
Unit 1 Complex Numbers

Introduction

Complex numbers are an extension of the real numbers designed to solve equations that have no solutions within the realm of real numbers. The history of mathematics shows that man has been developing and enlarging his concept of number according to the saying that "Necessity is the mother of invention". In the remote past they started with the set of counting numbers and invented, by stages, the negative numbers, rational numbers, irrational numbers etc. Since square of a positive as well as negative number is a positive number, the square root of a negative number does not exist in the realm of real numbers. Therefore, square roots of negative numbers were given no attention for centuries together. However, recently, properties of numbers involving square roots of negative numbers have also been discussed in detail and such numbers have been found useful and have been applied in many branches of pure, applied, financial and computational mathematics.

Complex Numbers

The numbers of the form $z = a + ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, are called **complex numbers** and the set of all complex numbers is denoted by C.

For example, $3+4i, 2-\frac{5}{7}i, -7-2i$ etc. are complex numbers.

Recognition of Real and Imaginary Parts

Let $x^2+1=0 \quad \Rightarrow x^2=-1 \quad \Rightarrow x=\pm\sqrt{-1}$ (Does not belong to the set of real numbers)
We, therefore, for convenience call it **imaginary number** and denote it by i (read as iota).

Remarks:

- If $z = a + ib$ is a complex number, then
  a = real part of z = Re z ; b = imaginary part of z = Im z
  For example, if $z = 3 + 4i$, then Re z = 3 and Im z = 4.

Note:
Every real number is a complex number with 0 as its imaginary part.

The product of a non-zero real number a and i is also an **imaginary number** and is written as ai.

Thus $2i, -3i, \sqrt{5}i, -\frac{11}{2}i$ are all imaginary numbers.

Conjugate Complex Numbers: If $z = a + ib$ is a complex number, then $a - ib$ is called the complex conjugate of z and is denoted by $\bar{z}$. For example, $5 - 4i$ is complex conjugate of $5 + 4i$ and $-2 - 3i$ is complex conjugate of $-2+3i$.

Note:
A real number is self-conjugate, i.e., if $a \in \mathbb{R}$, then $\bar{a} = a$.

Operations on Complex Numbers

With a view to develop algebra of **complex numbers**, we state a few definitions.

The symbols a, b, c, d, k, where used, represent real numbers.

(i) Addition: $(a + ib) + (c + id) = (a + c) + i(b + d)$

(ii) Subtraction: $(a + ib) - (c + id) = (a + ib) + [-(c + id)]$
$= a + ib + (-c - id) = (a - c) + i(b - d)$

(iii) Multiplication: $(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc)$

(iv) If k is any real number, then $k(a + ib) = ka + ikb$

Complex Numbers as Ordered Pairs of Real Numbers

We can define complex numbers also by using ordered pairs.

Let C be the set of ordered pairs belonging to $\mathbb{R} \times \mathbb{R}$ which are subject to the following properties:

(i) Equality $(a,b) = (c,d) \Leftrightarrow a = c \wedge b = d$

(ii) Sum $(a,b) + (c,d) = (a+c, b+d)$

(iii) Difference $(a,b) - (c,d) = (a-c, b-d)$

(iv) Product $(a,b)(c,d) = (ac - bd, ad + bc)$

(iv) If k is any real number, then $k(a,b) = (ka, kb)$

Example 1: Find the sum, difference and product of the complex numbers $(8,9)$ and $(5,-6)$

Solution:

Sum $= (8,9) + (5,-6) = (8+5, 9-6) = (13,3)$

Difference $= (8,9) - (5,-6) = (8-5, 9-(-6)) = (5,15)$

Product $= (8,9).(5,-6) = (8 \cdot 5 - (9)(-6), (8)(-6) + 9 \cdot 5)$

$= (40 + 54, -48 + 45) = (94, -3)$

Properties of the Fundamental Operations on Complex Numbers

It can be easily verified that the set C satisfies all the field axioms i.e., it possesses the properties of real numbers. By way of explanation of some points we observe as follows:

(i) The additive identity in C is $(0,0)$.

(ii) Every complex number (a,b) has the additive inverse $(-a,-b)$ i.e., $(a,b) + (-a,-b) = (0,0)$

(iii) The multiplicative identity is $(1,0)$ i.e.,

$$(a,b)\cdot(1,0) = (a \cdot 1 - b \cdot 0,\ b \cdot 1 + a \cdot 0) = (a,b) = (1,0)\cdot(a,b)$$

(iv) Every non-zero complex number {i.e., number not equal to $(0,0)$} has a multiplicative inverse.

The multiplicative inverse of (a,b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

$$\because\ (a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0), \text{ the identity element}$$

$$= \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)(a,b)$$

(v) $(a,b)[(c,d) \pm (e,f)] = (a,b)(c,d) \pm (a,b)(e,f)$

Note:
The set C of complex numbers does not satisfy the order axioms. In fact, there is no sense in saying that one complex number is greater or less than the other.

Example 2: If $z_1 = (4,2)$ and $z_2 = (3,-1)$, then find $\frac{z_1}{z_2}$.

Solution:

Given $z_1 = (4,2) = 4 + 2i$ and $z_2 = (3,-1) = 3 - 1i$

Now, $\frac{z_1}{z_2} = \frac{4+2i}{3-i}$

Multiply up and down by $(3+i)$

$$\frac{z_1}{z_2} = \frac{4+2i}{3-i} \times \frac{3+i}{3+i}$$

$$= \frac{(4)(3)+(4)(i)+(2i)(3)+(2i)(i)}{(3)^2-(i)^2} = \frac{12+4i+6i+2i^2}{9-i^2}$$

$$\frac{z_1}{z_2} = \frac{12+10i-2}{9-(-1)} = \frac{10+10i}{10} = 1+i \qquad \because i^2 = -1$$

Argand Diagram

The figure representing one or more complex numbers on the complex plane is called an **Argand diagram**. The Argand diagram is a way of representing one or more complex numbers on the complex plane. Points on the x-axis represent real numbers whereas the points on the y-axis represent imaginary numbers.

Every complex number will be represented by one and only one point of the coordinate plane and every point of the plane will represent one and only one complex number. The components of the complex number will be the coordinates of the point representing it. In this representation the x-axis is called the real axis and the y-axis is called the imaginary axis. The coordinate plane itself is called the **complex plane** or **z-plane**.

In an Argand diagram, the complex number $x + iy$ is uniquely represented by the order pair (x,y).

In adjoining figure, the complex numbers $3+2i, -2+2i, -3-2i$ and $2-2i$ correspond to the order pairs $(3,2), (-2,2), (-3,-2)$ and $(2,-2)$ respectively have been represented geometrically by the point A, B, C and D.

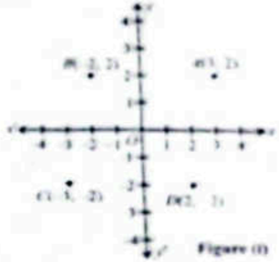

Figure (i)

Modulus of Complex Number:

The modulus of a complex number $x + iy$ is denoted by $|x+iy|$ and is defined as: $|x+iy| = \sqrt{x^2+y^2}$, i.e., if $z = x + iy$, then $|z| = \sqrt{x^2+y^2}$

OR

In other words, the modulus of a complex number is the distance from the origin to the point representing the number.

Note:
For a complex number $z = x + iy$, $|z| = \sqrt{x^2+y^2}$ is a real number.

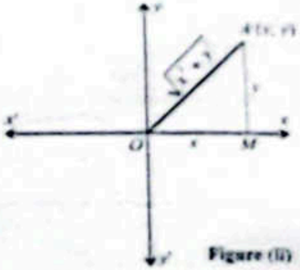

Figure (ii)

Example 3: If $z = \frac{(1+2i)^2}{2-i}$ then evaluate $|\bar{z}|$

Solution:

$$z = \frac{(1+2i)^2}{2-i} = \frac{1+4i+4i^2}{2-i} = \frac{-3+4i}{2-i} \times \frac{2+i}{2+i} \qquad \text{Multiply up and down by } (2+i)$$

$$= \frac{-6-3i+8i+4i^2}{2^2-i^2} = \frac{-6+5i-4}{4-(-1)} = \frac{-10+5i}{5} \qquad \because i^2 = -1$$

$\Rightarrow \qquad z = -2 + i$

Taking conjugate

$\bar{z} = \overline{-2+i} = -2 - i$

and $|\bar{z}| = |-2-i| = \sqrt{(-2)^2+(-1)^2} = \sqrt{4+1} \qquad \Rightarrow |\bar{z}| = \sqrt{5}$

Exercise 1.1

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1. Find the multiplicative inverse of each of the following complex numbers:

(i) $(-4, 7)$

Solution:

Let $z = (-4, 7)$

Here $a = -4, \quad b = 7$

Multiplicative inverse of z is

$$\frac{1}{z} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$= \left(\frac{-4}{(-4)^2+(7)^2}, \frac{-7}{(-4)^2+(7)^2}\right)$$

$$= \left(\frac{-4}{16+49}, \frac{-7}{16+49}\right)$$

$$= \left(-\frac{4}{65}, -\frac{7}{65}\right)$$

Alternate Method:

Let $z = (-4, 7) \qquad \Rightarrow z = -4 + 7i$

Multiplicative inverse of z is

$$\frac{1}{z}=\frac{1}{-4+7i}$$

$$=\frac{1}{-4+7i}\times\frac{-4-7i}{-4-7i}$$

$$=\frac{-4-7i}{(-4)^2-(7i)^2}=\frac{-4-7i}{16-49i^2}$$

$$=\frac{-4-7i}{16+49} \qquad \because i^2=-1$$

$$=\frac{-4-7i}{65}=-\frac{4}{65}-\frac{7i}{65}$$

$$=\left(-\frac{4}{65},-\frac{7}{65}\right)$$

(ii) $(\sqrt{2},-\sqrt{5})$

Solution:

Let $z=(\sqrt{2},-\sqrt{5})$

Here $a=\sqrt{2}$, $b=-\sqrt{5}$

Multiplicative inverse of z is

$$\frac{1}{z}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$$

$$=\left(\frac{\sqrt{2}}{(\sqrt{2})^2+(-\sqrt{5})^2},\frac{-(-\sqrt{5})}{(\sqrt{2})^2+(-\sqrt{5})^2}\right)$$

$$=\left(\frac{\sqrt{2}}{2+5},\frac{\sqrt{5}}{2+5}\right)$$

$$=\left(\frac{\sqrt{2}}{7},\frac{\sqrt{5}}{7}\right)$$

(iii) $(1,0)$

Solution:

Let $z=(1,0)$

Here $a=1$, $b=0$

Multiplicative inverse of z

$$\frac{1}{z}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$$

$$=\left(\frac{1}{(1)^2+(0)^2},\frac{-0}{(1)^2+(0)^2}\right)$$

$$=\left(\frac{1}{1+0},\frac{0}{1+0}\right)=(1,0)$$

2. Separate into real and imaginary parts (write as a simple complex number):

(i) $\dfrac{2-7i}{4+5i}$

Solution:

Multiply up and down by conjugate of $4+5i$

$$\frac{2-7i}{4+5i}=\frac{2-7i}{4+5i}\times\frac{4-5i}{4-5i}$$

$$=\frac{8-10i-28i+35i^2}{16-25i^2}$$

$$=\frac{8-38i-35}{16+25} \qquad \because i^2=-1$$

$$=\frac{-27-38i}{41}$$

$$=\frac{-27}{41}-\frac{38}{41}i$$

$$\text{Real part}=-\frac{27}{41}$$

$$\text{Imaginary part}=-\frac{38}{41}$$

(ii) $\dfrac{(-2+3i)^2}{1+i}$

Solution:

$$\frac{(-2+3i)^2}{1+i}=\frac{(-2)^2+(3i)^2+2(-2)(3i)}{1+i}$$

$$=\frac{4+9i^2-12i}{1+i}$$

$$=\frac{4-9-12i}{1+i} \qquad \because i^2=-1$$

Multiply up and down by conjugate of $1+i$

$$=\frac{-5-12i}{1+i}\times\frac{1-i}{1-i}$$

$$=\frac{-5+5i-12i+12i^2}{1-i^2}$$

$$=\frac{-5-7i-12}{1+1} \qquad \because i^2=-1$$

$$=\frac{-17-7i}{2}$$

$$=-\frac{17}{2}-\frac{7}{2}i$$

$$\text{Real part}=-\frac{17}{2}$$

$$\text{Imaginary Part}=-\frac{7}{2}$$

(iii) $\dfrac{i}{1+i}$

Solution:

Multiply up and down by conjugate of $1+i$

$$\frac{i}{1+i}=\frac{i}{1+i}\times\frac{1-i}{1-i}$$

$$\frac{i-i^2}{1-i^2}=\frac{i+1}{1+1} \qquad \because i^2=-1$$

$$=\frac{1+i}{2}$$

$$=\frac{1}{2}+\frac{1}{2}i$$

$$\text{Real part}=\frac{1}{2}$$

$$\text{Imaginary Part}=\frac{1}{2}$$

(iv) $\dfrac{(4+3i)^2}{4-3i}$

Solution:

$$\frac{(4+3i)^2}{4-3i}=\frac{(4)^2+(3i)^2+24i}{4-3i}$$

$$=\frac{16+9i^2+24i}{4-3i}$$

$$=\frac{16-9+24i}{4-3i} \qquad \because i^2=-1$$

Multiply up and down by conjugate of $4-3i$

$$=\frac{7+24i}{4-3i}\times\frac{4+3i}{4+3i}$$

$$=\frac{28+21i+96i+72i^2}{16-9i^2}$$

$$=\frac{28+117i-72}{16+9} \qquad \because i^2=-1$$

$$=\frac{-44+117i}{25}$$

$$=\frac{-44}{25}+\frac{117}{25}i$$

3. Prove that $\overline{z}=z$ iff z is real.

Solution:

Given: $\overline{z}=z$

To Prove: z is real

Proof:

Let $z=a+ib$, then $\overline{z}=a-ib$ ...(1)

As $\overline{z}=z$

$$a-ib=a+ib$$

$$-ib=ib$$

$$2ib=0$$

$$b=0 \text{ Put in eq. (1)}$$

$$z=a+0i=a$$

$\Rightarrow$ z is real.

Conversely:

Given: z is real

To Prove: $\overline{z}=z$

Proof:

As z is real, then

$$z=a+0i=a \qquad ...(2)$$

$$\overline{z}=a-0i=a \qquad ...(3)$$

From eq (2) and eq (3)

$$\overline{z}=z$$

4. For $z\in C$, show that:

(i) $\dfrac{z+\overline{z}}{2}=\operatorname{Re}(z)$

Solution:

Let $z=a+ib \quad \Rightarrow \overline{z}=a-ib$

$\operatorname{Re}(z)=a \quad , \quad \operatorname{Im}(z)=b$

$$\frac{z+\overline{z}}{2}=\frac{(a+ib)+(a-ib)}{2}$$

$$=\frac{a+ib+a-ib}{2}$$

$$=\frac{2a}{2}$$

$$=a$$

$$=\operatorname{Re}(z)$$

Hence $\dfrac{z+\overline{z}}{2}=\operatorname{Re}(z)$

(ii) $\dfrac{z-\overline{z}}{2i}=\operatorname{Im}(z)$

Solution:

Let $z=a+ib \quad \Rightarrow \overline{z}=a-ib$

$\operatorname{Re}(z)=a \quad , \quad \operatorname{Im}(z)=b$

$$\frac{z-\overline{z}}{2i}=\frac{(a+ib)-(a-ib)}{2i}$$

$$=\frac{a+ib-a+ib}{2i}$$

$$=\frac{2ib}{2i}$$

$$=b=\operatorname{Im}(z)$$

Hence $\dfrac{z-\overline{z}}{2i}=\operatorname{Im}(z)$

(iii) $|z|^2=z.\overline{z}$

Solution:

Let $z=a+ib \quad \Rightarrow \overline{z}=a-ib$

$$\text{L.H.S}=|z|^2$$

$$=(\sqrt{a^2+b^2})^2=a^2+b^2$$

$$\text{R.H.S}=z.\overline{z}$$

$$=(a+ib)(a-ib)$$

$$=(a)^2-(ib)^2$$

$$\begin{aligned} &= a^2 - i^2b^2 \\ &= a^2 + b^2 \qquad \because i^2 = -1 \end{aligned}$$

Hence proved

$$\text{L.H.S} = \text{R.H.S}$$

i.e., $|z|^2 = \overline{z}.\overline{z}$

5. If $z_1 = 2 + i, z_2 = 3 - 2i, z_3 = 1 + 3i$ then express $\dfrac{\overline{z_1}\,\overline{z_3}}{z_2}$ in the form of $a + ib$.

Solution:

$$\begin{aligned} \frac{\overline{z_1}.\overline{z_3}}{z_2} &= \frac{\overline{(2+i)}.\overline{(1+3i)}}{3-2i} \\ &= \frac{(2-i).(1-3i)}{3-2i} \\ &= \frac{2-6i-i+3i^2}{3-2i} \\ &= \frac{2-7i-3}{3-2i} \\ &= \frac{-1-7i}{3-2i} \\ &= \frac{-1-7i}{3-2i} \times \frac{3+2i}{3+2i} && \because i^2 = -1 \\ &= \frac{-3-2i-21i-14i^2}{(3)^2-(2i)^2} \\ &= \frac{-3-23i+14}{9+4} && \because i^2 = -1 \\ &= \frac{11-23i}{13} \\ &= \frac{11}{13} - \frac{23}{13}i \end{aligned}$$

6. If $z_1 = 2 + 7i$ and $z_2 = -5 + 3i$, then evaluate the following:

(i) $|2z_1 - 4z_2|$

Solution:

Given that:

$$\begin{aligned} z_1 &= 2+7i \ , \ z^2 = -5+3i \\ 2z_1 - 4z_2 &= 2(2+7i) - 4(-5+3i) \\ &= 4 + 14i + 20 - 12i \\ &= 24 + 2i \\ |2z_1 - 4z_2| &= \sqrt{(24)^2 + (2)^2} \\ &= \sqrt{576+4} = \sqrt{580} \\ &= \sqrt{4 \times 145} = 2\sqrt{145} \end{aligned}$$

(ii) $|3z_1 + 2\overline{z_1}|$

Solution:

$$\begin{aligned} 3z_1 + 2\overline{z_1} &= 3(2+7i) + 2(2-7i) \\ &= 6 + 21i + 4 - 14i = 10 + 7i \\ |3z_1 + 2\overline{z_1}| &= \sqrt{(10)^2 + (7)^2} \\ &= \sqrt{100+49} = \sqrt{149} \end{aligned}$$

(iii) $|-7z_2 + 2\overline{z_2}|$

Solution:

$$\begin{aligned} -7z_2 + 2\overline{z_2} &= -7(-5+3i) + 2(-5-3i) \\ &= 35 - 21i - 10 - 6i \\ &= 25 - 27i \\ |-7z_2 + 2\overline{z_2}| &= \sqrt{(25)^2 + (-27)^2} \\ &= \sqrt{625+729} = \sqrt{1354} \end{aligned}$$

(iv) $|(z_1 + z_2)^3|$

Solution:

$$\begin{aligned} (z_1 + z_2)^3 &= (2 + 7i - 5 + 3i)^3 \\ &= (-3 + 10i)^3 \end{aligned}$$

As we know

$$\begin{aligned} (a+b)^3 &= a^3 + b^3 + 3ab(a+b) \\ (z_1 + z_2)^3 &= (-3)^3 + (10i)^3 - 90i(-3 + 10i) \\ &= -27 + 1000i^3 + 270i - 900i^2 \end{aligned}$$

Put $i^3 = -i$ and $i^2 = -1$

$$\begin{aligned} (z_1 + z_2)^3 &= -27 - 1000i + 270i + 900 \\ &= 873 - 730i \\ |(z_1 + z_2)^3| &= \sqrt{(873)^2 + (-730)^2} \\ &= \sqrt{762129 + 532900} \\ &= \sqrt{1295029} = 109\sqrt{109} \end{aligned}$$

7. Show that $i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} = 0$, for all $n \in N$.

Solution:

$$\begin{aligned} \text{L.H.S} &= i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} \\ &= i^n \cdot i + i^n \cdot i^2 + i^n \cdot i^3 + i^n \cdot i^4 \\ &= i^n(i + i^2 + i^3 + i^4) \\ &= i^n(i + i^2 + i \cdot i^2 + (i^2)^2) \\ &= i^n(i - 1 + i(-1) + (-1)^2) && \because i^2 = -1 \\ &= i^n(i - 1 - i + 1) \\ &= i(0) \\ &= 0 = \text{R.H.S (Proved)} \end{aligned}$$

8. Find the least positive value of n, if $\left(\dfrac{1+i}{1-i}\right)^{2n} = 1$

Solution:

Multiply up and down by conjugate of $(1 - i)$

$$\begin{aligned} \text{Consider} \quad \frac{1+i}{1-i} &= \frac{1+i}{1-i} \times \frac{1+i}{1+i} \\ &= \frac{(1+i)^2}{1^2 - i^2} \end{aligned}$$

$$\begin{aligned} &= \frac{1+i^2+2i}{1+1} && \because i^2 = -1 \\ &= \frac{1-1+2i}{2} = \frac{2i}{2} \\ \frac{1+i}{1-i} &= i && \ldots(1) \end{aligned}$$

$$\begin{aligned} \text{Now,} \quad \left(\frac{1+i}{1-i}\right)^{2n} &= 1 && \text{(Given)} \\ (i)^{2n} &= 1 && \text{using (1)} \end{aligned}$$

For least +ve value of n, put

$$\begin{aligned} 2n &= 4 && \because i^4 = 1 \\ n &= \frac{4}{2} \end{aligned}$$

$\boxed{n = 2}$ As required.

9. Show that, the value of i^n for $n \in N$ and $n > 4$ is i^r, when r is the remainder when n is divided by 4.

Solution:

To prove: $i^n = i^r$

Where r is the remainder, when n is divided by 4, for $n \in N$ and $n > 4$.

By division algorithm, we have.

$$n = 4q + r \qquad \ldots(1)$$

Where q is quotient and r is the remainder when n is divided by 4.

$$\begin{aligned} \text{L.H.S} &= i^n \\ &= i^{4q+r} \text{ using (1)} \\ &= i^{4q} \cdot i^r \\ &= (i^4)^q \cdot i^r \\ &= (1)^q \cdot i^r && \because i^4 = 1 \\ &= 1 \cdot i^r \\ &= i^r = \text{R.H.S. (Proved)} \end{aligned}$$

Equality of Two Complex Numbers

The two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are said to be equal iff their real and imaginary parts are equal i.e., $a + bi = c + di \Leftrightarrow a = c$ and $b = d$.

Example 4: If $(3 + 2i)(x + iy) = 5 + 12i$, where $x, y \in R$, then find the values of x and y.

Solution:

Given that $\quad (3 + 2i)(x + iy) = 5 + 12i$

$$\begin{aligned} &\Rightarrow 3x + 3iy + 2ix + 2yi^2 = 5 + 12i \\ &\Rightarrow (3x - 2y) + (2x + 3y)i = 5 + 12i \end{aligned}$$

Comparing real and imaginary part, we have

$3x - 2y = 5 \ \ldots\text{(i)} \qquad$ and $\qquad 2x + 3y = 12 \qquad \ldots\text{(ii)}$

$3 \times$ Eq. (i) + $2 \times$ Eq. (ii)

$$\begin{aligned} 9x - 6y + 4x + 6y &= 15 + 24 \\ 13x &= 39 \\ x &= 3 \end{aligned}$$

Put $x = 3$ in equation (i), we have

$$\begin{aligned} 3(3) - 2y &= 5 \\ 9 - 2y &= -12 \\ -2y &= -4 \\ y &= 2 \end{aligned}$$

Thus, $x = 3, y = 2$

Square Root of a Complex Number:

The square root of a complex number is another complex number that, when squared, give the original complex number.

If a complex number $z = x + iy$, where $x, y \in R$, then the square root of the complex number $z = x + iy$ is given by

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right), \text{ where } |z| = \sqrt{x^2 + y^2} \geq 0 \text{ is modulus of } z.$$

Example 5: Find the square root of complex number $5 + 12i$ and also represent the square root on an Argand diagram.

Solution:

Let $z = 5 + 12i$

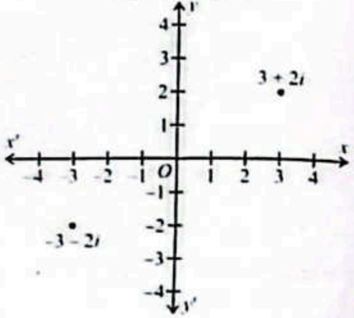
Here $x = 5$ and $y = 12$

$$|z| = |5+12i| = \sqrt{5^2+12^2} = \sqrt{169} = 13$$

The square root complex number z is:

$$\begin{aligned}\sqrt{z} &= \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)\\ &= \pm\left(\sqrt{\frac{13+5}{2}} + \frac{i12}{|12|}\sqrt{\frac{13-5}{2}}\right)\\ &= \pm(\sqrt{9}+i\sqrt{4}) = \pm(3+2i)\\ &= 3+2i \text{ or } -3-2i\end{aligned}$$

Argand Diagram



Exercise 1.2

1. Find the values of x and y in each of the following:

(i) $x + iy + 2 - 3i = i(5 - i)(3 + 4i)$

Solution:

$$\begin{aligned}x + iy + 2 - 3i &= i(5-i)(3+4i)\\ (x+2)+(y-3)i &= i(15+20i-3i-4i^2)\\ (x+2)+(y-3)i &= i(15+17i+4) && \because i^2=-1\\ (x+2)+(y-3)i &= 17i^2+19i\\ (x+2)+(y-3)i &= -17+19i && \because i^2=-1\end{aligned}$$

Comparing real and imaginary parts, we have

$$\begin{array}{r|r} x+2=-17 & y-3=19\\ x=-17-2 & y=19+3\\ x=-19 & y=22\end{array}$$

Thus, $x = -19$ and $y = 22$

(ii) $(x+iy)(1-i)=(2-3i)(-5+5i)\left(-i\frac{3}{5}\right)$

Solution:

$$\begin{aligned}(x+iy)(1-i) &= (2-3i)(-5+5i)\left(-\frac{3}{5}i\right)\\ x-ix+iy-i^2y &= -\frac{3}{5}i(-10+10i+15i-15i^2)\\ x-ix+iy+y &= -\frac{3}{5}i(-10+25i+15) && \because i^2=-1\\ (x+y)+(y-x)i &= -\frac{3}{5}i(25i+5)\\ (x+y)+(y-x)i &= -15i^2-3i\\ (x+y)+(y-x)i &= 15-3i\end{aligned}$$

Comparing real and imaginary parts, we have

$$x+y=15 \quad \text{...(1)}$$
$$-x+y=-3 \quad \text{...(2)}$$

Eq. (1) + Eq. (2)

$$2y=12 \quad \Rightarrow y=6$$

Put $y = 6$ in eq. (1), we have

$$\begin{aligned}x+6 &= 15\\ x &= 15-6 \Rightarrow x=9\end{aligned}$$

Thus, $x = 9$ and $y = 6$

(iii) $\frac{x}{2+i}+\frac{y}{3-i}=4+5i$

Solution:

$$\begin{aligned}\frac{x}{2+i}+\frac{y}{3-i} &= 4+5i\\ \frac{x}{2+i}\cdot\frac{2-i}{2-i}+\frac{y}{3-i}\cdot\frac{3+i}{3+i} &= 4+5i\\ \frac{2x-ix}{2^2-i^2}+\frac{3y+iy}{3^2-i^2} &= 4+5i\\ \frac{2x-ix}{4+1}+\frac{3y+iy}{9+1} &= 4+5i && \because i^2=-1\\ \frac{2x-ix}{5}+\frac{3y+iy}{10} &= 4+5i\\ \frac{4x-2ix+3y+iy}{10} &= 4+5i\\ \frac{4x+3y}{10}+\frac{-2x+y}{10}i &= 4+5i\end{aligned}$$

Comparing real and imaginary parts, we have

$$\frac{4x+3y}{10}=4 \quad \Rightarrow 4x+3y=40 \quad \text{...(1)}$$
$$\frac{-2x+y}{10}=5 \quad \Rightarrow -2x+y=50 \quad \text{...(2)}$$

Eq. (1) + 2 × Eq. (2)

$$\begin{aligned}4x+3y &= 40\\ \underline{-4x+2y} &\underline{= 100}\\ 5y &= 140 \quad \Rightarrow \quad y=28\end{aligned}$$

Put $y = 28$ in eq. (1), we have

$$\begin{aligned}4x+3(28) &= 40 && \Rightarrow 4x=40-84\\ 4x &= -44 && \Rightarrow x=-11\end{aligned}$$

Thus $x=-11$ and $y=28$

2. If $z_1=-13+24i$ and $z_2=x+yi$, find the values of x and y such that $z_1-z_2=-27+15i$

Solution:

Given that: $z_1=-13+24i, z_2=x+yi$

$$\begin{aligned}\text{As} \qquad z_1-z_2 &= -27+15i\\ \Rightarrow \quad (-13+24i)-(x+yi) &= -27+15i\\ -13+24i-x-yi &= -27+15i\\ (-x-13)+(24-y)i &= -27+15i\end{aligned}$$

Comparing real and imaginary parts, we have

$$\begin{array}{r|r} -x-13=-27 & 24-y=15\\ -x=-27+13 & -y=15-24\\ -x=-14 & -y=-9\\ x=14 & y=9\end{array}$$

Thus, $x = 14$ and $y = 9$

3. Find the value of x and y if:

(i) $(x+iy)^2=25+60i$

Solution:

$$\begin{aligned}(x+iy)^2 &= 25+60i\\ x^2+i^2y^2+2xyi &= 25+60i\\ (x^2-y^2)+(2xy)i &= 25+60i && \because i^2=-1.\end{aligned}$$

Comparing real and imaginary parts, we have

$$x^2-y^2=25 \quad \text{...(1)}$$
$$2xy=60 \quad \text{...(2)}$$

From eq. (2), we have

$$y=\frac{60}{2x} \quad \Rightarrow y=\frac{30}{x}$$

Put $y=\frac{30}{x}$ in eq (1)

$$\begin{aligned}x^2-\left(\frac{30}{x}\right)^2 &= 25\\ x^2-\frac{900}{x^2} &= 25\end{aligned}$$

Multiply by x^2, we have

$$\begin{aligned}x^4-900 &= 25x^2\\ x^4-25x^2-900 &= 0\\ x^4-45x^2+20x^2-900 &= 0\\ x^2(x^2-45)+20(x^2-45) &= 0\\ (x^2-45)(x^2+20) &= 0\end{aligned}$$

Either $x^2-45=0$ or $x^2+20=0$

$\Rightarrow$ $x^2=45$ but $x^2=-20$ (not possible)

$$x=\pm3\sqrt{5}$$

Put $x=3\sqrt{5}$ in eq (2)

$$\begin{aligned}2(3\sqrt{5})y &= 60 && \Rightarrow y=\frac{60}{6\sqrt{5}}\\ y &= \frac{10}{\sqrt{5}}\\ &= \frac{2\times5}{\sqrt{5}}=2\sqrt{5}\end{aligned}$$

Put $x=-3\sqrt{5}$ in eq (2)

$$\begin{aligned}2(-3\sqrt{5})y &= 60 && \Rightarrow y=\frac{60}{-6\sqrt{5}}\\ y &= \frac{-10}{\sqrt{5}}\\ &= -\frac{2\times5}{\sqrt{5}}=-2\sqrt{5}\end{aligned}$$

Thus, $x=3\sqrt{5}, y=2\sqrt{5}$ or $x=-3\sqrt{5}, y=-2\sqrt{5}$

(ii) $(x+iy)^2=64+48i$

Solution:

$$\begin{aligned}(x+iy)^2 &= 64+48i\\ x^2+i^2y^2+2xyi &= 64+48i\\ (x^2-y^2)+(2xy)i &= 64+48i && \because i^2=-1\end{aligned}$$

Comparing real and imaginary parts, we have

$$x^2-y^2=64 \quad \text{...(1)}$$
$$2xy=48 \quad \Rightarrow xy=24 \quad \text{...(2)}$$

From eq. (2), we have

$$y=\frac{24}{x}$$

Put this in eq. (1)

$$\begin{aligned}x^2-\left(\frac{24}{x}\right)^2 &= 64\\ x^2-\frac{576}{x^2} &= 64\end{aligned}$$

Multiply by x^2, we have

$$\begin{aligned}x^4-576 &= 64x^2\\ x^4-64x^2-576 &= 0\\ x^4-72x^2+8x^2-576 &= 0\\ x^2(x^2-72)+8(x^2-72) &= 0\\ (x^2-72)(x^2+8) &= 0\end{aligned}$$

Either $x^2-72=0$ or $x^2+8=0$

$\Rightarrow$ $x^2=72$ but $x^2=-8$ (not possible)

$$x=\pm6\sqrt{2}$$

Put $x=6\sqrt{2}$ in eq. (2)

$$(6\sqrt{2})y = 24 \qquad \Rightarrow y=\frac{24}{6\sqrt{2}}$$

$$y=\frac{4}{\sqrt{2}}=\frac{2\times 2}{\sqrt{2}}=2\sqrt{2}$$

Put $x=-6\sqrt{2}$ in eq. (2)

$$(-6\sqrt{2})y = 24 \qquad \Rightarrow y=\frac{24}{-6\sqrt{2}}$$

$$y=\frac{-4}{\sqrt{2}}=-\frac{2\times 2}{\sqrt{2}}=-2\sqrt{2}$$

Thus, $x=6\sqrt{2}, y=2\sqrt{2}$ or $x=-6\sqrt{2}, y=-2\sqrt{2}$

(iii) $(x+iy)^2=\dfrac{2i-3}{3+i}$

Solution:

$$\begin{aligned}
(x+iy)^2 &= \frac{2i-3}{3+i}\\
(x^2-y^2)+(2xy)i &= \frac{2i-3}{3+i}\times\frac{3-i}{3-i}\\
(x^2-y^2)+(2xy)i &= \frac{6i-2i^2-9+3i}{(3)^2-(i)^2}\\
(x^2-y^2)+(2xy)i &= \frac{2-9+9i}{9-i^2}\\
(x^2-y^2)+(2xy)i &= \frac{-7+9i}{9-(-1)} && \because i^2=-1\\
(x^2-y^2)+(2xy)i &= \frac{-7+9i}{10}\\
(x^2-y^2)+(2xy)i &= \frac{-7}{10}+\frac{9}{10}i
\end{aligned}$$

Comparing real and imaginary parts, we have

$$x^2-y^2=\frac{-7}{10} \qquad \ldots(1)$$

$$2xy=\frac{9}{10} \qquad \ldots(2)$$

From eq. (2), we have

$$y=\frac{9}{20x} \quad \text{Put this in eq. (1)}$$

$$x^2-\left(\frac{9}{20x}\right)^2=\frac{-7}{10}$$

$$x^2-\frac{81}{400x^2}=\frac{-7}{10}$$

Multiply by $400x^2$

$$400x^4-81=-280x^2$$

$$400x^4+280x^2-81=0$$

Use the quadratic formula

$$\begin{aligned}
x^2 &= \frac{-280\pm\sqrt{(280)^2-4(400)(-81)}}{2(400)}\\
x^2 &= \frac{-280\pm\sqrt{78400+129600}}{800}\\
x^2 &= \frac{-280\pm\sqrt{208000}}{800}\\
x^2 &= \frac{-280\pm 40\sqrt{130}}{800}\\
x^2 &= \frac{-7\pm\sqrt{130}}{20} && \text{Divide up and down by 40}\\
x &= \pm\sqrt{\frac{-7\pm\sqrt{130}}{20}}
\end{aligned}$$

Using $y=\dfrac{9}{20x}$

If $x=\sqrt{\dfrac{-7+\sqrt{130}}{20}}$ then $y=\dfrac{9}{20\sqrt{\dfrac{-7+\sqrt{30}}{20}}}$

If $x=-\sqrt{\dfrac{-7+\sqrt{130}}{20}}$ then $y=-\dfrac{9}{20\sqrt{\dfrac{-7+\sqrt{30}}{20}}}$

---

4. If $z_1=2+3i$ and $z_2=1-\alpha$, find the value of α such that $\operatorname{Im}(z_1z_2)=7$.

Solution:

Given that

$$\begin{aligned}
z_1 &= 2+3i\\
z_2 &= 1-\alpha\\
\operatorname{Im}(z_1 . z_2) &= 7
\end{aligned}$$

Consider

$$\begin{aligned}
z_1 . z_2 &= (2+3i) . (1-\alpha)\\
&= 2-2\alpha+3i-3\alpha i\\
z_1 . z_2 &= (2-2\alpha)+(3-3\alpha)i\\
\Rightarrow \operatorname{Im}(z_1 . z_2) &= 3-3\alpha\\
\Rightarrow 7 &= 3-3\alpha && \because \operatorname{Im}(z_1 . z_2)=7\\
3\alpha &= 3-7\\
3\alpha &= -4\\
\Rightarrow & \boxed{\alpha=\frac{-4}{3}}
\end{aligned}$$

---

5. If $z_1=x+yi$ and $z_2=a+bi$, find x, y, a and b such that $z_1+z_2=10+4i$ and $z_1-z_2=6+2i$.

Solution:

Given that:

$$\begin{aligned}
z_1 &= x+yi\\
z_2 &= a+bi\\
z_1+z_2 &= 10+4i && \ldots(1)\\
z_1-z_2 &= 6+2i && \ldots(2)
\end{aligned}$$

Eq. (1) + Eq. (2)

$$\begin{aligned}
z_1+z_2 &= 10+4i\\
z_1-z_2 &= 6+2i\\
\hline
2z_1 &= 16+6i\\
z_1 &= 8+3i && \text{Dividing by '2'}\\
\Rightarrow x+iy &= 8+3i && \because z_1=x+yi
\end{aligned}$$

Comparing real and imaginary parts, we have

$$x=8 \quad \text{and} \quad y=3$$

Eq. (1) – Eq. (2)

$$\begin{aligned}
z_1+z_2 &= 10+4i\\
\pm z_1 \mp z_2 &= \pm 6 \pm 2i\\
\hline
2z_2 &= 4+2i\\
z_2 &= 2+i\\
\Rightarrow a+bi &= 2+1i && \because z_2=a+bi
\end{aligned}$$

Comparing real and imaginary parts, we have

$$a=2 \quad \text{and} \quad b=1$$

---

6. Show that $\forall z_1, z_2 \in C, \overline{z_1z_2}$

Solution:

Let $z_1=a+ib$ and $z_2=c+id$

$\Rightarrow \overline{z_1}=a-ib$ and $\overline{z_2}=c-id$,

$$\begin{aligned}
\text{then} \quad z_1 . z_2 &= (a+ib) . (c+id)\\
&= (ac-bd)+i(ad+bc)\\
\text{L.H.S}=\overline{z_1} . \overline{z_2} &= (ac-bd)-i(ad+bc)\\
\text{R.H.S}=\overline{z_1} . \overline{z_2} &= (a-ib) . (c-id)\\
&= (ac-bd)+i(-ad-bc)\\
&= (ac-bd)-i(ad+bc)
\end{aligned}$$

Hence proved L.H.S = R.H.S

---

7. Find the square root of the following complex numbers:

(i) $-7-24i$

Solution:

Let $z=-7-24i$

Here $x=-7, y=-24$

$$\begin{aligned}
|z| &= \sqrt{x^2+y^2}\\
&= \sqrt{49+576}\\
&= \sqrt{625}=25
\end{aligned}$$

Square root of z is

$$\begin{aligned}
\sqrt{z} &= \pm\left(\sqrt{\frac{|z|+x}{2}}+i\frac{y}{|y|}\sqrt{\frac{|z|-x}{2}}\right)\\
&= \pm\left(\sqrt{\frac{25-7}{2}}+i\frac{-24}{|-24|}\sqrt{\frac{25+7}{2}}\right)\\
&= \pm\left(\sqrt{\frac{18}{2}}-i\frac{24}{24}\sqrt{\frac{32}{2}}\right)\\
&= \pm(3-i4)\\
&= +(3-4i) \quad \text{or} \quad -(3-4i)\\
&= 3-4i \quad \text{or} \quad -3+4i
\end{aligned}$$

(ii) $8-6i$

Solution:

Let $z=8-6i$

Here $x=8, y=-6$

$$\begin{aligned}
|z| &= \sqrt{8^2+(-6)^2}\\
&= \sqrt{64+36}\\
&= \sqrt{100}=10
\end{aligned}$$

Square root of z is

$$\begin{aligned}
\sqrt{z} &= \pm\left(\sqrt{\frac{|z|+x}{2}}+i\frac{y}{|y|}\sqrt{\frac{|z|-x}{2}}\right)\\
&= \pm\left(\sqrt{\frac{10+8}{2}}+i\frac{-6}{|-6|}\sqrt{\frac{10-8}{2}}\right)\\
&= \pm\left(\sqrt{\frac{18}{2}}-i\frac{6}{6}\sqrt{\frac{2}{2}}\right)\\
&= \pm(3-1i)\\
&= +(3-i) \quad \text{or} \quad -(3-i)\\
&= 3-i \quad \text{or} \quad -3+i
\end{aligned}$$

(iii) $-15-36i$

Solution:

Let $z=-15-36i$

Here $x=-15, y=-36$

$$\begin{aligned}
|z| &= \sqrt{(-15)^2+(-36)^2}\\
&= \sqrt{225+1296}\\
&= \sqrt{1521}=39
\end{aligned}$$

Square root of z is

$$\begin{aligned}
\sqrt{z} &= \pm\left(\sqrt{\frac{|z|+x}{2}}+i\frac{y}{|y|}\sqrt{\frac{|z|-x}{2}}\right)\\
&= \pm\left(\sqrt{\frac{39-15}{2}}+i\frac{-36}{|-36|}\sqrt{\frac{39+15}{2}}\right)\\
&= \pm\left(\sqrt{\frac{24}{2}}-i\frac{36}{36}\sqrt{\frac{54}{2}}\right)\\
&= \pm(\sqrt{12}-i\sqrt{27})\\
&= \pm(2\sqrt{3}-3\sqrt{3}i)\\
&= +(2\sqrt{3}-3\sqrt{3}i) \quad \text{or} \quad -(2\sqrt{3}-3\sqrt{3}i)\\
&= 2\sqrt{3}-3\sqrt{3}i \quad \text{or} \quad -2\sqrt{3}+3\sqrt{3}i
\end{aligned}$$

(iv) $119+120i$

Solution:

Let $z=119+120i$

Here $x=119, y=120$

$$\begin{aligned}
|z| &= \sqrt{(119)^2+(120)^2}\\
&= \sqrt{28561}=169
\end{aligned}$$

Square root of z is

$$\begin{aligned}\sqrt{z} &= \pm\left(\sqrt{\frac{|z|+x}{2}}+i\frac{y}{|y|}\sqrt{\frac{|z|-x}{2}}\right)\\ &= \pm\left(\sqrt{\frac{169+119}{2}}+i\frac{120}{|120|}\sqrt{\frac{169-119}{2}}\right)\\ &= \pm\left(\sqrt{\frac{288}{2}}+i\frac{120}{120}\sqrt{\frac{50}{2}}\right)\\ &= \pm(\sqrt{144}+i\sqrt{25})\\ &= \pm(12+5i)\\ &= +(12+5i) \quad\text{or}\quad -(12+5i)\\ &= 12+5i \quad\text{or}\quad -12-5i\end{aligned}$$

8. Find the square root of $13-20\sqrt{3}i$ and represent it on an Argand diagram.

Solution:

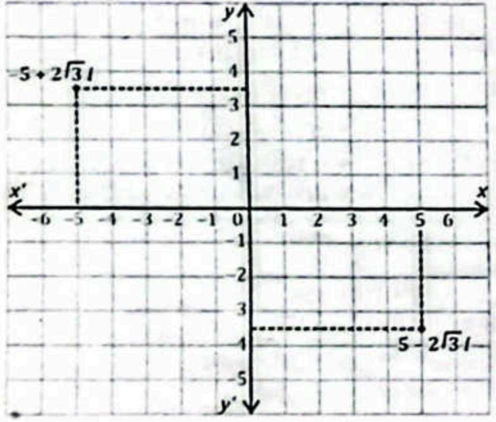
Let

$$\begin{aligned}z &= 13-20\sqrt{3}i\\ x &= 13,\ y=-20\sqrt{3}\\ |z| &= \sqrt{(13)^2+(-20\sqrt{3})^2}\\ &= \sqrt{169+1200}=37\end{aligned}$$

Square root of z is

$$\begin{aligned}\sqrt{z} &= \pm\left(\sqrt{\frac{|z|+x}{2}}+i\frac{y}{|y|}\sqrt{\frac{|z|-x}{2}}\right)\\ &= \pm\left(\sqrt{\frac{37+13}{2}}+i\frac{-20\sqrt{3}}{|-20\sqrt{3}|}\sqrt{\frac{37-13}{2}}\right)\\ &= \pm\left(\sqrt{25}-i\frac{20\sqrt{3}}{20\sqrt{3}}\sqrt{12}\right)\\ &= \pm(5-2\sqrt{3}i)\\ &= +(5-2\sqrt{3}i) \quad\text{or}\quad -(5-2\sqrt{3}i)\\ &= 5-2\sqrt{3}i \quad\text{or}\quad -5+2\sqrt{3}i\end{aligned}$$

Argand Diagram



9. Find the value of x and y if $(-7+i)(x+iy)+(-1-5i)=i(11-i)$

Solution:

$$\begin{aligned}(-7+i)(x+iy)+(-1-5i) &= i(11-i)\\ -7x-7yi+ix+i^2y-1-5i &= 11i-i^2\\ -7x-7yi+ix-y-1-5i &= 11i+1 \qquad \because i^2=-1\\ (-7x-y-1)+(-7y+x-5)i &= 1+11i\end{aligned}$$

Comparing real and imaginary parts, we have

$$\begin{aligned}-7x-y-1 &= 1 \qquad \Rightarrow -7x-y=2 \qquad \ldots(1)\\ -7y+x-5 &= 11 \qquad \Rightarrow x-7y=16 \qquad \ldots(2)\end{aligned}$$

Eq. (1) + 7 × Eq. (2)

$$\begin{aligned}-7x-\ \ y &= 2\\ 7x-49y &= 112\\ \hline -50y &= 114 \qquad \Rightarrow y=\frac{114}{-50}\\ y &= \frac{-57}{25} \quad \text{Put this in eq. (2)}\\ x-7\left(-\frac{57}{25}\right) &= 16 \qquad \Rightarrow x=16-\frac{399}{25}\\ x &= \frac{400-399}{25}=\frac{1}{25}\end{aligned}$$

Thus, $x=\frac{1}{25}$ and $y=\frac{-57}{25}$

10. Find the value of x and y if $(5-2i)(x+iy)+3=i(11-i)-4i$

Solution:

$$\begin{aligned}(5-2i)(x+iy)+3 &= i(11-i)-4i\\ 5x+5yi-2xi-2yi^2+3 &= 11i-i^2-4i\\ 5x+5yi-2xi+2y+3 &= 11i+1-4i \quad \because i^2=-1\\ (5x+2y+3)+(-2x+5y) &= 1+7i\end{aligned}$$

Comparing real and imaginary part, we have

$$\begin{aligned}5x+2y+3 &= 1 \qquad \Rightarrow 5x+2y=-2 \qquad \ldots(1)\\ -2x+5y &= 7 \qquad\qquad\qquad\qquad \ldots(2)\end{aligned}$$

2 × Eq. (1) + 5 × Eq. (2)

$$\begin{aligned}10x+\ \ 4y &= -4\\ -10x-25y &= 35\\ \hline 29y &= 31 \qquad \Rightarrow y=\frac{31}{29}\end{aligned}$$

Put $y=\frac{31}{29}$ in eq (1), we have

$$\begin{aligned}5x+2\left(\frac{31}{29}\right) &= -2\\ 5x &= -2-\frac{62}{29}\end{aligned}$$

$$\begin{aligned}5x &= \frac{-58-62}{29}\\ x &= \frac{-120}{5.29}\\ x &= \frac{-24}{29}\end{aligned}$$

Thus, $x=\frac{-24}{29}$ and $y=\frac{31}{29}$

11. Find the values of u and v if $\frac{u-2}{2+i}+\frac{v-3}{2-i}=4i$

Solution:

Given that: $\frac{u-2}{2+i}+\frac{v-3}{2-i}=4i$

$$\begin{aligned}\frac{(u-2)(2-i)+(v-3)(2+i)}{(2+i)(2-i)} &= 4i\\ \frac{2u-ui-4+2i+2v+vi-6-3i}{2^2-i^2} &= 4i\\ \frac{(2u+2v-10)+(-ui+vi-i)}{4+1} &= 4i \qquad \because i^2=-1\\ \frac{2(u+v-5)+(-u+v-1)i}{5} &= 4i\\ 2(u+v-5)+(-u+v-1)i &= 0+20i\end{aligned}$$

Equate the real and imaginary parts, we have

$$\begin{aligned}2(u+v-5) &= 0 & -u+v-1 &= 20\\ u+v &= 5 \quad \ldots(1) & -u+v &= 21 \quad \ldots(2)\end{aligned}$$

Equation (1) + Equation (2)

$$2v=26 \qquad \Rightarrow \boxed{v=13}$$

Put $v=13$ in eq (1), we have

$$\begin{aligned}u+13 &= 5 \qquad \Rightarrow u=5-13\\ &\boxed{u=-8}\end{aligned}$$

Hence, $u=-8$ and $v=13$

12. If $z_1=4+5i$ and $z_2=\alpha-2i$, find the value of α such that $\mathrm{Re}(z_1z_2)=20$.

Solution:

Given that: $z_1=4+5i,\ z_2=\alpha-2i,\ \mathrm{Re}(z_1z_2)=20$

$$\begin{aligned}z_1\cdot z_2 &= (4+5i)(\alpha-2i)\\ &= 4\alpha-8i+5\alpha i-10i^2\\ &= 4\alpha-8i+5\alpha i+10 \qquad \because i^2=-1\\ z_1\cdot z_2 &= (4\alpha+10)+(-8+5\alpha)i\\ \Rightarrow \quad \mathrm{Re}(z_1z_2) &= 4\alpha+10\\ \Rightarrow \quad 20 &= 4\alpha+10 \qquad \because \mathrm{Re}(z_1z_2)=20\\ 20-10 &= 4\alpha\\ 4\alpha &= 10 \qquad \Rightarrow \alpha=\frac{10}{4} \qquad \Rightarrow \alpha=\frac{5}{2}\end{aligned}$$

Complex Polynomial

A complex polynomial $P(z)$ is a polynomial function of the complex variable z with complex coefficients. It is expressed in the general form as

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$$

Where $a_n,a_{n-1},\ldots,a_1,a_0$ are complex numbers $(a_n\neq 0)$, and $n\geq 0$ is an integer representing the degree of the polynomial.

For examples:

- $P_1(z)=(1-i)z+3i$ (Linear complex polynomial)
- $P_2(z)=(5-4i)z^2+(2+i)z+(3-4i)$ (Quadratic complex polynomial)
- $P_3(z)=(2-i)z^3+2z^2i+(5+3i)$ (Cubic complex polynomial)

Note:
If $n=0$, then $P(z)$ becomes a constant polynomial.

Complex Polynomials as a Product of Linear Factors:

A fundamental property of complex polynomials is that they can always be factored into a product of linear factors. According to the **Fundamental theorem of algebra**, a polynomial of degree $n\geq 1$ has exactly n roots in complex numbers system C.

Corollary:

A corollary to this theorem states that any polynomial $P(z)$ of degree n can be factored completely into a constant a and n linear factor over C in the form

$$P(z)=a(z-z_1)(z-z_2)\ldots(z-z_n) \qquad (1)$$

where $z_1,z_2,\ldots,z_n$ are complex roots of the polynomial. Once we know the roots of a polynomial equation, we can apply equation (1) to factored the polynomial $P(z)$ into n linear factors. Specifically, if z_1 and z_2 are roots of the polynomial equation $P(z)$, then the equation must be $P(z)=(z-z_1)(z-z_2)$.

For examples, the polynomial $P(x)=x^2+4$ consists of real coefficient has no real roots, so it cannot be factored into linear polynomials with real coefficients. However, if we considered as a complex polynomial $P(z)=z^2+4$, we can easily be factored into two linear factors as:

$$z^2+4=(z+2i)(z-2i)$$

where $2i$ and $-2i$ are the complex roots of $z^2+4=0$

Note:
If $P(z)$ is a polynomial function, the values of z that satisfy $P(z)=0$ are called the zeros of the function $P(z)$ and roots of the polynomial equation $P(z)=0$.

Example 6: Factorize the polynomial $P(z)=z^2+(i-3)z-3i$

Solution:

$$\begin{aligned}P(z)&=z^2+(i-3)z-3i\\&=z^2+zi-3z-3i=z(z+i)-3(z+i)=(z+i)(z-3)\end{aligned}$$

Example 7: Factorize the polynomial $P(z)=z^2-4iz+12$

Solution:

$$\begin{aligned}P(z)&=z^2-4iz+12\\&=z^2-4iz-(-12)=z^2-4iz-i^2 12 && \because i^2=-1\\&=z^2-i6z+i2z-i^2 12=z(z-6i)+2i(z-6i)\\&=(z-6i)(z+2i)\end{aligned}$$

Example 8: Factorize the polynomial $P(z)=z^3+(1+i)z^2+iz$.

Solution:

$$\begin{aligned}P(z)&=z^3+(1+i)z^2+iz\\&=z[z^2+(1+i)z+i]=z[z^2+z+iz+i]\\&=z[z(z+1)+i(z+1)]=z[(z+1)(z+i)]\\&=z(z+1)(z+i)\end{aligned}$$

Key Concept:
The Rational Root Theorem is a mathematical tool used to find all possible rational roots of a polynomial equation with integer coefficients.

According to rational root theorem: If a polynomial $P(x)=a_n x^n+a_{(n-1)}x^{(n-1)}+\ldots+a_1x+a_0$ has integer coefficients, then every rational root $\frac{p}{q}$ (in simplest terms) satisfies:

(i) p is a factor of the constant term a_0. (ii) q is a factor of the leading coefficient a_n.

Example 9: Factorize the polynomial $P(z)=z^3-3z^2+z+5$.

Solution:

According to rational root theorem the possible root of the equation are ±1 and ±5.
On checking, we see that $z=-1$ is the root of the polynomial $P(z)$.

Because $P(-1)=(-1)^3-3(-1)^2+(-1)+5=0$

So, $z+1$ is a factor of the $P(z)$.

By using synthetic division, we have

$$\begin{array}{r|rrrl}-1 & 1 & -3 & 1 & 5\\ & & -1 & 4 & -5\\\hline & 1 & -4 & 5 & 0=\text{Remainder}\end{array}$$

Therefore, $P(z)=z^3-3z^2+z+5=(z+1)(z^2-4z+5)$ …(i)

Next, we will find the factors of z^2-4z+5

By using quadratic formula

$z^2-4z+5=0$, here $a=1, b=-4, c=5$

$$z=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}=\frac{4\pm\sqrt{16-20}}{2}=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm2i}{2}$$

$\Rightarrow \quad z=2\pm2i$

So, factorization of $z^2-4z+5=(z-(2+i))(z-(2-i))=(z-2-i)(z-2+i)$

Put in equation (i), we have

$z^3-3z^2+z+5=(z+1)(z-2-i)(z-2+i)$

Solution of Quadratic Equations by Completing the Square:

As we learned in previous classes, **completing the square** is a powerful and systematic method for solving quadratic equations. This technique involves rewriting a quadratic equation in the form $ax^2+bx+c=0$ into a perfect square trinomial, which can then be solved by taking the square root of both sides. This method is especially valuable when the quadratic equation does not factor easily. By completing the square, we can solve any quadratic equation, even those with irrational or complex roots, making it a more effective technique in algebra.

Example 10: Solve the equation $2z^2-12z+50=0$ by completing square method and hence express it as a product of its linear factors.

Solution:

$2z^2-12z+50=0$

$z^2-6z+25=0$ Dividing both sides by 2

$z^2-2(3)z=-25$

Add 3^2 on both sides, we have

$z^2-2(3)z+3^2=-25+3^2$

$(z-3)^2=-16 \quad \because a^2-2ab+b^2=(a-b)^2$

By taking square root on both sides, we have

$z-3=\pm\sqrt{-16}$

$z=3\pm4i$

$\Rightarrow z=3+4i$ or $z=3-4i$ (complex roots)

By Fundamental Theorem of Algebra, the given polynomial equation can be expressed as:

$$\begin{aligned}2z^2-12z+50&=2(z^2-6z+25)\\&=2(z-(3+4i))(z-(3-4i))=2(z-3-4i)(z-3+4i)\end{aligned}$$

Exercise 1.3

1. Factorize the following:

(i) a^2+4b^2

Solution:

$$\begin{aligned}a^2+4b^2&=a^2-(-1)4b^2\\&=a^2-4i^2b^2 && \because i^2=-1\\&=(a)^2-(2ib)^2\\&=(a+2ib)(a-2ib)\end{aligned}$$

(ii) $9a^2+16b^2$

Solution:

$$\begin{aligned}9a^2+16b^2&=9a^2-(-1)16b^2\\&=9a^2-16i^2b^2 && \because i^2=-1\\&=(3a)^2-(4ib)^2\\&=(3a+4ib)(3a-4ib)\end{aligned}$$

(iii) $3x^2+3y^2$

Solution:

$$\begin{aligned}3x^2+3y^2&=3[x^2+y^2]\\&=3[x^2-(-1)y^2]\\&=3[x^2-i^2y^2] && \because i^2=-1\\&=3[(x)^2-(iy)^2]\\&=3(x+iy)(x-iy)\end{aligned}$$

(iv) $144x^2+225y^2$

Solution:

$$\begin{aligned}144x^2+225y^2&=9(16x^2+25y^2)\\&=9(16x^2-(-1)25y^2)\\&=9(16x^2-25i^2y^2) && \because i^2=-1\\&=9((4x)^2-(5iy)^2)\\&=9(4x+5iy)(4x-5iy)\end{aligned}$$

(v) $z^2 - 2iz - 1$

Solution:

$$z^2 - 2iz - 1 = z^2 - 2iz + i^2 \qquad \because i^2 = -1$$
$$= z^2 - iz - iz + i^2$$
$$= z(z - i) - i(z - i)$$
$$= (z - i)(z - i)$$

(vi) $z^2 + 6z + 13$

Solution:

$$z^2 + 6z + 13$$

Add and subtract $(3)^2$, we have

$$= z^2 + 6z + 3^2 - 3^2 + 13$$
$$= (z^2 + 6z + 3^2) - 9 + 13$$
$$= (z + 3)^2 + 4$$
$$= (z + 3)^2 - (-4)$$
$$= (z + 3)^2 - 4i^2 \qquad \because i^2 = -1$$
$$= (z + 3)^2 - (2i)^2$$
$$= (z + 3 - 2i)(z + 3 + 2i)$$

(vii) $z^2 + 4z + 5$

Solution:

$$z^2 + 4z + 5$$

Add and subtract $(2)^2$, we have

$$= z^2 + 4z + 2^2 - 2^2 + 5$$
$$= (z^2 + 4z + 2^2) + 1$$
$$= (z + 2)^2 - (-1)$$
$$= (z + 2)^2 - i^2 \qquad \because i^2 = -1$$
$$= (z + 2 - i)(z + 2 + i)$$

(viii) $2z^2 - 22z + 65$

Solution:

$$2z^2 - 22z + 65 = 2\left(z^2 - 11z + \frac{65}{2}\right)$$

Add and subtract $\left(\frac{11}{2}\right)^2$, we have

$$= 2\left\{\left(z^2 - 11z + \left(\frac{11}{2}\right)^2\right) - \left(\frac{11}{2}\right)^2 + \frac{65}{2}\right\}$$
$$= 2\left\{\left(z - \frac{11}{2}\right)^2 - \frac{121}{4} + \frac{65}{2}\right\}$$
$$= 2\left\{\left(z - \frac{11}{2}\right)^2 + \frac{-121 + 130}{4}\right\}$$
$$= 2\left\{\left(z - \frac{11}{2}\right)^2 + \frac{9}{4}\right\}$$
$$= 2\left\{\left(z - \frac{11}{2}\right)^2 - \left(-\frac{9}{4}\right)\right\}$$
$$= 2\left\{\left(z - \frac{11}{2}\right)^2 - \frac{9}{4}i^2\right\}$$
$$= 2\left\{\left(z - \frac{11}{2}\right)^2 - \left(\frac{3}{2}i\right)^2\right\}$$
$$= 2\left(z - \frac{11}{2} + \frac{3}{2}i\right)\left(z - \frac{11}{2} - \frac{3}{2}i\right)$$
$$= 2\left(z - \frac{11 - 3i}{2}\right)\left(z - \frac{11 + 3i}{2}\right)$$

2. Factorize the following polynomial into its linear factors:

(i) $z^3 + 8$

Solution:

$$z^3 + 8 = z^3 + (2)^3$$

Formula: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$$= (z + 2)(z^2 - 2z + 4)$$
$$= (z + 2)(z^2 - 2z + 1^2 - 1^2 + 4)$$
$$= (z + 2)((z - 1)^2 + 3)$$
$$= (z + 2)((z - 1)^2 - 3i^2) \qquad \because i^2 = -1$$
$$= (z + 2)\left((z - 1)^2 - (\sqrt{3}i)^2\right)$$
$$= (z + 2)(z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i)$$
$$= (z + 2)(z - (1 - \sqrt{3}i))(z - (1 + \sqrt{3}i))$$

(ii) $z^3 + 27$

Solution:

$$z^3 + 27 = z^3 + 3^3$$

Formula: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$$= (z + 3)(z^2 - 3z + 9)$$
$$= (z + 3)\left(z^2 - 3z + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 9\right)$$
$$= (z + 3)\left(\left(z - \frac{3}{2}\right)^2 - \frac{9}{4} + 9\right)$$
$$= (z + 3)\left(\left(z - \frac{3}{2}\right)^2 + \frac{-9 + 36}{4}\right)$$
$$= (z + 3)\left(\left(z - \frac{3}{2}\right)^2 + \frac{27}{4}\right)$$
$$= (z + 3)\left(\left(z - \frac{3}{2}\right)^2 - \frac{27}{4}i^2\right) \qquad \because i^2 = -1$$
$$= (z + 3)\left(\left(z - \frac{3}{2}\right)^2 - \left(\frac{3\sqrt{3}}{2}i\right)^2\right)$$

$$= (z + 3)\left(z - \frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)\left(z - \frac{3}{2} - \frac{3\sqrt{3}}{2}i\right)$$
$$= (z + 3)\left(z - \frac{3 - 3\sqrt{3}i}{2}\right)\left(z - \frac{3 + 3\sqrt{3}i}{2}\right)$$

(iii) $z^3 - 2z^2 + 16z - 32$

Solution:

$$z^3 - 2z^2 + 16z - 32$$
$$= z^2(z - 2) + 16(z - 2)$$
$$= (z - 2)(z^2 + 16)$$
$$= (z - 2)(z^2 - 16i^2) \qquad \because i^2 = -1$$
$$= (z - 2)(z^2 - (4i)^2)$$
$$= (z - 2)(z - 4i)(z + 4i)$$

(iv) $z^4 + 21z^2 - 100$

Solution:

$$z^4 + 21z^2 - 100 = z^4 + 25z^2 - 4z^2 - 100$$
$$= z^2(z^2 + 25) - 4(z^2 + 25)$$
$$= (z^2 - 4)(z^2 + 25)$$
$$= (z^2 - 2^2)(z^2 - 25i^2) \qquad \because i^2 = -1$$
$$= (z - 2)(z + 2)(z^2 - (5i)^2)$$
$$= (z - 2)(z + 2)(z - 5i)(z + 5i)$$

(v) $z^4 - 16$

Solution:

$$z^4 - 16 = (z^2)^2 - 4^2$$
$$= (z^2 - 4)(z^2 + 4)$$
$$= (z^2 - 2^2)(z^2 - 4i^2) \qquad \because i^2 = -1$$
$$= (z - 2)(z + 2)(z^2 - (2i)^2)$$
$$= (z - 2)(z + 2)(z - 2i)(z + 2i)$$

(vi) $z^4 + 3z^2 - 4$

Solution:

$$z^4 + 3z^2 - 4$$
$$= z^4 + 4z^2 - z^2 - 4$$
$$= z^2(z^2 + 4) - 1(z^2 + 4)$$
$$= (z^2 - 1)(z^2 + 4)$$
$$= (z^2 - 1^2)(z^2 - 4i^2) \qquad \because i^2 = -1$$
$$= (z - 1)(z + 1)(z^2 - (2i)^2)$$
$$= (z - 1)(z + 1)(z + 2i)(z + 2i)$$

(vii) $z^4 + 5z^2 + 6$

Solution:

$$z^4 + 5z^2 + 6$$
$$= z^4 + 3z^2 + 2z^2 + 6$$
$$= z^2(z^2 + 3) + 2(z^2 + 3)$$
$$= (z^2 + 2)(z^2 + 3)$$
$$= (z^2 - 2i^2)(z^2 - 3i^2) \qquad \because i^2 = -1$$
$$= (z^2 - (\sqrt{2}i)^2)(z^2 - (\sqrt{3}i)^2)$$
$$= (z + \sqrt{2}i)(z - \sqrt{2}i)(z + \sqrt{3}i)(z - \sqrt{3}i)$$

(viii) $z^4 - 32z^2 - 3969$

Solution:

$$= z^4 - 32z^2 - 3969$$
$$= z^4 - 81z^2 + 49z^2 - 3969$$
$$= z^2(z^2 - 81) + 49(z^2 - 81)$$
$$= (z^2 - 81).(z^2 + 49)$$
$$= (z^2 - 9^2).(z^2 - (-49))$$
$$= (z - 9)(z + 9).(z^2 - 49i^2) \qquad \because i^2 = -1$$
$$= (z - 9).(z + 9).(z^2 - (7i)^2)$$
$$= (z - 9).(z + 9).(z - 7i)(z + 7i)$$

3. Find the roots of $z^4 + 7z^2 - 144 = 0$ and hence express it as a product of linear factors.

Solution:

$$z^4 + 7z^2 - 144 = 0$$
$$z^4 + 16z^2 - 9z^2 - 144 = 0$$
$$z^2(z^2 + 16) - 9(z^2 + 16) = 0$$
$$(z^2 - 9)(z^2 + 16) = 0$$
$$(z^2 - 3^2)(z^2 - 16i^2) = 0 \qquad \because i^2 = -1$$
$$(z - 3)(z + 3)(z^2 - (4i)^2) = 0$$
$$(z - 3)(z + 3)(z - 4i)(z + 4i) = 0$$
$$z - 3 = 0; z + 3 = 0; z - 4i = 0; z + 4i = 0$$
$$z = 3; z = -3; z = 4i; z = -4i$$

Roots: $3, -3, 4i, -4i$

Linear Factors: $(z - 3)(z + 3)(z - 4i)(z + 4i)$

4. Solve the following complex quadratic equation by completing square method:

(i) $2z^2 - 3z + 4 = 0$

Solution:

$$2z^2 - 3z + 4 = 0$$

Dividing both sides by 2

$$z^2 - \frac{3}{2}z + 2 = 0$$
$$z^2 - \frac{3}{2}z = -2$$

Add $\left(\frac{3}{4}\right)^2$ on both sides

$$z^2 - \frac{3}{2}z + \left(\frac{3}{4}\right)^2 = -2 + \left(\frac{3}{4}\right)^2$$
$$\left(z - \frac{3}{4}\right)^2 = -2 + \frac{9}{16}$$
$$\left(z - \frac{3}{4}\right)^2 = -\frac{23}{16}$$

$$\Rightarrow \quad z-\frac{3}{4} = \pm\sqrt{-\frac{23}{16}}$$

$$z = \frac{3}{4} \pm \frac{\sqrt{23}}{4}i$$

$$z = \frac{3 \pm \sqrt{23}i}{4}$$

(ii) $z^2 - 6z + 30 = 0$

Solution:

$$z^2 - 6z + 30 = 0$$

$$z^2 - 6z = -30$$

Add $(3)^2$ on both sides

$$z^2 - 6z + 3^2 = -30 + 3^2$$

$$(z-3)^2 = -21$$

$$\Rightarrow \quad z - 3 = \pm\sqrt{-21}$$

$$z = 3 \pm \sqrt{21}i$$

(iii) $3z^2 - 18z + 50 = 0$

Solution:

$$3z^2 - 18z + 50 = 0$$

Dividing both sides by 3

$$z^2 - 6z + \frac{50}{3} = 0$$

$$z^2 - 6z = -\frac{50}{3}$$

Add 3^2 on both sides

$$z^2 - 6z + 3^2 = -\frac{50}{3} + 3^2$$

$$(z-3)^2 = -\frac{50}{3} + 9$$

$$(z-3)^2 = -\frac{23}{3}$$

$$\Rightarrow \quad z - 3 = \pm\sqrt{-\frac{23}{3}}$$

$$z = 3 \pm \frac{\sqrt{23}}{\sqrt{3}}i$$

$$z = \frac{3\sqrt{3} \pm \sqrt{23}i}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}} \pm \frac{\sqrt{23}i}{\sqrt{3}}$$

$$z = 3 \pm \frac{\sqrt{23}i}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 3 \pm \frac{\sqrt{69}}{3}i$$

(iv) $z^2 + 4z + 13 = 0$

Solution:

$$z^2 + 4z + 13 = 0$$

$$z^2 + 4z = -13$$

Add $(2)^2$ on both sides

$$z^2 + 4z + 2^2 = -13 + 2^2$$

$$(z+2)^2 = -9$$

$$\Rightarrow \quad z + 2 = \pm\sqrt{-9}$$

$$z = -2 \pm 3i$$

(v) $2z^2 + 6z + 9 = 0$

Solution:

$$2z^2 + 6z + 9 = 0$$

Dividing both sides by 2

$$z^2 + 3z + \frac{9}{2} = 0$$

$$z^2 + 3z = \frac{-9}{2}$$

Add $\left(\frac{3}{2}\right)^2$ on both sides

$$z^2 + 3z + \left(\frac{3}{2}\right)^2 = -\frac{9}{2} + \left(\frac{3}{2}\right)^2$$

$$\left(z+\frac{3}{2}\right)^2 = -\frac{9}{2} + \frac{9}{4}$$

$$\left(z+\frac{3}{2}\right)^2 = \frac{-18+9}{4}$$

$$\Rightarrow \quad z + \frac{3}{2} = \pm\sqrt{\frac{-9}{4}}$$

$$z = -\frac{3}{2} \pm \frac{3}{2}i$$

$$z = \frac{-3 \pm 3i}{2}$$

$$z = \frac{-3}{2} \pm \frac{3}{2}i$$

(vi) $3z^2 - 5z + 7 = 0$

Solution:

$$3z^2 - 5z + 7 = 0$$

Dividing both sides by 3

$$z^2 - \frac{5}{3}z + \frac{7}{3} = 0$$

$$z^2 - \frac{5}{3}z = \frac{-7}{3}$$

Add $\left(\frac{5}{6}\right)^2$ on both sides

$$z^2 - \frac{5}{3}z + \left(\frac{5}{6}\right)^2 = -\frac{7}{3} + \left(\frac{5}{6}\right)^2$$

$$\left(z-\frac{5}{6}\right)^2 = -\frac{7}{3} + \frac{25}{36}$$

$$\left(z-\frac{5}{6}\right)^2 = \frac{-84+25}{36}$$

$$\Rightarrow \quad z - \frac{5}{6} = \pm\sqrt{\frac{-59}{36}}$$

$$z = \frac{5}{6} \pm \frac{\sqrt{59}}{6}i$$

$$z = \frac{5 \pm \sqrt{59}i}{6}$$

5. Solve the following equations:

(i) $2z^4 - 32 = 0$

Solution:

$$2z^4 - 32 = 0$$

Dividing both sides by 2

$$z^4 - 16 = 0$$

$$(z^2)^2 - (4)^2 = 0$$

$$(z^2 - 4)(z^2 + 4) = 0$$

$$(z^2 - 2^2)(z^2 - 4i^2) = 0 \qquad \because i^2 = -1$$

$$(z-2)(z+2)(z^2 - (2i)^2) = 0$$

$$(z-2)(z+2)(z-2i)(z+2i) = 0$$

$$\Rightarrow \quad z-2=0; z+2=0; z-2i=0; z+2i=0$$

$$\Rightarrow \quad z = 2, -2, 2i, -2i$$

(ii) $3z^5 - 243z = 0$

Solution:

$$3z^5 - 243z = 0$$

Dividing both sides by 3

$$z^5 - 81z = 0$$

$$z(z^4 - 81) = 0$$

$$z(z^2)^2 - 9^2) = 0$$

$$z(z^2 - 9)(z^2 + 9) = 0$$

$$z(z^2 - 3^2)(z^2 - 9i^2) = 0 \qquad \because i^2 = -1$$

$$z(z-3)(z+3)(z^2 - (3i)^2) = 0$$

$$z(z-3)(z+3)(z-3i)(z+3i) = 0$$

$$\Rightarrow \quad z=0; z-3=0; z+3=0; z-3i=0; z+3i=0$$

$$\Rightarrow \quad z = 0, 3, -3, 3i, -3i$$

(iii) $5z^5 - 5z = 0$

Solution:

$$5z^5 - 5z = 0$$

Dividing both sides by 3

$$z^5 - z = 0$$

$$z(z^4 - 1) = 0$$

$$z((z^2)^2 - 1^2) = 0$$

$$z(z^2 - 1)(z^2 + 1) = 0$$

$$z(z^2 - 1^2)(z^2 - i^2) = 0 \qquad \because i^2 = -1$$

$$z(z-1)(z+1)(z-i)(z+i) = 0$$

$$\Rightarrow \quad z=0; z-1=0; z+1=0; z-i=0; z+i=0$$

$$\Rightarrow \quad z = 0, 1, -1, i, -i$$

(iv) $z^3 - 5z^2 + z - 5 = 0$

Solution:

$$z^3 - 5z^2 + z - 5 = 0$$

$$z^2(z-5) + 1(z-5) = 0$$

$$(z-5)(z^2 + 1) = 0$$

$$(z-5)(z^2 - i^2) = 0 \qquad \because i^2 = -1$$

$$(z-5)(z-i)(z+i) = 0$$

$$\Rightarrow \quad z-5=0; z-i=0; z+i=0$$

$$\Rightarrow \quad z = 5, i, -i$$

(v) $4z^4 - 25z^2 - 21 = 0$

Solution:

$$4z^4 - 25z^2 - 21 = 0$$

$$4(z^2)^2 - 25(z^2)^1 - 21 = 0$$

Use Quadratic formula

$$z^2 = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(4)(-21)}}{2(4)}$$

$$z^2 = \frac{25 \pm \sqrt{625+336}}{8}$$

$$z^2 = \frac{25 \pm \sqrt{961}}{8}$$

$$z^2 = \frac{25 \pm 31}{8}$$

$$z^2 = \frac{25+31}{8} \quad , \quad z^2 = \frac{25-31}{8}$$

$$z^2 = \frac{56}{8} \quad , \quad z^2 = \frac{-6}{8}$$

$$z^2 = 7 \quad , \quad z^2 = \frac{-3}{4}$$

$$z = \pm\sqrt{7} \quad , \quad z = \pm\frac{\sqrt{-3}}{2} = \pm\frac{\sqrt{3}i}{2}$$

$$\Rightarrow \quad z = \sqrt{7}, -\sqrt{7}, \frac{\sqrt{3}i}{2}, \frac{-\sqrt{3}i}{2}$$

(vi) $z^3 + z^2 + z + 1 = 0$

Solution:

$$z^3 + z^2 + z + 1 = 0$$

$$z^2(z+1) + 1(z+1) = 0$$

$$(z+1)(z^2 + 1) = 0$$

$$(z+1)(z^2-i^2)=0 \qquad \because i^2=-1$$
$$(z+1)(z-i)(z+i)=0$$
$\Rightarrow \quad z+1=0;\ z-i=0;\ z+i=0$

$\Rightarrow \quad z=-1, i, -i$

6. Find a polynomial $P(z)$ of degree 3 with zeros $3, -2i, 2i$ and satisfying $P(1)=20$.

Solution:

Let required polynomial $= P(z)$

Since $P(z)$ is a polynomial of degree 3 with zeros $3, -2i, 2i$, therefore

$z=3, -2i, 2i$ (zeros)

$\Rightarrow \quad z-3, z+2i, z-2i$ (factors)

By Fundamental Theorem of Algebra

$$\begin{aligned} P(z) &= a(z-3)(z+2i)(z-2i) \\ &= a(z-3)(z^2-(2i)^2) \\ &= a(z-3)(z^2-4i^2) \\ &= a(z-3)(z^2+4) \qquad \because i^2=-1 \\ &= a(z^3+4z-3z^2-12) \\ P(z) &= a(z^3-3z^2+4z-12) \qquad \ldots(1) \end{aligned}$$

Given that: $P(1)=20$

$$\Rightarrow \quad a(1^3-3(1)^2+4(1)-12)=20$$
$$a(1-3+4-12)=20$$
$$a=\frac{20}{-10} \Rightarrow a=-2$$

Put $a=-2$ in eq. (1), we have

$$P(z)=-2(z^3-3z^2+4z-12)$$
$$P(z)=-2z^3+6z^2-8z+24$$

which is a required polynomial.

7. Find a polynomial $P(z)$ of degree 4 with zeros $2i, -2i, 1, -1$, and satisfying $P(2)=240$.

Solution:

Let required polynomial $= P(z)$

Since $P(z)$ is a polynomial of degree 4 with zeros $2i, -2i, 1, -1$, therefore

$z=2i, -2i, 1, -1$ (zeros)

$\Rightarrow \quad z-2i, z+2i, z-1, z+1$ (factors)

By Fundamental Theorem of Algebra

$$\begin{aligned} P(z) &= a(z-2i)(z+2i)(z-1)(z+1) \\ &= a(z^2-(2i)^2)(z^2-1^2) \\ &= a(z^2-4i^2)(z^2-1) \\ &= a(z^2+4)(z^2-1) \qquad \because i^2=-1 \\ &= a(z^4-z^2+4z^2-4) \\ P(z) &= a(z^4+3z^2-4) \qquad \ldots(1) \end{aligned}$$

Given that: $P(2)=240$

$$\Rightarrow \quad a(2^4+3(2)^2-4)=240$$
$$a(16+12-4)=240$$
$$a=\frac{240}{24} \Rightarrow a=10$$

Put $a=10$ in eq. (1), we have

$$P(z)=10(z^4+3z^2-4)$$
$$P(z)=10z^4+30z^2-40$$

which is required polynomial.

8. Find a polynomial $P(z)$ of degree 4 with zeros $4, -4, 1+i, 1-i$ and satisfying $P(2)=72$.

Solution:

Let required polynomial $= P(z)$

Since $P(z)$ is a polynomial of degree 4 with zeros $4, -4, 1+i, 1-i$, therefore

$z=4, -4, 1+i, 1-i$ (zeros)

$\Rightarrow \quad z-4, z+4, z-1-i, z-1+i$ (factors)

By Fundamental Theorem of Algebra

$$\begin{aligned} P(z) &= a(z-4)(z+4)((z-1)-i)((z-1)+i) \\ &= a(z^2-4^2)((z-1)^2-i^2) \\ &= a(z^2-16)(z^2+1-2z+1) \qquad \because i^2=-1 \\ &= a(z^2-16)(z^2-2z+2) \\ &= a(z^4-2z^3+2z^2-16z^2+32z-32) \\ P(z) &= a(z^4-2z^3-14z^2+32z-32) \qquad \ldots(1) \end{aligned}$$

Given that: $P(2)=72$

$$\Rightarrow \quad a(2^4-2(2)^3-14(2)^2+32(2)-32)=72$$
$$a(16-16-56+64-32)=72$$
$$\Rightarrow \quad a=\frac{72}{-24} \Rightarrow a=-3$$

Put $a=-3$ in eq. (1), we have

$$P(z)=-3(z^4-2z^3-14z^2+32z-32)$$
$$P(z)=-3z^4+6z^3+42z^2-96z+96$$

which is a required polynomial.

Three Cube Roots of Unity

Let x be a cube root of unity, then $x=(1)^{1/3} \Rightarrow x^3=1$

$\Rightarrow \quad x^3-1^3=0$

$\Rightarrow \quad (x-1)(x^2+x+1)=0 \qquad \because a^3-b^3=(a-b)(a^2+ab+b^2)$

Either $\quad x-1=0 \Rightarrow x=1$

or $\quad x^2+x+1=0$ (Here $a=1, b=1, c=1$)

By using quadratic formula, we have

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}i}{2} \qquad (\because \sqrt{-1}=i)$$

Note:
We know that the numbers containing i are called **imaginary numbers**. So $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$ are called imaginary cube roots of unity.

Thus, the three cube roots of unity are:

$$1, \frac{-1+\sqrt{3}i}{2} \text{ and } \frac{-1-\sqrt{3}i}{2}$$

Properties of Cube Roots of Unity

(i) Each complex cube root of unity is square of the other

If $\frac{-1+\sqrt{3}i}{2}=\omega$, then $\frac{-1-\sqrt{3}i}{2}=\omega^2$, and if $\frac{-1+\sqrt{3}i}{2}=\omega$, then $\frac{-1+\sqrt{3}i}{2}=\omega^2$ [ω is read as omega]

(ii) The sum of all the three cube roots of unity is zero i.e., $1+\omega+\omega^2=0$

(iii) The product of all the three cube roots of unity is unity i.e., $1.\omega.\omega^2=\omega^3=1$, as a consequence of which, each imaginary cube root of unity is the reciprocal of the other, that is, $\omega=\frac{1}{\omega^2}=\omega^{-2}, \omega^2=\frac{1}{\omega}=\omega^{-1}$.

Four Fourth Roots of Unity

Let x be the fourth root of unity, then $x=(1)^{1/4} \qquad \Rightarrow x^4=1$

$\Rightarrow \quad (x^2)^2-1^2=0$

$\Rightarrow \quad (x^2-1)(x^2+1)=0$

Either $\quad x^2-1=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

and $\quad x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x=\pm i$

Hence four fourth roots of unity are: $1, -1, i, -i$.

Properties of four Fourth Roots of Unity

We have found that the four fourth roots of unity are: $1, -1, +i, -i$

(i) Sum of all the four fourth roots of unity is zero

$\because \quad 1+(-1)+i+(-i)=0$

(ii) The real fourth roots of unity are additive inverses of each other.

1 and −1 are the real fourth roots of unity and $1+(-1)=0=(-1)+1$

(iii) Both the imaginary fourth roots of unity are conjugate of each other.

i and $-i$ are imaginary fourth roots of unity, which are obviously conjugates of each other.

(iv) Product of all the fourth roots of unity is −1 i.e., $1\times(-1)\times i\times(-i)=-1$

Example 11: Prove that: $(x^3+y^3)=(x+y)(x+\omega y)(x+\omega^2 y)$

Solution:

$$\begin{aligned} \text{R.H.S} &= (x+y)(x+\omega y)(x+\omega^2 y) \\ &= (x+y)(x+\omega^2 xy+\omega xy+\omega^3 y^2) \\ &= (x+y)[x+(\omega+\omega^2)xy+\omega^3 y^2] \\ &= (x+y)(x^2-xy+y^2)=x^3+y^3 \qquad \{\because \omega^3=1, \omega+\omega^2=-1\} \\ &= \text{L.H.S (Proved)} \end{aligned}$$

Exercise 1.4

1. Find the three cube roots of:

(i) 8

Solution:

Let x be the cube root of 8, then

$$x=(8)^{\frac{1}{3}}$$

$x^3=8 \quad \Rightarrow x^3-8=0 \Rightarrow (x)^3-(2)^3=0$

$\Rightarrow (x-2)(x^2+2x+4)=0$

Either $\quad x-2=0 \quad$ or $\quad x^2+2x+4=0 \quad \Rightarrow x=2$

or $\quad x^2+2x+4=0$

Here $\quad a=1, b=2, c=4$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-2\pm\sqrt{(2)^2-4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm \sqrt{4 \times 3}i}{2} = \frac{-2 \pm 2\sqrt{3}\,i}{2}$$

$$= 2\left(\frac{-1 \pm \sqrt{3}\,i}{2}\right)$$

$$x = 2\left(\frac{-1+\sqrt{3}\,i}{2}\right) \quad ; \quad x = 2\left(\frac{-1-\sqrt{3}\,i}{2}\right)$$

$$x = 2\omega \quad ; \quad x = 2\omega^2$$

Hence three cube roots of 8 are $2, 2\omega, 2\omega^2$

(ii) -8

Solution:

Let x be the cube root of -8, then

$$x = (-8)^{\frac{1}{3}}$$

$$x^3 = -8 \quad \Rightarrow x^3 + 8 = 0 \ \Rightarrow (x)^3 + (2)^3 = 0$$

$$\Rightarrow (x+2)(x^2 - 2x + 4) = 0$$

Either $x + 2 = 0$ or $x^2 - 2x + 4 = 0$

$\Rightarrow$ $x = -2$

or $x^2 - 2x + 4 = 0$

Here $a = 1$, $b = -2$, $c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} = \frac{2 \pm \sqrt{4-16}}{2} \quad \because i^2 = -1$$

$$x = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm \sqrt{4 \times 3}i}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2} = 2\left(\frac{1 \pm \sqrt{3}i}{2}\right)$$

$$x = 2\left(\frac{1+\sqrt{3}\,i}{2}\right) \quad ; \quad x = 2\left(\frac{1-\sqrt{3}\,i}{2}\right)$$

$$x = -2\left(\frac{-1-\sqrt{3}\,i}{2}\right) \quad ; \quad x = -2\left(\frac{-1+\sqrt{3}\,i}{2}\right)$$

$$x = -2\omega^2 \quad ; \quad x = -2\omega$$

Hence three cube roots of -8 are $-2, -2\omega, -2\omega^2$

(iii) -27

Solution:

Let x be the cube root of -27, then

$$x = (-27)^{\frac{1}{3}}$$

$$x^3 = -27 \Rightarrow x^3 + 27 = 0 \qquad \Rightarrow (x)^3 + (3)^3 = 0$$

$$\Rightarrow (x+3)(x^2 - 3x + 9) = 0$$

Either $x + 3 = 0$ or $x^2 - 3x + 9 = 0$

$\Rightarrow$ $x = -3$

or $x^2 - 3x + 9 = 0$

Here $a = 1$, $b = -3$, $c = 9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9-36}}{2} = \frac{3 \pm \sqrt{-27}}{2}$$

$$= \frac{3 \pm \sqrt{9 \times 3}i}{2} = \frac{3 \pm 3\sqrt{3}i}{2} = 3\left(\frac{1 \pm \sqrt{3}i}{2}\right)$$

$$x = 3\left(\frac{1+\sqrt{3}i}{2}\right) \quad ; \quad x = 3\left(\frac{1-\sqrt{3}i}{2}\right) \quad \because i^2 = -1$$

$$x = -3\left(\frac{-1-\sqrt{3}i}{2}\right) \quad ; \quad x = -3\left(\frac{-1+\sqrt{3}i}{2}\right)$$

$$x = -3\omega^2 \quad ; \quad x = -3\omega$$

Hence three cube roots of -27 are $-3, -3\omega, -3\omega^2$

(iv) 64

Solution:

Let x be the cube root of 64, then

$$x = (64)^{1/3}$$

$$x^3 = 64 \qquad \Rightarrow x^3 = 4^3 \qquad \Rightarrow x^3 - 4^3 = 0$$

$$\Rightarrow (x-4)(x^2 + 4x + 16) = 0$$

Either $x - 4 = 0$ or $x^2 + 4x + 16 = 0$

$\Rightarrow$ $x = 4$

or $x^2 + 4x + 16 = 0$

Here $a = 1, b = 4, c = 16$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-64}}{2} = \frac{-4 \pm \sqrt{-48}}{2}$$

$$x = \frac{-4 \pm \sqrt{16 \times 3}\,i}{2} = \frac{-4 \pm 4\sqrt{3}i}{2} = \frac{4(-1 \pm \sqrt{3}i)}{2}$$

$$x = 4\frac{(-1+\sqrt{3}i)}{2} \quad ; \quad x = 4\frac{(-1-\sqrt{3}i)}{2}$$

$$x = 4\omega \quad ; \quad x = 4\omega^2$$

Hence three cube roots of 64 are $4, 4\omega, 4\omega^2$

(v) -125

Solution:

Three cube roots of -125 :

Let x be the cube root -125 , then

$$x = (-125)^{\frac{1}{3}}$$

$$x^3 = -125$$

$$\Rightarrow \quad x^3 + 125 = 0$$

$$\Rightarrow \quad x^3 + 5^3 = 0$$

$$(x+5)(x^2 - 5x + 25) = 0$$

Either $x + 5 = 0$ or $x^2 - 5x + 25 = 0$

$\Rightarrow$ $x = -5$

or $x^2 - 5x + 25 = 0$

Here $a = 1, b = -5, c = 25$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(25)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25-100}}{2} = \frac{5 \pm \sqrt{-75}}{2}$$

$$x = \frac{5 \pm \sqrt{25 \times 3}i}{2} = \frac{5 \pm 5\sqrt{3}i}{2} = 5\left(\frac{1 \pm \sqrt{3}i}{2}\right)$$

$$x = 5\left(\frac{1+\sqrt{3}i}{2}\right) \quad ; \quad x = 5\left(\frac{1-\sqrt{3}i}{2}\right)$$

$$x = -5\left(\frac{-1-\sqrt{3}i}{2}\right) \quad ; \quad x = -5\left(\frac{-1+\sqrt{3}i}{2}\right)$$

$$x = -5\omega^2 \quad \text{and} \quad x = -5\omega$$

Hence three cube roots of -125 are $-5, -5\omega, -5\omega^2$

2. Find the four fourth roots of 16, 81, 625. Also show that their sum is zero in each case.

Solution:

Four fourth root of 16:

Let x be the fourth root of 16, then

$$x = (16)^{1/4}$$

$$x^4 = 16 \Rightarrow (x^2)^2 - 4^2 = 0 \Rightarrow (x^2 - 4)(x^2 + 4) = 0$$

Either $x^2 - 4 = 0$ or $x^2 + 4 = 0$

$$x^2 = 4 \quad ; \quad x^2 = -4$$

$$\sqrt{x^2} = \sqrt{4} \quad ; \quad \sqrt{x^2} = \sqrt{-4}$$

$$x = \pm 2 \quad ; \quad x = \pm 2i \quad \because \sqrt{-1} = i$$

Hence four fourth roots of 16 are $\pm 2, \pm 2i$

Four fourth root of 81:

Let x be the fourth root of 81, then

$$x = (81)^{1/4}$$

$$x^4 = 81$$

$$(x^2)^2 - 9^2 = 0$$

$$(x^2 - 9)(x^2 + 9) = 0$$

Either $x^2 - 9 = 0$ or $x^2 + 9 = 0$

$$x^2 = 9 \quad ; \quad x^2 = -9$$

$$\sqrt{x^2} = \sqrt{9} \quad ; \quad \sqrt{x^2} = \sqrt{-9}$$

$$x = \pm 3 \quad ; \quad x = \pm 3i \quad \because \sqrt{-1} = i$$

Hence four fourth root of 81 are $\pm 3, \pm 3i$

Four fourth root of 625:

Let x be the fourth root of 625, then

$$x = (625)^{1/4}$$

$$x^4 = 625$$

$$(x^2)^2 - 25^2 = 0$$

$$(x^2 - 25)(x^2 + 25) = 0$$

Either $x^2 - 25 = 0$ or $x^2 + 25 = 0$

$$x^2 = 25 \quad ; \quad x^2 = -25$$

$$\sqrt{x^2} = \sqrt{25} \quad ; \quad \sqrt{x^2} = \sqrt{-25}$$

$$x = \pm 5 \quad ; \quad x = \pm 5i \quad \because \sqrt{-1} = i$$

Hence four fourth roots of 625 are $\pm 5, \pm 5i$.

3. If 1, ω, ω² are the cube roots of unity, show that $1 + \omega^n + \omega^{2n} = 3$ where n is a multiple of 3 respectively.

Solution:

To prove: $1 + \omega^n + \omega^{2n} = 3$

Since n is a multiple of 3 , Therefore

$$n = 3k \ \ldots (1) \text{ where } k \in Z.$$

$$\begin{aligned}
\text{L.H.S} &= 1 + \omega^n + \omega^{2n} \\
&= 1 + \omega^{3k} + \omega^{2(3k)} && \text{using (1)} \\
&= 1 + (\omega^3)^k + (\omega^3)^{2k} \\
&= 1 + (1)^k + (1)^{2k} && \because \omega^3 = 1 \\
&= 1 + 1 + 1 \\
&= 3 = \text{R.H.S (Proved)}
\end{aligned}$$

4. Evaluate:

(i) $\left(\frac{1+\sqrt{-3}}{2}\right)^7 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7$

Solution:

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^7 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7$$

$$\begin{aligned}
&= \left(\frac{-1+\sqrt{3}i}{2}\right)^7 + \left(\frac{-1-\sqrt{3}i}{2}\right)^7 && \because \sqrt{-1} = i \\
&= (\omega)^7 + (\omega^2)^7 = \omega^7 + \omega^{14} \\
&= \omega^6 . \omega^1 + \omega^{12} . \omega^2 = (\omega^3)^2 . \omega + (\omega^3)^4 . \omega^2 \\
&= (1)^2 . \omega + (1)^4 . \omega^2 && \because \omega^3 = 1 \\
&= \omega + \omega^2 = -1 && \omega + \omega^2 = -1
\end{aligned}$$

(ii) $(1 + \sqrt{-3})^5 + (-1 - \sqrt{-3})^5$

Solution:

$$\begin{aligned}
&(-1+\sqrt{-3})^5 + (-1-\sqrt{-3})^5 \\
&= (-1+\sqrt{3}i)^5 + (-1-\sqrt{3}i)^5 && \because \sqrt{-1} = i \\
&= (2\omega)^5 + (2\omega^2)^5 && \because 2\omega = -1+\sqrt{3}i \text{ and } 2\omega^2 = -1-\sqrt{3}i \\
&= 32\omega^5 + 32\omega^{10} = 32(\omega^3 . \omega^2 + \omega^9 . \omega^1) \\
&= 32\{(\omega^3)^1 . \omega^2 + (\omega^3)^3 . \omega\} \\
&= 32\{(1)^1 . \omega^2 + (1)^3 . \omega\} && \because \omega^3 = 1 \\
&= 32(\omega^2 + \omega) = 32(-1) && \because \omega^2 + \omega = -1 \\
&= -32
\end{aligned}$$

5. Show that $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots$ to $2n$ factors $=2^{2n}$

Solution:

$$
\begin{aligned}
\text{L.H.S} &= (1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots 2n \text{ factors}\\
&= (1-\omega+\omega^2)(1-\omega^2+\omega.\omega^3)(1-\omega.\omega^3+\omega^2.(\omega^3)^2)(1-\omega^2.(\omega^3)^2+\omega.(\omega^3)^5)\ldots 2n \text{ factors}\\
&= (1-\omega+\omega^2)(1-\omega^2+\omega(1))(1-\omega(1)+\omega^2(1)^2)(1-\omega^2(1)^2+\omega(1)^5)\ldots 2n \text{ factors} \qquad \because \omega^3=1\\
&= (1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)(1+\omega^2+\omega)\ldots 2n \text{ factors}\\
&= ((1+\omega^2)-\omega)((1+\omega)-\omega^2)((1+\omega^2)-\omega)((1+\omega)-\omega^2)\ldots 2n \text{ factors}
\end{aligned}
$$

Put $1+\omega=-\omega^2$ and $1+\omega^2=-\omega$

$$
\begin{aligned}
&= (-\omega-\omega)(-\omega^2-\omega^2)(-\omega-\omega)(-\omega^2-\omega^2)\ldots 2n \text{ factors}\\
&= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)\ldots 2n \text{ factors}\\
&= (4\omega^3)(4\omega^3)\ldots n \text{ factors}\\
&= (4\omega^3)^n\\
&= (4.1)^n \qquad \because \omega^3=1\\
&= (2^2)^n\\
&= 2^{2n} = \text{R.H.S (Proved)}
\end{aligned}
$$

6. Prove that $\left(\frac{i+\sqrt{3}}{2}\right)^8+\left(\frac{i-\sqrt{3}}{2}\right)^8=-1.$

Solution:

$$
\begin{aligned}
\text{L.H.S} &= \left(\frac{i+\sqrt{3}}{2}\right)^8+\left(\frac{i-\sqrt{3}}{2}\right)^8\\
&= i^8\left(\frac{1+\frac{\sqrt{3}}{i}}{2}\right)^8+i^8\left(\frac{1-\frac{\sqrt{3}}{i}}{2}\right)^8
\end{aligned}
$$

As $i^8=(i^4)^2=(1)^2=1$ and $\frac{1}{i}=\frac{1}{i}\times\frac{i}{i}=-i$, so

$$
\begin{aligned}
\text{L.H.S} &= 1\cdot\left(\frac{1-\sqrt{3}i}{2}\right)^8+\left(\frac{1+\sqrt{3}i}{2}\right)^8\\
&= (-1)^8\left(\frac{-1+\sqrt{3}i}{2}\right)^8+(-1)^8\left(\frac{-1-\sqrt{3}i}{2}\right)^8\\
&= 1\cdot\omega^8+1\cdot(\omega^2)^8\\
&= \omega^8+\omega^{16}\\
&= \omega^2.\omega^6+\omega.\omega^{15}\\
&= \omega^2.(\omega^3)^2+\omega.(\omega^3)^5\\
&= \omega^2.(1)^2+\omega(1)^5 \qquad \because \omega^3=1\\
&= \omega^2+\omega\\
&= -1 = \text{R.H.S (Proved)}
\end{aligned}
$$

7. Evaluate $\sum_{k=0}^{5}\omega^{2k}$, where ω is an imaginary cube root of unity.

Solution:

$$
\begin{aligned}
\sum_{k=0}^{5}\omega^{2k} &= \omega^{2(0)}+\omega^{2(1)}+\omega^{2(2)}+\omega^{2(3)}+\omega^{2(4)}+\omega^{2(5)}\\
&= \omega^0+\omega^2+\omega^4+\omega^6+\omega^8+\omega^{10}\\
&= 1+\omega^2+\omega.\omega^3+(\omega^3)^2+\omega^2.(\omega^3)^2+\omega.(\omega^3)^3\\
&= 1+\omega^2+\omega.(1)+(1)^2+\omega^2(1)^2+\omega(1)^3 \because \omega^3=1\\
&= 1+\omega^2+\omega+1+\omega^2+\omega\\
&= 2+2\omega+2\omega^2\\
&= 2(1+\omega+\omega^2)\\
&= 2(0)\\
\sum_{k=0}^{5}\omega^{2k} &= 0
\end{aligned}
$$

8. If ω is an imaginary cube roots of unity, prove that $\frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}=\omega$

Solution:

Since ω is an imaginary cube roots of unity. Therefore

$$\omega^3=1 \quad \Rightarrow \omega.\omega^2=1$$

$$\Rightarrow \quad \omega=\frac{1}{\omega^2} \text{ and } \omega^2=\frac{1}{\omega} \qquad \ldots(1)$$

$$\text{L.H.S} = \frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}$$

Multiply and divide the numerator by ω.

$$
\begin{aligned}
\text{L.H.S} &= \frac{\frac{\omega}{\omega}(a+b\omega^2+c\omega)}{a\omega^2+b\omega+c}\\
&= \frac{\omega\left(a\cdot\frac{1}{\omega}+b\omega+c\right)}{a\omega^2+b\omega+c}\\
&= \frac{\omega(a\omega^2+b\omega+c)}{a\omega^2+b\omega+c} \qquad \because \frac{1}{\omega}=\omega^2 \text{ from (1)}\\
&= \omega\\
&= \text{R.H.S (Proved)}
\end{aligned}
$$

9. If ω is a cube root of unity, prove that $\frac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}}=\omega$

Solution:

Since ω is a cube root of unity. Therefore $\omega^3=1$

$$\Rightarrow \quad \omega.\omega^2=1$$

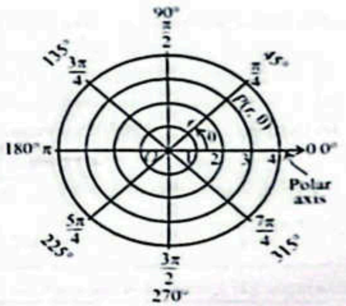
$$\Rightarrow \quad \omega=\frac{1}{\omega^2} \text{ and } \omega^2=\frac{1}{\omega}$$

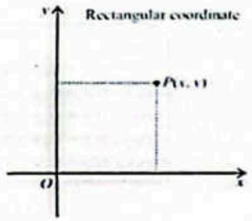
$$
\begin{aligned}
\text{L.H.S} &= \frac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}}\\
&= \frac{\omega^{12}(a+b\omega^5+c\omega^7)}{\omega^{14}(a+b\omega^8+c\omega^{16})}\\
&= \frac{1(a+b\omega^2\cdot\omega^3+c\omega\cdot(\omega^3)^2)}{\omega^2(a+b\omega^2\cdot(\omega^3)^2+c\cdot\omega(\omega^3)^5)}\\
&= \frac{a+b\omega^2(1)+c\omega(1)^2}{\omega^2(a+b\omega^2(1)^2+c\omega(1)^5)} \qquad \because \omega^3=1\\
&= \frac{a+b\omega^2+c\omega}{\omega^2(a+b\omega^2+c\omega)}\\
&= \frac{1}{\omega^2}\\
&= \omega = \text{R.H.S (Proved)} \qquad \because \frac{1}{\omega^2}=\omega \text{ from (1)}
\end{aligned}
$$

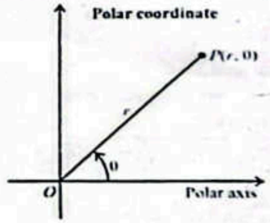
Polar Coordinates System

Polar coordinates are often more convenient than Cartesian coordinates in situations involving circular or rotational symmetry, or when a problem depends on distance from a fixed point and angle relative to a reference direction. Just as the Cartesian coordinate system uses an ordered pair (x, y) to describe the position of a point, the polar coordinate system determines the position of a point using a directed distance r from a fixed origin O (called the pole) and an angle θ that the line connecting the origin to the point makes with the polar axis (typically aligned with the positive x-axis).

In polar coordinate system the location of a point P can be described by polar coordinates in the form (r, θ), where r and θ are real numbers.



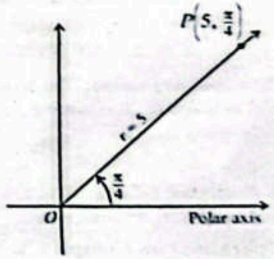




While r is typically considered non-negative ($r \geq 0$), it is also possible for r to be negative ($r < 0$). The value of r changes depending on its sign, and this affects the position of the point in the plane.

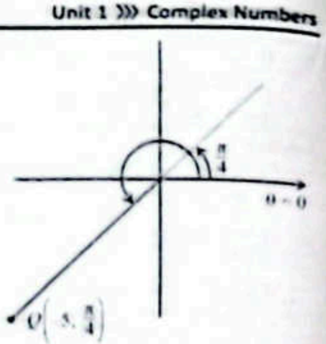
Case - 1:

When $r > 0$, the angle θ is the measure of any angle in standard position whose terminal side lies along the line connecting the origin to the point P, measured from the polar axis (positive x-axis). For example, the polar coordinates $\left(5, \frac{\pi}{4}\right)$ represent a point 5 units away from pole at an angle of $\frac{\pi}{4}$ radians.



Case - 2:

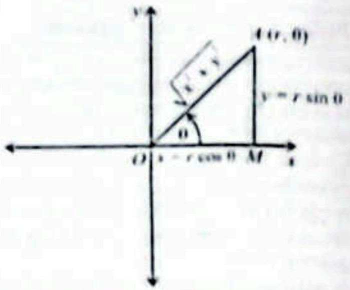
When $r < 0$, the angle θ is the measure of any angle in standard position whose terminal side lies along the line connecting the origin to the point Q, but the point Q is located $|r|$ units in the opposite direction (i.e., $\theta + \pi$) from the polar axis (positive x axis). For example, the polar coordinates $\left(-5, \frac{\pi}{4}\right)$ represent a point 5 units away from the pole, but in the direction of $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$ radians.



Note:
$(-5, \pi/4)$ and $\left(5, \frac{5\pi}{4}\right)$ represent the same point in the plane.

The Polar Form of a Complex Number

Consider the adjoining diagram representing the complex number $z = x + iy$. From the diagram, we see that $x = r\cos\theta$ and $y = r\sin\theta$, where $r = |z|$ is modulus and θ is called an argument of z.



Definition: Polar form of a complex number $z = x + iy$ is given by

$$z = x + iy = r\cos\theta + ir\sin\theta$$

where r = modulus of $z = |z| = \sqrt{x^2 + y^2}$ and θ = argument of $z = \tan^{-1}\frac{y}{x}$

Example 12: Express the complex number $1 + i\sqrt{3}$ in polar form.

Solution:

Let $z = 1 + i\sqrt{3}$

Here $x = 1$ and $y = \sqrt{3}$

Polar form of complex number z is:

$z = r\cos\theta + ir\sin\theta \quad \ldots(i)$

Where, $r = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

and $\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\sqrt{3}}{1} = 60^\circ$

Putting values in equation(i), we have

$z = r\cos\theta + ir\sin\theta$

$z = 2\cos 60^\circ + i2\sin 60^\circ$

Note:
(i) If $x = 0, y > 0$ then $\theta = 90^\circ$.
(ii) If $x = 0, y < 0$ then $\theta = -90^\circ$.
(iii) If $x = 0, y = 0$ then θ is undefined.
(iv) If $y = 0, x > 0$ then $\theta = 0^\circ$.
(v) If $y = 0, x < 0$ then $\theta = 180^\circ$.

Principal Argument: The principal argument θ of a complex number $z = a + bi$ is the angle between the positive real axis and the line joining (a, b) to the origin in the Argand plane.

$$\text{Arg } z = \theta = \tan^{-1}\left(\frac{b}{a}\right) \quad (a \neq 0)$$

It is denoted by Arg. It is a single, specific value of the argument, typically chosen within a standard range: $\arg z \in (-\pi, \pi]$ or $-\pi < \theta \leq \pi$.

Operations on Complex Numbers in Polar Form

Addition and Subtraction of Complex number in Polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form.

The addition and subtraction of these complex numbers can be computed simply as

Addition: $z_1 + z_2 = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$

Subtraction: $z_1 - z_2 = r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplication of Complex number in Polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form.

The product of these complex numbers can be derived by multiplying them directly and simplifying

Multiplication: $z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\right)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \qquad \because i^2 = -1$$

$$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{(Using trigonometric identities)}$$

Thus, multiplying two complex numbers in polar form involves multiplying their moduli and summing their arguments i.e., $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

Example 13: Find the product of $5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

Solution:

Let $z_1 = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

Here $r_1 = 5$ and $\theta_1 = \frac{\pi}{6}$, while $r_2 = 4$ and $\theta_2 = \frac{3\pi}{2}$

By using the product formula of complex numbers

$$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

$$= 5 \times 4\left[\cos\left(\frac{\pi}{6} + \frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{6} + \frac{3\pi}{2}\right)\right]$$

$$= 20\left[\cos\left(\frac{\pi + 9\pi}{6}\right) + i\sin\left(\frac{\pi + 9\pi}{6}\right)\right] = 20\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Thus, the required product is $20\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

Division of Complex Number in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form.

The formula for division of these numbers in polar form can be derived as below

Division: $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \quad \text{Multiply up and down by conjugate of } \cos\theta_2 + i\sin\theta_2$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad \text{(Using trigonometric identities)}$$

Thus, the modulus of the division of two complex numbers equals the quotient of their moduli, while the arguments of the quotient is the difference between their arguments.

Thus, when dividing two complex numbers, the modulus of the result is the ratio of their moduli, and the argument of the result is the difference between their arguments i.e., $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Example 14: Divide $\frac{2}{7}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$ by $\frac{3}{5}\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

Solution:

Let $z_1=\frac{2}{7}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$ and $z_2=\frac{3}{5}\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

Here, $r_1=\frac{2}{7}$ and $\theta_1=\frac{7\pi}{6}$, while $r_2=\frac{3}{5}$ and $\theta_2=-\frac{\pi}{2}$.

By using the quotient formula of complex numbers

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

$$=\frac{2}{7}\times\frac{5}{3}\left[\cos\left(\frac{7\pi}{6}-\left(-\frac{\pi}{2}\right)\right)+i\sin\left(\frac{7\pi}{6}-\left(-\frac{\pi}{2}\right)\right)\right]=\frac{10}{21}\left[\cos\left(\frac{7\pi}{6}+\frac{\pi}{2}\right)+i\sin\left(\frac{7\pi}{6}+\frac{\pi}{2}\right)\right]$$

$$=\frac{10}{21}\left[\cos\left(\frac{7\pi+3\pi}{6}\right)+i\sin\left(\frac{7\pi+3\pi}{6}\right)\right]$$

$$\frac{z_1}{z_2}=\frac{10}{21}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$$

Thus, the required quotient is $\frac{10}{21}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$.

Example 15: If $z=x+iy$, then write the equation $|3z-i|=|3\bar{z}+7|$ in term of x and y.

Solution:

Given that: $|3z-i|=|3\bar{z}+7|$ ... (i)

$$|3z-i|=|3(x+iy)-i|=|3x+i(3y-1)|=\sqrt{(3x)^2+(3y-1)^2}$$

$$|3\bar{z}+7|=|\overline{3x+3iy}+7|=|3x-3iy+7|=|3x+7+i(-3y)|=\sqrt{(3x+7)^2+(-3y)^2}$$

Putting these values in (i), we have

$$\sqrt{(3x)^2+(3y-1)^2}=\sqrt{(3x+7)^2+(-3y)^2}$$

Taking square on both sides

$$(3x)^2+(3y-1)^2=(3x+7)^2+(-3y)^2$$

$$9x^2+9y^2-6y+1=9x^2+42x+49+9y^2$$

$\Rightarrow$ $-6y+1=42x+49$

$\Rightarrow$ $-6y=42x+48$

or $y=-7x-8$ Dividing both sides by (-6)

The equation $y=-7x-8$ represents a straight line in the complex plane.

Example 16: Show that $(x+2)^2+y^2=8$ if $\arg\left(\frac{z+2i}{z-2i}\right)=\frac{3\pi}{4}$ for $z=x+iy$.

Solution:

$$\frac{z+2i}{z-2i}=\frac{x+iy+2i}{x+iy-2i}=\frac{x+i(y+2)}{x+i(y-2)}$$

$$=\frac{x+i(y+2)}{x+i(y-2)}\times\frac{x-i(y-2)}{x-i(y-2)}$$ Multiply up and down by conjugate of $x+i(y-2)$

$$=\frac{x^2-ix(y-2)+ix(y+2)-i^2(y^2-2^2)}{x^2-i^2(y-2)^2}$$

$$=\frac{x^2-ixy+2ix+ixy+2ix+y^2-2^2}{x^2+(y-2)^2}\quad\because i^2=-1$$

$$\frac{z+2i}{z-2i}=\frac{(x^2+y^2-4)+4ix}{x^2+(y-2)^2}=\frac{x^2+y^2-4}{x^2+(y-2)^2}+i\frac{4x}{x^2+(y-2)^2}$$

As $\arg\left(\frac{z+2i}{z-2i}\right)=\frac{3\pi}{4}$

$\Rightarrow$ $\tan^{-1}\left(\dfrac{\frac{4x}{x^2+(y-2)^2}}{\frac{x^2+y^2-4}{x^2+(y-2)^2}}\right)=\frac{3\pi}{4}$ $\Rightarrow$ $\frac{4x}{x^2+y^2-4}=\tan\frac{3\pi}{4}=-1$

$\Rightarrow$ $4x=-1(x^2+y^2-4)$ $\Rightarrow$ $x^2+4x+y^2=4$

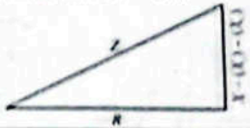
$(x^2+4x+2^2)+y^2=4+2^2$ Add 2^2 on both sides

$(x+2)^2+y^2=8$

Complex Numbers in the Real World (Voltage, Current and Resistance)

Ohm's Law is a fundamental principle in physics that describes the relationship between voltage V, current I and resistance R in an electrical circuit. Mathematically Ohm's Law can be expressed by the formula $V=IR$.

when dealing with alternating current (AC) circuits, resistance generalizes to impedance (Z). Resistance in a circuit is due to inductor (X_L) and capacitor (X_C). Their difference is reactance $X=(X_L)-(X_C)$. Geometrically it is shown in the adjacent figure. Here $Z=R+iX$ Then for AC circuits, Ohm's Law in Terms of Impedance is expressed by the formula $V=I\cdot Z$.



Example 17: If the impedance of circuit is $11(\cos 55.35°+i\sin 55.35°)$ ohms at a voltage of $25(\cos 30°+i\sin 30°)V$, find the value of current in the circuit.

Solution:

Given that Voltage: $V=25(\cos 30°+i\sin 30°)$

Impedance: $Z=11(\cos 55.35°+i\sin 55.35°)$

As we know $V=IZ$, where V is voltage, I denote the current and Z is impedance.

$\Rightarrow$ $I=\frac{V}{Z}$

Putting values, we have

$$I=\frac{25(\cos 30°+i\sin 30°)}{11(\cos 55.35°+i\sin 55.35°)}$$

$$I=\frac{25}{11}[\cos(30°-55.35°)+i\sin(30°-55.35°)]\qquad\because \arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$$

$I=2.27[\cos(-25.35°)+i\sin(-25.35°)]$

Express into rectangular form

$I=2.27[0.90+i(-0.42)]=2.04-0.95i$

Thus, current is $2.04-0.95i\,A$.

Cryptography: It is the science of securing information by transforming readable messages called plaintext into secrete code called ciphertext using mathematical algorithms and encryption keys. It consists of two main processes i.e., encryption to lock message with complex math, and decryption to unlock it with the right key.

Example 18: The word "MATH" is to be encrypted by multiplying a complex number $k=2+3i$ and then decrypted back to its original form using the concept of multiplicative inverse in complex numbers.

Each letter of the alphabet is assigned a numerical value as follows:

$$A=1, B=2, C=3, \ldots, Z=26$$

Solution: First, we assign each letter in the word "MATH" a complex number with zero imaginary part. The encryption and decryption are shown in the table below

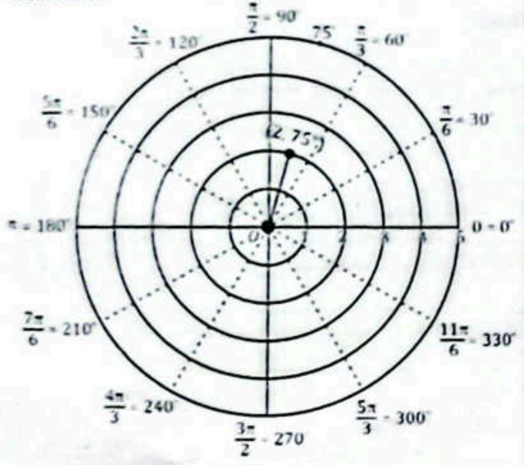
Letter	Complex Number (z)	(z) encrypted = $z \times k$	(z) decrypted = (z) encrypted /k	Letter
M	$13 + 0i$	$(13 + 0i)(2 + 3i) = 26 + 39i$	$(26 + 39i)/(2 + 3i) = 13 + 0i$	M
A	$1 + 0i$	$(1 + 0i)(2 + 3i) = 2 + 3i$	$(2 + 3i)/(2 + 3i) = 1 + 0i$	A
T	$20 + 0i$	$(20 + 0i)(2 + 3i) = 40 + 60i$	$(40 + 60i)/2 + 3i = 20 + 0i$	T
H	$8 + 0i$	$(8 + 0i)(2 + 3i) = 16 + 24i$	$16 + 24i/2 + 3i = 8 + 0i$	H

Exercise 1.5

1. Plot the following points:

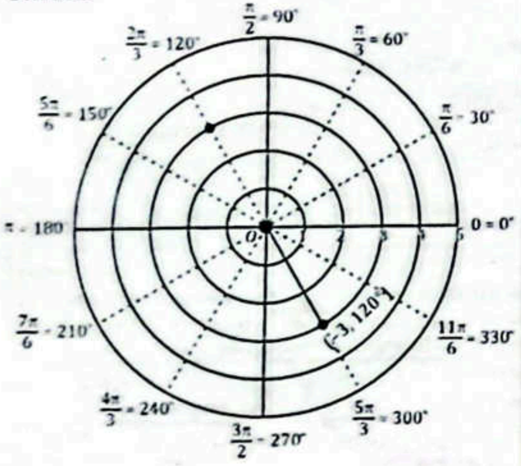
(i) $(2, 75°)$

Solution:



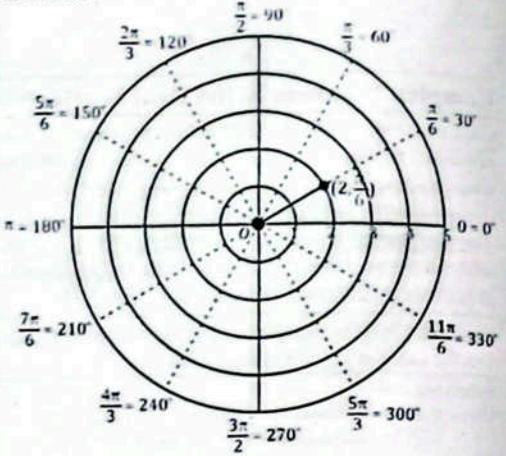
(ii) $(-3, 120°)$

Solution:



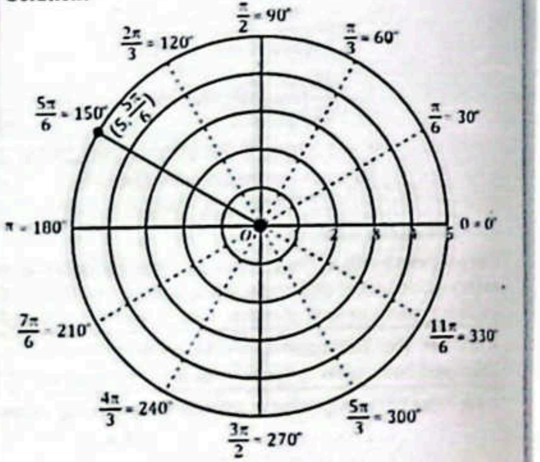
(iii) $\left(2, \frac{\pi}{6}\right)$

Solution:



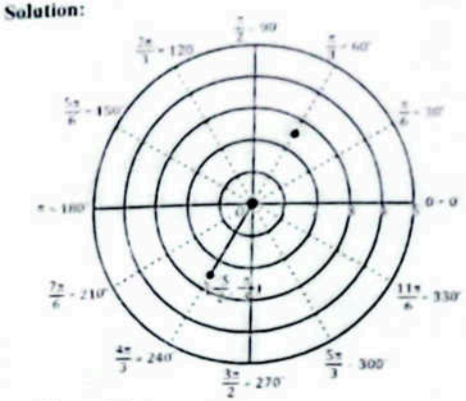
(iv) $\left(5, \frac{5\pi}{6}\right)$

Solution:



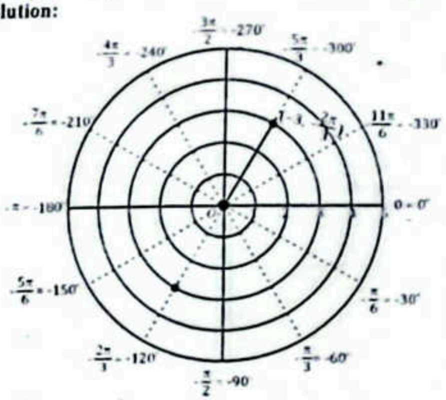
(v) $\left(-\frac{5}{2}, \frac{\pi}{3}\right)$

Solution:



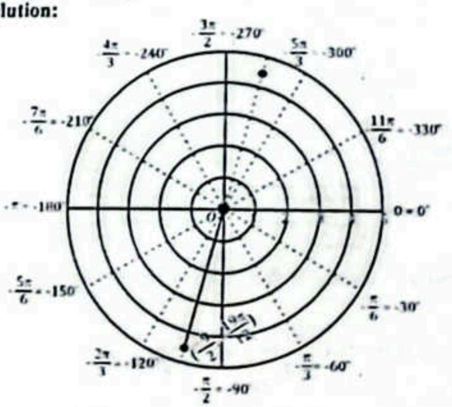
(vi) $\left(-3, -\frac{2\pi}{3}\right)$

Solution:



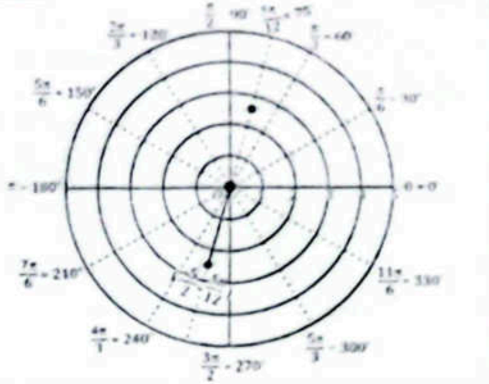
(vii) $\left(\frac{9}{2}, -\frac{19\pi}{12}\right)$

Solution:



(viii) $\left(-\frac{5}{2}, \frac{5\pi}{12}\right)$

Solution:



2. Express the following complex numbers in polar form:

(i) $4 + 3i$

Solution:

Let $z = 4 + 3i$

Here $x = 4, y = 3$

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25} = 5$$

Since $x > 0$ and $y > 0$, so θ lies in first quadrant.

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{4} = 36.87°$$

Polar form of z is

$$z = r(\cos\theta + i\sin\theta)$$
$$= 5(\cos(36.87°) + i\sin(36.87°))$$

(ii) $1 + i$

Solution:

Let $z = 1 + i$

Here $x = 1, y = 1$

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{1 + 1} = \sqrt{2}$$

Since $x > 0$ and $y > 0$, so θ lies in first quadrant.

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{1}{1}\right) = 45° = \frac{\pi}{4}$$

Polar form of z is

$$z = r(\cos\theta + i\sin\theta)$$
$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(iii) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Solution:

Let $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Here $x=\frac{1}{2}, y=\frac{\sqrt{3}}{2}$

$$r=\sqrt{x^2+y^2}$$
$$=\sqrt{\frac{1}{4}+\frac{3}{4}}$$
$$=\sqrt{\frac{4}{4}}=1$$

Since $x>0$ and $y>0$, so θ lies in first quadrant.

$$\theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{\sqrt{3}/2}{1/2}$$
$$=\tan^{-1}(\sqrt{3})=60°=\frac{\pi}{3}$$

Polar form of z is

$$z=r(\cos\theta+i\sin\theta)$$
$$z=1\cdot\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

(iv) $-\frac{5}{2}-\frac{5\sqrt{3}}{2}i$

Solution:

Let $z=-\frac{5}{2}-\frac{5\sqrt{3}}{2}i$

Here $x=-\frac{5}{2}, y=-\frac{5\sqrt{3}}{2}$

$$r=\sqrt{x^2+y^2}$$
$$=\sqrt{\frac{25}{4}+\frac{75}{4}}$$
$$=\sqrt{\frac{100}{4}}=5$$

Since $x<0$ and $y<0$, so θ lies in third quadrant.

$$\theta=\pi+\tan^{-1}\left|\frac{y}{x}\right|$$
$$=\pi+\tan^{-1}\left(\frac{5\sqrt{3}/2}{5/2}\right)$$
$$=\pi+\tan^{-1}(\sqrt{3})$$
$$=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$$

Polar form of z is

$$z=r(\cos\theta+i\sin\theta)$$
$$z=5\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$

(v) $\frac{1-i}{1+i}$

Solution:

Let $z=\frac{1-i}{1+i}=\frac{1-i}{1+i}\times\frac{1-i}{1-i}$

$$z=\frac{(1-i)^2}{1^2-i^2}=\frac{1+i^2-2i}{1+1} \qquad \because i^2=-1$$
$$z=\frac{1-1-2i}{2}=\frac{-2i}{2}=-i$$
$$z=0-1i$$

Here $x=0, y=-1<0$

$\Rightarrow$ $\theta=-90°=-\frac{\pi}{2}$ $\quad\because$ z lies on –ve imaginary axis

$$r=\sqrt{x^2+y^2}=\sqrt{0^2+(-1)^2}=\sqrt{1}=1$$

Polar form of z is

$$z=r(\cos\theta+i\sin\theta)$$
$$z=1\cdot\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$$

(vi) $\frac{\sqrt{3}+i}{1+\sqrt{3}i}$

Solution:

Let $z=\frac{\sqrt{3}+i}{1+\sqrt{3}i}$

$$=\frac{\sqrt{3}+i}{1+\sqrt{3}i}\times\frac{1-\sqrt{3}i}{1-\sqrt{3}i}=\frac{\sqrt{3}-3i+i-\sqrt{3}i^2}{1^2-(\sqrt{3}i)^2}$$
$$=\frac{\sqrt{3}-2i+\sqrt{3}}{1+3} \qquad \because i^2=-1$$
$$=\frac{2\sqrt{3}-2i}{4}=\frac{2(\sqrt{3}-i)}{4}=\frac{\sqrt{3}-i}{2}$$
$$z=\frac{\sqrt{3}}{2}-\frac{1}{2}i$$

Here $x=\frac{\sqrt{3}}{2}, y=-\frac{1}{2}$

$$r=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{3}{4}+\frac{1}{4}}=\sqrt{\frac{4}{4}}=1$$

Since $x>0$ and $y<0$, so θ lies in fourth quadrant

$$\theta=-\tan^{-1}\frac{|y|}{x}=-\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$=-\tan\frac{1}{\sqrt{3}}=-30°=-\frac{\pi}{6}$$

Polar form of z is

$$z=r(\cos\theta+i\sin\theta)$$
$$z=1\cdot\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$$

<u>**Alternate Method:**</u>

Let $\frac{z_1}{z_2}=\frac{\sqrt{3}+i}{1+\sqrt{3}i}$

Here $z_1=\sqrt{3}+1i$

$\Rightarrow$ $r_1=\sqrt{(\sqrt{3})^2+1^2}=\sqrt{3+1}=2$

and $\theta_1=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{1}{\sqrt{3}}=30°$

Now, $z_2=1+\sqrt{3}i$

$\Rightarrow$ $r_2=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{1+3}=2$

and $\theta_2=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{\sqrt{3}}{1}=60°$

Polar form of $\frac{z_1}{z_2}$ is

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$$
$$=\frac{2}{2}[\cos(30°-60°)+i\sin(30°-60°)]$$
$$=1\cdot(\cos(-30°)+i\sin(-30°))$$
$$\frac{z_1}{z_2}=1\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$$

(vii) $\frac{3+4i}{4+3i}$

Solution:

Let $z=\frac{3+4i}{4+3i}$

$$=\frac{3+4i}{4+3i}\times\frac{4-3i}{4-3i}=\frac{12-9i+16i-12i^2}{4^2-(3i)^2}$$
$$=\frac{12+7i+12}{16+9}=\frac{24+7i}{25} \qquad \because i^2=-1$$
$$z=\frac{24}{25}+\frac{7}{25}i$$

Here $x=\frac{24}{25}, y=\frac{7}{25}$

$$r=\sqrt{\left(\frac{24}{25}\right)^2+\left(\frac{7}{25}\right)^2}=\sqrt{\frac{576}{625}+\frac{49}{625}}=\sqrt{\frac{625}{625}}=\sqrt{1}=1$$

As $x>0$ and $y>0$, so θ lies in first quadrant.

$$\theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\left(\frac{\frac{7}{25}}{\frac{24}{25}}\right)=\tan^{-1}\left(\frac{7}{24}\right)$$

Polar form of z is

$$z=r(\cos\theta+i\sin\theta)$$
$$=1\cdot\left(\cos\left(\tan^{-1}\frac{7}{24}\right)+i\sin\left(\tan^{-1}\frac{7}{24}\right)\right)$$

---

3. Convert each of the complex number z in the rectangular form $+iy$:

(i) $4\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

Solution:

Let $z=4\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

$$=4(\cos300°+i\sin300°)$$
$$=4\left(\frac{1}{2}+i\left(-\frac{\sqrt{3}}{2}\right)\right)$$
$$=2-2\sqrt{3}i$$

(ii) $\frac{3}{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

Solution:

Let $z=\frac{3}{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

$$=\frac{3}{2}(\cos210°+i\sin210°)$$
$$=\frac{3}{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$$
$$=-\frac{3\sqrt{3}}{4}-\frac{3}{4}i$$

(iii) $|z|=7, \arg(z)=\frac{23\pi}{12}$

Solution:

Given that: $r=|z|=7, \theta=\frac{23\pi}{12}$

Polar form of z is

$$z=r(\cos\theta+i\sin\theta)$$
$$=7\left(\cos\frac{23\pi}{12}+i\sin\frac{23\pi}{12}\right)$$
$$=7(\cos345°+i\sin345°)$$
$$=7(0.97+i(-0.26))$$
$$6.79-1.82i$$

(iv) $|z| = 11, \arg(z) = -\frac{11\pi}{12}$

Solution:

Given that: $r = 11, \theta = \frac{-11\pi}{12}$

Polar form of z is

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= 11\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right) \\ &= 11\left(\cos\frac{11\pi}{12} - i\sin\frac{11\pi}{12}\right) \qquad \because \begin{cases} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{cases} \\ &= 11(\cos 165^\circ - i\sin 165^\circ) \\ &= 11(-0.97 - i(0.26)) \\ &= -10.67 - 2.86i \end{aligned}$$

(v) $|z| = \frac{10}{3}, \arg(z) = -\frac{17\pi}{12}$

Solution:

Given that: $r = \frac{10}{3}, \theta = \frac{-17\pi}{12}$

Polar form of z is

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= \frac{10}{3}\left(\cos\left(-\frac{17\pi}{12}\right) + i\sin\left(-\frac{17\pi}{12}\right)\right) \\ &= \frac{10}{3}\left(\cos\frac{17\pi}{12} - i\sin\frac{17\pi}{12}\right) \qquad \because \begin{cases} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{cases} \\ &= \frac{10}{3}(\cos 255^\circ - i\sin 255^\circ) \\ &= \frac{10}{3}(-0.26 - i(-0.97)) \\ &= -\frac{10}{3}(0.26) + \frac{10}{3}(0.97)i \\ &= -0.87 + 3.23i \end{aligned}$$

(vi) $2\cos(-33^\circ) + i2\sin(-33^\circ)$

Solution:

Let

$$\begin{aligned} z &= 2\cos(-33^\circ) + i2\sin(-33^\circ) \\ &= 2\cos 33^\circ - i2\sin 33^\circ \qquad \because \begin{cases} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{cases} \\ &= 2(0.84) - i2(0.54) \\ &= 1.68 - 1.08i \end{aligned}$$

4. **If** $z_1 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ **and** $z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ **then find:**

(i) $z_1 + z_2$

Solution:

$$\begin{aligned} z_1 + z_2 &= 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) + 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= 9(\cos 225^\circ + i\sin 225^\circ) + 5(\cos 60^\circ + i\sin 60^\circ) \\ &= 9\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) + 5\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= -6.36 - 6.36i + 2.5 + 4.33i \\ &= (-6.36 + 2.5) + (-6.36 + 4.33)i \\ &= -3.86 - 2.03i \end{aligned}$$

(ii) $z_1 - z_2$

Solution:

$$\begin{aligned} z_1 - z_2 &= 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) - 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= 9(\cos 225^\circ + i\sin 225^\circ) - 5(\cos 60^\circ + i\sin 60^\circ) \\ &= 9(-0.7071 - 0.7071i) - 5(0.5 + 0.866i) \\ &= -6.36 - 6.36i - 2.5 - 4.33i \\ &= (-6.36 - 2.5) - (6.36 + 4.33)i \\ &= -8.86 - 10.69i \end{aligned}$$

(iii) $z_1 \cdot z_2$

Solution:

$$z_1 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Here $r_1 = 9, \theta_1 = \frac{5\pi}{4}$ and $r_2 = 5, \theta_2 = \frac{\pi}{3}$

Putting values in the product formula

$$\begin{aligned} z_1 . z_2 &= r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\ &= (9)(5)\left[\cos\left(\frac{5\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{5\pi}{4} + \frac{\pi}{3}\right)\right] \\ &= 45\left[\cos\left(\frac{15\pi + 4\pi}{12}\right) + i\sin\left(\frac{15\pi + 4\pi}{12}\right)\right] \\ &= 45\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) \end{aligned}$$

(iv) $\frac{z_1}{z_2}$

Solution:

$$z_1 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Here $r_1 = 9, \theta_1 = \frac{5\pi}{4}$ and $r_2 = 5, \theta_2 = \frac{\pi}{3}$

Putting values in the quotient formula

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \\ &= \frac{9}{5}\left[\cos\left(\frac{5\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{5\pi}{4} - \frac{\pi}{3}\right)\right] \\ &= \frac{9}{5}\left[\cos\left(\frac{15\pi - 4\pi}{12}\right) + i\sin\left(\frac{15\pi - 4\pi}{12}\right)\right] \\ &= \frac{9}{5}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \end{aligned}$$

5. **If** $z_1 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$ **and** $z_2 = 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$ **then find the following and express the result into** $x + iy$ **form.**

(i) $z_1 + z_2$

Solution:

$$\begin{aligned} z_1 + z_2 &= 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) + 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \\ &= 7(\cos 345^\circ + i\sin 345^\circ) + 11(\cos 165^\circ + i\sin 165^\circ) \\ &= 7(0.9659 - 0.2588i) + 11(-0.9659 + 0.2588i) \\ &= 6.7613 - 1.8116i - 10.6249 + 2.8468i \\ &= (6.7613 - 10.6249) + (-1.8116 + 2.8468)i \\ &= -3.86 + 1.04i \end{aligned}$$

(ii) $z_1 - z_2$

Solution:

$$\begin{aligned} z_1 + z_2 &= 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) - 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \\ &= 7(\cos 345^\circ + i\sin 345^\circ) - 11(\cos 165^\circ + i\sin 165^\circ) \\ &= 7(0.9659 - 0.2588i) - 11(-0.9659 + 0.2588i) \\ &= 6.7613 - 1.8116i + 10.6249 - 2.8468i \\ &= (6.7613 + 10.6249) - (1.8116 + 2.8468)i \\ &= 17.39 - 4.66i \end{aligned}$$

(iii) $z_1 \cdot z_2$

Solution:

$$z_1 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$$

$$z_2 = 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

Here $r_1 = 7, \theta_1 = \frac{23\pi}{12}$ and $r_2 = 11, \theta_2 = \frac{11\pi}{12}$

Putting values in the product formula

$$\begin{aligned} z_1 . z_2 &= r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\ &= (7)(11)\left[\cos\left(\frac{23\pi}{12} + \frac{11\pi}{12}\right) + i\sin\left(\frac{23\pi}{12} + \frac{11\pi}{12}\right)\right] \\ &= 77\left[\cos\frac{34\pi}{12} + i\sin\frac{34\pi}{12}\right] \\ &= 77(\cos 510^\circ + i\sin 50^\circ) \\ &= 77(-0.866 + 0.5i) \\ &= -66.68 + 38.50i \end{aligned}$$

(iv) $\frac{z_1}{z_2}$

Solution:

$$z_1 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$$

$$z_2 = 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

Here $r_1 = 7, \theta_1 = \frac{23\pi}{12}$ and $r_2 = 11, \theta_2 = \frac{11\pi}{12}$

Putting values in the product formula

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \\ &= \frac{7}{11}\left[\cos\left(\frac{23\pi}{12} - \frac{11\pi}{12}\right) + i\sin\left(\frac{23\pi}{12} - \frac{11\pi}{12}\right)\right] \\ &= \frac{7}{11}(\cos\pi + i\sin\pi) \\ &= \frac{7}{11}(-1 + 0i) \\ &= \frac{-7}{11} + 0i \end{aligned}$$

6. **If z_1 and z_2 are two complex numbers, show that:**

(i) $Arg(z_1 z_2) = Arg\, z_1 + Arg\, z_2$

(ii) $Arg\left(\frac{z_1}{z_2}\right) = Arg\, z_1 - Arg\, z_2$

Solution:

Polar form of complex numbers z_1 and z_2 are

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$$

(i) To prove: $\text{Arg}(z_1 \cdot z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \\ &= r_1 r_2 \cdot e^{i\theta_1 + i\theta_2} \\ z_1 \cdot z_2 &= r_1 r_2 \cdot e^{i(\theta_1 + \theta_2)} \end{aligned}$$

$$\Rightarrow \text{Arg}(z_1 \cdot z_2) = \theta_1 + \theta_2$$

$$\text{Arg}(z_1 \cdot z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$$

(ii) To prove: $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$$

$$= \frac{r_1}{r_2} e^{i\theta_1 - i\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\Rightarrow \quad \mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

$$\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$$

---

7. Divide $z_1 = 6(\cos 150° + i\sin 150°)$ by $z_2 = 3(\cos 30° + i\sin 30°)$ and express in $x + iy$ form.

Solution:

$$z_1 = 6(\cos 150° + i\sin 150°)$$

$$z_2 = 3(\cos 30° + i\sin 30°)$$

Here $r_1 = 6, \theta_1 = 150°$ and $r_2 = 3, \theta_2 = 30°$

Putting values in the quotient formula

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$= \frac{6}{3}\left[\cos(150° - 30°) + i\sin(150° - 30°)\right]$$

$$= 2(\cos 120° + i\sin 120°)$$

$$= 2\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$$

---

8. Multiply $z_1 = 2(\cos 60° + i\sin 60°)$ and $z_2 = 5(\cos 90° + i\sin 90°)$ and express in $x + iy$ form.

Solution:

$$z_1 = 2(\cos 60° + i\sin 60°)$$

$$z_2 = 5(\cos 90° + i\sin 90°)$$

Here $r_1 = 2, \theta_1 = 60°$ and $r_2 = 5, \theta_2 = 90°$

Putting values in the quotient formula

$$z_1 . z_2 = r_1 . r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$= (2)(5)[\cos(60° + 90°) + i\sin(60° + 90)°]$$

$$= 10(\cos 150° + i\sin 150°)$$

$$= 10\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right) = -5\sqrt{3} + 5i$$

---

9. Find the modulus and argument of $z = -2 - 2i$.

Solution:

Given that: $z = -2 - 2i$

Here $x = -2, y = -2$

Modulus of z is

$$|z| = \sqrt{x^2 + y^2}$$

$$= \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$$

Since $x < 0$ and $y < 0$, so θ lies in third quadrant

Argument of z is

$$\theta = \pi + \tan^{-1}\left|\frac{y}{x}\right| = \pi + \tan^{-1}\left(\frac{2}{2}\right)$$

$$= \pi + \tan^{-1}(1) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

General value of $\theta = \frac{5\pi}{4} + 2n\pi$

---

10. Write the equation $\arg(\bar{z} - 2 + i) = \frac{2\pi}{3}$ in cartesian form, if $z = x + iy$.

Solution:

Given that: $z = x + iy \quad \Rightarrow \bar{z} = x - iy$

As $\quad \arg(\bar{z} - 2 + i) = \frac{2\pi}{3}$

$$\Rightarrow \arg(x - iy - 2 + i) = \frac{2\pi}{3}$$

$$\arg((x - 2) + (-y + 1)i) = \frac{2\pi}{3}$$

$$\tan^{-1}\left(\frac{-y + 1}{x - 2}\right) = \frac{2\pi}{3} \qquad \because \arg(a + ib) = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\frac{-y + 1}{x - 2} = \tan\left(\frac{2\pi}{3}\right)$$

$$\frac{-y + 1}{x - 2} = \tan(120°)$$

$$\frac{-y + 1}{x - 2} = -\sqrt{3}$$

$$-y + 1 = -\sqrt{3}(x - 2)$$

$$-y + 1 = -\sqrt{3}x + 2\sqrt{3}$$

$$\sqrt{3}x - 2\sqrt{3} + 1 = y$$

$$\Rightarrow \quad y = \sqrt{3}x - 2\sqrt{3} + 1 \quad \text{(Cartesian form)}$$

---

11. If $z = x + iy$ and $\arg\left(\frac{\bar{z} - 1 + 2i}{\bar{z} + 1 - 2i}\right) = \frac{9\pi}{4}$, show that $x^2 + y^2 - 4x + 2y - 5 = 0$.

Solution:

Given that: $z = x + iy \quad \Rightarrow \bar{z} = x - iy$

and $\quad \arg\left(\frac{\bar{z} - 1 + 2i}{\bar{z} + 1 - 2i}\right) = \frac{9\pi}{4}$

$$\Rightarrow \quad \arg\left(\frac{x - iy - 1 + 2i}{x - iy + 1 - 2i}\right) = \frac{9\pi}{4}$$

$$\arg\left(\frac{(x - 1) + (-y + 2)i}{(x + 1) + (-y - 2)i}\right) = \frac{9\pi}{4}$$

As we know: $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

$$\Rightarrow \quad \arg((x - 1) + (-y + 2)i) - \arg((x + 1) - (y + 2)i) = \frac{9\pi}{4}$$

As we know: $\arg(a + ib) = \tan^{-1}\frac{b}{a}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{-y + 2}{x - 1}\right) - \tan^{-1}\left(\frac{-y - 2}{x + 1}\right) = \frac{9\pi}{4}$$

As we know: $\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A - B}{1 + AB}\right)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{-y + 2}{x - 1} - \frac{-y - 2}{x + 1}}{1 + \frac{-y + 2}{x - 1} . \frac{-y - 2}{x + 1}}\right) = \frac{9\pi}{4}$$

$$\frac{\frac{(-y + 2)(x + 1) + (y + 2)(x - 1)}{(x - 1)(x + 1)}}{\frac{(x - 1)(x + 1) + (-y + 2)(-y - 2)}{(x - 1)(x + 1)}} = \tan\left(\frac{9\pi}{4}\right)$$

$$\frac{-xy - y + 2x + 2 + xy - y + 2x - 2}{x^2 - 1 + y^2 + 2y - 2y - 4} = \tan(405°)$$

$$\frac{4x - 2y}{x^2 + y^2 - 5} = 1$$

$$\Rightarrow \quad 4x - 2y = x^2 + y^2 - 5$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 2y - 5 = 0 \text{ (Proved)}$$

---

12. If $z = x + iy$ and $\arg(z - 2 - 3i) - \arg(z + 2 + 3i) = 2\pi$, show that $2y = 3x$.

Solution:

Given that: $z = x + iy$

$$\arg(z - 2 - 3i) - \arg(z + 2 + 3i) = 2\pi$$

$$\arg(x + iy - 2 - 3i) - \arg(x + iy + 2 + 3i) = 2\pi$$

$$\arg((x - 2) + (y - 3)i) - \arg((x + 2) + (y + 3)i) = 2\pi$$

As we know: $\arg(a + ib) = \tan^{-1}\left(\frac{b}{a}\right)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y - 3}{x - 2}\right) - \tan^{-1}\left(\frac{y + 3}{x + 2}\right) = 2\pi$$

As we know: $\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A - B}{1 + AB}\right)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{y - 3}{x - 2} - \frac{y + 3}{x + 2}}{1 + \frac{y - 3}{x - 2} . \frac{y + 3}{x + 2}}\right) = 2\pi$$

$$\frac{\frac{(x + 2)(y - 3) - (x - 2)(y + 3)}{(x - 2)(x + 2)}}{\frac{(x - 2)(x + 2) + (y - 3)(y + 3)}{(x - 2)(x + 2)}} = \tan(360°)$$

$$\frac{xy - 3x + 2y - 6 - xy - 3x + 2y + 6}{x^2 - 2^2 + y^2 - 3^2} = 0$$

$$-6x + 4y = 0$$

$$-3x + 2y = 0 \qquad \text{Dividing by 2}$$

$$\Rightarrow \quad 2y = 3x \text{ (Proved)}$$

---

13. Solve the equation:
$|z - 2i| = |z + 2|$ for $z = x + iy$.

Solution:

Given that: $z = x + iy \quad \Rightarrow \bar{z} = x - iy$

$$|z - 2i| = |\bar{z} + 2|$$

$$\Rightarrow |x + iy - 2i| = |x - iy + 2|$$

$$|x + (y - 2)i| = |(x + 2) - iy|$$

$$\sqrt{x^2 + (y - 2)^2} = \sqrt{(x + 2)^2 + (-y)^2} \qquad \because |a + ib| = \sqrt{a^2 + b^2}$$

Squaring both sides, we have

$$x^2 + y^2 + 4 - 4y = x^2 + 4 + 4x + y^2$$

$$\Rightarrow \quad -4y = 4x$$

$$y = \frac{4}{-4}x$$

$$y = -x$$

---

14. For $z = x + iy$, solve the equation
$|5z + 4 + i| = |5\bar{z} - 3 + 2i|$

Solution:

Given that: $z = x + iy \Rightarrow \bar{z} = x - iy$

$$|5z + 4 + i| = |5\bar{z} - 3 + 2i|$$

Put $z = x + iy$ and $\bar{z} = x - iy$

$$|5(x + iy) + 4 + i| = |5(x - iy) - 3 + 2i|$$

$$|5x + 5iy + 4 + i| = |5x - 5iy - 3 + 2i|$$

$$|(5x + 4) + (5y + 1)i| = |(5x - 3) + (-5y + 2)i|$$

$$\sqrt{(5x + 4)^2 + (5y + 1)^2} = \sqrt{(5x - 3)^2 + (-5y + 2)^2}$$

Squaring both sides

$$\cancel{25x^2} + 16 + 40x + \cancel{25y^2} + 1 + 10y = \cancel{25x^2} + 9 -$$

$$30x + \cancel{25y^2} + 4 - 20y$$

$$40x + 30x + 10y + 20y + 17 - 13 = 0$$

$$70x + 30y + 4 = 0$$

$$\Rightarrow \quad 30y = -70x - 4$$

$$y = -\frac{7}{3}x - \frac{2}{15} \qquad \text{Dividing by '2'}$$

---

15. Determine the set of points $z = x + iy$ that satisfy the equation $|3\bar{z} - 2 + i| = |3z + i|$.

Solution:

Given that: $z = x + iy \Rightarrow \bar{z} = x - iy$

$$|3\bar{z} - 2 + i| = |3z + i|$$

Put $z=x+iy$ and $\bar{z}=x-iy$

$$|3(x-iy)-2+i| = |3(x+iy)+i|$$
$$|3x-3iy-2+i| = |3x+3iy+i|$$
$$|(3x-2)+i(-3y+1)| = |3x+(3y+1)i|$$
$$\sqrt{(3x-2)^2+(-3y+1)^2} = \sqrt{(3x)^2+(3y+1)^2}$$

Squaring both sides

$$\cancel{9x^2}+4-12x+\cancel{9y^2}+\cancel{1}-6y=\cancel{9x^2}+\cancel{9y^2}+\cancel{1}+6y$$
$$6y+6y=-12x+4$$
$$12y=-12x+4$$

Dividing by '12' on both sides

$$\frac{12y}{12}=\frac{-12x}{12}+\frac{4}{12}$$
$$\boxed{y=-x+\frac{1}{3}}$$

---

16. If $z=x+iy$ and $w=\dfrac{1-iz}{z-i}$, show that $|w|=1\Rightarrow z$ is real.

Solution:

Given that: $z=x+iy$, $w=\dfrac{1-iz}{z-i}$

To prove: If $|w|=1\Rightarrow z$ is real.

As $|w|=1$

$$\Rightarrow \left|\frac{1-iz}{z-i}\right|=1$$
$$\frac{|1-iz|}{|z-i|}=1 \qquad \because \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$$
$$|1-iz|=|z-i|$$
$$|1-i(x+iy)|=|x+iy-i| \qquad \because z=x+iy$$
$$|1-ix-i^2y|=|x+i(y-1)|$$
$$|(1+y)-ix|=|x+i(y-1)| \qquad \because i^2=-1$$
$$\sqrt{(1+y)^2+(-x)^2}=\sqrt{x^2+(y-1)^2}$$

Squaring both sides

$$\cancel{1}+\cancel{y^2}+2y+\cancel{x^2}=\cancel{x^2}+\cancel{y^2}+\cancel{1}-2y$$
$$2y+2y=0$$
$$4y=0 \Rightarrow \boxed{y=0}$$

Put $y=0$ in z, we have

$$z=x+iy=x+0i=x \text{ (real)}$$

Hence, if $|w|=1$, then z is real.

---

17. If z_1 and z_2 are different complex numbers with $|z_2|=1$, find $\left|\dfrac{z_2-z_1}{1-\bar{z}_1z_2}\right|$.

Solution:

As $|z_2|=1\Rightarrow|z_2|^2=1\Rightarrow z_2\bar{z}_2=1$ ...(1)

and $|z_2-z_1|=|\overline{z_2-z_1}|=|\bar{z}_2-\bar{z}_1|$

$$\left|\frac{z_2-z_1}{1-\bar{z}_1z_2}\right|=\left|\frac{z_2-z_1}{z_2\bar{z}_2-\bar{z}_1z_2}\right| \qquad \because z_2\bar{z}_2=1 \text{ from (1)}$$
$$=\left|\frac{z_2-z_1}{z_2(\bar{z}_2-\bar{z}_1)}\right|$$
$$=\left|\frac{z_2-z_1}{z_2(\overline{z_2-z_1})}\right| \qquad \because \overline{(z_2-z_1)}=\bar{z}_2-\bar{z}_1$$
$$=\frac{|z_2-z_1|}{|z_2(\overline{z_2-z_1})|} \qquad \because \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$$
$$=\frac{|z_2-z_1|}{|z_2||\overline{z_2-z_1}|} \qquad \because |z_1\cdot z_2|=|z_1||z_2|$$
$$=\frac{|z_2-z_1|}{(1)|z_2-z_1|} \qquad \because |z_2|=1, |\bar{z}|=|z|$$
$$=\frac{1}{1}=1$$

---

18. An AC source supplies a voltage of $V=120\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ volts to a circuit with impedance $Z=\dfrac{1+i\sqrt{3}}{2}$ ohms. Calculate the current in polar form.

Solution:

Given that Voltage: $V=120\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

Impedance: $Z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

First, we express z in polar fam

$$r=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{\frac{4}{4}}=\sqrt{1}=1$$
$$\theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)=\tan^{-1}\left(\sqrt{3}\right)=60^\circ=\frac{\pi}{3}$$
$$z=r(\cos\theta+i\sin\theta)=1\cdot\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

As we know

$V=IZ$, where V is a voltage, I denote the current and Z is impedance.

$$\Rightarrow \quad I=\frac{V}{Z}$$
$$I=\frac{120\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{1\cdot\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$$

$$=120\left(\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\right)$$
$$=120\left(\cos\left(\frac{3\pi-4\pi}{12}\right)+i\sin\left(\frac{3\pi-4\pi}{12}\right)\right)$$
$$=120\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)$$
$$I=120\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right) \text{ (As required)}$$

---

19. An AC circuit has an impedance of $Z=3-6i$ ohms and is connected to a voltage source of $V=90+30i$ volts. Find the current in both rectangular and polar form.

Solution:

Given that voltage: $V=90+30i$ volts

Impedance: $Z=3-6i$ ohms

Current: $I=?$

As we know

$$V=IZ$$
$$I=\frac{V}{Z}=\frac{90+30i}{3-6i}=\frac{30(3+i)}{3(1-2i)}$$
$$=\frac{10(3+i)}{1-2i}\times\frac{1+2i}{1+2i}$$
$$=\frac{10\left(3+6i+i+2i^2\right)}{1^2-4i^2}=\frac{10(3+7i-2)}{1+4} \qquad \because i^2=-1$$
$$I=\frac{10(1+7i)}{5}=2(1+7i)=2+14i$$
$$I=2+14i \text{ (Rectangular form)}$$

Now, we express I in polar form

Here $x=2$, $y=14$

$$r=\sqrt{2^2+14^2}=\sqrt{4+196}=\sqrt{200}=10\sqrt{2}$$
$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{14}{2}\right)=\tan^{-1}(7)=81.87^\circ$$

Polar form of I is

$$z=r(\cos\theta+i\sin\theta)$$
$$z=10\sqrt{2}(\cos(81.87^\circ)+i\sin(81.87^\circ))$$

---

20. Encrypt the word "CODE" by multiplying the complex encryption key $k=2-i$. Then decrypt it back to the original word.

Solution:

Complex encryption key: $k=2-i$

Word: CODE

As $k=2-i$

$$k^{-1}=\frac{1}{2-i}\times\frac{2+i}{2+i}$$
$$k^{-1}=\frac{2+i}{2^2-i^2}=\frac{2+i}{4+1}=\frac{1}{5}(2+i)$$

The encryption and decryption shown in the tables below:

Letter	Complex Number (E)	z encrypted $=z\times k$
C	$3+0i$	$(3+0i)(2-i)=6-3i$
O	$15+0i$	$(15+0i)(2-i)=30-15i$
D	$4+0i$	$(4+0i)(2-i)=8-4i$
E	$5+0i$	$(5+0i)(2-i)=10-5i$

z decrypted $=$ (z encrypted) k^{-1}	Letter
$(6-3i)\frac{1}{5}(2+i)=\frac{1}{5}(12+6i-6i-3i^2)$ $=\frac{1}{5}(12+3)=3$	C
$(30-15i)\frac{1}{5}(2+i)=\frac{1}{5}(60+30i-30i-15i^2)$ $=\frac{1}{5}(60+15)=15$	O
$(8-4i)\frac{1}{5}(2+i)=\frac{1}{5}(16+8i-8i-4i^2)$ $=\frac{1}{5}(20)=4$	D
$(10-5i)\frac{1}{5}(2+i)=\frac{1}{5}(20+10i-10i-5i^2)$ $=\frac{1}{5}(20+5)=5$	E

---

21. Consider the complex encryption key $k=3-3i$. Encrypt the word "QUIZ", and then recover the original word using the inverse of the key.

Solution:

Complex encryption key: $k=3-3i$

Word: QUIZ

As $z=3-3i \quad\Rightarrow\quad z^{-1}=\dfrac{1}{3-3i}\times\dfrac{3+3i}{3+3i}$

$$z^{-1}=\frac{3(1+i)}{3^2-9i^2}=\frac{3(1+i)}{9+9}$$
$$=\frac{3(1+i)}{18}=\frac{1}{6}(1+i)$$

The encryption and decryption shown in the tables below:

Letter	Complex Number (z)	z encrypted $=z\times k$
Q	$17+0i$	$(17+0i)(3-3i)=51-51i$
U	$21+0i$	$(21+0i)(3-3i)=63-63i$
I	$9+0i$	$(9+0i)(3-3i)=27-27i$
Z	$26+0i$	$(26+0i)(3-3i)=78-78i$

z decrypted = (z encrypted) k^{-1}	Letter
$(51-51i)\frac{1}{6}(1+i)=\frac{1}{6}(51+51i-51i-51i^2)$ $=\frac{1}{6}(51+51)$ $=17$	Q
$(63-63i)\frac{1}{6}(1+i)=\frac{1}{6}(63+63i-63i-63i^2)$ $=\frac{1}{6}(63+63)$ $=21$	U
$(27-27i)\frac{1}{6}(1+i)=\frac{1}{6}(27+27i-27i-27i^2)$ $=\frac{1}{6}(27+27)$ $=9$	I
$(78-78i)\frac{1}{6}(1+i)=\frac{1}{6}(78+78i-78i^2-78i^2)$ $=\frac{1}{6}(78+78)$ $=26$	Z

22. Encrypt the word "CLASS" by adding the complex encryption key $k=-3+4i$. Then decrypt it back to the original word.

Solution:

Complex encryption key: $k=-3+4i$

Word: CLASS

The encryption and decryption shown in the tables below:

Letter	Complex Number (z)	z encrypted = $z+k$
C	$3+0i$	$3+0i-3+4i=0+4i$
L	$12+0i$	$12+0i-3+4i=9+4i$
A	$1+0i$	$1+0i-3+4i=-2+4i$
S	$19+0i$	$19+0i-3+4i=16+4i$
S	$19+0i$	$19+0i-3+4i=16+4i$

z decrypted = (z encrypted) − k	Letter
$0+4i+3-4i=3+0i$	C
$9+4i+3-4i=12+0i$	L
$-2+4i+3-4i=1+0i$	A
$16+4i+3-4i=19+0i$	S
$16+4i+3-4i=19+0i$	S

Formula Sheet

1. Addition: $(a+ib)+(c+id)=(a+c)+i(b+d)$
2. Subtraction: $(a+ib)-(c+id)=(a-c)+i(b-d)$
3. Multiplication: $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$
4. If k is any real number, then $k(a+ib)=ka+ikb$
5. $a+bi=c+di \Leftrightarrow a=c$ and $b=d$.
6. $|x+iy|=\sqrt{x^2+y^2}$
7. $\sqrt{x+iy}=\pm\left(\sqrt{\frac{|z|+x}{2}}+\frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$, where $|z|=\sqrt{x^2+y^2}\geq 0$ is modulus of z.
8. If $\frac{-1+\sqrt{3}i}{2}=\omega$, then $\frac{-1-\sqrt{3}i}{2}=\omega^2$ Or If $\frac{-1-\sqrt{3}i}{2}=\omega$, then $\frac{-1+\sqrt{3}i}{2}=\omega^2$
9. If 1, ω, ω^2 are the cube roots of unity, then $1+\omega+\omega^2=0$ and $\omega^3=1$.
10. Polar form of $x+iy=r\cos\theta+ir\sin\theta$
11. If $z=a+bi$, then $\arg z=\theta=\tan^{-1}\left(\frac{b}{a}\right)$, $(a\neq 0)$ and for principal argument θ: $-\pi<\theta\leq\pi$
12. If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$ be two complex number in polar form, then
    - i) Addition: $z_1+z_2=r_1(\cos\theta_1+i\sin\theta_1)+r_2(\cos\theta_2+i\sin\theta_2)$
    - ii) Subtraction: $z_1-z_2=r_1(\cos\theta_1+i\sin\theta_1)-r_2(\cos\theta_2+i\sin\theta_2)$
    - iii) Multiplication: $z_1\cdot z_2=r_1\cdot r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$
        - $\arg(z_1\cdot z_2)=\arg(z_1)+\arg(z_2)$

- iv) Division: $\frac{z_1}{z_2}=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$
    - $\arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$

Multiple Choice Questions (MCQs)

Exercise 1.1

1. If $z=x+iy$ is a complex number then, y is called ----- of z.

(A) Re(z) (B) Im(z) (C) complex part (D) all of these

2. Conjugate of complex number $-2+3i$ is -----

(A) $2-3i$ (B) $2+3i$ (C) $-2-3i$ (D) $-2+3i$

3. The modulus of a complex number $x+iy$ is equal -----

(A) $\sqrt{x+y}$ (B) $\sqrt{x^2-y^2}$ (C) $\sqrt{x^2+y^2}$ (D) none of these

4. Multiplicative inverse of complex number $(\sqrt{2},-\sqrt{5})$ is -----

(A) $\left(\frac{\sqrt{2}}{\sqrt{7}},\frac{\sqrt{5}}{\sqrt{7}}\right)$ (B) $\left(-\frac{\sqrt{2}}{\sqrt{7}},-\frac{\sqrt{5}}{\sqrt{7}}\right)$ (C) $\left(\frac{\sqrt{2}}{7},\frac{\sqrt{5}}{7}\right)$ (D) $\left(-\frac{\sqrt{2}}{7},-\frac{\sqrt{5}}{7}\right)$

5. $i^2.i^4.i^6.i^8$ equals to:

(A) 1 (B) 0 (C) −1 (D) i

Exercise 1.2

6. If $(x+iy)^2=a+ib$, then $x^2-y^2=$ -----

(A) a^2-b^2 (B) a^2+b^2 (C) a (D) b

7. If z_1 & z_2 are two complex numbers then $\overline{z_1z_2}=$

(A) $\overline{z_1}\,\overline{z_2}$ (B) z_1z_2 (C) $\overline{z_1}.z_2$ (D) none of these

8. If z_1 & z_2 are two complex numbers then $\overline{z_1+z_2}=$

(A) $\overline{z_1}+\overline{z_2}$ (B) $\overline{z_1}-\overline{z_2}$ (C) z_1+z_2 (D) none of these

9. Square root of a complex number $z=x+iy$, where $x,y\in R$ is -----

(A) $\pm\left(\sqrt{\frac{|z|+x}{2}}+i\sqrt{\frac{|z|-x}{2}}\right)$ (B) $\pm\left(\sqrt{\frac{|z|+x}{2}}+\frac{i}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$

(C) $\pm\left(\sqrt{\frac{|z|+x}{2}}+\frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$ (D) none of these

10. Square root of complex number $5+12i$ is -----

(A) $3+2i$ (B) $-3-2i$ (C) $\pm(3+2i)$ (D) $-3+2i$

Exercise 1.3

11. Factors of the complex polynomial $P(z)=z^2+(i-3)z-3i$ are:

(A) $(z-i)(z-3)$ (B) $(z+i)(z+3)$ (C) $(z+i)(z-3)$ (D) $(z-i)(z+3)$

12. If p(z) is a polynomial function, the values of z that satisfy $P(z)=0$ are called the ----- of the function $P(z)$.

(A) roots (B) zeros (C) solutions (D) values

13. The complex roots of polynomial $z^2+4=0$ are -----

(A) $2i, -3i$ (B) $2i, -2i$ (C) $4, -4$ (D) $\sqrt{2}, -\sqrt{2}$