Exercise 6.7

Q1. Find equation of the tangent and normal to each of the following at the indicated point.

i. $y^2 = 4ax$ at $(at^2, 2at)$

Solution

$$y^2 = 4ax \quad \text{-------} \quad (1)$$

Diff eq. (i) both sides w.r.t 'x', we have

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$$

$$2y\frac{dy}{dx} = (4a) \qquad \frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{\frac{dy}{dx}}{P(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$$

So slope of tangent line at P = $\frac{1}{t}$ and slope of normal line at P = - t .

Now equation of tangent at P is $y - 2at = \frac{1}{t}(x - at^2)$ or $yt - 2at^2 = x - at^2$

Or $yt = x + at^2$

Equation of normal at P is

$$y - 2at = -t(x - at^2)$$

$$y - 2at = -tx + at^3$$

$$y = -tx + at^3 + 2at$$

ii. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **at** $(a\cos\theta, b\sin\theta)$

Solution

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{-----------} \ (1)$$

Diff eq. (i) both sides w.r.t 'x', we have

$$\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \quad \frac{1}{a^2}(2x)+\frac{1}{b^2}(2y)\frac{dy}{dx}=0$$

$$\frac{1}{b^2}(2y)\frac{dy}{dx}=-\frac{2x}{a^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{xb^2}{ya^2}$$

$$\frac{\frac{dy}{dx}}{P(a\cos\theta, b\sin\theta)}=\frac{a\cos\theta\,.\,b^2}{b\sin\theta\,.\,a^2}=\frac{a\cos\theta}{b\sin\theta}$$

So slope of tangent line of P = $\frac{-b\cos\theta}{a\sin\theta.}$ And slope of normal line at P = $\frac{a\cos\theta}{b\sin\theta.}$

Now equation of tangent at P is

$$y-b\sin\theta=\frac{-b\cos\theta}{a\sin\theta.}(x-a\cos\theta)$$

$$\frac{\sin\theta}{b}(y-b\sin\theta)=\frac{-\cos\theta}{a}(x-a\cos\theta)$$

$$\frac{y}{b}\sin\theta-\sin^2\theta=-\frac{x}{a}\cos\theta+\cos^2\theta$$

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\sin^2\theta+\cos^2\theta$$

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Equation of normal at P is

$$y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta.}(x-a\cos\theta)$$

$\Rightarrow \quad \frac{b}{\sin\theta}(y - b\sin\theta) = \frac{a}{\cos\theta}(x - a\cos\theta)$

$\Rightarrow \quad a\sin\theta\, x - b\cos\theta\, y = \sin\theta\cos\theta\,(a^2 - b^2)$

iii. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **at $(a\sec\theta, b\tan\theta)$**

Solution

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ----------- (1)

Diff eq. (i) both sides w.r.t 'x', we have

$\frac{d}{dx}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$

$\Rightarrow \quad \frac{1}{a^2}(2x) - \frac{1}{b^2}(2y)\frac{dy}{dx} = 0$

$\frac{1}{b^2}(2y)\frac{dy}{dx} = \frac{2x}{a^2}$

$\frac{dy}{dx} = \frac{xb^2}{ya^2}$

$$\frac{dy}{dx} = \frac{a\sec\theta\,.\,b^2}{b\tan\theta\,.\,a^2} = \frac{b\sec\theta}{a\tan\theta}$$

So slope of tangent line of P = $\frac{b\sec\theta}{a\tan\theta.}$

And slope of normal line at P = $-\frac{a\tan\theta}{b\sec\theta.}$

Now equation of tangent at P is

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta.}(x - a\sec\theta)$

$\frac{\tan\theta}{b}(y - b\tan\theta) = \frac{\sec\theta}{a.}(x - a\sec\theta)$

$\frac{y}{b}\tan\theta - \tan^2\theta = \frac{x}{a}\sec\theta - \sec^2\theta$

$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = \sec^2\theta - \tan^2\theta$

$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Equation of normal at P is

$$y - b\tan\theta = \frac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$$

$b\sec\theta\, y - b^2\sec\theta\tan\theta = -a\tan\theta\, x + a^2\sec\theta\tan\theta$

$a\tan\theta\, x + b\sec\theta\, y = a^2\sec\theta\tan\theta + b^2\sec\theta\tan\theta$

$a\tan\theta\, x + b\sec\theta\, y = \sec\theta\,(a^2 - b^2)$

Q2. Write equation of the tangent to the given conic at the indicated point.

i. $3x^2 = -16y$ at the points whose ordinate is -3

Solution

Given

$3x^2 = -16y$ ----------- (1)

Put $y = -3$

$3x^2 = -16(-3)$

$3x^2 = 16(3)$

$x^2 = 16$

$\Rightarrow$ $x = \pm 4$

thus points on the conic are (4,-3) and (-4,-3) to find equation of tangent at the point (4,-3) 0n the conic (1).

Replace x^2 by 4x

And y by $\frac{1}{2}(y - 3)$

We have

$3(4x) = -16\left(\frac{1}{2}\right)(y-3)$

$12x = -8(y-3)$

$3x = -2(y-3)$

$3x + 2y - 6 = 0$

To find equation of tangent at the point (-4,-3) on the conic (1).

Replace x^2 by -4x

And y by $\frac{1}{2}(y-3)$

We have

$3(-4x) = -16\left(\frac{1}{2}\right)(y-3)$

$-12x = -8(y-3)$

$3x = 2(y-3)$

$3x - 2y + 6 = 0$

ii. $3x^2 - 7y^2 = 20$ at the point where y=-1

Solution

Given

$3x^2 - 7y^2 = 20$ ----------- (1)

Put $y = -1$ in eq. (1), we have

$3x^2 - 7(1)^2 = 20$

$3x^2 - 7 = 20$

$3x^2 = 27$

$x^2 = 9$

$x = \pm 3$

so indicated points are P(3,-1) and (-3,-1)

diff. eq. (i) w.r.t 'x', we have

$$\frac{d}{dx}(3x^2 - 7y^2) = \frac{d}{dx}(20)$$

$$\Rightarrow \quad 3(2x) - 7(2y)\frac{dy}{dx} = 0$$

$$6x - 14y\frac{dy}{dx} = 0$$

$$14y\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{14y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x}{7y}$$

$$\frac{\frac{dy}{dx}}{P(3,-1)} = \frac{3(3)}{7(-1)}$$

$$= \frac{9}{-7} = -\frac{9}{7} = m_1$$

And $$\frac{\frac{dy}{dx}}{P(-3,-1)} = \frac{3(-3)}{7(-1)}$$

$$= \frac{-9}{-7} = \frac{9}{7} = m_2$$

Equation of tangent at point P(3,-1) is $y - (-1) = -\frac{9}{7}(x - 3)$

$$7y + 7 = -9x + 27$$

$9x + 7y - 20 = 0$ at (3,-1)

And equation of tangent at point P(-3,-1)

$$y - (-1) = \frac{9}{7}(x - (-3))$$

$$7y + 7 = 9x + 27$$

$9x - 7y + 20 = 0$ at (-3,-1)

Q3. Find equation of tangent which passes through the given points to the given conics.

i. $x^2 + y^2 = 25$ **through (7,-1)**

Solution

The given equation is $x^2 + y^2 = 25$ ------------- (1)

Differentiating (1) w.r.t. 'x', we have

$$2x + 2y\frac{dy}{dx} = 0 \qquad 2y\frac{dy}{dx} = -2x$$

$$y\frac{dy}{dx} = -x \qquad \frac{dy}{dx} = -\frac{x}{y}$$

$\Rightarrow$ $$\frac{dy}{dx}(7,-1) = -\frac{7}{-1} = 7 = m$$

This is the slope of the tangent to (1) at the equation of tangent at this point is

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = 7(x - 7)$$

$$y + 1 = 7x - 49$$

$$7x - y - 50 = 0$$

ii. $y^2 = 12x$ **through (1,4)**

Solution

The given equation is

$y^2 = 12x$ ------------- (1)

Differentiating (1) w.r.t. 'x', we have

$$2y\frac{dy}{dx} = 12$$

$$\frac{dy}{dx} = \frac{6}{y}$$

⇒ $\frac{dy}{dx}(1{,}4) = \frac{6}{4} = \frac{3}{2} = m$

This is the slope of the tangent to (1) at (1,4)

The equation of tangent at this point is

$$y - y_1 = m(x - x_1)$$

$$y - (4) = \frac{3}{2}(x - 1)$$

$$2y - 8 = 3x - 3$$

$$3x - 2y + 5 = 0$$

iii. $x^2 - 2y^2 = 2$ **through (1,-2)**

Solution

$$x^2 - 2y^2 = 2$$

$$2x - 4y\frac{dy}{dx} = 0$$

$$4y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{x}{2y}$$

$$\frac{dy}{dx}(1, -2) = \frac{1}{2(-2)} = -\frac{1}{4} = m$$

This is the slope of the tangent to (1) at(1,-2)

the equation of tangent at this point is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{1}{4}(x - 1)$$

$$4y + 8 = -x + 1$$

$$x + 4y + 7 = 0$$

Q4. Find equations of the normal to the parabola $y^2 = 8x$ which are parallel to the line $2x + 3y = 10$

Solution

$y^2 = 8x$ ------------ (1)

$2x + 3y = 2$ ------- (2)

From (1) $2y\frac{dy}{dx} = 8$

$\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$

Slope of tangent = $\frac{4}{y}$

Slope of normal = $\frac{y}{4}$

From (2) slope of line = $-\frac{2}{3}$

Since normal to parabola (1) iss parallel to (2)is parabola to line (1), we have

$\Rightarrow$ $-\frac{y}{4} = -\frac{2}{3} = \frac{8}{3}$

Put in (1)

$(\frac{8}{3})^2 = 8x$

$\frac{64}{9} = 8x$

$x = \frac{8}{9}$

$(x, y) = \left(\frac{8}{9}, \frac{8}{3}\right)$

Required normal is

$y - y_1 = m(x - x_1)$

$y - \frac{8}{3} = \frac{\frac{8}{3}}{4}\left(x - \frac{8}{9}\right)$

$$y - \frac{8}{3} = \frac{2}{3}\left(x - \frac{8}{9}\right)$$

$$y - \frac{8}{3} = \frac{2}{3}x + \frac{16}{27}$$

$$27y - 72 = -18x + 16$$

$$18x + 27y - 88 = 0$$

Q5. Find equation of tangent to the ellipse $\frac{x^2}{4} + y^2 = 1$ which are parallel to line $2x - 4x + 5 = 0$.

Solution

Slope of required tangent is $m = \frac{-2}{-4} = \frac{1}{2}$

Equation of tangents are $y = mx \pm \sqrt{a^2m^2 + b^2}$

$$Y = \frac{1}{2}x \pm \sqrt{\left(\frac{1}{2}\right)^2 4 + 1}$$

$$Y = \frac{1}{2}x \pm \sqrt{\left(\frac{1}{4}\right) 4 + 1}$$

$$y = \frac{1}{2}x \pm \sqrt{2} \qquad 2y = \pm\sqrt{2}$$

this is the two tangents are $x - 2y + 2\sqrt{2} = 0$

and $x - 2y - 2\sqrt{2} = 0$

Q6. Find equation of tangent to the conic $9x^2 - 4y^2 = 36$ parallel to $5x - 2x + 7 = 0$.

Solution

Given

$$9x^2 - 4y^2 = 36 \quad \text{--------------- (1)}$$

$$5x - 2x + 7 = 0. \quad \text{-------------- (2)}$$

From (1)

$$18x - 8y\frac{dy}{dx} = 0$$

$$8y\frac{dy}{dx} = 18x$$

$$\frac{dy}{dx} = \frac{18x}{8y} = \frac{9x}{4y}$$

Slope of tangent = $\frac{9x}{4y}$

Slope of line (2) $= \frac{5}{-2} = \frac{5}{2} = m$

$$\frac{9x}{4y} = \frac{5}{2}$$

$$x = \frac{5}{2} \times \frac{4y}{9} = \frac{10y}{9} \quad \text{---------- (3)}$$

put in (1)

$$9\left[\frac{10y}{9}\right]^2 - 4y^2 = 36$$

$$\frac{100y^2}{9} - 4y^2 = 36$$

$$100y^2 - 36y^2 = 36 \times 9$$

$$64y^2 = 36 \times 9$$

$$y^2 = 36 \times \frac{9}{64} = \frac{81}{64}$$

$$y = \pm\frac{9}{4}$$

$$y = \frac{9}{4} \qquad x = \frac{10}{9}\left[\frac{9}{9}\right] = \frac{5}{2}$$

$$(x, y) = [\frac{5}{2}, \frac{9}{4}]$$

When $\quad y = -\frac{9}{4} \qquad x = \frac{10}{9}\left[\frac{-9}{4}\right] = -\frac{5}{2}$

$$(x, y) = [-\frac{5}{2}, -\frac{9}{4}]$$

Thus there are two points on the conic (1) where tangents are parallel to (2).

Equation of tangent at $[\frac{5}{2}, \frac{9}{4}]$ is

$$y - \frac{9}{4} = \frac{9\left(\frac{5}{2}\right)}{4\left(\frac{9}{4}\right)} \quad (x - \frac{5}{2})$$

$$y - \frac{9}{4} = \frac{\frac{45}{2}}{9}\left(x - \frac{5}{2}\right)$$

$$4y - 9 = 10x - 25$$

$$10x - 4y - 16 = 0$$

$$5x - 2y - 8 = 0$$

Equation of tangent at $[-\frac{5}{2}, -\frac{9}{4}]$ is

$$y - [-\frac{9}{4}] = \frac{9\left(-\frac{5}{2}\right)}{4\left(-\frac{9}{4}\right)} \quad (x - [-\frac{5}{2}])$$

$$y + \frac{9}{4} = \frac{-\frac{45}{2}}{-9}\left(x + \frac{5}{2}\right)$$

$$y + \frac{9}{4} = \frac{5}{2}\left(x + \frac{5}{2}\right)$$

$$4y + 9 = 10x + 25$$

$$10x - 4y + 16 = 0$$

$$5x - 2y + 8 = 0$$

Q7. Find equation of common tangents to the given conics.

Solution

$x^2 = 80y$ ---------------- (1)

$x^2 + y^2 = 81$ ---------------- (2)

Let equation of common tangent to

$y = mx + c$ ------------- (3)

putting it in (1)

$$x^2 = 80(mx + c) = 80mx + 80c$$

$\therefore \quad x^2 - 80mx - 80c = 0$

The root of this quadratic equation will be equal if its discriminant is equal to zero i.e.,

$$(-80m)^2 - 4(1)(-80c) = 0$$

$$6400m^2 + 320c = 0$$

$$320(20m^2 + c) = 0 \qquad 20m^2 + c = 0$$

$$c = -20m^2$$

putting it in (3)

$$y = mx - 20m^2$$

putting it in (2)

$$x^2 + (mx - 20m^2)^2 = 81$$

$$x^2 + m^2x^2 + 400m^4 - 40m^3x - 81 = 0$$

$$(1 + m^2)x^2 - 40m^3x + (400m^4 - 81) = 0$$

The root of this quadratic equation will be equal if its discriminant is equal to zero i.e.,

$$(-40m^3)^2 - 4(1 + m^2)(400m^4 - 81) = 0$$

$$1600m^6 - 4(400m^6 + 400m^4 - 81m^2 - 81) = 0$$

$$1600m^6 - 1600m^6 - 1600m^4 + 324m^2 + 324 = 0$$

$$4(400m^4 - 81m^2 - 81) = 0$$

$$400m^4 - 81m^2 - 81 = 0$$

$$t = \frac{-(-81) \pm \sqrt{(-81)^2 - 4(400)(-81)}}{2(400)}$$

$$t = \frac{-(-81) \pm \sqrt{6561 + 129600}}{800}$$

$$t = \frac{-(-81) \pm \sqrt{136161}}{800} = \frac{81 \pm 369}{800}$$

$t = \frac{450}{800}$ $\qquad$ $t = -\frac{288}{800}$

$t = \frac{9}{16}$ $\qquad$ $t = -\frac{9}{25}$

$m^2 = t = \frac{9}{16}$ $\qquad$ $m^2 = t = \frac{9}{16}$

$Y = \pm\frac{3}{4}x - 20\left(\frac{9}{16}\right) = \pm\frac{3}{4}x - \frac{45}{4} = 3x - 45$

$4y = \pm 3x - 45$ $\qquad$ $\pm 3x - 4y - 45 = 0$

ii. $\mathbf{y^2 = 16x}$ **and** $\mathbf{x^2 = 2y}$

Solution

$y^2 = 16x$ ---------------- (1) $\qquad a = \frac{16}{4} = 4$

$x^2 = 2y$ ---------------- (2)

Let equation of common tangent to

$y = mx + c$ ------------- (3)

this line will be tangent to (1)

$c = \frac{a}{m} = \frac{4}{m}$ -------- (5)

solving (2) and (4) simultaneously

$$x^2 = 2\left(mx + \frac{4}{m}\right)$$

$$x^2 = 2mx + \frac{8}{m}$$

$\therefore \quad x^2 - 2mx - \frac{8}{m} = 0$ ------------- (5)

Line will be tangent if line touches the parabola at just one point i.e. the roots of equation (5)are equal i.e..

$$(-2m)^2 - 4(1)\left(-\frac{8}{m}\right) = 0$$

$$4m^2 + \frac{32}{m} = 0$$

$$m^2 + \frac{8}{m} = 0 \qquad m^3 = -8$$

$$m = -2$$

thus $\quad c = \frac{4}{m} = \frac{4}{-2} = -2$

putting the value of m and c in we have

$$Y = -2x - 2 \qquad 2x + y + 2 = 0$$

Q8. Find the points of intersection of the given conics

Solution

i. $\frac{x^2}{18} + \frac{y^2}{8} = 1 \qquad \frac{x^2}{3} - \frac{y^2}{3} = 81$

$$\frac{8x^2 + 18y^2}{144} = 1 \qquad \frac{x^2 - y^2}{3} = 1$$

$$8x^2 + 18y^2 = 144 \qquad x^2 - y^2 = 3$$

$8x^2 + 18y^2 = 144$ ------------ (1)

$8x^2 - 8y^2 = 24$ -------------- (2)

$$26y^2 = 120 \qquad y^2 = \frac{120}{26} = \frac{60}{13} = \pm\sqrt{\frac{60}{13}}$$

putting it in (2)

$$x^2 - \frac{60}{13} = 3 \Rightarrow \quad x^2 = 3 + \frac{60}{13} = \frac{99}{13}$$

$$\Rightarrow x = \pm\sqrt{\frac{99}{13}}$$

ii. $\mathbf{x^2 + y^2 = 8}$ **and** $\mathbf{x^2 - y^2 = 1}$

Solution

Let $x^2 + y^2 = 8$ ---------- (1)

$x^2 - y^2 = 1$ ---------- (2)

Adding eq. (1) and (2) we get

$$2x^2 = 9 \qquad x^2 = \frac{9}{2}$$

$$x = \pm\sqrt{\frac{9}{2}} \quad \Rightarrow \quad x = \pm\frac{3}{\sqrt{2}}$$

Substituting the value of 'x', in eq. (1)

$$\frac{9}{2} + y^2 = 8$$

$$y^2 = 8 - \frac{9}{2} \qquad y^2 = \frac{16-9}{2}$$

$$y^2 = \frac{7}{2} \quad \Rightarrow \quad y = \pm\sqrt{\frac{7}{2}}$$

So points of intersection $\left[\pm\sqrt{\frac{9}{2}}, \pm\sqrt{\frac{7}{2}}\right]$ or $\left[\pm\frac{3}{\sqrt{2}}, \pm\sqrt{\frac{7}{2}}\right]$

iii. $\mathbf{3x^2 - 4y^2 = 12}$ **and** $\mathbf{3y^2 - 2x^2 = 7}$

Solution

$3x^2 - 4y^2 = 12$ ---------- (1)

$3y^2 - 2x^2 = 7$ ---------- (2)

multiplying eq.(1) by 2 and eq. (2) by 3, then adding, we get

$$6x^2 - 8y^2 = 24$$

$$\underline{-6x^2 + 9y^2 = 24}$$

$$y^2 = 45$$

$$y = \pm\sqrt{45}$$

Substituting the value of 'y', in eq. (1), we get

$$3x^2 - 4(\pm\sqrt{45})^2 = 12$$

$3x^2 - 4(45) = 12$ or $3x^2 - 180 = 12$

$3x^2 = 192$ or $x^2 = 64$

$\Rightarrow$ $x = \pm 8$

so points of intersection $(\pm 8, \pm\sqrt{45})$

iv. $\mathbf{3x^2 + 5y^2 = 60}$ **and** $\mathbf{9x^2 + y^2 = 124}$

Solution

Given conics are

$3x^2 + 5y^2 = 60$ ------------- (1)

$9x^2 + y^2 = 124$ ------------- (2)

multiplying eq.(1) by 3and subtract from eq. (2).

$$9x^2 + y^2 = 124$$

$$\underline{\pm 9x^2 \pm 15y^2 = -180}$$

$-14\,y^2 = -56$ $\quad$ $14\,y^2 = 56$

$\Rightarrow$ $y^2 = \frac{56}{14} = 4$ $\quad$ $y = \pm 2$

Substituting the value of 'y', in eq. (1), we get

$$3x^2 + 5(4) = 60 \quad \text{or} \quad 3x^2 = 60 - 20 = 40$$

$$\Rightarrow \quad x^2 = \frac{40}{3} \quad \text{or} \quad x = \pm\frac{2\sqrt{10}}{3}$$

so points of intersection are $(\pm\frac{2\sqrt{10}}{3}, \pm 2)$.

v. $\frac{x^2}{9} + \frac{y^2}{9} = 1$ **and** $\frac{x^2}{2} - \frac{y^2}{4} = 1$

Solution

Given conic equations are

$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$

$$\frac{x^2}{2} - \frac{y^2}{4} = 1$$

multiplying eq.(2) by $\frac{1}{2}$ and subtract from eq. (1).

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\underline{\pm\frac{x^2}{4} \pm\frac{y^2}{8} = -\frac{1}{2}}$$

$$\frac{y^2}{9} - \frac{y^2}{8} = 1 - \frac{1}{2}$$

$$y^2\left(\frac{1}{9} - \frac{1}{8}\right) = \frac{2-1}{2} \qquad y^2\left(\frac{8-9}{72}\right) = \frac{1}{2}$$

$$y^2(-\frac{1}{72}) = \frac{1}{2} \qquad y^2 = \frac{-72}{2}$$

$$\Rightarrow \quad y^2 = -36 \qquad y = \pm 6i \text{ (imaginary)}$$

Thus intersection o given conics is not possible.