Exercise 7.5

Q1. Find the volume of parallelepiped for which the given vector are three edges.

Solution

(i) $\underline{U} = 3\underline{i} + 0\underline{j} + 2\underline{k}$;

$\underline{V} = \underline{i} + 2\underline{j} + \underline{k}$;

$\underline{W} = 0\underline{i} - \underline{j} + 4\underline{k}$

Volume of parallelepiped = $[\underline{a} \quad \underline{b} \quad \underline{c}]$

$$= \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

Expand from R1

$= 3(8+1)+0+2(-1-0)$

$= 27-2$

$= 25$ cubic unit

(ii) Volume of parallelepiped = $[\underline{a} \quad \underline{b} \quad \underline{c}]$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

Expand from R1

$= 1(-1-6)+4(1+4)-1(-3+2)$

$= -7+20+1$

$= 14$ cubic unit

(iii) $\hat{a} = \underline{i} - 2\underline{j} + 3\underline{k}$;

$\hat{b} = 2\underline{i} - \underline{j} - \underline{k}$;

$\hat{c} = 0\underline{i} - \underline{j} + \underline{k}$

Volume of parallelepiped = $[\underline{a} \quad \underline{b} \quad \underline{c}]$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

Expand from R1

$= 1(-1+1)+2(2+3)+3(2-0)$

$= 0+4+6 = 10$ cubic unit

Q2. Verify that

$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$

If $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}$; $\underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$; $\underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$

Solution

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$$

$= 3(3+10)+1(3+4)+5(20-6)$

$= 39+8+70 = 117$ cubic unit ------------- (1)

$$\underline{b}.\underline{c} \times \underline{a} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

$= 4(25+1)+3(10-3)-2(-2-15)$

$= 104-21+34 = 117$ cubic unit ------------- (2)

$$\underline{\mathbf{c}} . \underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$= 2(2-15)-5(-6-20)+1(9+4)$

$= -26 +130+13$

$= 117$ cubic unit ------------ (3)

From (1), (2) and (3)

$\underline{a} . \underline{b} \times \underline{c} = \underline{b}.\underline{c} \times \underline{a} = \underline{c}.\underline{a} \times \underline{b}$

Q3. Prove that the vectors $\underline{i}-2\underline{j}+3\underline{k}$;$-2\underline{i}+3\underline{j}-4\underline{k}$ and $\underline{i}-3\underline{j}+5\underline{k}$ are coplanar.

Solution

Let $\underline{a} = \underline{i}-2\underline{j}+3\underline{k}$;

$\underline{b} = -2\underline{i}+3\underline{j}-4\underline{k}$;

$\underline{c} = \underline{i}-3\underline{j}+5\underline{k}$

Consider

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ -3 & -3 & 5 \end{vmatrix}$$

$= 1(15\text{-}12)+2(\text{-}10+4)+3(6\text{-}3)$

$= 1(3)+2(\text{-}6)+3(3)$

$= 3\text{-}12+9 = 12\text{-}12$

$= 0$

Hence $\underline{a}, \underline{b}, \underline{c}$ are coplanar

Q4. Find the α such that the vector is coplanar

Solution

(i) Let $\underline{a} = \underline{i} - \underline{j} + \underline{k}$;

$\underline{b} = \underline{i} - 2\underline{j} - 3\underline{k}$;

$\underline{c} = 3\underline{i} - \alpha \underline{j} + 5\underline{k}$

$[\underline{a} \quad \underline{b} \quad \underline{c}] = 0$

$$= \begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix}$$

$= 0$

(ii) Let $\underline{a} = \underline{i} - 2\alpha\,\underline{j} + \underline{k}$;

$\underline{b} = \underline{i} - \underline{j} + 2\underline{k}$;

$\underline{c} = \alpha\,\underline{i} - \underline{j} + \underline{k}$

$[\underline{a} \quad \underline{b} \quad \underline{c}] = 0$

$$= \begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix}$$

$= 0$

$\Rightarrow \quad 1(-1+2)+2\alpha(1-2\alpha)-1(-1+\alpha)=0$

$\Rightarrow \quad 4\alpha^2 - \alpha - 2 = 0$

$$\alpha = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-2)}}{2(4)}$$

$$= \frac{1 \pm \sqrt{1+32}}{8}$$

$\Rightarrow \quad \alpha = \frac{1 \pm \sqrt{33}}{8}$

Q5. (a) Find the value of

Solution

(i) $2\underline{i} \times 2\,\underline{j} \,.\, \underline{k} = 4(\underline{i} \times \underline{j}) \,.\, \underline{k}$

$= 4\underline{k} \,.\, \underline{k} \quad \therefore \quad \underline{k} \,.\, \underline{k} = 1$

$= 4$

(ii) $3\underline{j}.(\underline{k}\times\underline{i}) = 3\underline{j}.\underline{j} \therefore \underline{j}.\underline{j}=1$

$= 3(1)$

$= 3$

(iii) $[\underline{k}\ \underline{i}\ \ \underline{j}] = \underline{k}.(\underline{i}\times\ \underline{j})$

$= \underline{k}.\underline{k}$

$= 1$

(iv) $[\underline{i}\ \underline{j}\ \ \underline{k}] = \underline{i}.(\underline{j}\times\ \underline{k})$

$= \underline{i}.\underline{i}$

$=0$

(b) Prove that $\underline{U}.(\underline{V}\times\underline{W}) + \underline{V}.(\underline{W}\times\underline{U}) + \underline{W}.(\underline{U}\times\underline{V}) = 3\,\underline{U}.(\underline{V}\times\underline{W})$

Solution

$L.H.S = \underline{U}.(\underline{V}\times\underline{W}) + \underline{V}.(\underline{W}\times\underline{U}) + \underline{W}.(\underline{U}\times\underline{V})$ --------- (1)

Consider

$\underline{V}.(\underline{W}\times\underline{U}) = \times\,\underline{W}\times\underline{V}$

$= \times\ \underline{U}\times\underline{W}$

$= \times \underline{U} \times \underline{V}$

$= \times \underline{V} \times \underline{W}$

Put values in (i)

$L.H.S = \underline{U}.(\underline{V} \times \underline{W}) + \underline{U}.(\underline{V} \times \underline{W}) + \underline{U}.(\underline{V} \times \underline{W})$

$= 3(\underline{U}.(\underline{V} \times \underline{W}))$

$= R.H.S$

Q6. Find the values of tetrahedron with the vertices.

(i) (0,1,2) , (3,2,1) , (1,2,1) and (5,5,6)

Solution

Let A(0,1,2), B(5,5,6) , C(1,2,1)and D(3,2,1)

Then $\hat{a} = \vec{AB} = (5-0)\ \underline{i}+(2-1)\underline{j} + (6-2)\underline{k}$

$= 5\underline{i}+4\ \underline{j} + 4\underline{k}$

$\hat{b} = \vec{AC} = (1-0)\ \underline{i}+(2-1)\underline{j} + (1-2)\underline{k}$

$= \underline{i}+\ \underline{j} - \underline{k}$

$\hat{c} = \vec{AD} = (3-0)\ \underline{i}+(2-1)\underline{j} + (1-2)\underline{k}$

$= 3\underline{i}+\ \underline{j} - \underline{k}$

Volume of tetrahedral $= \frac{1}{6}[\underline{a}\quad \underline{b}\quad \underline{c}]$

$$= \frac{1}{6}\begin{vmatrix} 5 & 4 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & -1 \end{vmatrix}$$

Expand from R1

$$= \frac{1}{6}[5(-1+1) - 4(-1+3) + 4(1-3)]$$

$$= \frac{1}{6}[5(0) - 4(2) + 4(-2)]$$

$$= \frac{1}{6}(0\text{-}8\text{-}8)$$

$$= \frac{-16}{6}$$

$$= \left|\frac{-8}{3}\right|$$

$$= \frac{8}{3}$$

(ii) Let A(2,1,8), B(3,2,9) , C(2,1,4)and D(3,3,10)

Then $\hat{a} = A\vec{B}$ = (3,2,9) – (2,1,8) = (2,1,4)

$\hat{b} = A\vec{C}$ = (2,1,4) – (2,1,8) =(0,0,4)

$\hat{c} = A\vec{D}$ = (3,3,10) – (2,1,8)=(1,2,2)

Volume of tetrahedral = $\frac{1}{6}[\underline{a} \quad \underline{b} \quad \underline{c}]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= \frac{1}{6}[1(0+8) - 1(0+4) + 1(0-0)]$$

$= \frac{1}{6}(8-4+0)$

$= \frac{1}{6}[4]$

$= \frac{2}{3}$

Q7. Find the work done , if the point at which the constant force $\underline{F} = 4\,\underline{i} + 3\,\underline{j} + 5\underline{k}$ is, applied to an object, moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$.

Solution

Let $\underline{d} = P_2 - P_1 = (2,4,6) - (3,1,-2)$

$= (2-3, 4-1, 6+2)$

$\underline{d} = -\,\underline{i} + 3\,\underline{j} + 8\underline{k}$

Work done $= \underline{F} \, . \, \underline{d}$

$= (4\,\underline{i} + 3\,\underline{j} + 5\underline{k}) \, . \, (-\,\underline{i} + 3\,\underline{j} + 8\underline{k})$

$= 4(-1) + 3(3) + 5(8)$

$= -4 + 9 + 40 = 45$

Q8. A particle, acted by constant force $4\,\underline{i} + \underline{j} - 3\underline{k}$ and $3\underline{i} - \underline{j} - \underline{k}$ is, displacement from A(1,2,3) to B(5,4,1). Find the work done.

Solution

Here $\underline{d} = \vec{AB}$

$= (5,4,1) - (1,4,3)$

$= 4\underline{i} + 2\underline{j} - 2\underline{k}$

Here $\vec{F_1} = 4\underline{i} + \underline{j} - 3\underline{k}$ and $\vec{F_2} = 3\underline{i} - \underline{j} - \underline{k}$

Total force $= \vec{F} = \vec{F_1} + \vec{F_2}$

$= 7\underline{i} + 0\underline{j} - 4\underline{k}$

Work done $= \underline{F} \cdot \underline{d}$

$= (7\underline{i} + 0\underline{j} - 4\underline{k}) \cdot (4\underline{i} + 2\underline{j} - 2\underline{k})$

$= 7(4) + 0(2) + (-4)(-2)$

$= 28 + 0 + 8 = 36$

Q9. A particle is displaced from the point A(5,-5,-7) to the point B(6,2,-2) under the action of constant forces defined by $(10\underline{i} - \underline{j} + 11\underline{k})$, $(4\underline{i} + 5\underline{j} + 9\underline{k})$ and $(-2\underline{i} + \underline{j} - 9\underline{k})$. show that the total work done by the forces I 67 units.

Solution

Here $\vec{F_1} = 10\underline{i} - \underline{j} + 11\underline{k}$, $\vec{F_2} = 4\underline{i} + 5\underline{j} + 9\underline{k}$

And $\vec{F_3} = -2\underline{i} + \underline{j} - 9\underline{k}$

Now $\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} = 12\underline{i} + 5\underline{j} + 11\underline{k}$

Here $\underline{d} = \vec{AB} = (6,2,-2) - (5,-5,-7)$

$= \underline{i} + 7\underline{j} + 5\underline{k}$

Now work done = $\underline{F} \cdot \underline{d}$

$= (12\underline{i} + 5\underline{j} + 11\underline{k}) \cdot (\underline{i} + 7\underline{j} + 5\underline{k})$

$= 12(1) + 5(7) + 11(5)$

$= 12 + 35 + 55$

$= 102$

Q10. A force of magnitude 6 unit acing parallel to2 $\underline{i}$ -2 $\underline{j}$ +2 $\underline{k}$ displacement, the point of application from (1,2,3) to (5,3,7). Find the work done.

Solution

Let $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$

$$|\underline{a}| = \sqrt{(2)^2 + (-2)^2 + (1)^2}$$

$$= \sqrt{4 + 4 + 1}$$

$$= 3$$

Let $\underline{U}$ is a unit vector parallel to $\underline{a}$ then

$$\underline{U} = \frac{\underline{a}}{|\underline{a}|} = \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$$

Here give force = $\underline{F} = 6\underline{U} = \frac{6(2\underline{i} - 2\underline{j} + \underline{k})}{3}$

$= 2(2\underline{i} - 2\underline{j} + \underline{k})$

$= 4\underline{i} - 4\underline{j} + 2\underline{k}$

And $\underline{d} = (5-1)\underline{i} + (3-2)\underline{j} + (7-3)\underline{k}$

$= 4\underline{\mathbf{i}} + \underline{j} + 4\underline{k}$

Now work done= $\underline{F} \cdot \underline{d} = (4\underline{\mathbf{i}} - \mathbf{4}\underline{j} + 2\underline{k})(4\underline{\mathbf{i}} + \underline{j} + 4\underline{k})$

$= 4(4)-4(1)+2(4)$

$= 16-4+8 = 20$

Q11. A force $\underline{F} = 3\underline{i} + 2\underline{j} - 4\underline{k}$ is applied at the point (1,-1,2). Find moment of force about the point (2,-1,3).

Solution

$\vec{F} = 3\underline{i} + 2\underline{j} - 4\underline{k}$

P = (1,-1,2) and A (2,-1,3)

Now $\quad \vec{AP} = (1,-1,2) - (2,-1,3)$

$= (1-2, -1+1, 2-3)$

$= (-1,0,-1)$

$$\vec{AP} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

Expand

$= (0+2)\,\underline{I} - (4+3)\,\underline{j} + (-2)\,\underline{k}$

$= 2\underline{i} - 7\underline{j} - 2\underline{k}$

Q12. A force $\underline{F} = 4\underline{i} - 3\underline{k}$ passes through the point A(2,-2,5). Find moment of force about the point B(1,-3,1).

Solution

$\vec{F} = 4\underline{i} + 0\underline{j} - 3\underline{k}$

A (2,-2,5) and B (1,-3,1)

Now $\vec{BA}$ = (2,-2,5) - (1,-3,1)

= (2-1, -2+3, 5-1)

= (1,1,4)

$$\vec{BA} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix}$$

Expand

= (-3) $\underline{i}$ –(-3-16) $\underline{j}$ +(0-4) $\underline{k}$

= $-3\underline{i} + 19\underline{j} - 4\underline{k}$

Q13. A force $\underline{F} = 2\underline{i} + 1\underline{j} - 3\underline{k}$ acting at a point A(1,-2,1). Find moment of force about the point B(2,0,-2).

Solution

$\vec{F} = 2\underline{i} + \underline{j} - 3\underline{k}$

A (1,-2,1) and B (2,0,-2)

Now $\vec{BA} = (1,-2,1) - (2,0,-2)$

$= (1-2, -2-0, 1+2)$

$= (-1,-2,3)$

$$\vec{BA} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$$

Expand

$= (6-3)\,\underline{i} - (3-6)\,\underline{j} + (-1+4)\,\underline{k}$

$= -3\underline{i} + 3\underline{j} + 3\underline{k}$

Q14. Find the moment about A(1,1,1) of each of the concurrent forces $\underline{i} - 2\underline{j}$, $3\underline{i} + 2\underline{j} - \underline{k}$, $5\underline{j} + 2\underline{k}$ where P (2,0,1) is the point of concurrency.

Solution

Let $\vec{F_1} = \underline{i} - 2\underline{j} + 0\underline{k}$,

$\vec{F_2} = 3\underline{i} + 2\underline{j} - \underline{k}$

$\vec{F_3} = 0\underline{i} + 5\underline{j} + 2\underline{k}$

Resultant force $\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3}$

$= \underline{i} - 2\underline{j} + 0\underline{k} + 3\underline{i} + 2\underline{j} - \underline{k} + 0\underline{i} + 5\underline{j} + 2\underline{k}$

$= 4\underline{i} + 5\underline{j} + \underline{k}$

Hence, measure $\vec{R}$ about A = $\vec{AP} \times \vec{A}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 4 & 5 & 1 \end{vmatrix}$$

$= (-1-0)\,\underline{i} - (1-0)\,\underline{j} + (5+4)\,\underline{k}$

$= -\underline{i} - \underline{j} + 9\underline{k}$

Q15. A force $\underline{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$ is, applied at P(1,-2,3). Find ITS moment about the point Q(2,1,1).

Solution

$\vec{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$

$\vec{PQ} = (1-2, -2-1, 3-1)$

$= (-1,-3,2)$

$$\vec{PQ} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix}$$

Expand $= (9-8)\,\underline{i} - (3-14)\,\underline{j} + (-4+21)\,\underline{k}$

$= \underline{i} + 11\underline{j} + 17\underline{k}$