Exercise 7.3

Q1. Find the cosine of the angle between $\underline{U}$ and $\underline{V}$.

(i) $\underline{U} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{V} = 2\underline{i} - 3\underline{j} + \underline{k}$

(iii) $\underline{U} = [-3, 5]$, $\underline{V} = [6, -2]$

(ii) $\underline{U} = \underline{i} - 3\underline{j} + 4\underline{k}$, $\underline{V} = 4\underline{i} - \underline{j} + 3\underline{k}$

(iv) $\underline{U} = [2, -3, 1]$, $\underline{V} = [2, 4, 1]$

Solution

(i) A dot product between two vectors is given by

$$\underline{U} \cdot \underline{V} = |\underline{U}||\underline{V}| \cos\theta$$

$\Rightarrow$ $\cos\theta = \frac{\underline{U} \cdot \underline{V}}{|\underline{U}||\underline{V}|}$ -------------- (A)

Now $|\underline{U}| = \sqrt{(3)^2 + (1)^2 + (-1)^2}$

$= \sqrt{9 + 1 + 1}$

$= \sqrt{11}$

$|\underline{V}| = \sqrt{(2)^2 + (-1)^2 + (1)^2}$

$= \sqrt{4 + 1 + 1}$

$= \sqrt{6}$

$\underline{U} \cdot \underline{V} = (3\underline{i} + \underline{j} - \underline{k})(2\underline{i} - 3\underline{j} + \underline{k})$

$= 3(2)+(1)(-1)+(-1)(1)$

$= 6-1-1 \quad = 4$

Putting all these values in (A)

$$\text{Cos}\,\theta = \frac{4}{\sqrt{11}.\sqrt{6}}$$

$$= \frac{4}{\sqrt{66}}$$

(ii) $\underline{\mathbf{U}} = \underline{\mathbf{i}} - 3\underline{\mathbf{j}} + 4\underline{\mathbf{k}}$, $\underline{\mathbf{V}} = 4\underline{\mathbf{i}} - \underline{\mathbf{j}} + 3\underline{\mathbf{k}}$

$$|\underline{U}| = \sqrt{(1)^2 + (-3)^2 + (4)^2}$$

$$= \sqrt{1 + 9 + 16}$$

$$= \sqrt{26}$$

$$|\underline{V}| = \sqrt{(4)^2 + (-1)^2 + (3)^2}$$

$$= \sqrt{16 + 1 + 9}$$

$$= \sqrt{26}$$

Now $\underline{U} . \underline{V} = (\underline{i} - 3\underline{j} + 4\underline{k})(4\underline{i} - \underline{j} + 3\underline{k})$

$$= 1(4)+(-3)(-1)+(4)(3)$$

$$= 4+3+12$$

$$= 19$$

As $\cos\theta = \frac{\underline{U}.\underline{V}}{|\underline{U}||\underline{V}|}$

$$\text{Cos}\,\theta = \frac{19}{\sqrt{26}.\sqrt{26}}$$

$$= \frac{19}{26}$$

(iii) $\underline{\mathbf{U}} = [-3, 5]$, $\underline{\mathbf{V}} = [6, -2]$

$\underline{\mathbf{U}} = -3\underline{\mathbf{i}} + 5\underline{\mathbf{j}}$, $\underline{\mathbf{V}} = 6\underline{\mathbf{i}} - 2\underline{\mathbf{j}}$

$|\underline{U}| = \sqrt{(-3)^2 + (5)^2}$

$= \sqrt{9 + 25}$

$= \sqrt{34}$

$|\underline{V}| = \sqrt{(6)^2 + (-2)^2}$

$= \sqrt{36 + 4}$

$= \sqrt{40}$

$\Rightarrow \quad |\underline{V}| = 2\sqrt{10}$

$\underline{U} . \underline{V} = (-3\underline{i} + 5\underline{j})(6\underline{i} - 2\underline{j})$

$=(- 3)(6)+(5)(-2)$

$= -18-10 \quad = -28$

$\Rightarrow \quad \underline{U} . \underline{V} = -28$

$\Rightarrow \quad \cos\theta = \frac{\underline{U} . \underline{V}}{|\underline{U}||\underline{V}|}$

$\text{Cos}\,\theta = \frac{-28}{\sqrt{34}.2\sqrt{10}}$

$= \frac{-14}{\sqrt{34}.\sqrt{10}}$

$= \frac{-14}{2\sqrt{85}}$

(iv) $\underline{U} = [2, -3, 1]$, $\underline{V} = [2, 4, 1]$

$\underline{U} = 2\underline{i} - 3\underline{j} + \underline{k}$, $\underline{V} = 2\underline{i} + 4\underline{j} + \underline{k}$

$|\underline{U}| = \sqrt{(2)^2 + (-3)^2 + (1)^2}$

$= \sqrt{4 + 9 + 1}$

$= \sqrt{14}$

$|\underline{V}| = \sqrt{(2)^2 + (4)^2 + (1)^2}$

$= \sqrt{4 + 16 + 1}$

$= \sqrt{21}$

Now $\underline{U} \cdot \underline{V} = (2\underline{i} - 3\underline{j} + \underline{k})(2\underline{i} + 4\underline{j} + \underline{k})$

$= 2(2)+(-3)(4)+(1)(1)$

$= 4-12+1$

$= -7$

As $\cos\theta = \frac{\underline{U}.\underline{V}}{|\underline{U}||\underline{V}|}$

$\text{Cos}\,\theta = \frac{-7}{\sqrt{14}.\sqrt{21}}$

$= \frac{-7}{7\sqrt{6}}$

$= \frac{-1}{\sqrt{6}}$

Q2. Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when.

Solution

$\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$

$|\underline{a}| = \sqrt{(1)^2 + (-1)^2}$

$= \sqrt{1 + 1}$

$= \sqrt{2}$

$|\underline{b}| = \sqrt{(1)^2 + (1)^2}$

$= \sqrt{1 + 1}$

$= \sqrt{2}$

$\underline{a} \cdot \underline{b} = (\underline{i} - \underline{k})(\underline{j} + \underline{k})$

$= 1(0)+(0)(1)+(-1)(1)$

$= 0+0-1$

$= -1$

Now projection of $\underline{a}$ along $\underline{b}$ $= \frac{\underline{a}.\underline{b}}{|\underline{b}|} = \frac{-1}{\sqrt{2}}$

also projection of $\underline{b}$ along $\underline{a}$ $= \frac{\underline{a}.\underline{b}}{|\underline{a}|} = \frac{-1}{\sqrt{2}}$

(ii) $\underline{\mathbf{a}} = 3\underline{\mathbf{i}} + \underline{\mathbf{j}} - \underline{\mathbf{k}}$, $\underline{\mathbf{b}} = -2\underline{\mathbf{i}} - \underline{\mathbf{j}} + \underline{\mathbf{k}}$

$|\underline{a}| = \sqrt{9 + 1 + 1}$

$= \sqrt{11}$

$|\underline{b}| = \sqrt{4 + 1 + 1}$

$= \sqrt{6}$

$\underline{a} \cdot \underline{b} = (3\underline{i} + \underline{j} - \underline{k})(-2\underline{i} - \underline{j} + \underline{k})$

$= 3(-2)+(1)(-1)+(-1)(1)$

$= -6-1-1$

$= -8$

Now projection of $\underline{a}$ along $\underline{b}$ $= \frac{\underline{a}.\underline{b}}{|\underline{b}|} = \frac{-8}{\sqrt{6}}$

also projection of $\underline{b}$ along $\underline{a}$ $= \frac{\underline{a}.\underline{b}}{|\underline{a}|} = \frac{-8}{\sqrt{11}}$

Q3. Find a real number α so that $\underline{U}$ and $\underline{V}$ are perpendicular.

Solution

(i) We are given $\underline{U}$ and $\underline{V}$ are perpendicular.

$\Rightarrow \quad \underline{U} . \underline{V} = 0$

$\Rightarrow \quad (2\alpha\underline{i} + \underline{j} - \underline{k}) . (\underline{i} + \alpha\underline{j} + 4\underline{k}) = 0$

$\Rightarrow \quad 2\alpha + \alpha - 4 = 0$

$\Rightarrow \quad 3\alpha - 4 = 0$

$\Rightarrow \quad \alpha = \frac{4}{3}$

(ii) We are given $\underline{U}$ and $\underline{V}$ are perpendicular.

$\Rightarrow \quad \underline{U} . \underline{V} = 0$

$\Rightarrow \quad (\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}) . (\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$

$(\alpha)(1) + (2\alpha)(\alpha) + (-1)(3) = 0$

$2(\alpha)^2 + \alpha - 3 = 0$

$\alpha(2\alpha + 3) - 1(2\alpha + 3) = 0$

$\Rightarrow \quad (\alpha - 1)(2\alpha + 3) = 0$

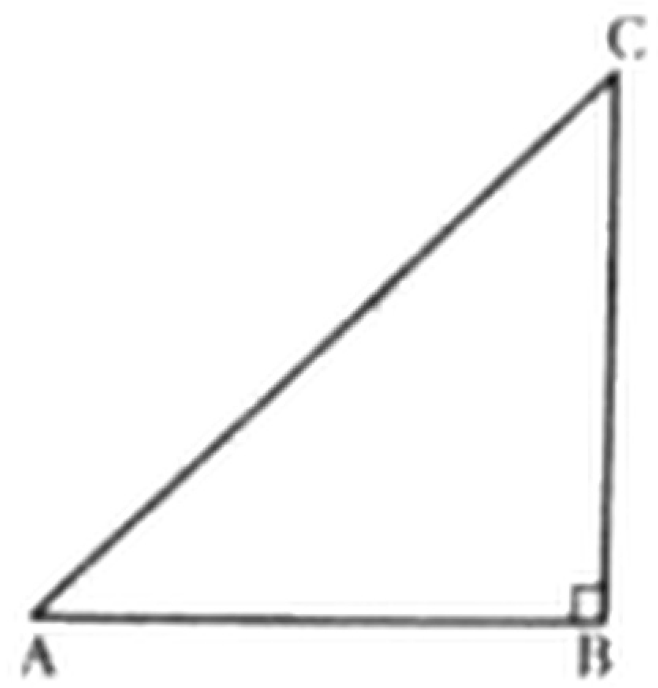
$$\alpha - 1 = 0 \qquad 2\alpha + 3 = 0$$

$$\alpha = 1 \qquad \alpha = \frac{-3}{2}$$

$$\Rightarrow \qquad \alpha = 1, \frac{-3}{2}$$

Q4. Find the number "z" so that the triangle with vertices A(1,-1,0),B(-2,2,1) and C(0,2,z) is right triangle with right angle C etc.

Solution



$\vec{AC} = (0,2,Z)-(1,-1,0)$

$= (0-1,2+1,z-0)$

$\vec{AC} = -\underline{\mathbf{i}} + \mathbf{3}\underline{\mathbf{j}} + z\underline{\mathbf{k}}$

$\vec{BC} = (0,2,Z)-(-z,2,1)$

$= (0+2,2-2,z-1)$

$\vec{BC} = 2\underline{i} + 0\underline{j} + (z-1)\underline{k}$

$\vec{AC}$ is perpendicular to $\vec{BC}$

$\Rightarrow \quad \vec{AC} \cdot \vec{BC} = 0$

$\Rightarrow \quad \left(-\underline{i} + 3\underline{j} + z\underline{k}\right)\left(2\underline{i} + 0\underline{j} + (z-1)\underline{k}\right) = 0$

$(-1)(2)+(3)(0)+z(z-1) = 0$

$-2 + 0 + z^2 - z = 0$

$z^2 - z - 2 = 0$

$z^2 - 2z + z = 0$

$z(z-2)+1(z-2) = 0$

$(z-2)(z+1) = 0$

$\Rightarrow \quad z-2 = 0, \qquad z+1 = 0$

$z = 2, \qquad z = -1$

$\Rightarrow \quad z = 2, -1$

Q5. If V is a vector for which

$\underline{V}.\underline{i} = 0.$ $\qquad \underline{V}.\underline{j} = 0,$ $\qquad \underline{V}.\underline{k} = 0$, find $\underline{V}$.

Let $\underline{V} = (a\underline{i} + b\underline{j} + c\underline{k})$ ------------------- (1)

$\underline{V} . \underline{i} = (a\underline{i} + b\underline{j} + c\underline{k}).\underline{i} = 0$

$\Rightarrow \quad \left(a\underline{i} + b\underline{j} + c\underline{k}\right) . \left(\underline{i} + 0\underline{j} + 0\underline{k}\right) = 0$

$\Rightarrow \quad a(1) + 0 + 0 = 0$

⇒ $a = 0$

⇒ Again $\underline{V}.\underline{j} = 0$

⇒ $0+b(1)+0 = 0$

⇒ $b = 0$

Also $\underline{V}.\underline{k} = 0$

⇒ $c = 0$

put a, b, c in (1)

⇒ $\underline{V} = 0\underline{i} + 0\underline{j} + 0\underline{k}$

$\underline{V} = 0$ (Null Vector)

Q6. i. Show that the vectors $(3\underline{i} - 2\underline{j} + \underline{k})$, $(\underline{i} - 3\underline{j} + 5\underline{k})$ and $(2\underline{i} + \underline{j} - 4\underline{k})$ form a right angle.

ii. Show that the set of points, P=(1,3,2), Q=(4,1,4) and R(6,5,5) form a right triangle.

Solution

(i) let $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$

$\underline{b} = \underline{i} - 3\underline{j} + 5\underline{k}$

And $\underline{c} = 2\underline{i} + \underline{j} - 4\underline{k}$

$\underline{b}+\underline{c} = 3\underline{i} - 2\underline{j} + \underline{k}$

Now $\underline{a}.\underline{c} = (3\underline{i} - 2\underline{j} + \underline{k})\ .\ (2\underline{i} + \underline{j} - 4\underline{k})$

$= 6-2-4=6-6$

$\Rightarrow \quad \underline{a} \perp \underline{c}$

(ii) P(1,3,2), Q(4,1,4) and R (6,5,5)

$\vec{PQ} = (4,1,4)-(1,3,2)$

$= (4-1,1-3,4-2)$

$= 3\underline{i} - 2\underline{j} + 2\underline{k}$

$\vec{QR} = (6,5,5) - (4,1,4)$

$= (6-4, 5-1, 5-4)$

$= 2\underline{i} + 4\underline{j} + \underline{k}$

$\vec{PR} = (6,5,5) -(1,3,2)$

$= (6-1, 5-3, 5-2)$

$= 5\underline{i} + 2\underline{j} + 3\underline{k}$

Now $\vec{PQ} + \vec{QR} = (3,-2,2)+(2,,4,1)$

$= (3+2,-2+4,2+1)$

$= 5\underline{i} + 2\underline{j} + 3\underline{k}$

$= \vec{PR}$

Now $\vec{PQ} . \vec{QR} = (3\underline{i} - 2\underline{j} + 2\underline{k}). (2\underline{i} + 4\underline{j} + \underline{k})$

$= 3(2)+(-2)(4)+2(1)$

$= 6-8+2$

$= 0$

$\Rightarrow \quad \vec{PQ} \perp \vec{QR}$

So, P,Q,R are vertices of a right triangle.

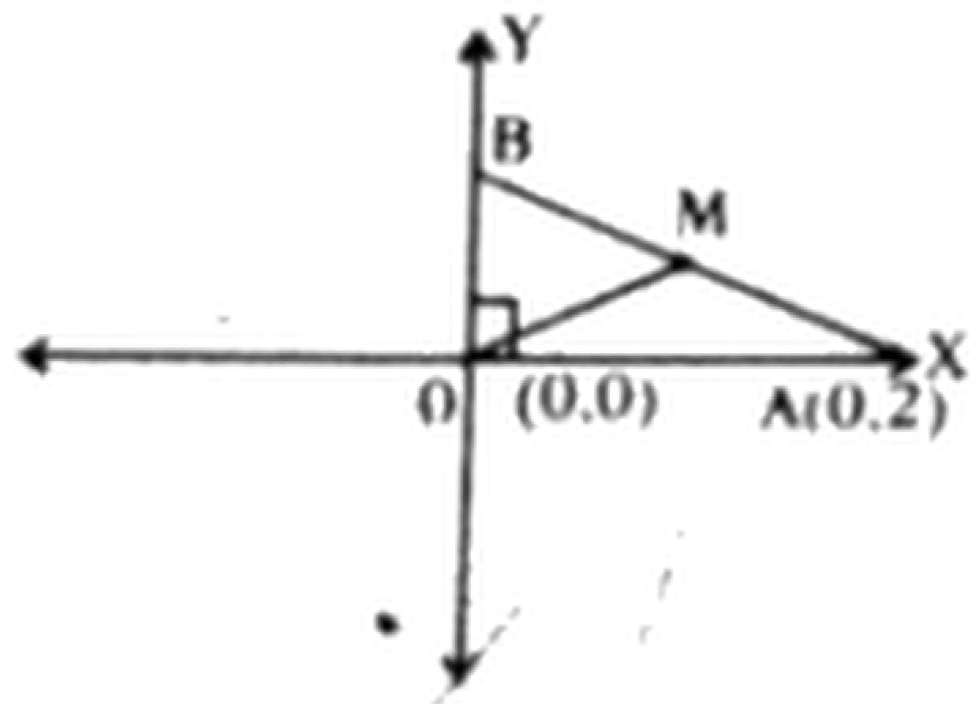
Q7. Show that mid-point of hypotenuse of a right angle is equidistant from its vertices.

Solution

Let consider a right-angle triangle OAB as show clearly AB is a hypotenuse

Let M be the mid-point of AB, here we have to show $\vec{AM} = \vec{MB} = \vec{PQ} \perp \vec{OM}$

Now we find $\vec{AM}$



$\vec{AM} = \left(\frac{a}{2}, \frac{a}{2}\right) - (a, 0)$

$= \left(\frac{a}{2} - 0, \frac{b}{2} - 0\right)$

$$= \left(\frac{a}{2}, \frac{b}{2} \right)$$

$$A\vec{M} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \sqrt{\frac{a^2 + b^2}{4}} \qquad \text{-----------} \ (1)$$

$$B\vec{M} = \left(-\frac{a}{2}, \frac{b}{2} \right) - (0, b)$$

$$= \left(-\frac{a}{2} - 0, \frac{b}{2} - b \right)$$

$$= \left(\frac{a}{2}, \quad \frac{b}{2} \right)$$

$$B\vec{M} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \sqrt{\frac{a^2 + b^2}{4}} \qquad \text{--------------} \ (2)$$

$$O\vec{M} = \left(\frac{a}{2}, \frac{a}{2} \right) - (0,0)$$

$$= \left(\frac{a}{2} - 0, \frac{b}{2} - 0 \right)$$

$$= \left(\frac{a}{2}, \frac{b}{2} \right)$$

$$O\vec{M} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \sqrt{\frac{a^2 + b^2}{4}} \qquad \text{--------------} \ (3)$$

From (1), (2) and (3)

$\vec{OM} = \vec{BM} = \vec{AM}$ proved.

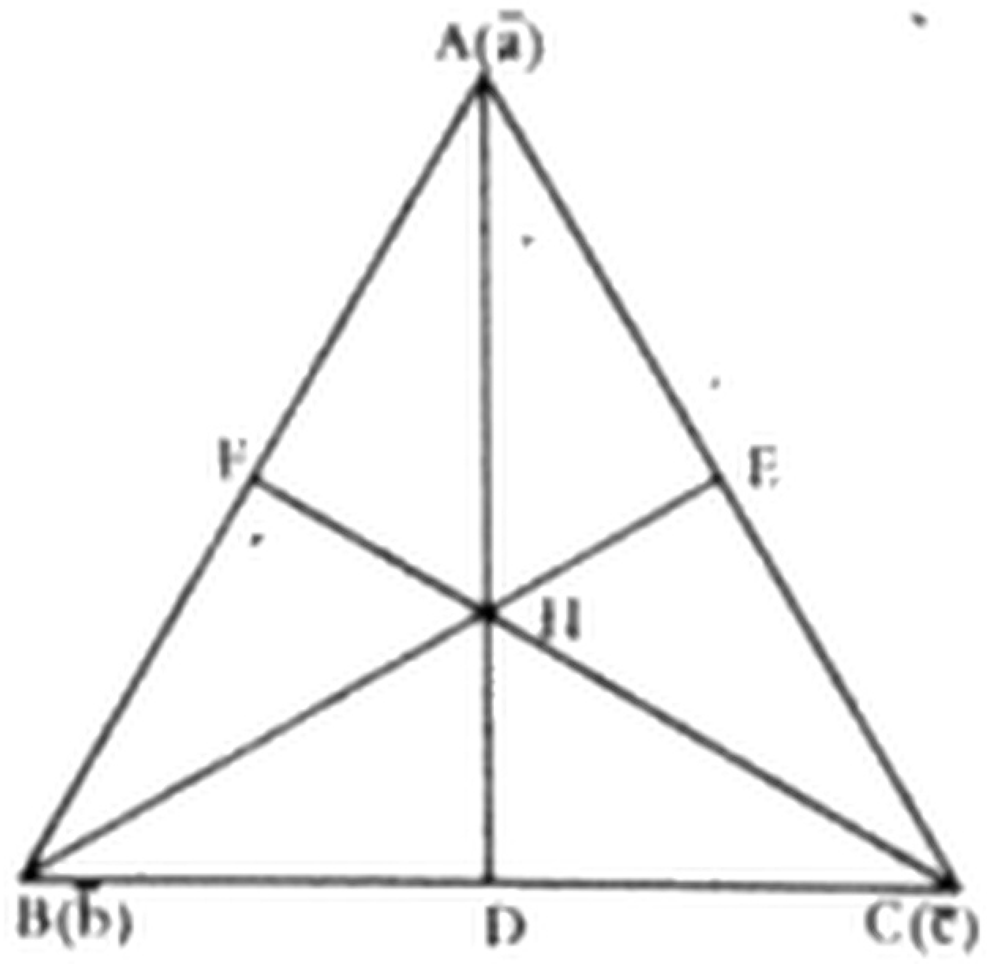
Q8. Prove that the perpendicular bisectors of the sides of a triangle are concurrent

Solution

Suppose K(0) is the intersection of two perpendicular bisector, KD and KE of a ΔABC then choose the vertices A($\underline{a}$), B($\underline{b}$), C($\underline{c}$).

$\vec{CD} \perp \vec{BC}$ and

$\vec{KE} \perp \vec{CA}$



We have $\frac{1}{2}(\underline{c} + \underline{b}).(\underline{c} - \underline{b}) = 0$

$\Rightarrow$ $c^2 - b^2 = 0$ ---------- (1)

And $\frac{1}{2}(\underline{a} + \underline{c}).(\underline{a} - \underline{c}) = 0$

$\Rightarrow$ $a^2 - c^2 = 0$ ------------ (2)

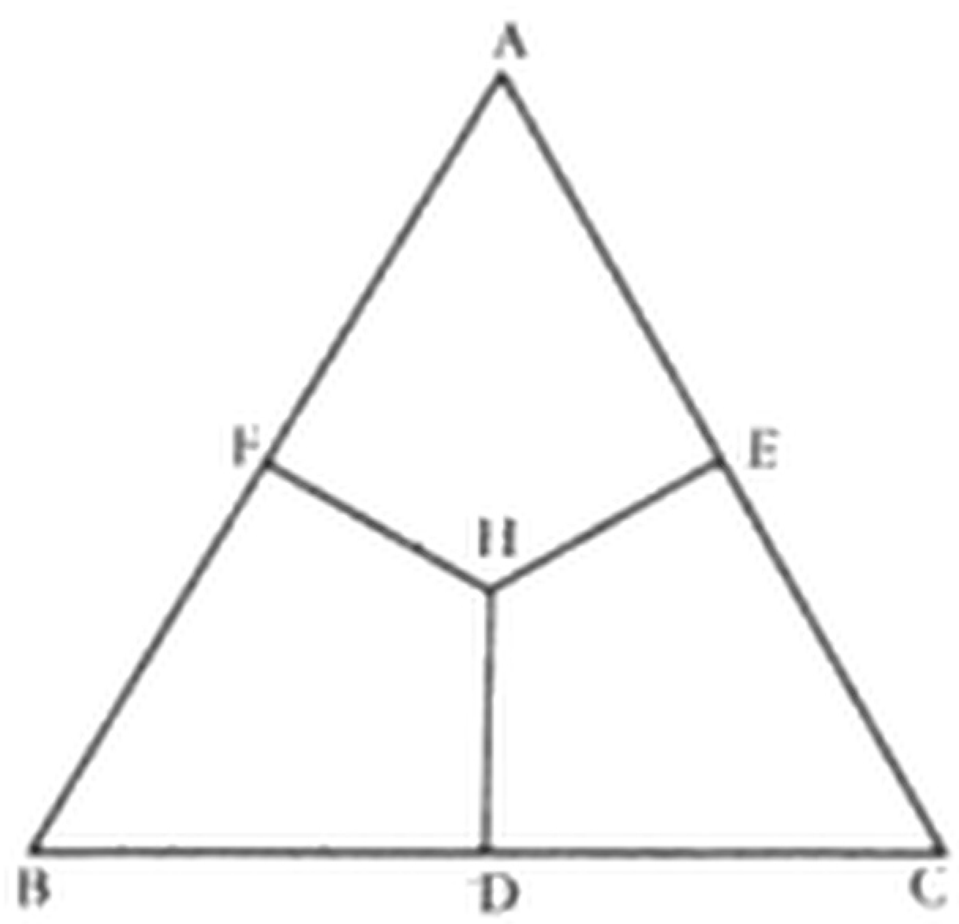
Adding (1) and (2) $a^2 - b^2 = 0$

$\frac{1}{2}(\underline{a}+\underline{b}).(\underline{a}-\underline{b})=0$ ------------- (3)

$\Rightarrow$ $\vec{HF} \perp \vec{AB}$

Q9. Prove that the altitudes of a triangle are concurrent.

Solution



Let A,B,C are the PVs of a triangle with vertices $\underline{a}, \underline{b}, \underline{c}$ respectively.

Where H is the intersection of two altitudes AD and BE

$\because$ $\vec{AD} \perp \vec{BC}$

$\Rightarrow$ $\vec{AH} \perp \vec{BC}$

$-0.(\vec{c}-\vec{b}) = 0$

$\Rightarrow$ a.b= $\vec{a}.\vec{b}$ ------------ (1)

Similarly, $\vec{BE} \perp \vec{CA}$, so $\vec{BH} \perp \vec{CA}$

-b.($\vec{a}.\vec{c}$) = 0 or $\vec{b}.\vec{c} = \vec{c}.\vec{b}$ ------------ (2)

From (1) and (2)

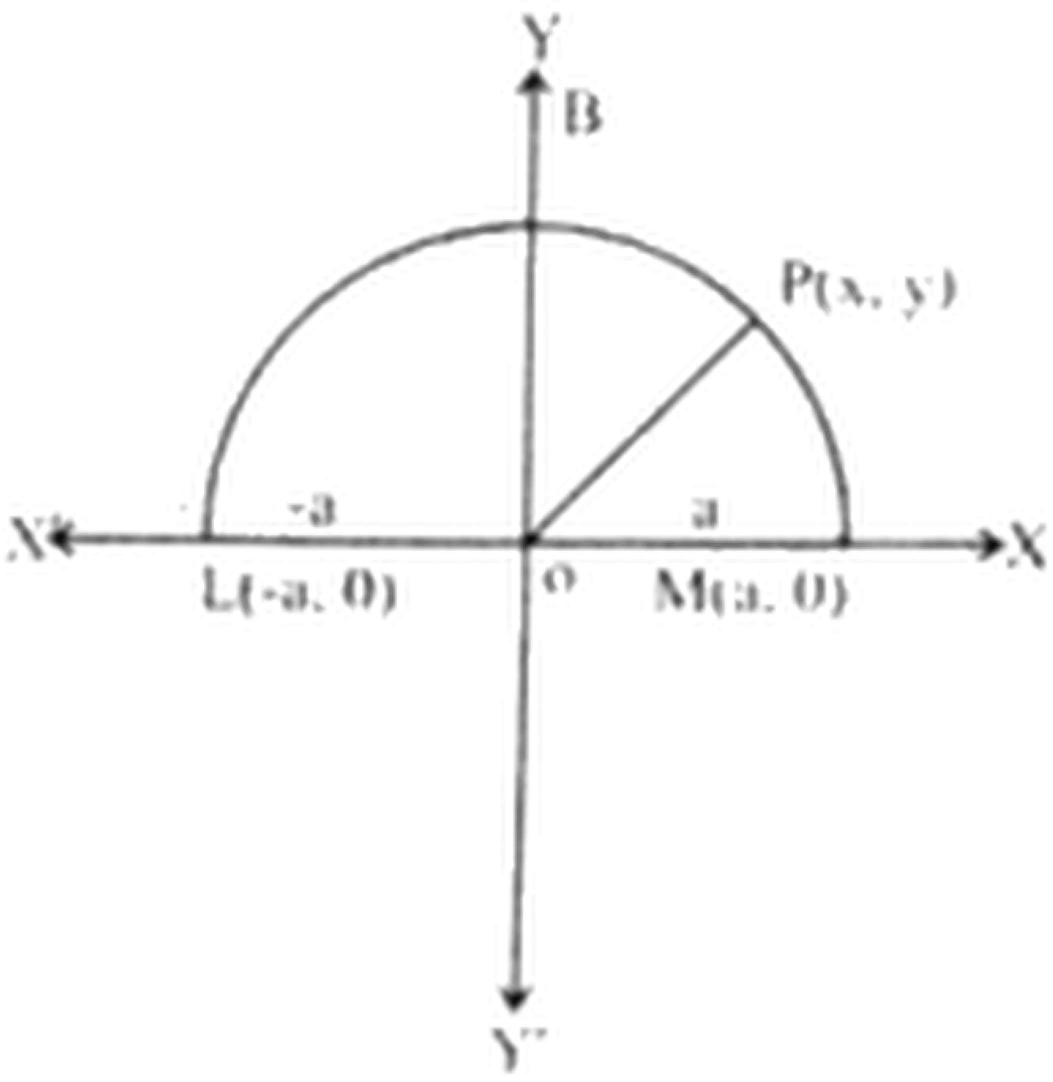
$$\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$$

Or $\quad \vec{c}(\vec{a} \cdot \vec{c}) = 0$

$$\vec{HC} \perp \vec{AB}$$

Q10. Prove that the angle in a semi-circle is a right triangle.

Solution



Let O be the center, LM be the diameter and P be any point on the semi-circle of radius.

Now $\vec{LP} = \vec{LO} = \vec{a} + \vec{r}$

$$\vec{MP} = \vec{MO} + \vec{OP} = \vec{a} + \vec{r}$$

$$\vec{LP} \cdot \vec{MP} = (\vec{a} + \vec{r})$$

$$= \vec{a} \cdot \vec{r} + \vec{a} \cdot \vec{r}$$

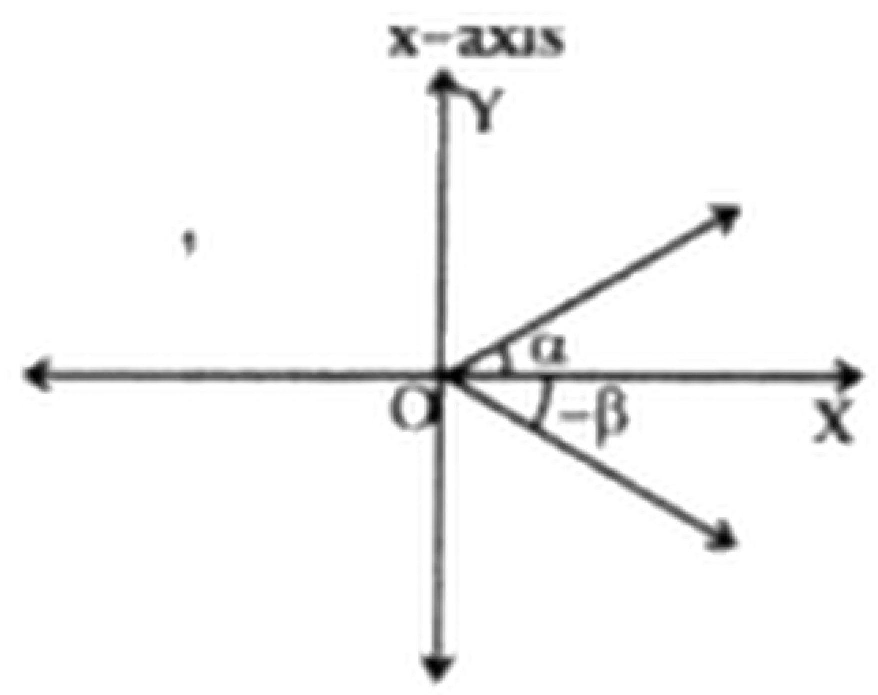
$= \quad a^2 + r^2$

$= 0$

Hence $m\angle LMP = 90°$

Q11. Prove that cos (α + β) = cos α . cos β - sin α . sin β

Solution



Let $\hat{a}$ and $\hat{b}$ the unit vectors making angle α and $-\beta$ with x-axis.

$\vec{a} = \vec{OA} = \cos\alpha\, \underline{i} + \sin\alpha \underline{j}$

$\dot{b} = \vec{OB} = (0)(-b)\, \underline{i} + \sin(-\alpha)\, \underline{j}$

$= \cos\beta\ \underline{i} + \sin\beta\, \underline{j}$

$\therefore \qquad \angle AOB = \alpha + \beta$

$\hat{a} \cdot \hat{b} = (\cos\alpha\, \underline{i} + \sin\alpha \underline{j}).(\cos\beta\ \underline{i} + \sin\beta\, \underline{j})$

$= \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$

$\Rightarrow \qquad \cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$

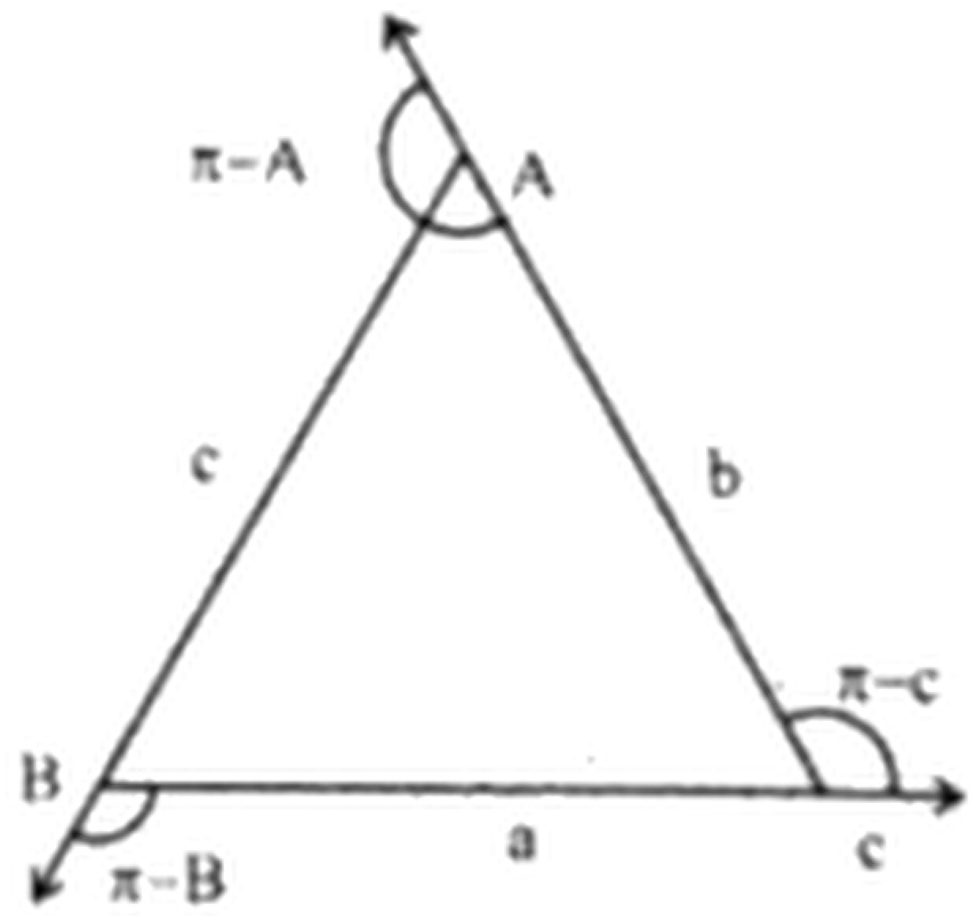
Q12. Prove that in any angle ABC

(i) b = c cos A+ a cos C

(ii) c = a cos B +b cos A

(iii) $b^2 = a^2 + c^2 - 2ac \cos B$

(iv) $c^2 = a^2 + b^2 - 2ab \cos C$



(i) $\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{b} = -\underline{a} - \underline{c}$

Taking dot product with $\underline{b}$;

$\underline{b} \cdot \underline{b} = -\underline{b} \cdot \underline{a} - \underline{b} \cdot \underline{c}$

$|\underline{b}|\,|\underline{b}| \cos 0° = -|\underline{b}|\,|\underline{a}| \cos(\pi - C) - |\underline{b}|\,|\underline{c}| \cos(\pi - A)$

(b)(b)(1) = -(b)(a)(-cos C)- (b)(c)(-cos A)

$b^2 = ba \cos C + bc \cos A$

b = a cos C + c cos A Proved

(ii) $\underline{\mathbf{a}} + \underline{\mathbf{b}} + \underline{\mathbf{c}} = \mathbf{0}$

$$\underline{c} = -\underline{a} - \underline{b}$$

Taking dot product with $\underline{b}$;

$$\underline{c} - \underline{c} = -\underline{c} \cdot \underline{a} - \underline{b} \cdot \underline{c}$$

$$|\underline{c}|\,|\underline{c}|\cos 0^\circ = -|\underline{c}|\,|\underline{a}|\cos(\pi - B) - |\underline{b}|\,|\underline{c}|\cos(\pi - A)$$

$$(c)(c)(1) = -(c)(a)(-\cos B) - (b)(c)(-\cos A)$$

$$c^2 = ca\cos B + bc\cos A$$

$$c = a\cos B + c\cos A$$ Proved

(iii) $\underline{\mathbf{a}} + \underline{\mathbf{b}} + \underline{\mathbf{c}} = \mathbf{0}$

$$\underline{b} = -\underline{a} - \underline{c} = -(\underline{a} + \underline{c})$$

Taking dot product with $\underline{b}$;

$$\underline{b} . \underline{b} = [-(\underline{a} + \underline{c})][-(\underline{a} + \underline{c})] = (\underline{a} + \underline{c})(\underline{a} + \underline{c})$$

$$|\underline{b}|\,|\underline{b}|\cos 0^\circ = \underline{a} . \underline{a} + \underline{a} . \underline{c} + \underline{c} . \underline{a} + \underline{c} . \underline{c}$$

$$= |\underline{a}|\,|\underline{a}|\cos 0^\circ + \underline{a} . \underline{c} + \underline{c} . \underline{a} + |\underline{c}|\,|\underline{c}|\cos 0^\circ$$

$$(b)(b)(1) = (a)(a)(1) + \underline{a} . \underline{c} + \underline{c} . \underline{a} + (c)(c)(1)$$

$$b^2 = a^2 + c^2 + 2|\underline{c}|\,|\underline{a}|\cos(\pi - B)$$

$$= a^2 + c^2 + 2ac(-\cos B)$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$ Proved

(iv) $\underline{a} + \underline{b} + \underline{c} = 0$

$$\underline{c} = -\underline{a} - \underline{b} = -(\underline{a} + \underline{b})$$

$$\underline{c}.\underline{c} = [-(\underline{a} + \underline{b})][-(\underline{a} + \underline{b})] = (\underline{a} + \underline{b})(\underline{a} + \underline{b})$$

$$|\underline{c}||\underline{c}|\cos 0^\circ = \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{b}.\underline{a} + \underline{b}.\underline{b}$$

$$= |\underline{a}||\underline{a}|\cos 0^\circ + \underline{a}.\underline{b} + \underline{b}.\underline{a} + |\underline{b}||b|\cos 0^\circ$$

$$(c)(c)(1) = (a)(a)(1) + \underline{a}.\underline{b} + \underline{b}.\underline{a} + (b)(b)(1)$$

$$c^2 = a^2 + b^2 + 2|\underline{b}||\underline{a}|\cos(\pi - C)$$

$$= a^2 + b^2 + 2ab(-\cos C)$$

$$c^2 = a^2 + b^2 - 2ab\cos C \text{ Proved}$$