\sqrt{4\cos^2 2\theta}}$$
$$= \sqrt{2+2\cos 2\theta}$$
$$= \sqrt{2(1+\cos^2 2\theta)}$$
$$= \sqrt{2\cdot 2\cos^2\theta} \quad \text{using (1)}$$
$$= \sqrt{4\cos^2\theta}$$
$$= 2\cos\theta = \text{R.H.S (Proved)}$$

4. Reduce $\sin^4\theta$ to an expression involving only function of multiples of θ, raised to the first power.

Solution:

Consider: $\sin^4\theta = (\sin^2\theta)^2$

$$= \left(\frac{1-\cos 2\theta}{2}\right)^2$$

Using $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\sin^4\theta = \frac{1-2\cos 2\theta+\cos^2 2\theta}{4}$$
$$= \frac{1}{4}\left\{1-2\cos 2\theta+\left(\frac{1+\cos 4\theta}{2}\right)\right\}$$

Using $\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$\Rightarrow \cos^2 2\theta = \frac{1+\cos 4\theta}{2}$$

$$\sin^4\theta = \frac{1}{4}\left\{\frac{2-4\cos 2\theta+1+\cos 4\theta}{2}\right\}$$
$$\sin^4\theta = \frac{3-4\cos 2\theta+\cos 4\theta}{8} \quad \text{(As required)}$$

5. Find the values of $\sin\theta$ and $\cos\theta$ without using table or calculator, when θ is:
(i) 18° (ii) 36° (iii) 54° (iv) 72°
Hence prove that:

$$\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$$

Hint:

Let $\theta = 18^\circ$	Let $\theta = 36^\circ$
$5\theta = 90^\circ$	$5\theta = 180^\circ$
$(3\theta+2\theta) = 90^\circ$	$3\theta+2\theta = 180^\circ$
$3\theta = 90^\circ - 2\theta$	$3\theta = 180^\circ - 2\theta$
$\sin 3\theta = \sin(90^\circ - 2\theta)$ etc.	$\sin 3\theta = \sin(180^\circ - 2\theta)$ etc.

(i) 18°

Solution:

$$\theta = 18^\circ$$
$$5\theta = 90^\circ \quad \text{(Multiply by '5')}$$
$$2\theta + 3\theta = 90^\circ \quad \Rightarrow 2\theta = 90^\circ - 3\theta$$
$$\sin 2\theta = \sin(90^\circ - 3\theta)$$
$$2\sin\theta\,.\cos\theta = \cos 3\theta \quad \because \sin(1\times 90^\circ - 3\theta) = \cos 3\theta$$
$$2\sin\theta\,.\cos\theta = 4\cos^3\theta - 3\cos\theta$$
$$2\sin\theta = 4\cos^2\theta - 3$$

Dividing by $\cos\theta$ $\quad (\because \cos\theta \neq 0, \text{ when } \theta = 18^\circ)$

$$2\sin\theta = 4(1-\sin^2\theta) - 3 \quad \because \cos^2\theta = 1-\sin^2\theta$$
$$2\sin\theta = 4 - 4\sin^2\theta - 3$$
$$4\sin^2\theta + 2\sin\theta - 1 = 0$$

Here $a = 4$, $b = 2$, $c = -1$

$$\sin\theta = \frac{-2\pm\sqrt{4+16}}{8} \quad \text{using quadratic formula}$$
$$\sin\theta = \frac{-2\pm\sqrt{20}}{8} = \frac{2(-1\pm\sqrt{5})}{8} = \frac{-1\pm\sqrt{5}}{4}$$

Since $\theta = 18^\circ$ lies in I-quadrant, so $\sin\theta \neq \dfrac{-1-\sqrt{5}}{4}$.

Therefore

$$\sin\theta = \frac{-1+\sqrt{5}}{4} \quad \Rightarrow \boxed{\sin 18^\circ = \frac{\sqrt{5}-1}{4}}$$

$$\cos 18^\circ = \sqrt{1-\sin^2 18^\circ} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}$$
$$= \sqrt{1-\frac{5+1-2\sqrt{5}}{16}} = \sqrt{\frac{16-6+2\sqrt{5}}{16}}$$
$$\boxed{\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$

(ii) 36°

Solution:

$$\theta = 36^\circ$$
$$5\theta = 180^\circ \quad \text{(Multiply by 5)}$$
$$2\theta + 3\theta = 180^\circ \quad \Rightarrow \quad 2\theta = 180^\circ - 3\theta$$
$$\sin 2\theta = \sin(180^\circ - 3\theta)$$
$$2\sin\theta\,.\cos\theta = \sin(2\times 90^\circ - 3\theta)$$
$$2\sin\theta\cos\theta = \sin 3\theta$$
$$2\sin\theta\cos\theta = 3\sin\theta - 4\sin^3\theta$$
$$2\cos\theta = 3 - 4\sin^2\theta$$

Dividing by $\sin\theta$ $\quad (\because \sin\theta \neq 0, \text{ where } \theta = 36^\circ)$

$$2\cos\theta = 3 - 4(1-\cos^2\theta)$$
$$4\cos^2\theta - 2\cos\theta - 1 = 0$$

Here $a = 4$, $b = -2$, $c = -1$

$$\cos\theta = \frac{2\pm\sqrt{4+16}}{8} \quad \text{using quadratic formula}$$
$$\cos\theta = \frac{2\pm\sqrt{20}}{8} = \frac{2(1\pm\sqrt{5})}{8} = \frac{1\pm\sqrt{5}}{4}$$

Since $\theta = 36^\circ$ lies in I-quad., so $\cos\theta \neq \dfrac{1-\sqrt{5}}{4}$. Therefore,

$$\cos\theta = \frac{1+\sqrt{5}}{4} \quad \Rightarrow \boxed{\cos 36^\circ = \frac{1+\sqrt{5}}{4}}$$

$$\sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \sqrt{1-\left(\frac{1+\sqrt{5}}{4}\right)^2}$$
$$= \sqrt{1-\frac{1+5+2\sqrt{5}}{16}} = \sqrt{\frac{16-6-2\sqrt{5}}{16}}$$
$$\boxed{\sin 36^\circ = \sqrt{\frac{10-2\sqrt{5}}{4}}}$$

(iii) 54°

Solution:

$$\sin 54^\circ = \sin(90^\circ - 36^\circ) = \cos 36^\circ = \frac{1+\sqrt{5}}{4}$$
$$\Rightarrow \boxed{\sin 54^\circ = \frac{\sqrt{5}+1}{4}}$$
$$\cos 54^\circ = \cos(90^\circ - 36^\circ) = \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$$
$$\Rightarrow \boxed{\cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

(iv) 72°

Solution:

$$\sin 72^\circ = \sin(90^\circ - 18^\circ) = \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$$
$$\Rightarrow \boxed{\sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$
$$\cos 72^\circ = \cos(90^\circ - 18^\circ) = \sin 18^\circ = \frac{\sqrt{5}-1}{4}$$
$$\Rightarrow \boxed{\cos 72^\circ = \frac{\sqrt{5}-1}{4}}$$

Now, we will prove that:

$$\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$$

$$\text{L.H.S} = \cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ$$
$$= \cos 36^\circ \cos 72^\circ \cos(90^\circ + 18^\circ)\cos(90^\circ + 54^\circ)$$
$$= \cos 36^\circ \cos 72^\circ(-\sin 18^\circ)(-\sin 54^\circ)$$
$$\because \begin{cases}\cos(90^\circ + 18^\circ) = -\sin 18^\circ \\ \cos(90^\circ + 54^\circ) = -\sin 54^\circ\end{cases}$$
$$= \frac{\sqrt{5}+1}{4}\times\frac{\sqrt{5}-1}{4}\times\left(-\frac{\sqrt{5}-1}{4}\right)\times\left(-\frac{\sqrt{5}+1}{4}\right)$$
$$= \frac{(\sqrt{5})^2-1^2}{16}\times\left(\frac{(\sqrt{5})^2-1^2}{16}\right)$$
$$= \frac{4}{16}\times\frac{4}{16} = \frac{1}{16}$$
$$= \text{R.H.S (Proved)}$$

Express the Product (of sines and cosines) as Sums or Differences (of sines and cosines):

i) Show that $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

Proof

$$\text{R.H.S} = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$= 2\sin\alpha\cos\beta = \text{L.H.S (Proved)}$$

ii) Show that $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

Proof

$$\text{R.H.S} = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$
$$= \{\sin\alpha\cos\beta + \cos\alpha\sin\beta\} - \{\sin\alpha\cos\beta - \cos\alpha\sin\beta\}$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta - \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= 2\sin\alpha\cos\beta = \text{L.H.S (Proved)}$$

iii) Show that $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

Proof

$$\text{R.H.S} = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$= 2\cos\alpha\cos\beta = \text{L.H.S (Proved)}$$

iv) Show that $-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

Proof

$$\begin{aligned} \text{R.H.S} &= \cos(\alpha+\beta) - \cos(\alpha-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta - \{\cos\alpha\cos\beta + \sin\alpha\sin\beta\} \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= -2\sin\alpha\sin\beta = \text{L.H.S (Proved)} \end{aligned}$$

Express the Sums or Differences (of sines and cosines) as Product (of sines and cosines):

i) Show that:

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

Proof

We know that

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta) \quad ...(1)$$

Let $\alpha+\beta = P$ and $\alpha-\beta = Q$

By adding

$$\begin{aligned} \alpha+\beta &= P \\ \alpha-\beta &= Q \\ \hline 2\alpha &= P+Q \\ \alpha &= \frac{P+Q}{2} \end{aligned}$$

By subtracting

$$\begin{aligned} \alpha+\beta &= P \\ \pm\alpha\mp\beta &= \pm Q \\ \hline 2\beta &= P-Q \\ \beta &= \frac{P-Q}{2} \end{aligned}$$

Now putting values in eq (1)

$$2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2} = \sin P + \sin Q$$

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

ii) Show that:

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

Proof

We know that

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta) \quad ...(2)$$

Let $\alpha+\beta = P$, $\alpha-\beta = Q$

By adding

$$\begin{aligned} \alpha+\beta &= P \\ \alpha-\beta &= Q \\ \hline 2\alpha &= P+Q \\ \alpha &= \frac{P+Q}{2} \end{aligned}$$

By subtracting

$$\begin{aligned} \alpha+\beta &= P \\ \pm\alpha\mp\beta &= \pm Q \\ \hline 2\beta &= P-Q \\ \beta &= \frac{P-Q}{2} \end{aligned}$$

Now putting values in eq (2)

$$2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2} = \sin P - \sin Q$$

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

iii) Show that:

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

Proof

We know that

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta) \quad ...(3)$$

Let $\alpha+\beta = P$ and $\alpha-\beta = Q$

By adding

$$\begin{aligned} \alpha+\beta &= P \\ \alpha-\beta &= Q \\ \hline 2\alpha &= P+Q \\ \alpha &= \frac{P+Q}{2} \end{aligned}$$

By subtracting

$$\begin{aligned} \alpha+\beta &= P \\ \pm\alpha\mp\beta &= \pm Q \\ \hline 2\beta &= P-Q \\ \beta &= \frac{P-Q}{2} \end{aligned}$$

Now putting values in eq (3)

$$2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2} = \cos P + \cos Q$$

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

iv) Show that:

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

Proof

We know that

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta) \quad ...(4)$$

Let $\alpha+\beta = P$, $\alpha-\beta = Q$

By adding

$$\begin{aligned} \alpha+\beta &= P \\ \alpha-\beta &= Q \\ \hline 2\alpha &= P+Q \\ \alpha &= \frac{P+Q}{2} \end{aligned}$$

By subtracting

$$\begin{aligned} \alpha+\beta &= P \\ \pm\alpha\mp\beta &= \pm Q \\ \hline 2\beta &= P-Q \\ \beta &= \frac{P-Q}{2} \end{aligned}$$

Now putting values in eq (4)

$$-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2} = \cos P - \cos Q$$

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

Example 14: Express $2\sin7\theta\cos3\theta$ as a sum or difference.

Solution:
$$\begin{aligned} 2\sin7\theta\cos3\theta &= \sin(7\theta+3\theta) + \sin(7\theta-3\theta) \\ &= \sin10\theta + \sin4\theta \end{aligned}$$

Example 15: Prove without using table / calculator, that $\sin19^\circ\cos11^\circ + \sin71^\circ\sin11^\circ = \frac{1}{2}$

Solution:
$$\begin{aligned} \text{L.H.S} &= \sin19^\circ\cos11^\circ + \sin71^\circ\sin11^\circ \\ &= \frac{1}{2}[2\sin19^\circ\cos11^\circ + 2\sin71^\circ\sin11^\circ] \\ &= \frac{1}{2}[2\sin19^\circ\cos11^\circ - (-2\sin71^\circ\sin11^\circ)] \\ &= \frac{1}{2}[\{\sin(19^\circ+11^\circ) + \sin(19^\circ-11^\circ)\} - \{\cos(71^\circ+11^\circ) - \cos(71^\circ-11^\circ)\}] \\ &= \frac{1}{2}[\sin30^\circ + \sin8^\circ - \cos82^\circ + \cos60^\circ] = \frac{1}{2}\left[\frac{1}{2} + \sin8^\circ - \cos(90^\circ-8^\circ) + \frac{1}{2}\right] \\ &= \frac{1}{2}\left[\frac{1}{2} + \sin8^\circ - \sin8^\circ + \frac{1}{2}\right] \qquad (\because \cos82^\circ = \cos(90^\circ-8^\circ) = \sin8^\circ) \\ &= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2}\right] \\ &= \frac{1}{2} = \text{R.H.S} \end{aligned}$$

Hence, $\sin19^\circ\cos11^\circ + \sin71^\circ\sin11^\circ = \frac{1}{2}$

Example 16: Express $\sin5x + \sin7x$ as a product.

Solution:
$$\begin{aligned} \sin5x + \sin7x &= 2\sin\frac{5x+7x}{2}\cos\frac{5x-7x}{2} \\ &= 2\sin6x\cos(-x) = 2\sin6x\cos x \quad (\because \cos(-\theta) = \cos\theta) \end{aligned}$$

Example 17: Express $\cos\theta + \cos3\theta + \cos5\theta + \cos7\theta$ as a product.

Solution: $\cos\theta + \cos3\theta + \cos5\theta + \cos7\theta$
$$\begin{aligned} &= (\cos3\theta + \cos\theta) + (\cos7\theta + \cos5\theta) \\ &= 2\cos\frac{3\theta+\theta}{2}\cos\frac{3\theta-\theta}{2} + 2\cos\frac{7\theta+\theta}{2}\cos\frac{7\theta-5\theta}{2} \\ &= 2\cos2\theta\cos\theta + 2\cos6\theta\cos\theta = 2\cos\theta(\cos6\theta + \cos2\theta) \\ &= 2\cos\theta\left[2\cos\frac{6\theta+2\theta}{2}\cos\frac{6\theta-2\theta}{2}\right] \\ &= 2\cos\theta(2\cos4\theta\cos2\theta) = 4\cos\theta\cos2\theta\cos4\theta \end{aligned}$$

Example 18: Show that $\cos20^\circ\cos40^\circ\cos80^\circ = \frac{1}{8}$

Solution: L.H.S $= \cos20^\circ\cos40^\circ\cos80^\circ$
$$\begin{aligned} &= \frac{1}{4}(4\cos20^\circ\cos40^\circ\cos80^\circ) \\ &= \frac{1}{4}[(2\cos40^\circ\cos20^\circ)\cdot2\cos80^\circ] \end{aligned}$$

$$=\frac{1}{4}[(\cos(40^\circ+20^\circ)+\cos(40^\circ-20^\circ))\cdot 2\cos80^\circ]$$
$$=\frac{1}{4}[(\cos60^\circ+\cos20^\circ)\cdot 2\cos80^\circ]$$
$$=\frac{1}{4}\left[\left(\frac{1}{2}+\cos20^\circ\right)\cdot 2\cos80^\circ\right] \qquad \because \cos60^\circ=\frac{1}{2}$$
$$=\frac{1}{4}(\cos80^\circ+2\cos80^\circ\cos20^\circ)$$
$$=\frac{1}{4}[\cos80^\circ+\{\cos(80^\circ+20^\circ)+\cos(80^\circ-20^\circ)\}]$$
$$=\frac{1}{4}(\cos80^\circ+\cos100^\circ+\cos60^\circ)$$
$$=\frac{1}{4}[\cos80^\circ+\cos(180^\circ-80^\circ)+\cos60^\circ]$$
$$=\frac{1}{4}\left[\cos80^\circ-\cos80^\circ+\frac{1}{2}\right] \qquad [\because \cos(180^\circ-80^\circ)=\cos(2\times90^\circ-80^\circ)=-\cos80^\circ]$$
$$=\frac{1}{4}\left(\frac{1}{2}\right)=\frac{1}{8}=\text{R.H.S}$$

Hence, $\cos20^\circ\cos40^\circ\cos80^\circ=\frac{1}{8}$

Exercise 10.4

1. Express the following products as sums or differences:

(i) 2 sin 3θ cos θ

Solution:

Using Formula $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

$$2\sin3\theta\cos\theta=\sin(3\theta+\theta)+\sin(3\theta-\theta)$$
$$=\sin4\theta+\sin2\theta$$

(ii) 2 cos 5θ sin 3θ

Solution:

Using Formula $2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$

$$2\cos5\theta\sin3\theta=\sin(5\theta+3\theta)-\sin(5\theta-3\theta)$$
$$=\sin8\theta-\sin2\theta$$

(iii) sin 5θ cos 2θ

Solution:

$$\sin5\theta\cos2\theta=\frac{1}{2}[2\sin5\theta\cos2\theta]$$

Using Formula $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

$$=\frac{1}{2}[\sin(5\theta+2\theta)+\sin(5\theta-2\theta)]$$
$$=\frac{1}{2}[\sin7\theta+\sin3\theta]$$

(iv) 2 sin 7θ sin 2θ

Solution:

Using Formula $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$

$$2\sin7\theta\sin2\theta=\cos(7\theta-2\theta)-\cos(7\theta+2\theta)$$
$$=\cos5\theta-\cos9\theta$$

(v) cos(x + y) sin(x − y)

Solution:

$$\cos(x+y)\sin(x-y)=\frac{1}{2}[2\cos(x+y)\sin(x-y)]$$
$$=\frac{1}{2}[\sin(x+y+x-y)-\sin(x+y-x+y)]$$
$$=\frac{1}{2}[\sin2x-\sin2y]$$

(vi) cos(2x + 30°) cos(2x − 30°)

Solution:

$$\cos(2x+30^\circ)\cos(2x-30^\circ)$$
$$=\frac{1}{2}[2\cos(2x+30^\circ)\cos(2x-30^\circ)]$$

Using Formula $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$

$$=\frac{1}{2}[\cos(2x+30^\circ+2x-30^\circ)+\cos(2x+30^\circ-2x+30^\circ)]$$
$$=\frac{1}{2}[\cos4x+\cos60^\circ]$$

(vii) sin 12° sin 46°

Solution:

$$\sin12^\circ\sin46^\circ=\frac{1}{2}[2\sin12^\circ\sin46^\circ]$$

Using Formula $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$

$$=\frac{1}{2}[\cos(12^\circ-46^\circ)-\cos(12^\circ+46^\circ)]$$
$$=\frac{1}{2}[\cos34^\circ-\cos58^\circ] \qquad \text{using } \cos(-\theta)=\cos\theta$$

(viii) sin(x + 45°) sin(x − 45°)

Solution:

$$\sin(x+45^\circ)\sin(x-45^\circ)=\frac{1}{2}[2\sin(x+45^\circ)\sin(x-45^\circ)]$$

Using Formula $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$

$$=\frac{1}{2}[\cos(x+45^\circ-x+45^\circ)-\cos(x+45^\circ+x-45^\circ)]$$
$$=\frac{1}{2}[\cos90^\circ-\cos2x]$$

2. Express the following sums or differences as products:

(i) sin 5θ + sin 3θ

Solution:

Using Formula $\sin P+\sin Q=2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$$\sin5\theta+\sin3\theta=2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$$
$$=2\sin4\theta\cos\theta$$

(ii) sin 8θ − sin 4θ

Solution:

Using Formula $\sin P-\sin Q=2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

$$\sin8\theta-\sin4\theta=2\cos\left(\frac{8\theta+4\theta}{2}\right)\sin\left(\frac{8\theta-4\theta}{2}\right)$$
$$=2\cos6\theta\sin2\theta$$

(iii) cos 6θ + cos 3θ

Solution:

Using Formula $\cos P+\cos Q=2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$$\cos6\theta+\cos3\theta=2\cos\left(\frac{6\theta+3\theta}{2}\right)\cos\left(\frac{6\theta-3\theta}{2}\right)$$
$$=2\cos\frac{9\theta}{2}.\cos\frac{3\theta}{2}$$

(iv) cos 7θ − cos θ

Solution:

Using Formula $\cos P-\cos Q=-2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

$$\cos7\theta-\cos\theta=-2\sin\left(\frac{7\theta+\theta}{2}\right)\sin\left(\frac{7\theta-\theta}{2}\right)$$
$$=-2\sin4\theta.\sin3\theta$$

(v) cos 12° + cos 48°

Solution:

Using Formula $\cos P+\cos Q=2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$$\cos12^\circ+\cos48^\circ=2\cos\left(\frac{12^\circ+48^\circ}{2}\right)\cos\left(\frac{12^\circ-48^\circ}{2}\right)$$
$$=2\cos30^\circ\cos(-18^\circ)$$
$$=2\cos30^\circ\cos18^\circ \qquad \because \cos(-\theta)=\cos\theta$$

(vi) sin(x + 30°) + sin(x − 30°)

Solution:

$$\sin(x+30^\circ)+\sin(x-30^\circ)$$

Using Formula $\sin P+\sin Q=2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$$=2\sin\left(\frac{x+30^\circ+x-30^\circ}{2}\right).\cos\left(\frac{x+30^\circ-x+30^\circ}{2}\right)$$
$$=2\sin\left(\frac{2x}{2}\right).\cos\left(\frac{60^\circ}{2}\right)=2\sin x\cos30^\circ$$

3. Prove the following identities:

(i) $\frac{\sin3x-\sin x}{\cos x-\cos3x}=\cot2x$

Solution:

$$\text{L.H.S}=\frac{\sin3x-\sin x}{\cos x-\cos3x}$$
$$=\frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}$$
$$=\frac{2\cos2x\,.\,\sin x}{2\sin2x\,.\,\sin x} \qquad \because \sin(-\theta)=-\sin\theta$$
$$=\cot2x=\text{R.H.S (Proved)}$$

(ii) $\frac{\sin8x+\sin2x}{\cos8x+\cos2x}=\tan5x$

Solution:

$$\text{L.H.S}=\frac{\sin8x+\sin2x}{\cos8x+\cos2x}$$

$$=\frac{2\sin\left(\frac{8x+2x}{2}\right)\cos\left(\frac{8x-2x}{2}\right)}{2\cos\left(\frac{8x+2x}{2}\right)\cos\left(\frac{8x-2x}{2}\right)}$$

$$=\frac{2\sin 5x\,.\cos 3x}{2\cos 5x\,.\cos 3x}$$

$$=\tan 5x = \text{R.H.S (Proved)}$$

(iii) $\frac{\sin A-\sin B}{\sin A+\sin B}=\tan\frac{A-B}{2}\cot\frac{A+B}{2}$

Solution:

$$\text{L.H.S} = \frac{\sin A-\sin B}{\sin A+\sin B}$$

$$=\frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

$$=\cot\left(\frac{A+B}{2}\right)\cdot\tan\left(\frac{A-B}{2}\right)$$

$$=\tan\left(\frac{A-B}{2}\right)\cdot\cot\left(\frac{A+B}{2}\right)$$

= R.H.S (Proved)

(iv) $\frac{\sin 80^\circ+\sin 40^\circ}{\cos 80^\circ+\cos 40^\circ}=\sqrt{3}$

Solution:

$$\text{L.H.S} = \frac{\sin 80^\circ+\sin 40^\circ}{\cos 80^\circ+\cos 40^\circ}$$

$$=\frac{2\sin\left(\frac{80^\circ+40^\circ}{2}\right)\cdot\cos\left(\frac{80^\circ-40^\circ}{2}\right)}{2\cos\left(\frac{80^\circ+40^\circ}{2}\right)\cdot\cos\left(\frac{80^\circ-40^\circ}{2}\right)}$$

$$=\frac{2\sin 60^\circ\cdot\cos 20^\circ}{2\cos 60^\circ\cdot\cos 20^\circ}$$

$$=\frac{\sin 60^\circ}{\cos 60^\circ}$$

$$=\tan 60^\circ=\sqrt{3}=\text{R.H.S (Proved)}$$

4. Prove that:

(i) $\cos 15^\circ+\cos 105^\circ+\cos 195^\circ+\cos 160^\circ+\cos 285^\circ = 0$

Solution:

$$\text{L.H.S} = \cos 15^\circ+\cos 105^\circ+\cos 195^\circ+\cos 285^\circ$$

$$=2\cos\left(\frac{15^\circ+105^\circ}{2}\right)\cdot\cos\left(\frac{15^\circ-105^\circ}{2}\right)+2\cos\left(\frac{195^\circ+285^\circ}{2}\right)\cos\left(\frac{195^\circ-285^\circ}{2}\right)$$

$$=2\cos 60^\circ\cdot\cos(-45^\circ)+2\cos 240^\circ\cdot\cos(-45^\circ)$$

$$=2\cdot\frac{1}{2}\cos 45^\circ+2\cos(2\times 90^\circ+60^\circ)\cos 45^\circ$$

$$=\cos 45^\circ+2(-\cos 60^\circ)\cdot\cos 45^\circ$$

$$=\frac{1}{\sqrt{2}}-2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$

$$=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0=\text{R.H.S (Proved)}$$

(ii) $\frac{\sin 2\theta+\sin 4\theta+\sin 6\theta+\sin 8\theta}{\cos 2\theta+\cos 4\theta+\cos 6\theta+\cos 8\theta}=\tan 5\theta$

Solution:

$$\text{L.H.S} = \frac{\sin 2\theta+\sin 4\theta+\sin 6\theta+\sin 8\theta}{\cos 2\theta+\cos 4\theta+\cos 6\theta+\cos 8\theta}$$

$$=\frac{(\sin 2\theta+\sin 8\theta)+(\sin 4\theta+\sin 6\theta)}{(\cos 2\theta+\cos 8\theta)+(\cos 4\theta+\cos 6\theta)}$$

$$=\frac{2\sin\left(\frac{2\theta+8\theta}{2}\right)\cos\left(\frac{2\theta-8\theta}{2}\right)+2\sin\left(\frac{4\theta+6\theta}{2}\right)\cos\left(\frac{4\theta-6\theta}{2}\right)}{2\cos\left(\frac{2\theta+8\theta}{2}\right)\cos\left(\frac{2\theta-8\theta}{2}\right)+2\cos\left(\frac{4\theta+6\theta}{2}\right)\cos\left(\frac{4\theta-6\theta}{2}\right)}$$

$$=\frac{2\sin 5\theta\cdot\cos(-3\theta)+2\sin 5\theta\cdot\cos(-3\theta)}{2\cos 5\theta\cdot\cos(-3\theta)+2\cos 5\theta\cdot\cos(-3\theta)}$$

$$=\frac{2\sin 5\theta(\cos 3\theta+\cos 3\theta)}{2\cos 5\theta(\cos 3\theta+\cos 3\theta)} \qquad \because \cos(-\theta)=\cos\theta$$

$$=\frac{\sin 5\theta}{\cos 5\theta}=\tan 5\theta=\text{R.H.S (Proved)}$$

(iii) $\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-\cos^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\sin\alpha$

Solution:

$$\text{L.H.S} = \cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-\cos^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Let $A=\frac{\pi}{4}-\frac{\alpha}{2}$ and $B=\frac{\pi}{4}+\frac{\alpha}{2}$

By adding	By subtracting
$A=\frac{\pi}{4}-\frac{\alpha}{2}$	$A=\frac{\pi}{4}-\frac{\alpha}{2}$
$B=\frac{\pi}{4}+\frac{\alpha}{2}$	$\pm B=\pm\frac{\pi}{4}\pm\frac{\alpha}{2}$
$A+B=\frac{\pi}{4}+\frac{\pi}{4}$	$A-B=-\frac{\alpha}{2}-\frac{\alpha}{2}$
$A+B=\frac{\pi}{2}=90^\circ$	$A-B=-\alpha$
$\frac{A+B}{2}=45^\circ$	$\frac{A-B}{2}=-\frac{\alpha}{2}$

$$\text{L.H.S} = \cos^2 A-\cos^2 B$$

$$=(\cos A+\cos B)(\cos A-\cos B)$$

$$=\left(2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right)\left(-2\sin\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)\right)$$

Put $\frac{A+B}{2}=45^\circ$ and $\frac{A-B}{2}=-\frac{\alpha}{2}$

$$=\left(2\cos 45^\circ\cdot\cos\left(-\frac{\alpha}{2}\right)\right)\left(-2\sin 45^\circ\cdot\sin\left(-\frac{\alpha}{2}\right)\right)$$

$$=\left(2\cdot\frac{1}{\sqrt{2}}\cdot\cos\frac{\alpha}{2}\right)\left(2\cdot\frac{1}{\sqrt{2}}\sin\frac{\alpha}{2}\right) \qquad \because \sin(-\theta)=-\sin\theta$$

$$=\frac{4}{(\sqrt{2})^2}\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}$$

$$=2\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}$$

$$=\sin\alpha=\text{R.H.S (Proved)}$$

(iv) $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{2}\cos 2\theta$

Solution:

$$\text{L.H.S} = \sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)$$

$$=\frac{1}{2}\left[2\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)\right]$$

Using $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$

$$=\frac{1}{2}\left[\cos\left(\frac{\pi}{4}-\theta-\frac{\pi}{4}-\theta\right)-\cos\left(\frac{\pi}{4}-\theta+\frac{\pi}{4}+\theta\right)\right]$$

$$=\frac{1}{2}\left[\cos(-2\theta)-\cos\left(\frac{\pi}{2}\right)\right]$$

$$=\frac{1}{2}[\cos 2\theta-0] \qquad \because \cos\left(\frac{\pi}{2}\right)=0$$

$$=\frac{1}{2}\cos 2\theta=\text{R.H.S (Proved)}$$

(v) $\frac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\tan 4\theta$

Solution:

$$\text{L.H.S} = \frac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}$$

$$=\frac{(\sin 7\theta+\sin\theta)+(\sin 5\theta+\sin 3\theta)}{(\cos 7\theta+\cos\theta)+(\cos 5\theta+\cos 3\theta)}$$

Using $\sin P+\sin Q=2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

and $\cos P+\cos Q=2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$$=\frac{2\sin\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right)+2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}{2\cos\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right)+2\cos\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}$$

$$=\frac{2\sin 4\theta\cos 3\theta+2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta+2\cos 4\theta\cos\theta}$$

$$=\frac{2\sin 4\theta(\cos 3\theta+\cos\theta)}{2\cos 4\theta(\cos 3\theta+\cos\theta)}$$

$$=\frac{\sin 4\theta}{\cos 4\theta}=\tan 4\theta=\text{R.H.S (Proved)}$$

5. Prove that:

(i) $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=\frac{1}{16}$

Solution:

$$\text{L.H.S} = \cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ$$

$$=\cos 20^\circ\cos 40^\circ.\left(\frac{1}{2}\right).\cos 80^\circ \qquad \because \cos 60^\circ=\frac{1}{2}$$

$$=\frac{1}{2}.\cos 20^\circ\cos 80^\circ\cos 40^\circ$$

Multiplying & dividing by 2

$$=\frac{1}{2}\cdot\cos 20^\circ\cdot\frac{1}{2}(2\cos 80^\circ\cos 40^\circ)$$

Using Formula $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$

$$=\frac{1}{4}\cos 20^\circ\{\cos(80^\circ+40^\circ)+\cos(80^\circ-40^\circ)\}$$

$$=\frac{1}{4}\cos 20^\circ\{\cos 120^\circ+\cos 40^\circ\}$$

$$=\frac{1}{4}\cos 20^\circ\left[-\frac{1}{2}+\cos 40^\circ\right]$$

$$\because \cos 120^\circ=\cos(1\times 90^\circ+30^\circ)=-\sin 30^\circ=-\frac{1}{2}$$

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{4}\cos 40^\circ\cos 20^\circ$$

Multiply and divide by 2

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{4}.\frac{1}{2}(2\cos 40^\circ\cos 20^\circ)$$

Using Formula $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{8}\{\cos(40^\circ+20^\circ)+\cos(40^\circ-20^\circ)\}$$

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{8}\{\cos 60^\circ+\cos 20^\circ\}$$

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{8}\cos 60^\circ+\frac{1}{8}\cos 20^\circ$$

$$=\frac{1}{8}\cos 60^\circ=\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)=\frac{1}{16}=\text{R.H.S (Proved)}$$

(ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}=\frac{3}{16}$

Solution:

$$\text{L.H.S}=\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}$$

$= \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

$= \sin 20^\circ \sin 40^\circ \left(\frac{\sqrt{3}}{2}\right) \sin 80^\circ \qquad \because \sin 60^\circ = \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{2} \cdot \sin 20^\circ \sin 80^\circ \sin 40^\circ$

Multiply and divide by 2

$= \frac{\sqrt{3}}{2} \cdot \sin 20^\circ \cdot \frac{1}{2}(2\sin 80^\circ \sin 40^\circ)$

Using Formula $2\sin\alpha \sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

$= \frac{\sqrt{3}}{4} \cdot \sin 20^\circ . \{\cos(80^\circ - 40^\circ) - \cos(80^\circ + 40^\circ)\}$

$= \frac{\sqrt{3}}{4} \sin 20^\circ . \{\cos 40^\circ - \cos 120^\circ\}$

$= \frac{\sqrt{3}}{4} \sin 20^\circ . \left\{\cos 40^\circ + \frac{1}{2}\right\}$

$\because \cos 120^\circ = \cos(90^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$= \frac{\sqrt{3}}{4} \cos 40^\circ \sin 20^\circ + \frac{\sqrt{3}}{8} \sin 20^\circ$

$= \frac{\sqrt{3}}{4} \cdot \frac{1}{2}(2\cos 40^\circ \sin 20^\circ) + \frac{\sqrt{3}}{8} \sin 20^\circ$

Using Formula $2\cos\alpha \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

$= \frac{\sqrt{3}}{8} \cdot \{\sin(40^\circ + 20^\circ) - \sin(40^\circ - 20^\circ)\} + \frac{\sqrt{3}}{8} \sin 20^\circ$

$= \frac{\sqrt{3}}{8} \sin 60^\circ - \frac{\sqrt{3}}{8} \sin 20^\circ + \frac{\sqrt{3}}{8} \sin 20^\circ$

$= \left(\frac{\sqrt{3}}{8}\right) \cdot \left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16}$ = R.H.S (Proved)

(iii) $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$

Solution:

L.H.S $= \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$

$= \sin 10^\circ \left(\frac{1}{2}\right) \sin 50^\circ \sin 70^\circ \qquad \because \sin 30^\circ = \frac{1}{2}$

$= \frac{1}{2} \cdot \sin 10^\circ \sin 70^\circ \sin 50^\circ$

$= \frac{1}{2} \cdot \sin 10^\circ \cdot \frac{1}{2}(2\sin 70^\circ \sin 50^\circ)$

Using Formula $2\sin\alpha \sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

$= \frac{1}{4} \sin 10^\circ \{\cos(70^\circ - 50^\circ) - \cos(70^\circ + 50^\circ)\}$

$= \frac{1}{4} \sin 10^\circ \{\cos 20^\circ - \cos 120^\circ\}$

$= \frac{1}{4} \sin 10^\circ \left\{\cos 20^\circ + \frac{1}{2}\right\}$

$\because \cos 120^\circ = \cos(90^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$= \frac{1}{4} \cos 20^\circ \sin 10^\circ + \frac{1}{8} \sin 10^\circ$

$= \frac{1}{4} \cdot \frac{1}{2}(2\cos 20^\circ \sin 10^\circ) + \frac{1}{8} \sin 10^\circ$

Using Formula $2\cos\alpha \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

$= \frac{1}{8}\{\sin(20^\circ + 10^\circ) - \sin(20^\circ - 10^\circ)\} + \frac{1}{8} \sin 10^\circ$

$= \frac{1}{8} \sin 30^\circ - \frac{1}{8} \sin 10^\circ + \frac{1}{8} \sin 10^\circ$

$= \frac{1}{8}\left(\frac{1}{2}\right) = \frac{1}{16}$ = R.H.S (Proved)

6. Prove that: $\frac{\sin 3\theta}{1 + 2\cos 2\theta} = \sin\theta$**; deduce the value of** $\sin 15^\circ$

Solution:

L.H.S $= \frac{\sin 3\theta}{1 + 2\cos 2\theta}$

Using Formulas $\cos 2\theta = 1 - 2\sin^2\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

L.H.S $= \frac{3\sin\theta - 4\sin^3\theta}{1 + 2(1 - 2\sin^2\theta)}$

$= \frac{\sin\theta(3 - 4\sin^2\theta)}{1 + 2 - 4\sin^2\theta}$

$= \frac{\sin\theta(3 - 4\sin^2\theta)}{(3 - 4\sin^2\theta)}$

$= \sin\theta$ = R.H.S (Proved)

Deduction:

Given: $\frac{\sin 3\theta}{1 + 2\cos 2\theta} = \sin\theta$

Put $\theta = 15^\circ$

$\frac{\sin 3(15^\circ)}{1 + 2\cos 2(15^\circ)} = \sin 15^\circ$

$\sin 15^\circ = \frac{\sin 45^\circ}{1 + 2\cos 30^\circ}$

$= \frac{\frac{1}{\sqrt{2}}}{1 + 2 \cdot \frac{\sqrt{3}}{2}}$

$= \frac{\frac{1}{\sqrt{2}}}{1 + \sqrt{3}}$

Rationalize the denominator

$= \frac{1}{\sqrt{2}(1 + \sqrt{3})} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$

$= \frac{1 - \sqrt{3}}{\sqrt{2}(1^2 - (\sqrt{3})^2)}$

$= \frac{1 - \sqrt{3}}{\sqrt{2}(1 - 3)} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2} - \sqrt{6}}{(\sqrt{2})^2(-2)}$

$= \frac{\sqrt{6} - \sqrt{2}}{2(2)}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

7. Prove that: $\tan 75^\circ - \tan 15^\circ = 2\sqrt{3}$

Solution:

L.H.S $= \tan 75^\circ - \tan 15^\circ$

$= \tan(45^\circ + 30^\circ) - \tan(45^\circ - 30^\circ)$

Using Formula: $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}$

L.H.S $= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \cdot \tan 30^\circ} - \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \cdot \tan 30^\circ}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} - \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} \qquad \because \tan 45^\circ = 1, \tan 30^\circ = \frac{1}{\sqrt{3}}$

$= \frac{\frac{\sqrt{3} + 1}{\sqrt{3}}}{\frac{\sqrt{3} - 1}{\sqrt{3}}} - \frac{\frac{\sqrt{3} - 1}{\sqrt{3}}}{\frac{\sqrt{3} + 1}{\sqrt{3}}}$

$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} - \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$= \frac{(\sqrt{3} + 1)^2 - (\sqrt{3} - 1)^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$

$= \frac{4(\sqrt{3})(1)}{(\sqrt{3})^2 - 1^2} \qquad \because \begin{cases}(a+b)^2 - (a-b)^2 = 4ab \\ (a+b)(a-b) = a^2 - b^2\end{cases}$

$= \frac{4\sqrt{3}}{3 - 1}$

$= \frac{4\sqrt{3}}{2}$

$= 2\sqrt{3}$ = R.H.S (Proved)

8. Prove that: $\cos 15^\circ - \sin 15^\circ = \frac{1}{\sqrt{2}}$

Solution:

L.H.S $= \cos 15^\circ - \sin 15^\circ$

$= \cos(1 \times 90^\circ - 75^\circ) - \sin 15^\circ$

$= \sin 75^\circ - \sin 15^\circ$

Using Formula: $\sin P - \sin Q = 2\cos\frac{P + Q}{2} \cdot \sin\frac{P - Q}{2}$

L.H.S $= 2\cos\left(\frac{75^\circ + 15^\circ}{2}\right) \cdot \sin\left(\frac{75^\circ - 15^\circ}{2}\right)$

$= 2\cos 45^\circ . \sin 30^\circ$

$= 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{1}{\sqrt{2}}$ = R.H.S (Proved)

9. Prove that: $\frac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cos\alpha - \sin\beta\cos\beta} = \tan(\alpha + \beta)$

Solution:

L.H.S $= \frac{\sin^2\alpha - \sin^2\beta}{\sin\alpha \cdot \cos\alpha - \sin\beta\cos\beta}$

$= \frac{(\sin\alpha + \sin\beta)(\sin\alpha - \sin\beta)}{\frac{1}{2}(2\sin\alpha \cdot \cos\alpha - 2\sin\beta\cos\beta)}$

$= \frac{2(\sin\alpha + \sin\beta)(\sin\alpha - \sin\beta)}{\sin 2\alpha - \sin 2\beta}$

$= \frac{2\left(2\sin\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}\right)\left(2\cos\frac{\alpha+\beta}{2} \cdot \sin\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{2\alpha + 2\beta}{2}\right) \cdot \sin\left(\frac{2\alpha - 2\beta}{2}\right)}$

$= \frac{\left(2\sin\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha+\beta}{2}\right)\left(2\sin\frac{\alpha-\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}\right)}{\cos(\alpha+\beta) \cdot \sin(\alpha-\beta)}$

$= \frac{\sin 2\left(\frac{\alpha+\beta}{2}\right) \cdot \sin 2\left(\frac{\alpha-\beta}{2}\right)}{\cos(\alpha+\beta) \cdot \sin(\alpha-\beta)} \qquad \because 2\sin\theta\cos\theta = \sin 2\theta$

$= \frac{\sin(\alpha+\beta) \cdot \sin(\alpha-\beta)}{\cos(\alpha+\beta) \cdot \sin(\alpha-\beta)}$

$= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$= \tan(\alpha + \beta)$ = R.H.S (Proved)

10. Prove that:

$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma)$

$= 4\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\gamma+\alpha}{2}\right)$

Solution:

L.H.S $= \sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma)$

$= (\sin\alpha + \sin\beta) + (\sin\gamma - \sin(\alpha + \beta + \gamma))$

$= 2\sin\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2} + 2\cos\frac{\gamma+\alpha+\beta+\gamma}{2} \cdot \sin\frac{\gamma-\alpha-\beta-\gamma}{2}$

$= 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} + 2\cos\frac{\alpha+\beta+2\gamma}{2}\cdot\sin\left(\frac{-\alpha-\beta}{2}\right)$

$= 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} - 2\cos\frac{\alpha+\beta+2\gamma}{2}\cdot\sin\frac{\alpha+\beta}{2} \because \sin(-\theta) = -\sin\theta$

$= 2\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta+2\gamma}{2}\right)$

$= 2\sin\frac{\alpha+\beta}{2}\left\{-2\sin\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\sin\left(\frac{\frac{\alpha-\beta}{2}-\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\right\}$

$= -4\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta+\alpha+\beta+2\gamma}{2\cdot 2}\right)\cdot\sin\left(\frac{\alpha-\beta-\alpha-\beta-2\gamma}{2\cdot 2}\right)$

$= -4\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{2\alpha+2\gamma}{4}\right)\cdot\sin\left(\frac{-2\beta-2\gamma}{4}\right)$

$= -4\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{2(\alpha+\gamma)}{4}\right)\cdot\sin\left(\frac{-2(\beta+\gamma)}{4}\right)$

$= 4\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha+\gamma}{2}\right)\cdot\sin\left(\frac{\beta+\gamma}{2}\right) \because \sin(-\theta) = -\sin\theta$

$= 4\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\beta+\gamma}{2}\right)\cdot\sin\left(\frac{\gamma+\alpha}{2}\right)$

= R.H.S (Proved)

Formula Sheet

Fundamental Law of Trigonometry and Deductions

1. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$	2. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
3. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$	4. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
5. $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$	6. $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

Double Angle Identities

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$	2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$	3. $\cos 2\alpha = 2\cos^2\alpha - 1$
4. $\cos 2\alpha = 1 - 2\sin^2\alpha$	5. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$	

Half Angle Identities

1. $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$	2. $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$	3. $\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$
and $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$	and $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$	and $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$

Triple Angle Identities

1. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$	2. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$	3. $\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$

Product (of sines and cosines) as Sums or Differences (of sines and cosines)

1. $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$	2. $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$
3. $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$	4. $-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

Sums or Differences (of sines and cosines) as Product (of sines and cosines)

1. $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$	2. $\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$
3. $\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$	3. $\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

Multiple Choice Questions (MCQs)

Exercise 10.1

1. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ then $|PQ| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ is called - - - - -
   (A) quadratic formula (B) cramer's rule (C) distance formula (D) none of these
2. Distance between A(3, 8), B(5, 6) is - - - - -
   (A) $\sqrt{2}$ (B) $2\sqrt{2}$ (C) 2 (D) $\frac{1}{2}$
3. If θ is added to or subtracted from odd multiple of right angle, the trigonometric ratios change into - - -
   (A) reciprocals (B) co-ratios (C) both (A) and (B) (D) none of these
4. $\tan\left(\alpha - \frac{\pi}{2}\right) =$ - - - - -
   (A) $\sec\alpha$ (B) $\cot\alpha$ (C) $-\cot\alpha$ (D) $\tan\alpha$
5. $\cos\left(\frac{3\pi}{2} - \theta\right)$ is equal to:
   (A) $-\sin\theta$ (B) $\sin\theta$ (C) $\cos\theta$ (D) $-\cos\theta$
6. The value of cos 315° is - - - - -
   (A) 0 (B) 1 (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{\sqrt{2}}$

Exercise 10.2

7. The value of sin15° is - - - - -
   (A) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ (B) $\frac{\sqrt{3}}{2\sqrt{2}}$ (C) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (D) $\frac{\sqrt{3}-1}{\sqrt{2}}$
8. $\sin(45° + \alpha) =$ - - - - -
   (A) $\frac{1}{\sqrt{2}}(\sin\alpha - \cos\alpha)$ (B) $\frac{1}{2}(\cos\alpha - \sin\alpha)$ (C) $\frac{1}{2}\sin\alpha$ (D) $\frac{1}{\sqrt{2}}(\sin\alpha + \cos\alpha)$
9. $\cot(\alpha - \beta) =$ - - - - -
   (A) $\frac{\cot\alpha - \cot\beta}{1+\cot\alpha\cot\beta}$ (B) $\frac{\cot\alpha + \cot\beta}{1-\cot\alpha\cot\beta}$ (C) $\frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$ (D) $\frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$
10. If $r\cos\theta = 3$, $r\sin\theta = 4$ then r = - - - - -
    (A) 25 (B) − 25 (C) 5 (D) − 5
11. $\frac{\cos 19° + \sin 19°}{\cos 19° - \sin 19°} =$ - - - - -
    (A) tan19° (B) cot19° (C) tan64° (D) cot64°

Exercise 10.3

12. $\sin 2\alpha$ is equal to - - - - -
    (A) $1 - 2\sin^2\alpha$ (B) $2\cos^2\alpha - 1$ (C) $2\sin\alpha\cos\alpha$ (D) $\sin\alpha$
13. If $\sin 2\theta = 1$ then value of θ is - - - - -
    (A) 30° (B) 45° (C) 60° (D) 90°
14. $\cos^2\alpha - \sin^2\alpha =$ - - - - -
    (A) 1 (B) cos 2α (C) sin 2α (D) sin 3α
15. $\tan\frac{\alpha}{2} =$ - - - - -
    (A) $\pm\sqrt{\frac{1-\cos\alpha}{1-\cos\alpha}}$ (B) $\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ (C) $\pm\sqrt{\frac{1-\sin\alpha}{1-\cos\alpha}}$ (D) $\pm\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}$

16. $\cos 3\alpha =$ - - - - -

(A) $4\sin^3\alpha - 3\sin\alpha$ (B) $3\cos^3\alpha + 4\cos\alpha$ (C) $\cos^2\frac{3\alpha}{2} - \sin^2\frac{3\alpha}{2}$ (D) $4\cos^3\alpha - 3\cos\alpha$

Exercise 10.4

17. $2\sin 7\theta.\cos 3\theta$ is equal to - - - - -

(A) $\sin 10\theta + \sin 4\theta$ (B) $\sin 10\theta - \sin 4\theta$ (C) $4\cos^3\theta - 3\cos\theta$ (D) $\cos 10\theta - \cos 4\theta$

18. $\cos 48° + \cos 12° =$ - - - - -

(A) $2\cos 18°$ (B) $3\cos 18°$ (C) $\sqrt{3}\cos 18°$ (D) $\sqrt{2}\cos 18°$

19. $\cos 20° \cos 40° \cos 80° =$ - - - - -

(A) $\frac{1}{4}$ (B) $\frac{1}{8}$ (C) 1 (D) $\frac{1}{2}$

20. $\cos(\alpha + \beta) + \cos(\alpha - \beta) =$ - - - - -

(A) $2\sin\alpha.\cos\beta$ (B) $2\cos\alpha.\sin\beta$ (C) $2\cos\alpha.\cos\beta$ (D) $-2\sin\alpha.\sin\beta$

ANSWER KEY

1.	C	2.	B	3.	B	4.	C	5.	A	6.	D	7.	C	8.	D	9.	D	10.	C
11.	C	12.	C	13.	B	14.	B	15.	B	16.	D	17.	A	18.	C	19.	B	20.	C

Unit 11 Trigonometric Functions and their Graphs

Introduction

In this unit, students will explore key concepts essential for understanding the role of trigonometry in mathematics and its real-life applications. We will begin by learning how to determine the domain and range of trigonometric functions to understand their behavior. Next, we will discuss even and odd functions, along with their periodicity, which explains their repeating patterns.

Students will then learn how to graph and analyze sine, cosine, and tangent functions, following this, we will focus on calculating the maximum and minimum values of sinusoidal functions and examining their unique properties such as amplitude, frequency, and phase shifts.

Finally, students will apply these trigonometric concepts to solve practical problems in navigation, engineering, and physics, including calculating distances, optimizing solar panel angles, and analyzing forces in structures. Mastering these concepts will enable students to solve both theoretical and real-world problems using trigonometry.

Let us first find domains and ranges of trigonometric functions before drawing their graphs.

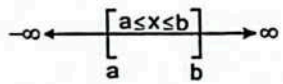
Interval in Set of Real Number '*R*'

The set of all numbers lying between two given real numbers is called an interval in *R*.

1. **Closed Interval $[a, b]$**

Closed interval from a to $b = \{x \mid x \in R;\ a \le x \le b\}$

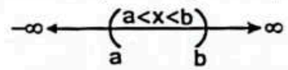
= set of all real numbers lying between a and b including the end points a and b.



2. **Open Interval (a, b) or $]a, b[$**

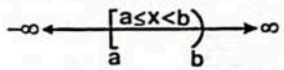
Open interval from a to $b = \{x \mid x \in R;\ a < x < b\}$

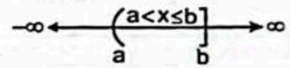
= set of all real numbers lying between a and b, excluding the end points a and b.



3. **Closed-Open Interval $[a, b)$ and Open-Closed Interval $(a, b]$**

$[a, b) = \{x \mid x \in R;\ a \le x < b\}$ and $(a, b] = \{x \mid x \in R;\ a < x \le b\}$





4. **Real Number Set *R* as an Open Interval**

The set *R* can be thought of as the open interval $(-\infty, \infty)$, so that

$R = (-\infty, \infty) = \{x \mid x \in R;\ -\infty < x < \infty\}$.

The infinite interval in *R* can be given by $(-\infty, a)$, $(a, +\infty)$, $(-\infty, a]$, $[a, \infty)$.