Exercise 7.4

Q1. Compute the cross product $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$ check your answer by showing that $\underline{a}$ and $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$.

(i) $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$; $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

Solution

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$$

$$\mathbf{\underline{a} \times \underline{b} = -3\underline{j} - 3\underline{k} \Rightarrow 0\underline{i} - 3\underline{j} - 3\underline{k}}$$

Now, $\underline{a}.(\underline{a} \times \underline{b}) = (2\underline{i} + \underline{j} - \underline{k}).(-3\underline{j} - 3\underline{k})$

$= (2\underline{i} + \underline{j} - \underline{k}).(0\underline{i} - 3\underline{j} - 3\underline{k})$

$= 2(0) + 1(-3) + (-1)(-3)$

$= 0-3+3 = 0 \Rightarrow 0\underline{i} + 0\underline{j} + 0\underline{k}$

$\Rightarrow$ **$\underline{a} \times \underline{b} \perp \underline{a}$**

Now, $\underline{b}.(\underline{a} \times \underline{b}) = (\underline{i} - \underline{j} + \underline{k}).(0\underline{i} - 3\underline{j} - 3\underline{k})$

$= 1(0) + (-1) + (-3) + 1(-3)$

As $\underline{b}(\underline{a} \times \underline{b}) = 3-3 = 0$

As $\underline{b}(\underline{a} \times \underline{b}) = 0 \Rightarrow \underline{a} \times \underline{b}$ is $\perp \underline{b}$

Now, $\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$

$= \underline{i}(1-1) - \underline{j}(-1-2) + \underline{k}(1+2)$

$\underline{b} \times \underline{a} = 0\underline{i} + 3\underline{j} + 3\underline{k}$

Now $(\underline{b} \times \underline{a})\ \underline{a} = (0\underline{i} + 3\underline{j} + 3\underline{k}).\ (2\underline{i} + \underline{j} - \underline{k})$

$= 0 \times 2 + 3 \times 1 + 3 \times (-1)$

$= 0+3-3$

$= 2(0) + (1-3) + (-1)(-3)$

$= 0 - 3 + 3 = 0$

As $(\underline{b} \times \underline{a}).\ \underline{a} = \underline{b} \times \underline{a} \perp \underline{a}$

Now $(\underline{b} \times \underline{a}).\ \underline{a} = (0\underline{i} + 3\underline{j} + 3\underline{k}).\ (\underline{i} - \underline{j} + \underline{k})$

$= 0 - 3 + 3$

As $(\underline{b} \times \underline{a}).\ \underline{b} = 0$

$\Rightarrow (\underline{b} \times \underline{a})$ is $\perp\ \underline{b}$

(ii) $\underline{a} = \underline{i} + \underline{j} + 0\underline{k}$; $\underline{b} = \underline{i} - \underline{j} + 0\underline{k}$

Solution

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$= \perp(0) - \underline{j}\ (0) + \underline{k}\ (-1-1)$

$\underline{a} \times \underline{b} = -2\underline{k}$ $=0\underline{i} + 0\underline{j} - 2\underline{k}$

Now, $\underline{a}.\ (\underline{a} \times \underline{b}) = (\underline{i} + \underline{j} + 0\underline{k}).\ (0\underline{i} + 0\underline{j} - 2\underline{k})$

$= 1 \times 0 + (-1).\ (0) + 0\ (-2)$

As $\underline{a}.\ (\underline{a} \times \underline{b}) = 0$

$\Rightarrow \quad \underline{b}$ is $\perp \underline{a} \times \underline{b}$

Now $\quad \underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$= \underline{i}\,(0) - \underline{j}\,(0) + \underline{k}\,(1+1)$

$\underline{b} \times \underline{a} = 0\underline{i} + 0\underline{j} + 2\underline{k}$

$\underline{a}.\,(\underline{b} \times \underline{a}) = (\underline{i} + \underline{j} + 0\underline{k})\,(0\underline{i} + 0\underline{j} + 2\underline{k})$

$= 0+0+0$

$\Rightarrow \quad \underline{a}.\,(\underline{b} \times \underline{a}) = 0 \Rightarrow \underline{b} \times \underline{a} \perp \underline{a}$

Again $\underline{b}.\,(\underline{b} \times \underline{a}) = (\underline{i} - \underline{j} + 0\underline{k})\,(0\underline{i} + 0\underline{j} + 2\underline{k})$

$= 0+0+0$

$\underline{b}.\,(\underline{b} \times \underline{a}) = 0 \Rightarrow \underline{b} \times \underline{a}$ is $\perp \underline{b}$

$b.\,(b \times a) = (\underline{i} - \underline{j} + 0\underline{k}).\,(0\underline{i} + 0\underline{j} + 2\underline{k})$

$= (1)\,(0) + (-1)\,(0) + (0)\,(2)$

$= 0+0+0 = 0$

$\Rightarrow b \perp b \times \underline{a}$

(iii) $\quad \underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}\ ;\ b = \underline{i} + \underline{j} + 0\underline{k}$

Solution

$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= (0\text{-}1)\,\underline{i} - \underline{j}\,(0\text{-}1) + (3+2)\,\underline{k}$

$\underline{a} \times \underline{b} = -\underline{i} + \underline{j} + 5\underline{k}$

$(\underline{a} \times \underline{b}) . \underline{a} = (-\underline{i} + \underline{j} + 5\underline{k}).(3\underline{i} - 2\underline{j} + \underline{k})$

$= -1+1+0$

$(\underline{a} \times \underline{b}) . \underline{b} = 0 \Rightarrow \underline{a} \times \underline{b}$ is $\perp \underline{b}$

Now $\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$

$= \underline{i}\,(1 \times 1 - 0 \times (-2)) - \underline{j}\,(1\times1-3\times0) + \underline{k}\,(1\times (-2))$

$\underline{b} \times \underline{a} = \underline{i} - \underline{j} - 5\underline{k}$

$(\underline{b} \times \underline{a}). \underline{a} = (\underline{i} - \underline{j} - 5\underline{k})\,(3\underline{i} - 2\underline{j} + \underline{k})$

$= 3+2-5 = 0$

$\Rightarrow \quad \underline{b} \times \underline{a} \perp \underline{a}$

$(\underline{b} \times \underline{a}) . \underline{b} = (\underline{i} - \underline{j} - 5\underline{k})\,(\underline{i} + \underline{j} + 0\underline{k})$

$= 1\times1-1\times1-5\text{-}0 = 0$

$\Rightarrow \quad \underline{b} \times \underline{a} \perp \underline{b}$

(iv) $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$; $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

Solution

$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$

$= \underline{i}\,(1+2) - \underline{j}\,(-4+4) + \underline{k}\,(-4-2)$

$\underline{a} \times \underline{b} = 3\underline{i} + 0\underline{j} - 6\underline{k}$

$(\underline{a} \times \underline{b}) \cdot \underline{a} = (3\underline{i} + 0\underline{j} - 6\underline{k}) \cdot (-4\underline{i} + \underline{j} + 2\underline{k})$

$(\underline{a} \times \underline{b}) \cdot \underline{a} = -12+0+12 = 0$

$\underline{a} \times \underline{b}$. is $\perp \underline{a}$

$(\underline{a} \times \underline{b}) \cdot \underline{a} = (3\underline{i} + 0\underline{j} - 6\underline{k}) \cdot (-4\underline{i} + \underline{j} + 2\underline{k})$

$= 3\times2+0\times1\ (-6)\ (1) = 6 - 6 = 0$

$(\underline{a} \times \underline{b}) \cdot \underline{b} = 0 \Rightarrow \underline{a} \times \underline{b} \perp \underline{b}$

Now $\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ -4 & 1 & -2 \end{vmatrix}$

$= \underline{i}\ (-2-1) - \underline{j}\ (-4+4) + \underline{k}\ (2+4)$

$\underline{b} \times \underline{a} = -3\underline{i} + 0\underline{j} + 6\underline{k}$

$(\underline{b} \times \underline{a}) \cdot \underline{a}. = (-3\underline{i} + 0\underline{j} + 6\underline{k})\ (4\underline{i} + \underline{j} + 2\underline{k})$

$= 12+0-12 = 0$

$\Rightarrow \underline{b} \times \underline{a} \perp \underline{a}$

$\Rightarrow (\underline{b} \times \underline{a}) \cdot \underline{b} = (-3\underline{i} + 0\underline{j} + 6\underline{k}) \cdot (2\underline{i} + \underline{j}. + \underline{k})$

$= -6 + 0.6 = 0$

$\Rightarrow \underline{b} \times \underline{a} \perp \underline{b}$

Q 2. Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$. Also find sine of the angle between them.

(i) $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$, $4\underline{i} + 3\underline{j} - \underline{k}$

Solution

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$

$= \underline{i}\,(6+9) \times \underline{j}\,(-2+12) + \underline{k}\,(6+24)$

$a \times \underline{b} = 15\underline{i} - 10\underline{j} + 30\underline{k}$

$|\underline{a} \times \underline{b}| = \sqrt{(15)^2 + (-10)^2 + (30)^2}$

$= \sqrt{1225}$

$= 35$

As $\underline{a} \times \underline{b}$ is vector $\perp$ to both $\underline{a}$ and $\underline{b}$

$= \frac{a \times b}{|a \times b|}$

$= \frac{15i - 10j + 30k}{35}$

Also $|\underline{a}| = \sqrt{4 + 36 + 9} = 7$;

And $|\underline{b}| = \sqrt{16 + 9 + 1} = \sqrt{26}$

If θ is the angle between $\underline{a}$ and $\underline{b}$ then $\sin\theta = \frac{|a \times b|}{|a| \times |b|}$

$= \frac{35}{7\sqrt{26}}$

$= \frac{5}{\sqrt{26}}$

(ii) $\underline{a} = -\underline{i} - \underline{j} - \underline{k}$, $= 2\underline{i} - 3\underline{j} + 4\underline{k}$

Solution

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$= \underline{i}\,(-4-3) - \underline{j}\,(-4+2) + \underline{k}\,(3+2)$

$\underline{a} \times \underline{b} = -7\underline{i} + 2\underline{j} + 5\underline{k}$

$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2}$

$= \sqrt{78}$

$|\underline{a}| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2}$

$= \sqrt{1+1+1}$

$= \sqrt{3}$

$|\underline{b}| = \sqrt{(2)^2 + (-3)^2 + (4)^2}$

$= \sqrt{4+9+16}$

$= \sqrt{29}$

Required unit vector n

$= \frac{a \times b}{|a \times b|}$,

$= \frac{-7\underline{i}+2\underline{j}+5\underline{k}}{\sqrt{78}}$

If θ is the angle between $\underline{a}$ and $\underline{b}$ then sinθ

$= \frac{|a \times b|}{|a||b|}$

$= \frac{\sqrt{78}}{\sqrt{3}\sqrt{29}}$

$= \frac{\sqrt{78}}{\sqrt{87}}$

$= \sqrt{\frac{78}{87}}$

$= \sqrt{\frac{26}{29}}$

(iii) $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k}$

$\underline{b} = -\underline{i} + 3\underline{j} - 2\underline{k}$

Solution

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$$

$= \underline{i}\,(4-4) - \underline{j}\,(-4+4) + \underline{k}\,(2-2)$

$= 0\underline{i} - 0\underline{j} + 0\underline{k}$

$\Rightarrow$ (Null vector)

$|\underline{a}| = \sqrt{(2)^2 + (-2)^2 + (4)^2}$

$= \sqrt{24}$

$|\underline{b}| = \sqrt{(-1)^2 + (1)^2 + (-2)^2}$

$= \sqrt{6}$

$= \frac{0}{\sqrt{24}\sqrt{6}}$

Hence $\sin\theta = \frac{|a \times b|}{|a||b|}$

$= 0$

(iv) $\underline{a} = \underline{i} + \underline{j}$

$\underline{b} = \underline{i} - \underline{j}$

Solution

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$= \underline{i}\ (0\text{-}0) - \underline{j}\ (0\text{-}0) + \underline{k}\ (\text{-}1\text{-}1)$

$\underline{a} \times \underline{b} = \text{-}2\underline{k}$

$|\ \underline{a} \times \underline{b}\ | = \sqrt{(-2)^2}$

$= \sqrt{4}$

$= 2$

Required unit vector

$= \frac{|a \times b|}{|a||b|}$

$= \frac{-2k}{2}$

$= \text{-}\underline{k}$

$|\ \underline{a}\ | = \sqrt{1+1+1}$

$= \sqrt{2}$;

$|\ \underline{b}\ | = \sqrt{1+1+0}$

$= \sqrt{2}$

If θ is the angle between $\underline{a}$ and $\underline{b}$ then

$\sin\theta = \frac{|a \times b|}{|a||b|}$

$= \frac{2}{\sqrt{2} \times \sqrt{2}}$

$= \frac{2}{\sqrt{4}}$

$= \frac{2}{2}$

$= 1$

Q3. Find the area of the triangle, determined by the point P, Q, and R.

(i) P(0,0,0) ; Q(2,3,2) ; R(-1,1,4)

Solution

Area of Δ PQR

$$= \frac{1}{2}|\vec{PQ} \times \vec{PR}| \qquad = \frac{1}{2}|\vec{QP} \times \vec{QR}| \qquad = \frac{1}{2}|\vec{RP} \times \vec{RQ}|$$

We can use any one formula

Here P[0,0,0] , Q[0,0,0] , R[-1,1,4]

$\vec{PQ} = [2,3,2] - [0,0,0] = [2,3,2]$

$\vec{PR} = [-1,1,4] - [0,0,0] = [-1,1,4]$

Now $\vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$

$= \underline{i}(12-2) - \underline{j}(8+2) + \underline{k}(2+3)$

$\vec{PQ} \times \vec{PR} = 10\underline{i} - 10\underline{j} + 5\underline{k}$

$\vec{PQ} \times \vec{PR} = \sqrt{(10)^2 + (-10)^2 + (5)^2}$

$= \sqrt{225}$

$= 15$

Area of Δ PQR $= \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$= \frac{1}{2}(15)$

$= \frac{15}{2}$ Sq. Unit

(ii) P (1,-1,-1) ; Q(2,0,-1) ; R(0,2,1)

Solution

P[1,-1,-1] ; Q[2,0,-1] ; R[0,2,1]

$\vec{PQ}$ = [2,0,-1] – [1,-1,-1] = [2-1,0+1,-1+1]

$\vec{PQ}$ = [1,1,0] = i + j + 0k

$\vec{PR}$ = p. v of R – p. v of P = [0,2,1] – [1,-1,-1]

= [0-1,2+1+1] . = [-1,3,2] = -i + 3j + 2k

Now $\vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$

= i (2-0) – j (2-0) + k (3+1)

$\vec{PQ} \times \vec{PR}$ = 2i – 2j + 4k

$|\vec{PQ} \times \vec{PR}| = \sqrt{(2)^2 + (-2)^2 + (4)^2}$

$= \sqrt{4 + 4 + 16}$

$= \sqrt{24}$

$= \sqrt{6 + 4}$

$= 2\sqrt{6}$

Area of Δ PQR = $\frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$= \frac{1}{2}2\sqrt{6}$

= $\sqrt{6}$ Sq. Unit

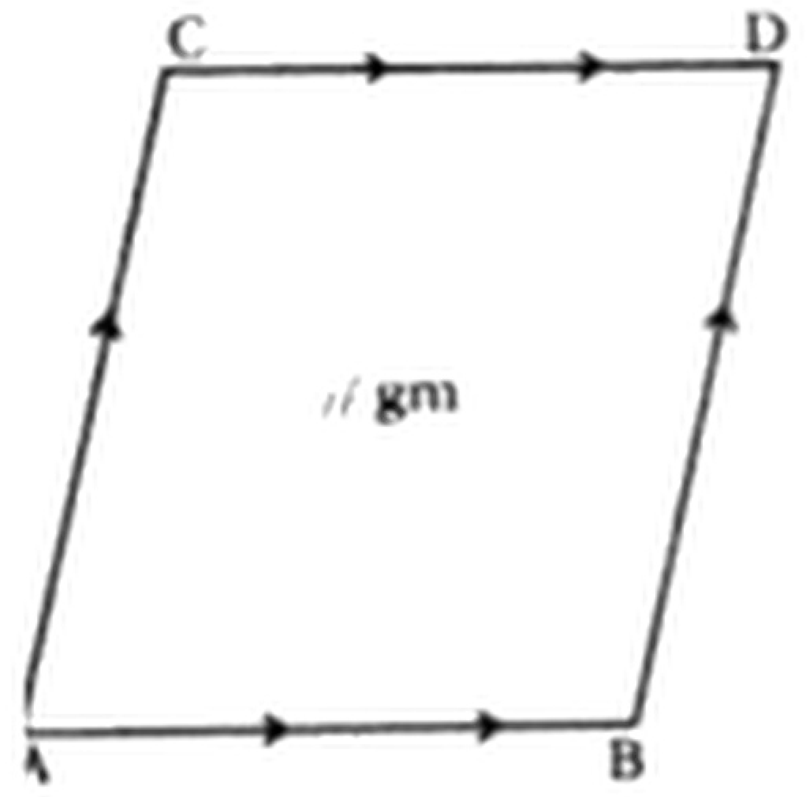
Q4. Find the area of parallelogram, whose vertices are:

(i) A[0,0,0] , B[1,2,3] , C[2,-1,1] , D[3,1,4]

Solution

First off all check which two sides are parallel

Here $\vec{AB} = \vec{CD}$ $\Rightarrow$ $\vec{AB}$ is parallel to $\vec{AC}$



So

$\Rightarrow$ ABCD is a parallelogram

Area of parallelogram ABCD = $|\vec{AB} \times \vec{AC}|$

$\vec{AB} = [1,2,3] - [0,0,0] = [1,2,3]$

$\vec{AC}$ = p.v of C – p.v of A

$= [2,-1,1] - [0,0,0] = [2,-1,1]$

$$|\vec{AB} \times \vec{AC}| = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

$= \underline{i}(2+3) - \underline{j}(1-6) + \underline{k}(-1-4)$

$= 5\underline{i} + 5\underline{j} - 5\underline{k}$

$|\vec{AB} \times \vec{AC}| = \sqrt{(5)^2 + (5)^2 + (-5)^2}$

$= \sqrt{25 + 25 + 25}$

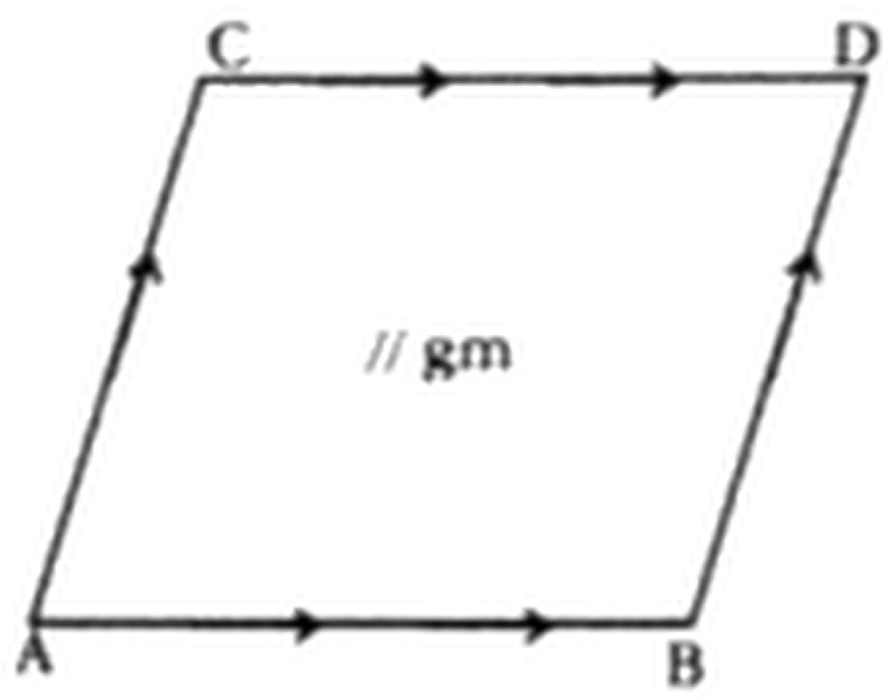
$= \sqrt{75}$

$= \sqrt{25 \times 3}$

$= 5\sqrt{3}$

(ii) A[1,2,-1], B[4,2,-3], C[6,-5,2], D[9,-5,0]

Solution



$\vec{AB}$ = p.v of B – p.v of A

= [4,2,-3] – [1,2,-1]

= [3,0,-2]

$\vec{CD}$ = p.v of D – p.v of C

= [9,-5,0] – [6,-5,2]

= [3,0,-2]

As $\vec{AB} = \vec{CD}$ ⇒ $\vec{AB}$ is parallel to $\vec{CD}$

So ABCD is a parallelogram

$\vec{AC} = [6,-5,2] - [1,2,-1]$

$= [6-1,-5-2,-2+1]$

$= [5,-7,3]$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 5 & -7 & 3 \end{vmatrix}$$

$= \underline{i}(0 - 14) - \underline{j}(9 + 10) + \underline{k}(-21 - 0)$

$= -14\underline{i} - 19\underline{j} - 21\underline{k}$

Area of parallelogram ABCD $= |\vec{AB} \times \vec{AC}|$

$|\vec{AB} \times \vec{AC}| = \sqrt{(-14)^2 + (-19)^2 + (-21)^2}$

$= \sqrt{196 + 361 + 441}$

$= \sqrt{998}$

(ii) A[-1,1,1], B[-1,2,2], C[-3,4,-5], D[-3,5,-4]

Solution

$\vec{AB} = [-1,2,2] - [-1,1,1]$

$= [-1+1,2-1,2-1]$

$= [0,1,1]$

$\vec{CD}$ = p.v of D – p.v of C

$= [-3,5,-4] - [-3,4,-5]$

$= [-3+3,5-4,-4+5]$

$= [0,1,1]$

As $\vec{AB} = \vec{CD}$

⇒ $\vec{AB}$ is parallel to $\vec{CD}$

So ABCD is a parallelogram

Now

$\vec{AC} = [-3,4,-5] - [-1,1,1]$

$= [-3+1,4-1,-5-1]$

$= [-2,3,-6]$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix}$$

$= \underline{i}(-6-3) - \underline{j}(0+2) + \underline{k}(0+2)$

$= -9\underline{i} - 2\underline{j} + 2\underline{k}$

Area of parallelogram ABCD $= |\vec{AB} \times \vec{AC}|$

$|\vec{AB} \times \vec{AC}| = \sqrt{(-9)^2 + (-2)^2 + (2)^2}$

$= \sqrt{81+4+4}$

$= \sqrt{89}$

Q5. Which vectors, if any, are perpendicular or parallel

(i) $\underline{U} = 5\underline{i} - \underline{j} + \underline{k}$; $\underline{V} = \underline{j} - 5\underline{k}$; $\underline{W} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

Solution

$\underline{U} . \underline{V} = (5\underline{i} - \underline{j} + \underline{k}) . (\underline{j} - 5\underline{k})$

$= 0 - 1 - 5$

$= -6 \neq 0$

$\underline{U}$ and $\underline{V}$ are not perpendicular to each other

$\underline{U} . \underline{W} = (5\underline{i} - \underline{j} + \underline{k}) . (-15\underline{i} + 3\underline{j} - 3\underline{k})$

$= -75 - 3 - 3 \neq 0$

$\underline{U}$ and $\underline{W}$ are not perpendicular to each other

$\underline{V} \cdot \underline{W} = (\underline{j} - 5\underline{k}) \cdot (-15\underline{i} + 3\underline{j} - 3\underline{k})$

$= 0+3+15 \neq 0$

$\underline{V}$ and $\underline{W}$ are not perpendicular to each other

Now $\underline{W} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

$= -3\ (5\underline{i} - \underline{j} + \underline{k})$

$\underline{W} = \underline{U}$

$\Rightarrow \underline{W} \parallel \underline{U}$

$\Rightarrow$ $\underline{W}$ and $\underline{U}$ are parallel except this neither pair is parallel

(ii) $\underline{U} = \underline{i} + 2\underline{j} - \underline{k}$; $\underline{V} = -\underline{i} + \underline{j} + \underline{k}$ and $\underline{W} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

Solution

$\underline{U} \cdot \underline{V} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (-\underline{i} + \underline{j} + \underline{k})$

$= -1+2-1 = 0$

$\underline{U}$ and $\underline{V}$ are perpendicular to each other

$\underline{U} \perp \underline{V}$

$\underline{U} \cdot \underline{W} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k})$

$= \frac{\pi}{2} - 2\pi - \frac{\pi}{2} = \pi$

$\underline{U}$ and $\underline{W}$ are not perpendicular to each other

$$\underline{V} . \underline{W} = (-\underline{i} + \underline{j} + \underline{k}) . (-\frac{\pi}{2}\underline{i} - \pi\,\underline{j} + \frac{\pi}{2}\underline{k})$$

$$= \frac{\pi}{2} - \pi + \frac{\pi}{2} = 0$$

$\underline{V}$ and $\underline{W}$ are perpendicular to each other

$\underline{V} \perp \underline{W}$

$$\underline{W} = -\frac{\pi}{2}\underline{i} - \pi\,\underline{j} + \frac{\pi}{2}\underline{k}$$

$$= \frac{\pi}{2}\left(\underline{i} + 2\,\underline{j} \quad \underline{k}\right)$$

$$= \frac{\pi}{2}\underline{U}$$

$\Rightarrow \underline{W} \parallel \underline{U}$

$\Rightarrow$ $\underline{W}$ and $\underline{U}$ are parallel except this neither pair is parallel

Q6. Prove that

$$\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + c \times (\underline{a} + \underline{b}) = 0$$

Solution

$$\text{L.H.S} = \underline{a} \times (b + \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$$

$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b}$$

As $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$

But $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

$$= \underline{a} \times b + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} - \underline{a} \times \underline{b} - \underline{a} \times \underline{c} - \underline{b} \times \underline{c}$$

$$= 0 = \text{R.H.S}$$

So proved

Q7. **If $\underline{a} + \underline{b} + \underline{c} = 0$, then prove that**

$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

Solution

$$\underline{a} + \underline{b} + \underline{c} = 0$$

Taking its cross product with a on both sides

$$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{a} \times 0$$

$$\underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$$

As $\underline{a} \times \underline{a} = 0$

$$0 + \underline{a} + 0 + \underline{b} \times \underline{c} = 0$$

$$\underline{a} \times \underline{b} = \underline{c} \times \underline{a} \quad \text{-----------------(A)}$$

Also (i) cross multiplication with $\underline{b}$

$$\Rightarrow \quad \underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{b} \times 0$$

$$\underline{b} \times \underline{a} + 0 + \underline{b} \times \underline{c} = 0$$

$$b \times \underline{c} = -\underline{b} \times \underline{a}$$

$$b \times c = a \times b \quad \text{-----------------------(B)}$$

Comparing (A) and (B) we get

$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$$

Q8. **Prove that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$**

Solution

Let $\hat{a}$ and $\hat{b}$ be the unit in xy-plane making angles α and β with x-axis respectively such that α > β

Then clearly

$$\hat{a} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$$

$$\hat{b} = \cos\beta\,\underline{i} + \sin\beta\,\underline{j}$$

and

$$\text{Now } \hat{a}\times\hat{b} = \begin{vmatrix} i & j & k \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$= \underline{i}(0) - \underline{j}(0) + \underline{k}(\cos\beta \quad \sin\alpha - \sin\beta\cos\alpha)$$

$$\Rightarrow \quad \hat{b}\times\hat{a} = (\cos\beta\sin\alpha - \sin\beta\cos\alpha)\,\hat{k} \text{ ------------------------(1)}$$

$$\text{Also} \quad \hat{b}\times\hat{a} = |\hat{b}|\,|\hat{a}|\,\sin(\alpha-\beta)\,\hat{k}$$

$$\text{As} \quad |\hat{b}| = |\hat{a}| = 1$$

$$\hat{b}\times\hat{a} = (1)(1)\sin(\alpha-\beta)\,\hat{k} \text{ ------------------- (2)}$$

Now Comparing (1) and (1) we get

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Q9. If $\underline{a}\times\underline{b} = 0$ and a.b = 0

What conclusion can be drawn about $\underline{a}$ or $\underline{b}$?

Solution

$\underline{a}\times\underline{b} = 0$ and a.b = 0

Now $\underline{a}\times\underline{b} = 0$ ⇒ $\underline{a}$ and $\underline{b}$ are parallel

And $\underline{a} . \underline{b} = 0$ $\Rightarrow \underline{a}$ and $\underline{b}$ are parallel

This is not possible that $\underline{a}$ and $\underline{b}$ are perpendicular and at the same time $\underline{a}$ and $\underline{b}$ are parallel.

So the given information i.e.

$\underline{a} \times \underline{b} = 0$ and $a.b = 0$ is possible only if at least one of the given vector $\underline{a}$ or $\underline{b}$ is a null vector.