Exercise 6.5

Q1. Find an equation for the ellipse with given data and sketch its graph.

i. Foci $(\pm 3, 0)$ and minor axis of length 10

Solution

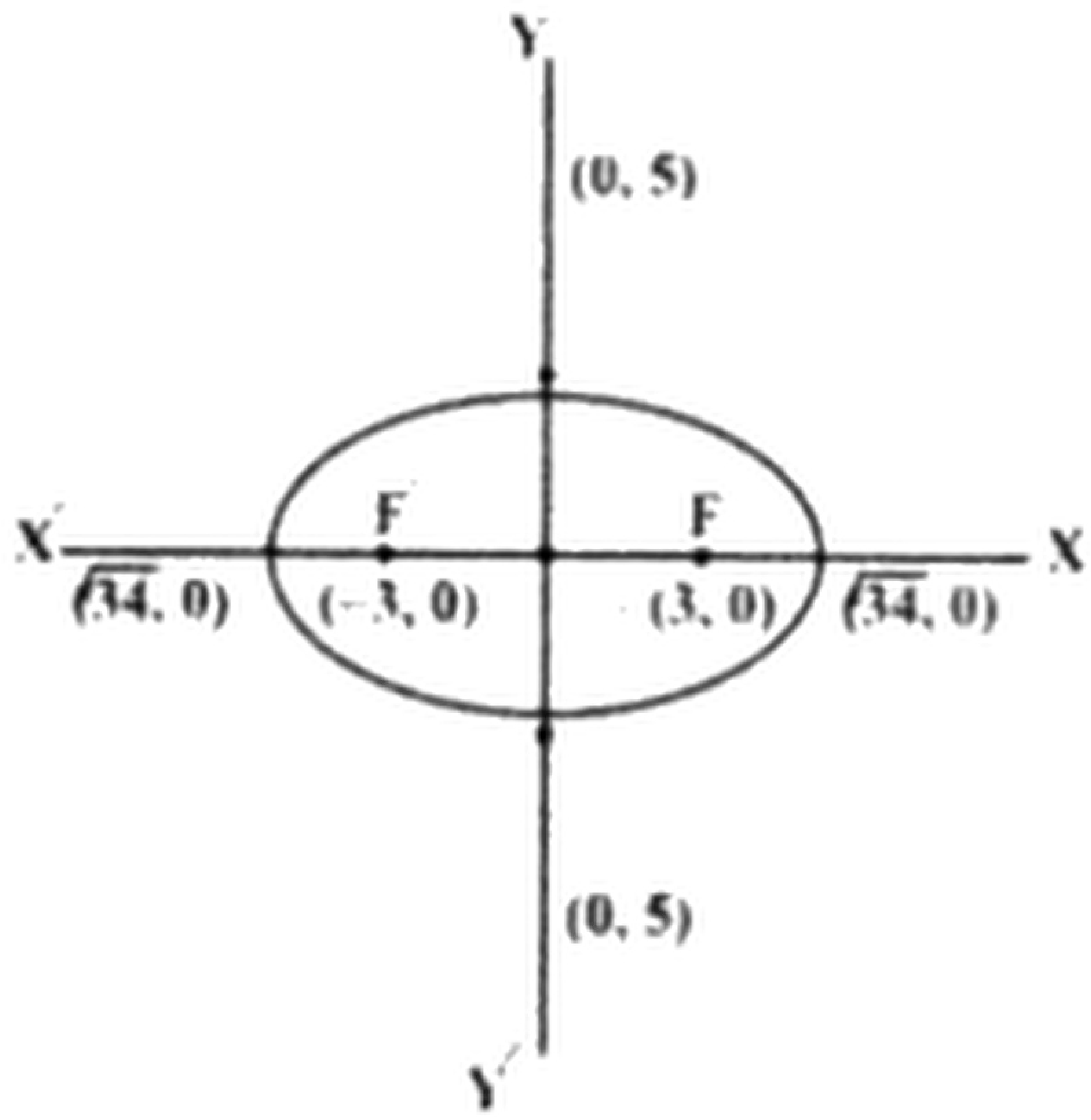
F(3,0), F'(-3,0), ae = 3

Here c = 3 , 2b = 10 i.e. b=5

$c^2 = a^2 - b^2 \Rightarrow (3^2) = a^2 - (5)^2$

$9 = a^2 - 25 \quad a^2 = 25 + 9 = 34$

Thus equation of ellipse is $\frac{x^2}{34} + \frac{y^2}{25} = 1$



ii. Foci (0,-1) and (0,-5) and major axis of length 6

Solution

F(0,-1), F'(0,-5)

The centre of the ellipse is located midway between the foci.

The centre is $\left(\frac{0+0}{2}, \frac{-1-5}{2}\right) = (0, -3)$

Since c represents the distance from the centre of each focus. c = 2 and $c^2 = 4$

Because the length of the major axis

$2a = 6$, $a = 3$, and $a^2 = 9$

$ae = 5 - 3 = 2$, $3e = 2 \Rightarrow e = \frac{2}{3}$

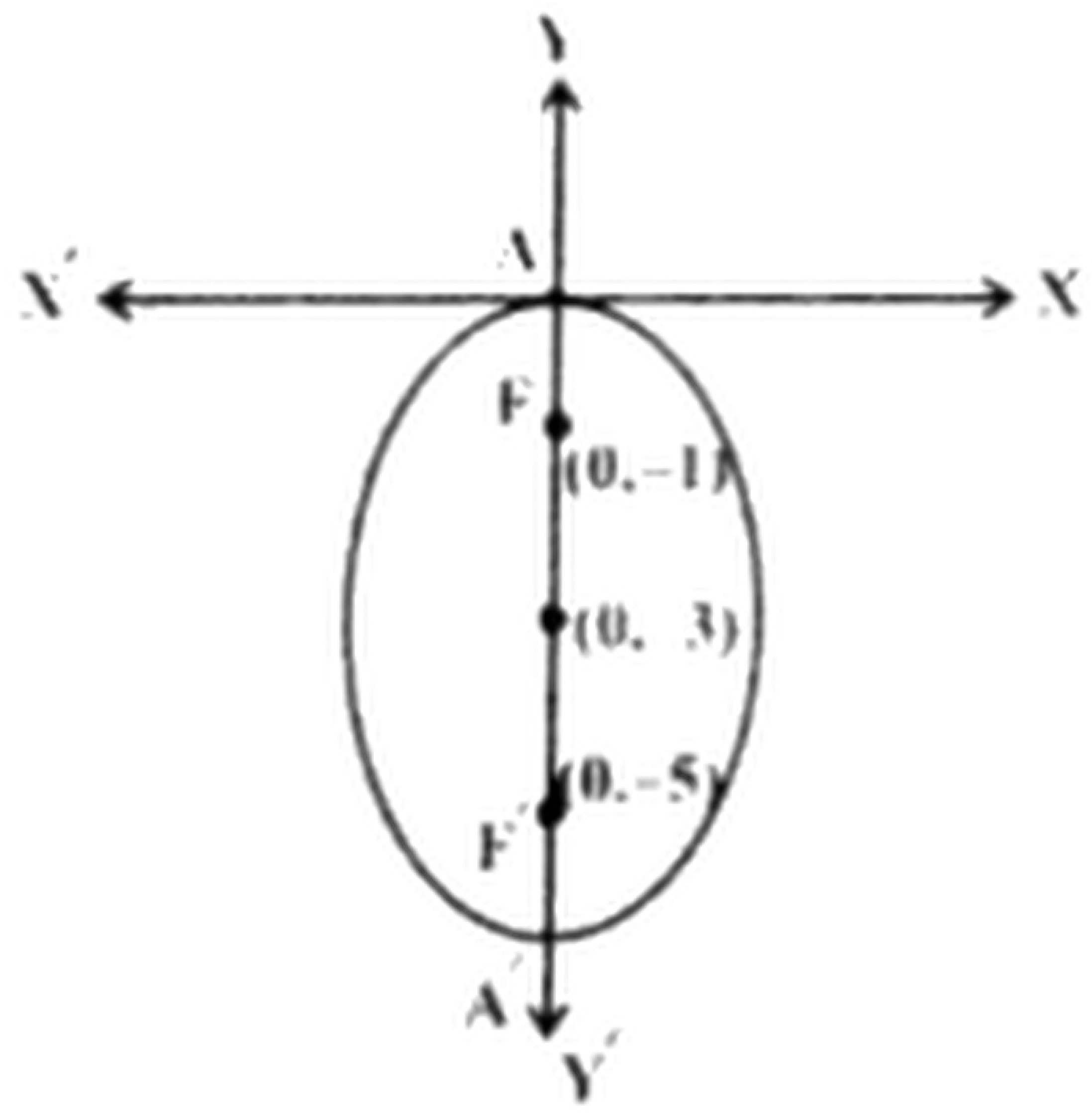
$e^2 = \frac{a^2 - b^2}{a^2} \Rightarrow \frac{4}{9} = \frac{9 - b^2}{9} \Rightarrow b^2 = 5$

From the location of the foci we can determine that major axis is vertical. Therefore

$\frac{(x-2)^2}{5} + \frac{(y-(-3))^2}{9} = 1$

$\frac{x^2}{5} + \frac{(y+3)^2}{9} = 1$ numerical

Sketch is



iii. Foci $(-3\sqrt{3}, 0)$ and vertices $(\pm 6, 0)$

Solution

$$ae = 3\sqrt{3} \Rightarrow e = \frac{3\sqrt{3}}{a} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

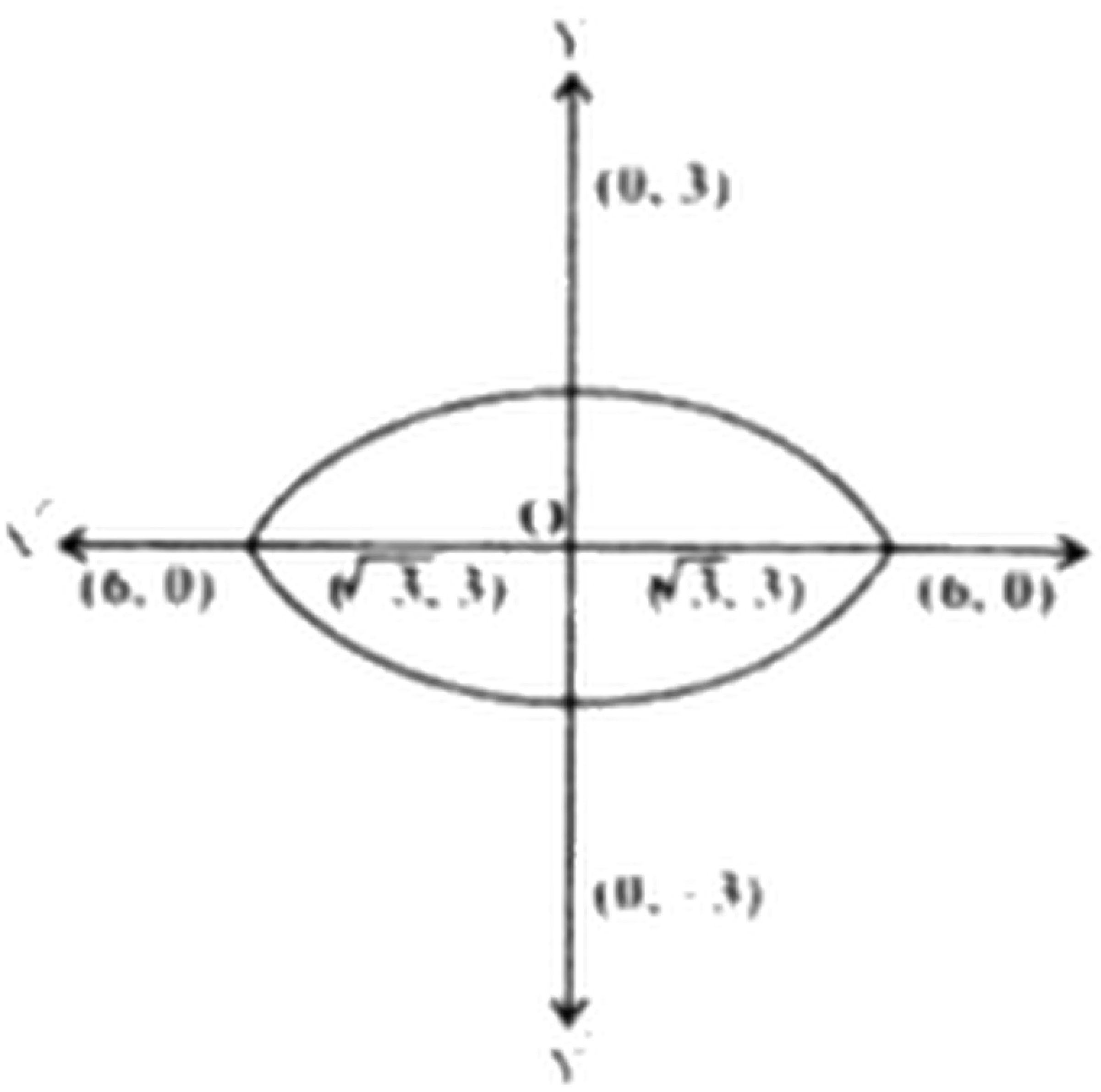
$$e^2 = \frac{a^2 - b^2}{a^2} \Rightarrow \left(3\sqrt{3}\right)^2 = (6)^2 - (b)^2$$

$$\Rightarrow 27 = 36 - b^2 = b^2 = 9$$

$$\Rightarrow b = 3$$

Thus equation of ellipse is $\frac{x^2}{36} + \frac{y^2}{9} = 1$

Sketch is



iv. Vertices (5,1),(-1,1) foci (3,1) and (1,1)

Solution

Vertices V_1 (5,1) and V_2 (-1,1); foci $F_1(3,1)$ and $F_2(1,1)$

$$ae = 4 - 2 = 2$$

$3e = 2 \Rightarrow e = \frac{2}{3}$

$e^2 = \frac{a^2 - b^2}{a^2}$

$\frac{4}{9} = \frac{9 - b^2}{9}$

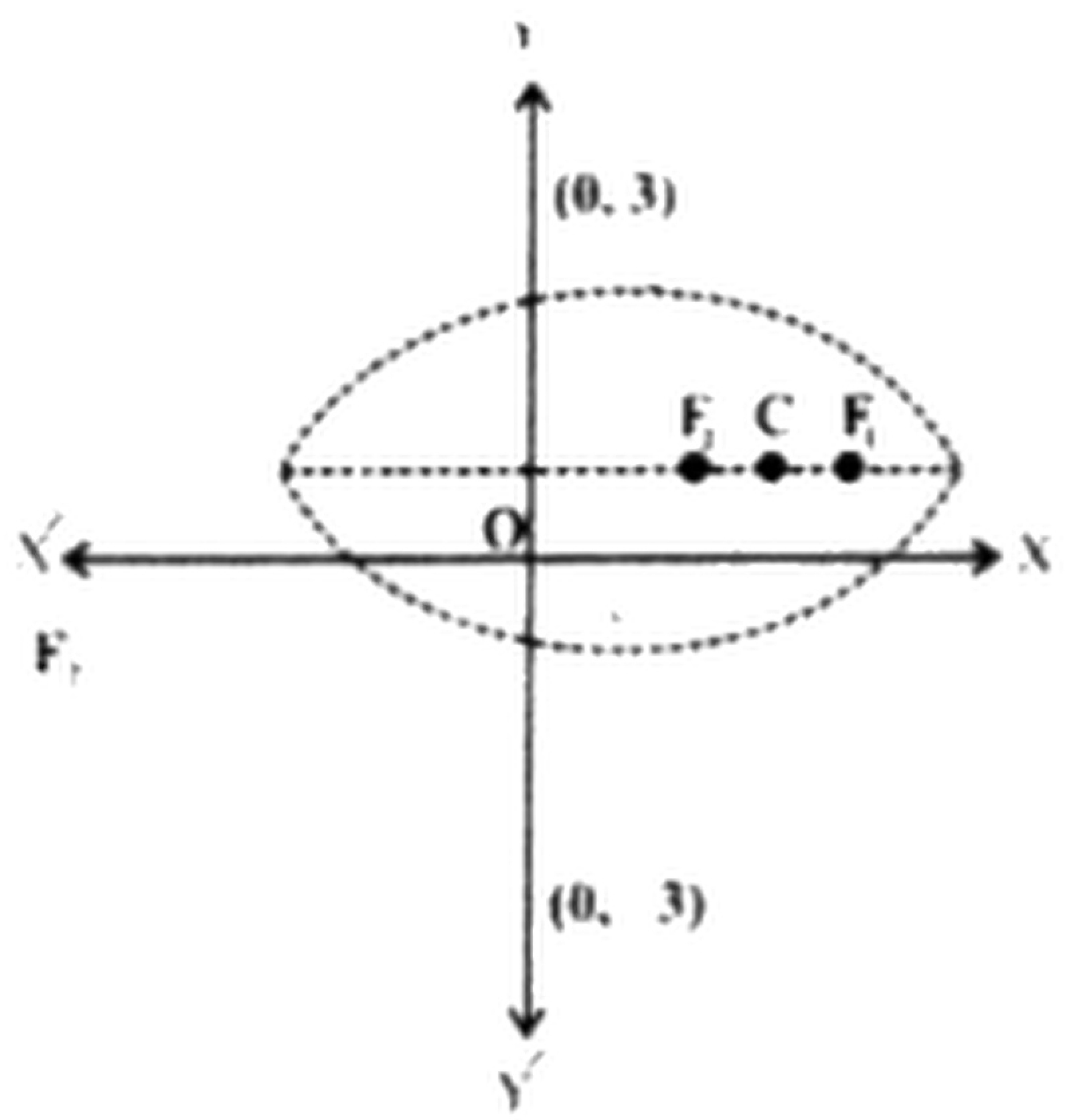
$b^2 = 5$

$b = \pm\sqrt{5}$

This equation of ellipse is

$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{8} = 1$

Sketch is



v. Foci $(\pm\sqrt{5}, 0)$ and passing through the points $\left(\frac{3}{2}, \sqrt{3}\right)$

Solution

F $(\sqrt{5}, 0)$,F'$(-\sqrt{5}, 0)$ through $\left(\frac{3}{2}, \sqrt{3}\right)$

$ae = (\sqrt{5}) \Rightarrow e = \frac{\sqrt{5}}{a}$

$$e^2 = \frac{a^2-b^2}{a^2} \Rightarrow \frac{5}{a^2} = \frac{a^2-b^2}{a^2} = a^2 - b^2 = 5 \Rightarrow a^2 = b^2 + 5$$

Equation of required ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since $\left(\frac{3}{2}, \sqrt{3}\right)$ lie on it therefore

$$\frac{^9/_4}{a^2} + \frac{3}{b^2} = 1 \Rightarrow \frac{9}{4a^2} + \frac{3}{b^2} = 1 \Rightarrow \frac{9b^2+12a^2}{4a^2b^2} = 1$$

Putting (1) in (3)

$$12(5+b^2) + 9b^2 = 4b^2(5 + b^2)$$

$$60+12b^2 + 9b^2 = 20b^2 + 4b^4$$

$$4b^4 - b^2 - 60 = 0$$

$$b^2 = t$$

$$4t^2 - t - 60 = 0$$

$$t = \frac{-(-1)\pm\sqrt{(-1)^2-4(4)(-60)}}{2(4)} = \frac{1\pm\sqrt{1+960}}{8}$$

$$t = \frac{1\pm31}{8} \Rightarrow t = \frac{1+31}{8} \ \& \ t = \frac{1-31}{8}$$

$$t = \frac{32}{8} \quad \text{and } t = \frac{-30}{8}$$

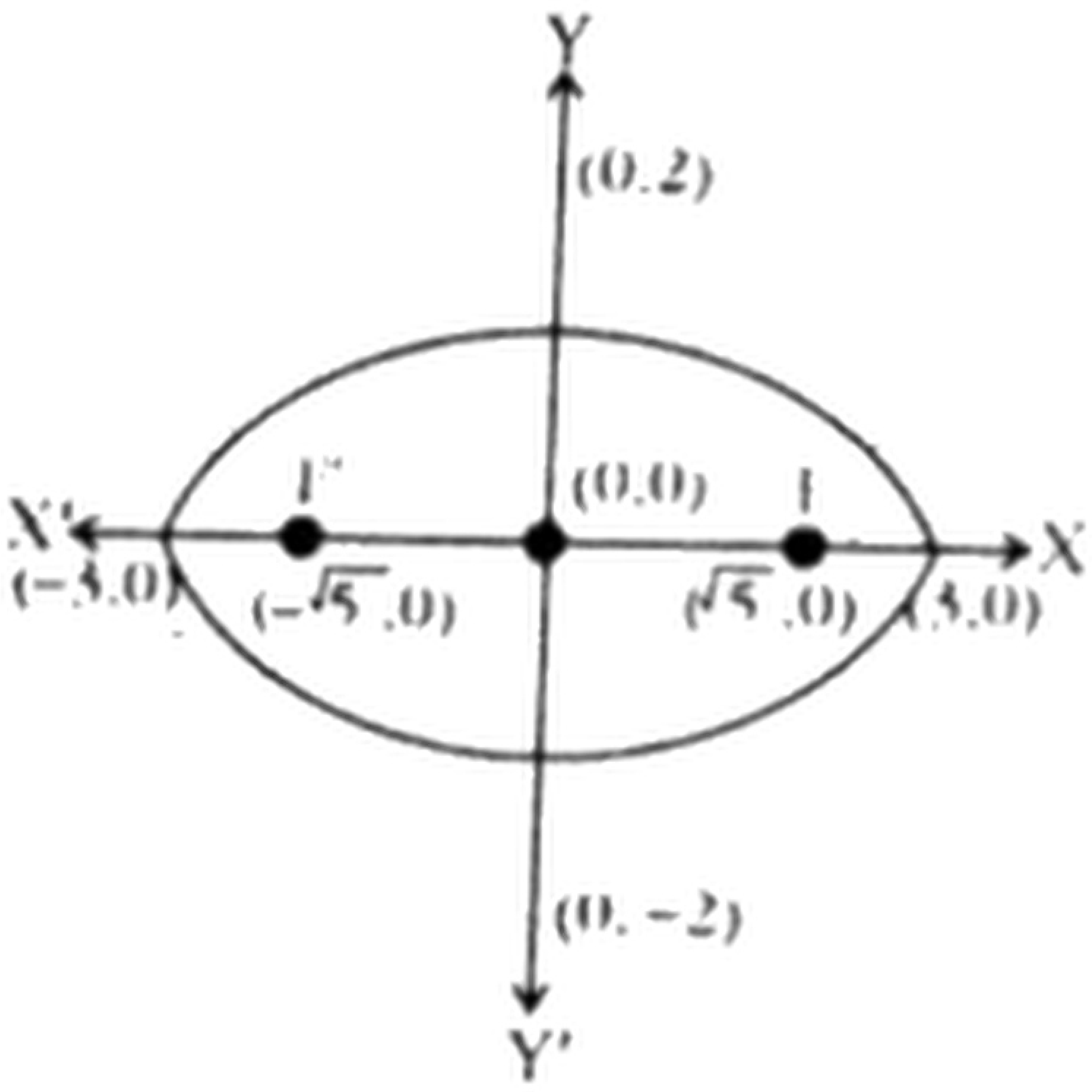
$$t = 4 \qquad\qquad t = \frac{-15}{4}$$

Gives imaginary roots hence discard

$$b^2 = t = 4 \Rightarrow b = \pm 2$$

Putting the values in (2)

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$



vi. Vertex $(0, \pm 5)$, eccentricity = $\frac{3}{5}$

Solution

Vertex $(0, \pm 5)$ gives

$$a = 5 \Rightarrow a^2 = 25$$

And $e = \frac{3}{5}$ (given)

As $c = ae$

Substituting the values of a and e , we have

$\Rightarrow$ $c = 5\left[\frac{3}{5}\right] \Rightarrow c = 3$

And $c^2 = a^2 - b^2$

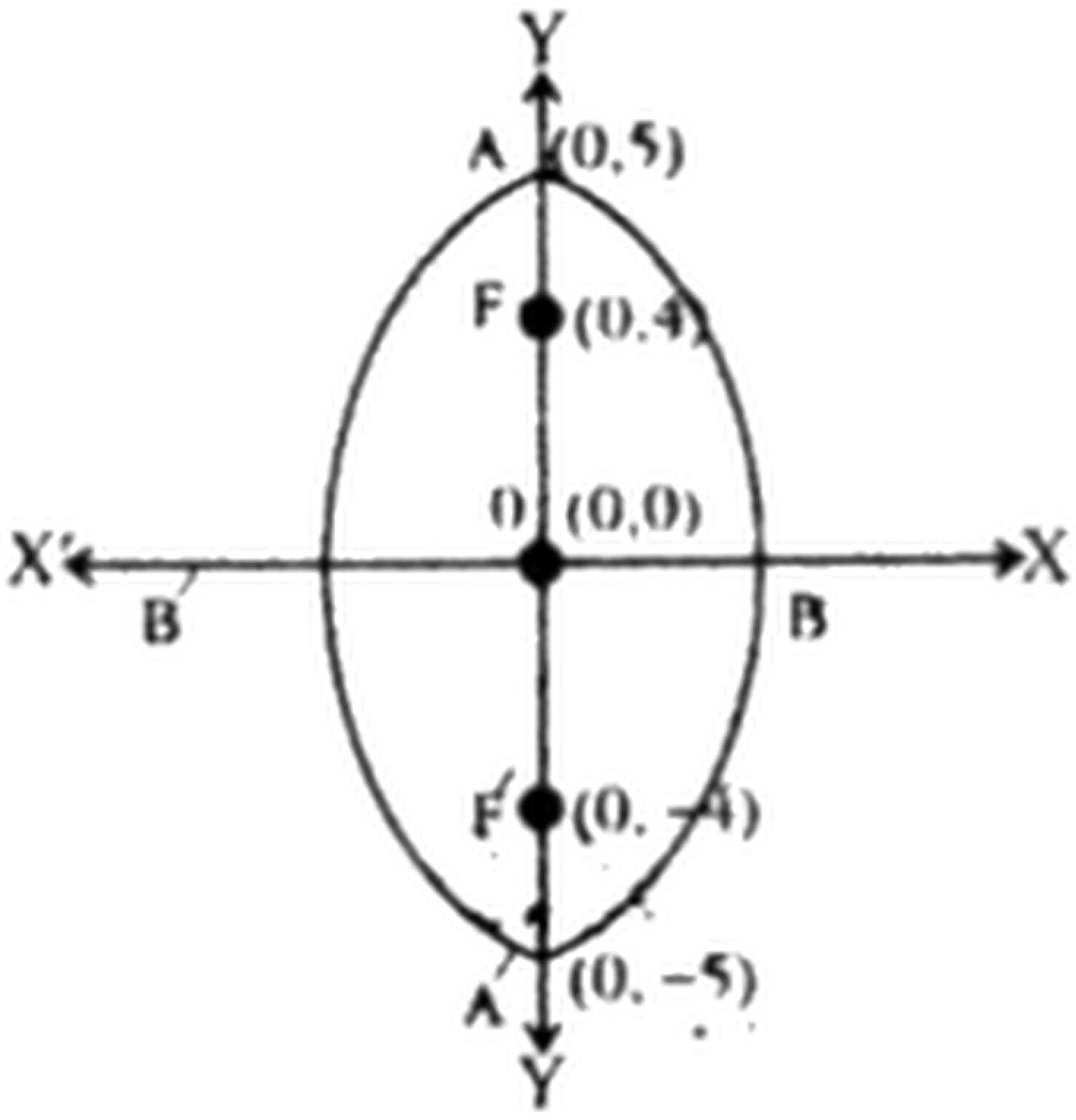
Substituting the values of a and c, we have

$\Rightarrow$ $(4)^2 = (5)^2 - b^2 \quad \Rightarrow 16 = 25 - b^2$

$\Rightarrow$ $b^2 = 25\text{-}16 \qquad \Rightarrow b^2 = 9$

Equation of ellipse is

$$\frac{y^2}{25} + \frac{x^2}{9} = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$$



vii. **Centre (0,0), foci (0,3), vertex (0,4)**

Solution

$$ae = 3 \Rightarrow 4c = 3 \Rightarrow c = \frac{3}{4}$$

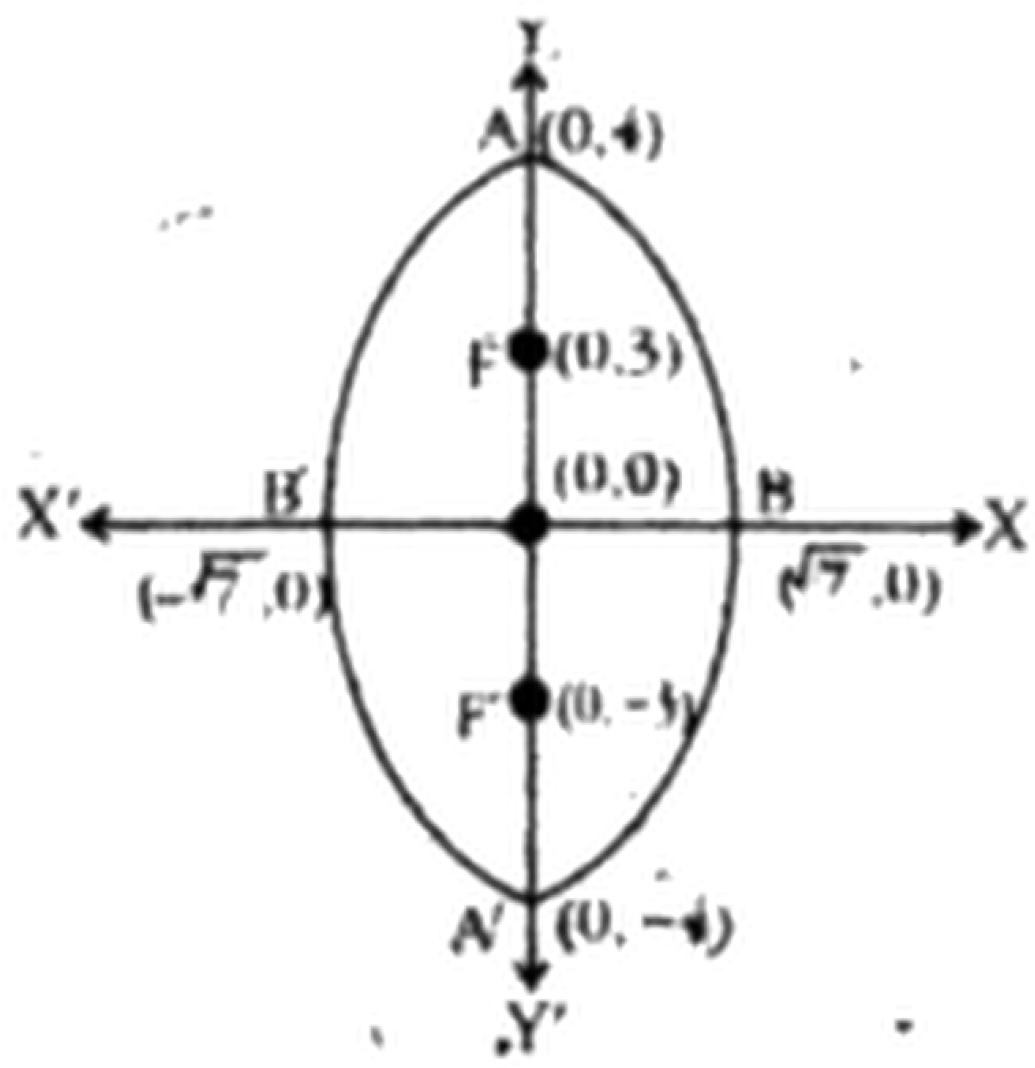
$$e^2 = \frac{a^2-b^2}{a^2}$$

$$\frac{4}{16} = \frac{16-b^2}{16}$$

$$b^2 = 7 \qquad \Rightarrow b = \pm\sqrt{7}$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{7} + \frac{y^2}{16} = 1$$



viii. Centre (2,2) major axis parallel to y-axis and of length 8 minor axis parallel to x-axis and of length 6 units.

Solution

Let equation of ellipse is

$\frac{(x-2)^2}{a^2}+\frac{(y-2)^2}{b^2}=1$, clearly centre at (2,2)

Here $2a = 8 \Rightarrow a=4$

$2b = 6 \Rightarrow b=6$

$c^2 = a^2 - b^2 \Rightarrow c^2 = 4^2 - 3^2$

$\Rightarrow$ $c^2 = 16 - 19 \Rightarrow c = \sqrt{7}$

Foci are given by

X-2 = 0, Y-2 = $\pm(a > b)$

Foci are

F(2,2+$\sqrt{7}$), (2,2-$\sqrt{7}$)

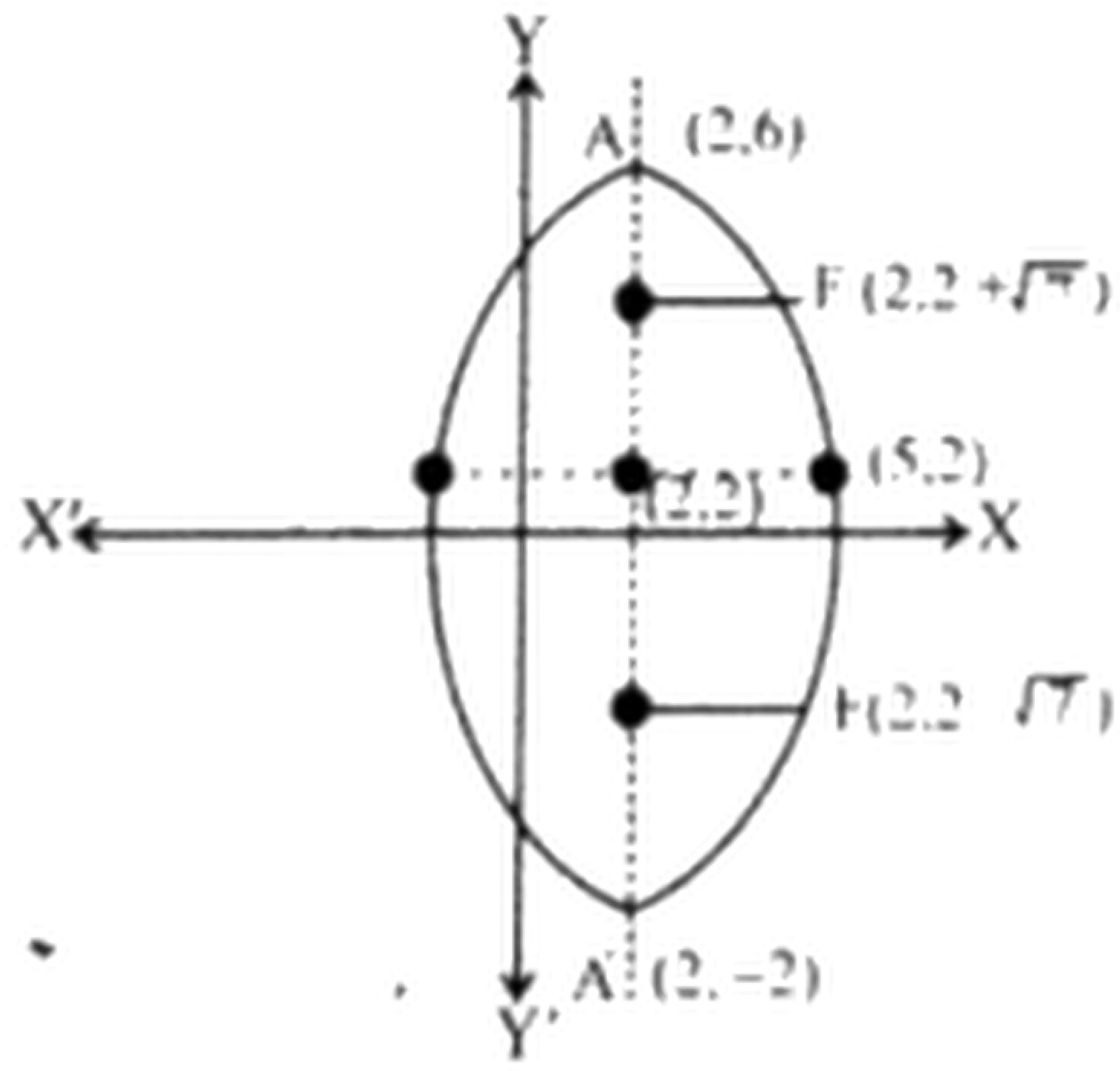
Vertices are x – 2 = 0, y – 2 = $\pm a$

Vertices are (2,2)(2,6)

Covertices are $x-2=\pm b, \quad y-2 = 0$

i.e. (-1,2) , (5,2)

sketch is



ix. Centre (0,0), symmetric with respect to both the axis and passing through the points (2,3) and (6,1).

Solution

Let equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

As it passing through (2,3), (6,1)

$$\frac{4}{a^2}+\frac{9}{b^2}=1 \quad \text{----------- (1)}$$

$$\frac{36}{a^2}+\frac{1}{b^2}=1 \quad \text{------------ (2)}$$

From (2) $\frac{1}{b^2}=1-\frac{36}{a^2}$ put in (1)

$$\frac{4}{a^2}+9\left[1-\frac{36}{a^2}\right]=1$$

$$\frac{4}{a^2}-\frac{36\times 9}{a^2}=1-9$$

$\Rightarrow \qquad \left[\frac{1}{a^2} - \frac{81}{a^2}\right] = -8$

$\frac{1-81}{a^2} = -2 \qquad \Rightarrow \frac{-80}{a^2} = -2$

$\Rightarrow \qquad a^2 = 40$

Put in (1)

$\Rightarrow \qquad \frac{9}{b^2} = 1 - \frac{1}{10} \qquad \Rightarrow \frac{9}{b^2} = \frac{9}{10}$

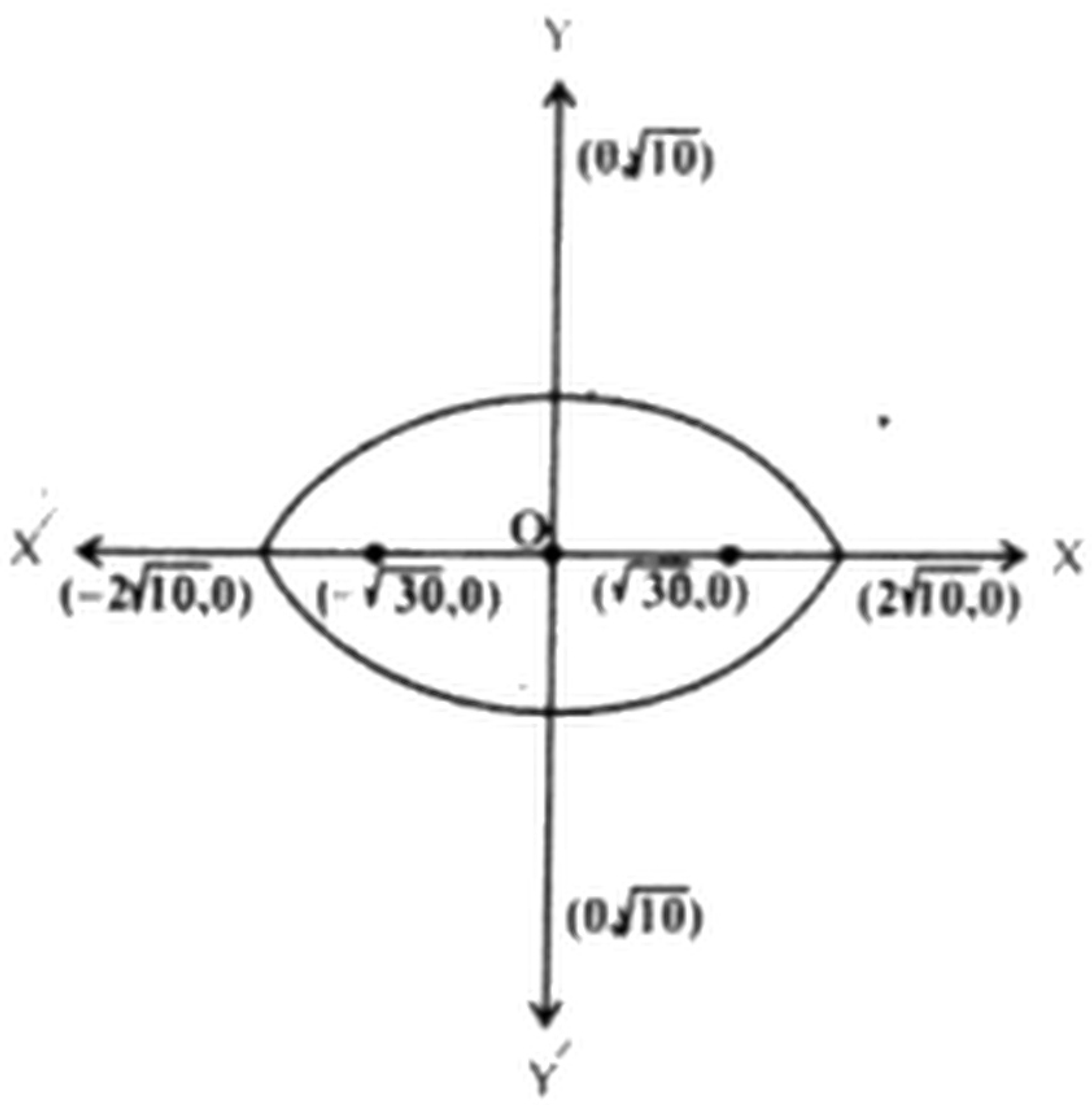
$\Rightarrow \qquad B^2 = 10 \qquad \Rightarrow c^2 = a^2 - b^2$

$\Rightarrow \qquad c^2 = 40 - 10 \qquad \Rightarrow c^2 = 30$

$\Rightarrow \qquad c = ae = \sqrt{30}$

Required equation of ellipse is

$$\frac{x^2}{40} + \frac{y^2}{10} = 1$$



x. **Centre (0,0) major axis horizontal, the points (3,1), (4,0) lies on the graph?**

Solution

Let equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ------------- (1) (a>b)

$\frac{(3)^2}{a^2} + \frac{(1)^2}{b^2} = 1$ ------------ (2)

$\frac{(4)^2}{a^2} + \frac{(0)^2}{b^2} = 1$ ---------- (3)

From equation (3)

$$\frac{16}{a^2} + 0 = 1 \text{ or } a^2 = 16$$

Substituting the values of a^2 in equation (2)

$\frac{9}{16} + \frac{1}{b^2} = 1$ or $\frac{1}{b^2} = 1 - \frac{9}{16}$

$\frac{1}{b^2} = \frac{16-9}{16}$ or $\frac{1}{b^2} = \frac{7}{16}$

Or $b^2 = \frac{16}{7}$

Putting the values of a^2 and b^2 in equation (1)

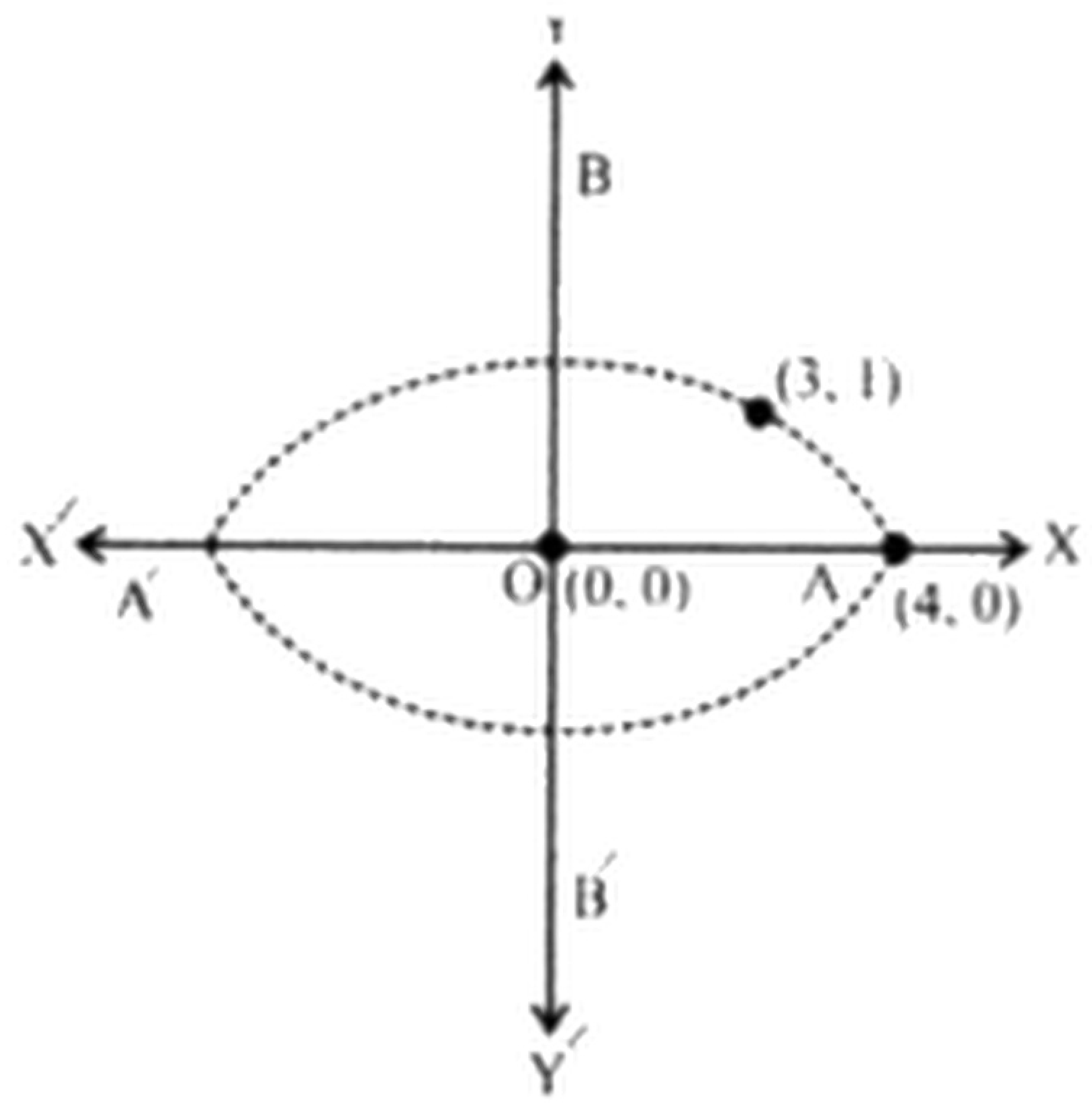
$\frac{x^2}{16} + \frac{y^2}{\left(\frac{16}{7}\right)} = 1$ or $\frac{x^2}{16} + \frac{7y^2}{16} = 1$

Sketch is

$c^2 = \frac{a^2-b^2}{a^2} = \frac{16-16/7}{16}$

$c^2 = \frac{6}{7}$ $c = \sqrt{\frac{6}{7}}$

$ac = 4\sqrt{\frac{6}{7}}$



Q2. Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is given:

i. $\mathbf{x^2 + 4y^2 = 16}$

Solution

To begin we place this equation in standard form by dividing the equation by 16

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \quad \text{or} \quad \frac{(x-0)^2}{16} + \frac{(y-0)^2}{4} = 1$$

The centre is at origin. Since the larger denominators is under x^2, the major axis is horizontal. To find the vertices more $\sqrt{16}$ = units horizontally from centre. To find covertice move $\sqrt{4} = 2$ unit vertically from centre. To find co-ordinates of foci move $\sqrt{16-4} = \sqrt{12} = 2\sqrt{3}$ units horizontally from centre.

The results we summarized a

Centre = (0,0)

Vertices = (4,0) and (-4,0)

Covertices = (0,2) and (0,-2)

Foci = $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$

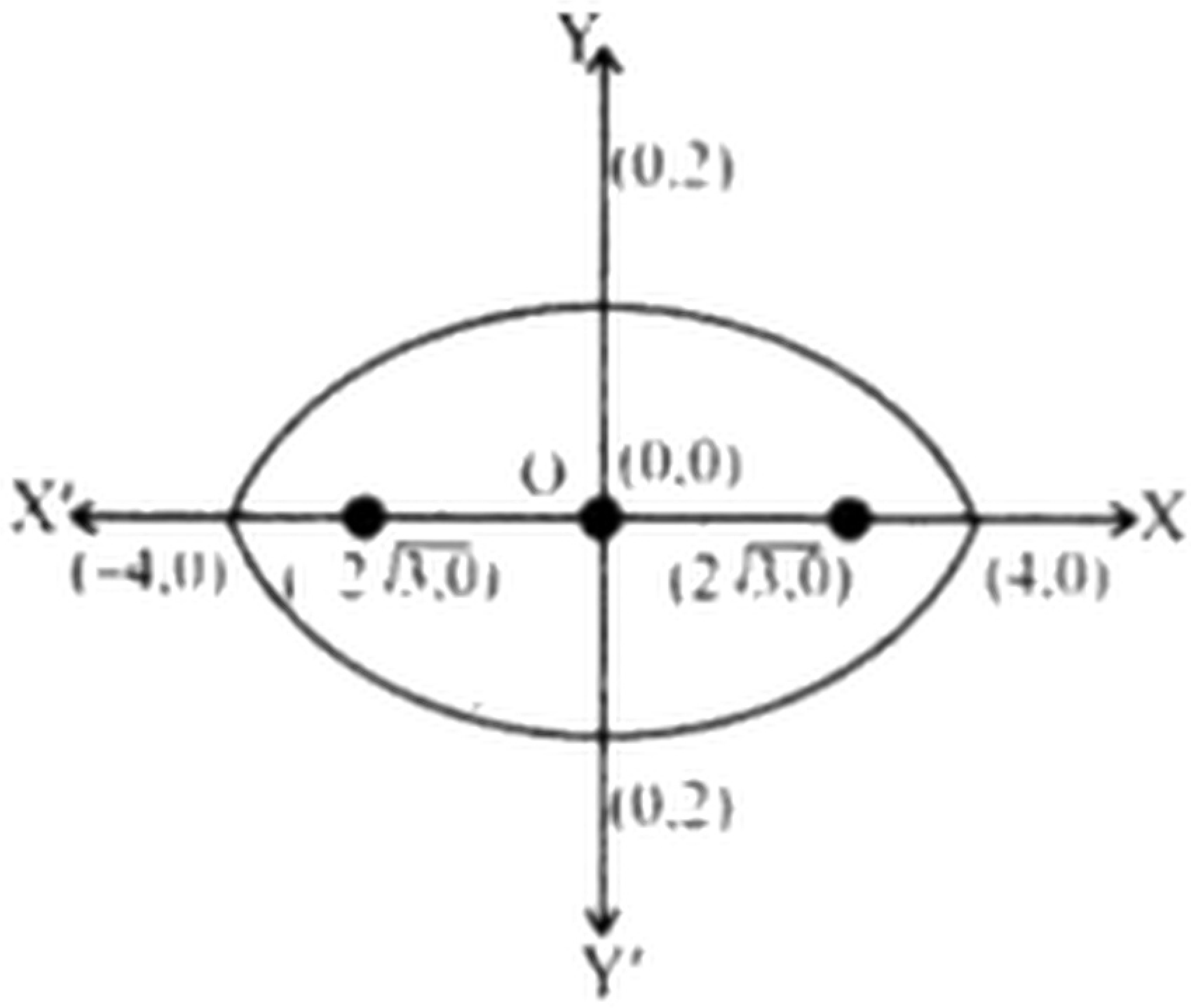
Sketch is

$$c^2 = \frac{a^2-b^2}{a^2} = \frac{16-4}{16} = \frac{3}{4}$$

$$c = \frac{\sqrt{3}}{2}$$

$$ac = 4\frac{\sqrt{3}}{2}$$

$$\frac{a}{c} = \frac{8}{\sqrt{3}}$$



ii. $\mathbf{9x^2 + y^2 = 18}$

Solution

The given equation maybe written as

$$\frac{x^2}{2} + \frac{y^2}{18} = 1$$

$$\frac{(x-0)^2}{2} + \frac{(y-0)^2}{18} = 1$$

Which is standard form of an ellipse.

Centre (0,0). Semi major axis

$$a = \sqrt{18} = 3\sqrt{2}$$

semi major axis

$b = \sqrt{2}$ from $c^2 = a^2 - b^2$ we have

$c^2 = 18 - 2 = 16$ or $c=4$

Foci F = (0,4), F(0,4)

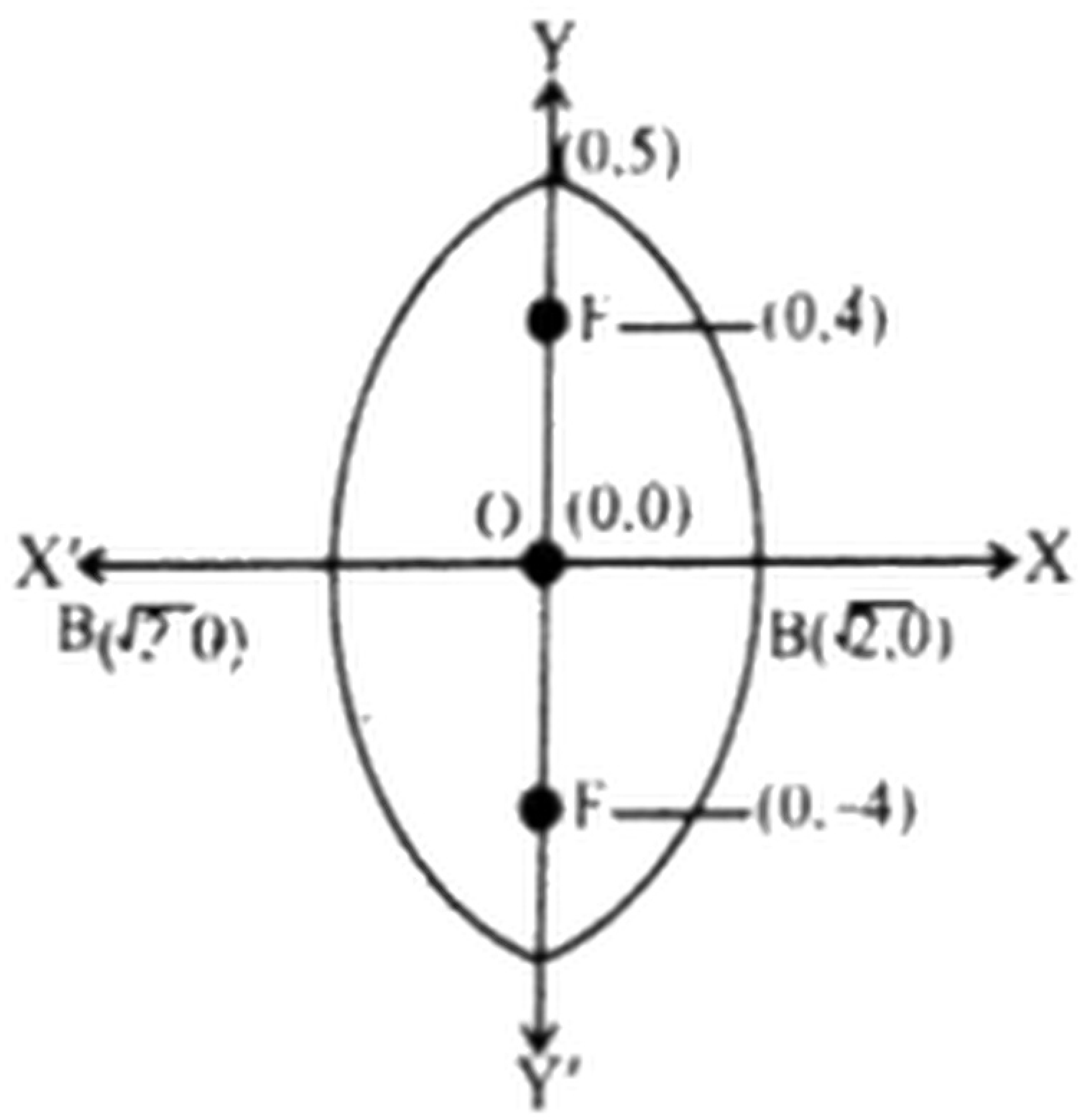
Vertices $A(0,-3\sqrt{2})$, $A'(0,3\sqrt{2})$

Covertices: $B(-\sqrt{2},0), B(\sqrt{2},0)$ the graphic are shown.

$$c^2 = \frac{a^2-b^2}{a^2} = \frac{18-2}{18} = \frac{8}{9}$$

$$c = \frac{2\sqrt{2}}{3}$$

$$ac = 4$$



iii. $25x^2 + 9y^2 = 225$

Solution

Standard form of equation and dividing by 225

$$\frac{25x^2}{225} + \frac{9y^2}{225} = 1 \text{ or } \frac{x^2}{9} + \frac{y^2}{25} = 1$$

$$\frac{(x-0)^2}{9} + \frac{(y-0)^2}{25} = 1$$

since the larger denominator I used under y^2 the major axis is vertical composing above equation with

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \text{Centre} = (0,0)$$

Semi major axis $a = \sqrt{25} = 5$

Semi minor axis $b = \sqrt{9} = \pm 3$

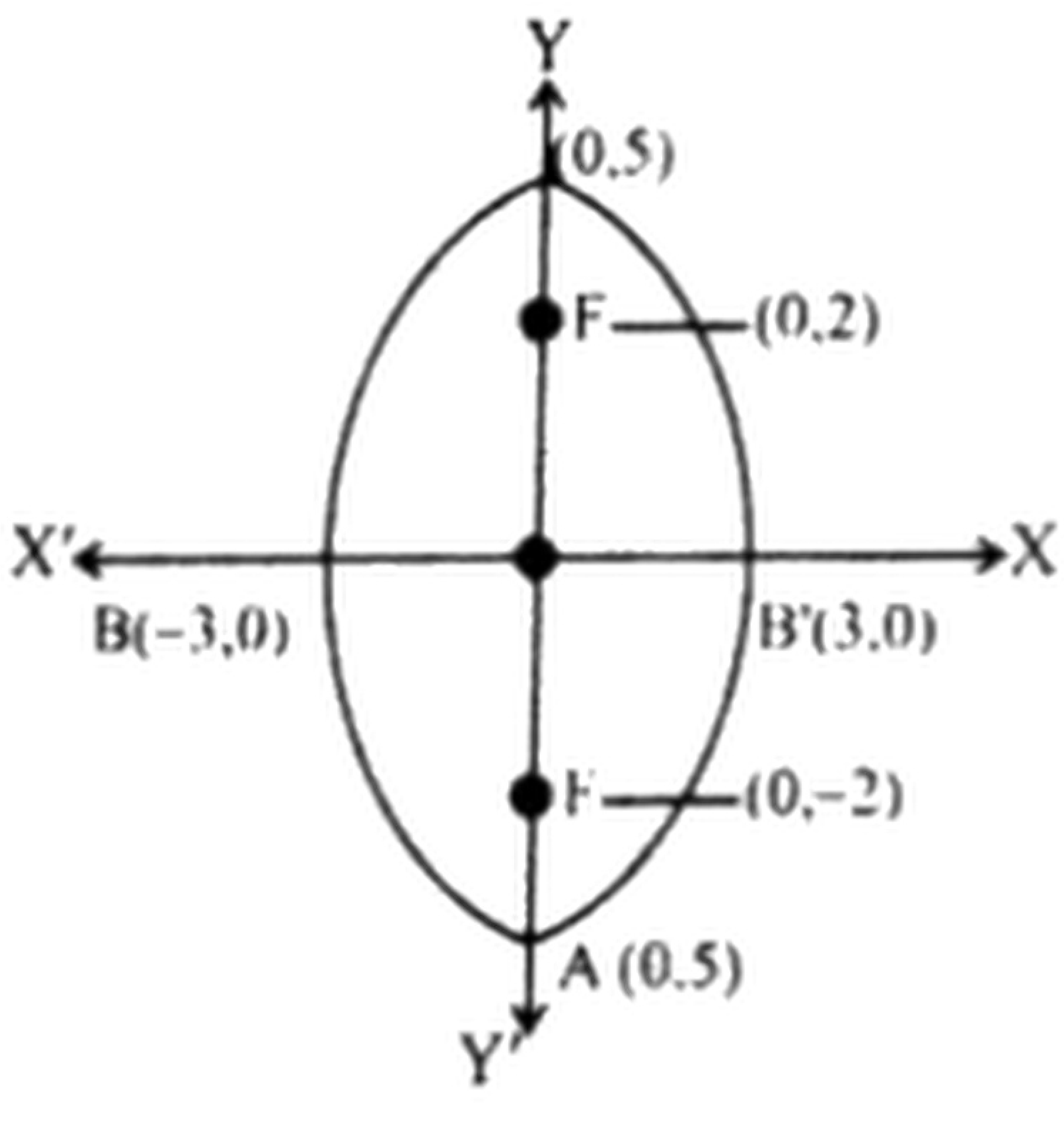
From $c^2 = a^2 - b^2 \Rightarrow c^2 = 25 - 9 = 16$

$c = \sqrt{4} = -2$

foci F(0,-2),F'(0,2) $c^2 = 25, a = \pm 5$

vertices = A(0,-5), A'(0,5) $ac = 4, \frac{a}{c} = \frac{25}{4}$

Convinces =B(-3,0), (3,0)



iv. $\frac{(2x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$

Solution

Standard form of equation

$$\frac{2^2\left(x-\frac{1}{2}\right)^2}{4} + \frac{(y+2)^2}{16} = 1$$

$$\frac{\left(x-\frac{1}{2}\right)^2}{1} + \frac{(y+2)^2}{16} = 1 \quad \text{------------ (1)}$$

If we set $x - \frac{1}{2} = x, y + 2 = y$ in (1)

$$\frac{x^2}{1} + \frac{y^2}{16} = 1 \quad \text{-------------- (2)}$$

since the larger denominator is used under y^2 the major axis is vertical along the line x = 0

i.e. along the line $x - \frac{1}{2} = 0$ (a line parallel to y)

Semi major axis a = $\sqrt{16} = 4$

Semi minor axis b = $\sqrt{1} = 1$

$$c = \sqrt{a^2 - b^2} = \sqrt{16-1} = \sqrt{15}$$

equation of (2) is

x = 0 , y = 0

i.e. $x - \frac{1}{2} = 0, y + 2 = 0$

i.e. $x = \frac{1}{2}, y = -2$ is the center of (1) the centre is $(\frac{1}{2}, -2)$

vertices of (2) are

x = 0 , $\frac{y^2}{16} = 1$

i.e. x = 0, y = ± 4

i.e. $x = \frac{1}{2}, y = -2 \pm 4$

$\left(\frac{1}{2}, -6\right), \left(\frac{1}{2}, 2\right)$ are vertices of (1)

Covertices of (2) are $x = \pm 1, y = 0$

$\Rightarrow \quad x - \frac{1}{2} = \pm 1, y + 2 = 0$

$\Rightarrow \quad x = \frac{1}{2} \pm 1, y = -2$

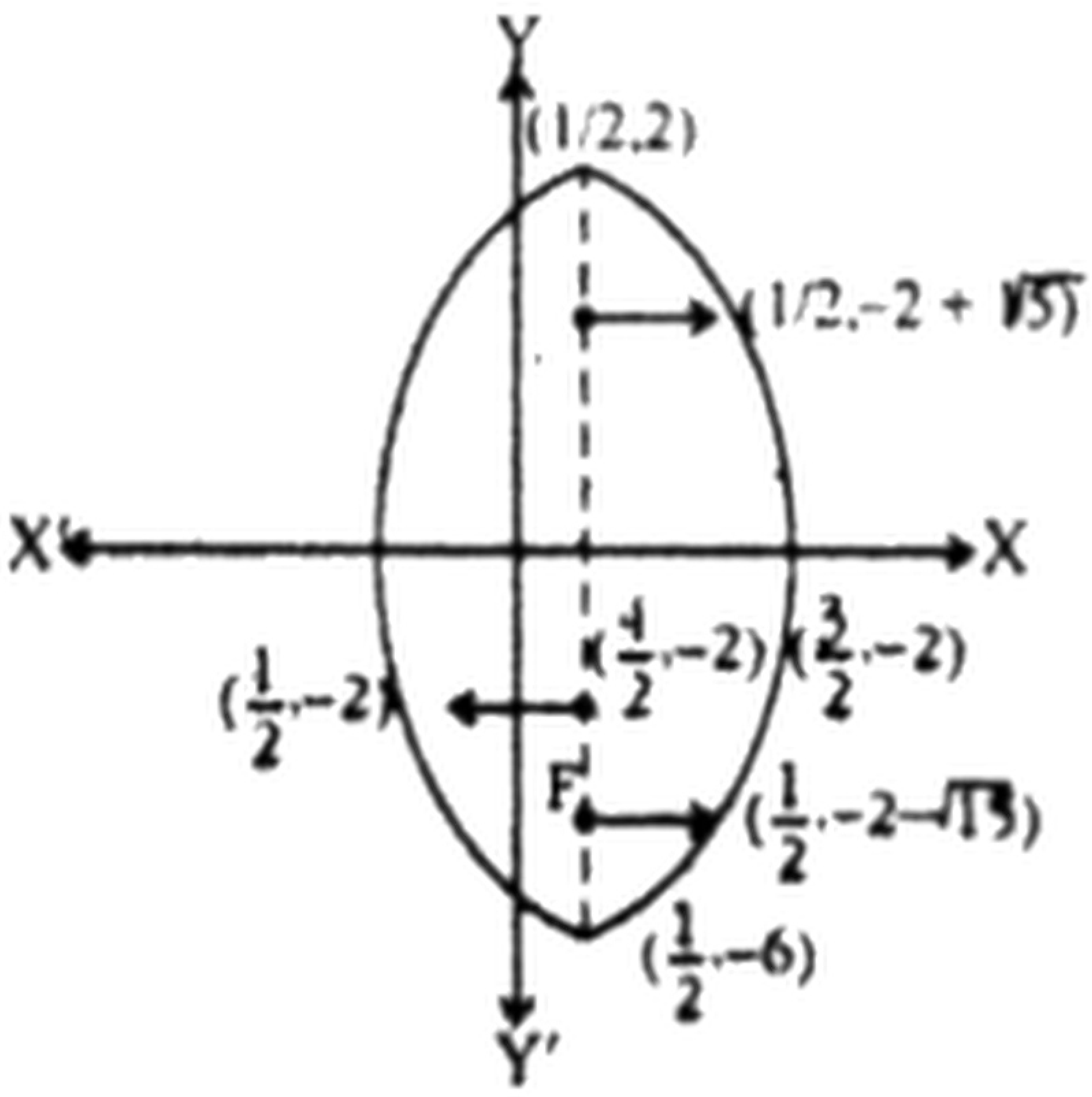
$\Rightarrow \quad \left(-\frac{1}{2}, 2\right), \left(\frac{3}{2}, -2\right)$ are covertices

$\Rightarrow$ (2) are $x = 0$ $y = \pm c$

i.e. $x - \frac{1}{2} = 0, y + 2 = \pm\sqrt{15}$

$x = \frac{1}{2}, y = -2 \pm \sqrt{15}$

i.e. $\left(\frac{1}{2}, -2 - \sqrt{15}\right), \left(\frac{1}{2}, -2 + \sqrt{15}\right)$ are foci of (1)



v. $\mathbf{x^2 + 16x + 4y^2 - 16y + 76 = 0}$

Solution

$$x^2 + 16x + 4(y^2 - 4y) = -76$$

$$(x^2 + 16x + 64) + 4(y^2 - 4y + 4) = -76+64+16$$

$$(x + 8)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x+8)^2}{4} + \frac{4(y-2)^2}{4} = 1$$

$$\frac{(x+8)^2}{4} + \frac{(y-2)^2}{1} = 1 \quad \text{------------ (1)}$$

If we set $x + 8 = x, y - 2 = y$ in (1)

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \quad \text{-------------- (2)}$$

since the larger denominator is used under x^2 the major axis is horizontal

i.e. along the line y = 0

along the line $y - 2 = 0$

Semi major axis $a = \sqrt{4} = 2$

Semi minor axis $b = \sqrt{1} = 1$

$$c = \sqrt{a^2 - b^2} = \sqrt{4 - 1} = \sqrt{3}$$

centre of (2) is

x = 0 , y = 0

i.e. $x + 8 = 0, y - 2 = 0$

i.e. $x = -8, y = 2$ is the center of (1) the centre is (-8,2)

vertices of (2) are $\frac{x^2}{4} = 1, y = 0$

i.e. $x^2 = 4$, $y = 0$

$\Rightarrow$ $x = \pm 2,$ $y = 0$

$\Rightarrow$ $x + 8 = \pm 2,$ $y - 2 = 0$

$x = -8 \pm 2,$ $y = 2$

$\Rightarrow \quad (-10,2),(-6,2)$ are vertices of (1)

Covertices of (2) are

$x = 0,\ y^2 = 1$

$x = 0\ ,\ y = \pm 1$

$\Rightarrow \quad x + 8 = 0,\ y - 2 = \pm 1$

$\Rightarrow \quad x = -8, y = 2 \pm 1$

$\Rightarrow \quad (-8,1),(-8,3)$ are covertices

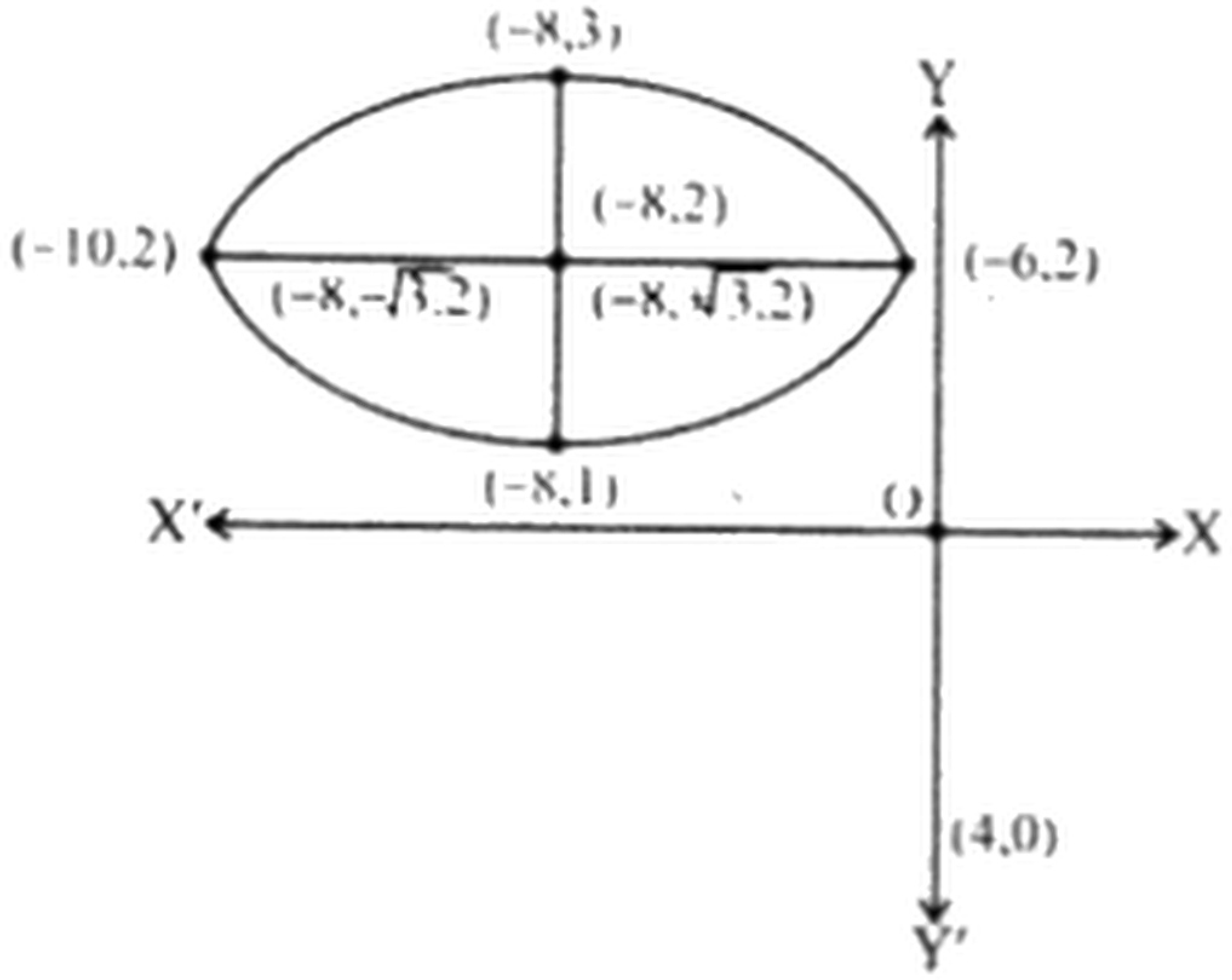
$\Rightarrow \quad$ (2) are

$x = \pm c \quad y = 0$

i.e. $\quad x + 8 = \pm\sqrt{3}, \qquad y - 2 = 0$

$x = -8 \pm \sqrt{3}, \qquad y = 2$

i.e. $\left(-8 - \sqrt{3}, 2\right), \left(-8 + \sqrt{3}, 2\right)$ are foci of (1)



vi. $\quad \mathbf{25x^2 + 4y^2 - 250x - 16y + 541 = 0}$

Solution

$$(25x^2 - 250x) + (4y^2 - 16y) = -541$$

$$25(x^2 - 10x) + 4(y^2 - 4y) = -76$$

$$25(x^2 - 10x + 25) + 4(y^2 - 4y + 4) = -541 + 625 + 16$$

$$25(x-5)^2 + 4(y-2)^2 = 100$$

$$\frac{25(x-5)^2}{100} + \frac{(y-2)^2}{25} = 1 \quad \text{----------} (1)$$

If we set $x - 5 = x, \quad y - 2 = y$ in (1)

$$\frac{x^2}{4} + \frac{y^2}{25} = 1 \quad \text{--------------} (2)$$

since the larger denominator is used under y^2 the major axis is vertical along the line x = 0

i.e. along the line $x - 5 = 0$,x = 5

Semi major axis a = $\sqrt{25} = 5$

Semi minor axis b = $\sqrt{4} = 2$

$$c = \sqrt{a^2 - b^2} = \sqrt{25-4} = \sqrt{21}$$

⇒ (2) is

x = 0 , y = 0

$x - 5 = 0, y - 2 = 0$

⇒ $x = 5, y = 2$

the center of (1) is (5,2) vertices of (2) we have

$x^2 = 0$, $y^2 = 25$

x = 0, y = ± 5

⇒ x - 5 = 0, y-2 = ± 5

i.e. x =5, y = 2 ± 5

$\Rightarrow$ (5,7), (5,3) are vertices

Covertices of (2) are

$x^2 = 4$, $y^2 = 0$

$x = \pm 2$, $y = 0$

$\Rightarrow$ $x - 5 = \pm 2$, $y - 2 = 0$

$\Rightarrow$ $x = 5 \pm 2$, $y = 2$

(7,2), (3,2) are covertices

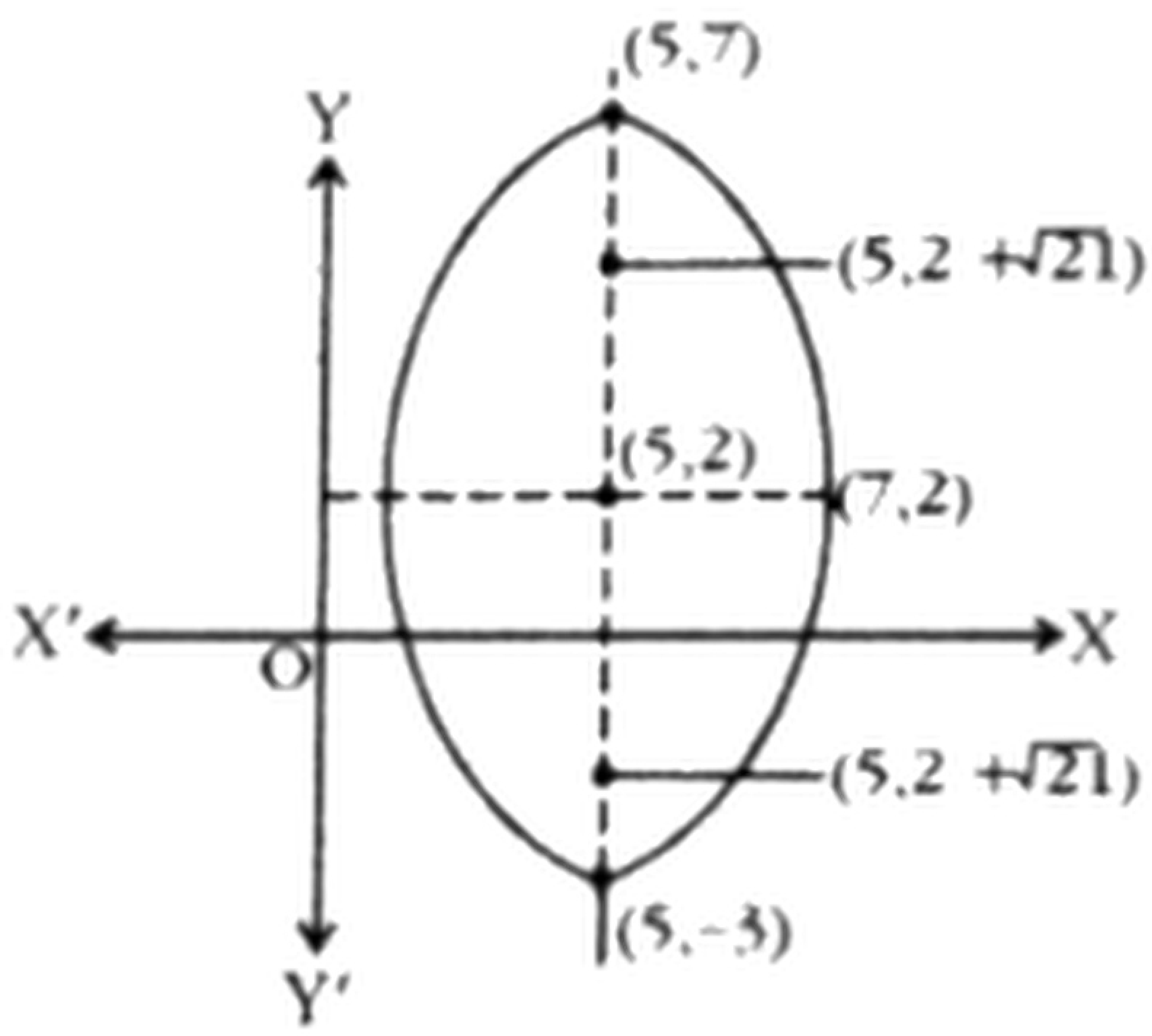
foci of (2) are

$x = 0$ $y = \pm c$

$x - 5 = 0$, $y - 2 = \pm\sqrt{21}$

$\Rightarrow$ $x = 5$, $y = 2 \pm \sqrt{21}$

$(5,2 - \sqrt{21}), (5,2 + \sqrt{21})$ are foci of (1)



Q3. Let 'a' be a positive number and $0 < c < a$

Let $F(-c, 0)$ and $F'(c, 0)$ be two given points.

Prove that the locus of points $P(x, y)$ such that $|PF| + |PF'| = 2a$, is an ellipse

Solution

$$|PF| + |PF'| = 2a$$

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Square both sides

$$(x+c)^2 + y^2 + (x-c)^2 + y^2 + 2\sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2} = 4a^2$$

$$x^2 + c^2 + 2xc + y^2 + x^2 + c^2 - 2xc + y^2 + 2\sqrt{\{(x+c)^2 + y^2\}\{(x-c)^2 + y^2\}} = 4a^2$$

$$2x^2 + 2y^2 + 2c^2 + 2\sqrt{(x^2 + y^2 + c^2)^2 - (2xc)^2} = 4a^2$$

$$\sqrt{(x^2 + y^2 + c^2)^2 - (2xc)^2} = 2a^2 - (x^2 + y^2 + c^2)$$

Let $x^2 + y^2 + c^2 = t$

$$\sqrt{t^2 - 4x^2c^2} = 2a^2 - t$$

$$t^2 - 4x^2c^2 = 4a^4 + t^2 - 4a^2t$$

$$4a^4 + 4x^2c^2 - 4a^2t = 0$$

$$a^4 + x^2c^2 - a^2(x^2 + y^2 + c^2) = 0$$

$$a^4 + x^2c^2 - a^2x^2 - a^2y^2 - a^2c^2 = 0$$

$$(c^2 - a^2)x^2 - a^2y^2 + a^2(a^2 - c^2) = 0$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

$$\frac{(a^2-c^2)x^2}{a^2(a^2-c^2)} + \frac{a^2y^2}{a^2(a^2-c^2)} = \frac{a^2(a^2-c^2)}{a^2(a^2-c^2)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{(a^2-c^2)} = 1$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2 - c^2$ which is an ellipse.

Q4. Use problem 3 to find equation of the ellipse as locus of points $P(x, y)$ such that. The sum of distance from the point (0,0) and (1,1) is 2?

Solution

By providing condition

$$\sqrt{(x-0)^2+(y-0)^2}+\sqrt{(x-1)^2+(y-1)^2}=2$$

$$\sqrt{x^2+y^2}+\sqrt{x^2-2x+1+y^2-2y+1}=2$$

Square both sides

$$x^2+y^2+x^2+y^2-2x-2y+2+2\sqrt{x^2+y^2}\sqrt{x^2+y^2-2x-2y+2}=4$$

$$2x^2+2y^2-2x-2y+2+2\sqrt{(x^2+y^2)(x^2+y^2-2x-2y+2)}=4$$

$$x^2+y^2-x-y+1+\sqrt{(x^2+y^2)(x^2+y^2-2x-2y+2)}=2$$

Let $x^2+y^2=t$

$$t-x-y+1+\sqrt{t(t-2x-2y+2)}=2$$

$$\sqrt{t^2-2tx-2ty+2t}=2-t+x+y-1$$

$$\sqrt{t^2-2tx-2ty+2t}=1-t+x+y$$

Square both sides

$$t^2-2tx-2ty+2t=(1-t)^2+(x+y)^2+2(1-t)(x+y)$$

$$t^2-2tx-2ty+2t=\ 1+\ t^2-2t+x^2+y^2+2xy+2(x+y-tx-ty)$$

$$t^2-2tx-2ty+2t=\ 1+\ t^2-2t+x^2+y^2+2xy+2x+2y-2tx-2ty)$$

$$2t=\ 1-2t+x^2+y^2+2xy+2x+2y$$

By replacing the value of t

$$2x^2+2y^2=1-2x^2-2y^2+x^2+y^2+2xy+2x+2y$$

$$3x^2+3y^2-2xy-2x-2y=0$$

Q5. Define the latus rectum of an ellipse. Prove that latus rectum of ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{2b^2}{a}$$

Solution

The chord through the foci and perpendicular to the axis of ellipse is called latus rectum ellipse.

The equation of the ellipse is

$$\frac{X^2}{a^2}+\frac{y^2}{b^2}=1$$

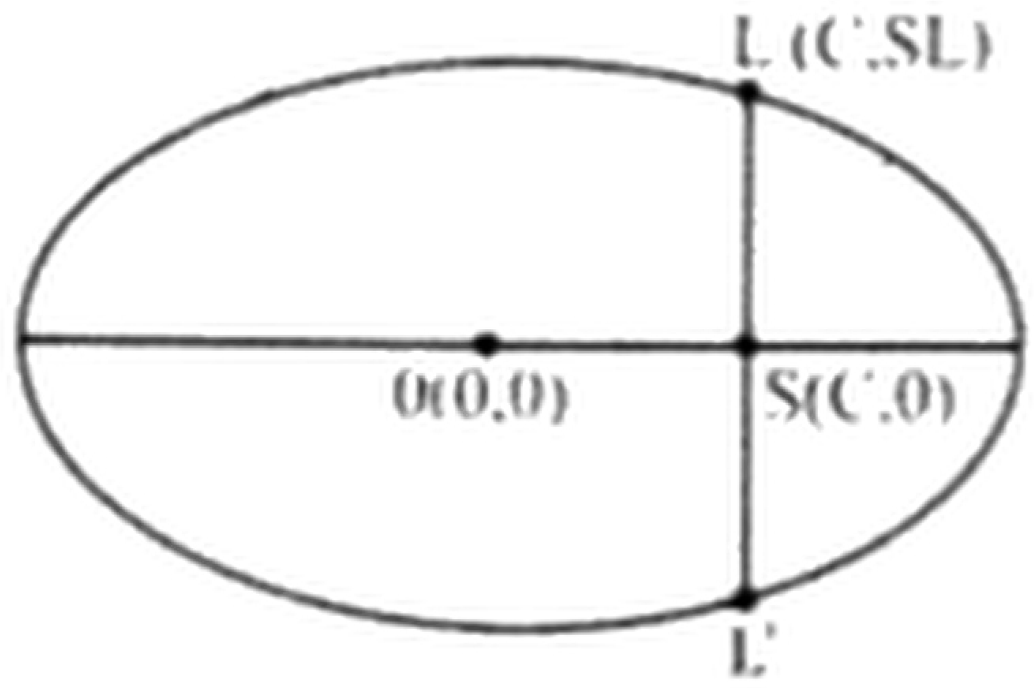
Let LSL be the latus rectum since the coordinates are (c,0)

Co-ordinates of L are (c,SL). But L lies on ellipse

$$\frac{c^2}{a^2}+\frac{(SL)^2}{b^2}=1$$

$$\frac{(SL)^2}{b^2}=1-\frac{c^2}{a^2}=\frac{a^2-c^2}{a^2}$$

$$=\frac{b^2}{a^2}\ (\therefore c^2=a^2-b^2)$$



$$(SL)^2=\frac{b^4}{a^2}$$

But $\quad LSL=|LS|+|SL|$

$$=|LS|+|LS|$$

$$2|LS|=\frac{2b^2}{a}$$

Here the length of latus rectum $=\frac{2b^2}{a}$

Q6. The major axis of an ellipse is standard form lies along the x-axis and has length$4\sqrt{2}$. The distance between the foci equals the length of the minor axis. Write an equation of ellipse.

Solution

Give length of major axis = $4\sqrt{2}$

$\Rightarrow \quad 2a = 4\sqrt{2} \qquad a = 2\sqrt{2}$

$\Rightarrow \quad a^2 = 8$

Also distance between foci = length of minor axis

i.e. 2c = 2b b = c ac = e

from $c^2 = a^2 - b^2$

$\Rightarrow \quad 2b^2 = a^2 \quad b^2 = \frac{a^2}{2}$

$\Rightarrow \quad a^2 = \frac{8}{2} = 4$

Hence equation of ellipse is

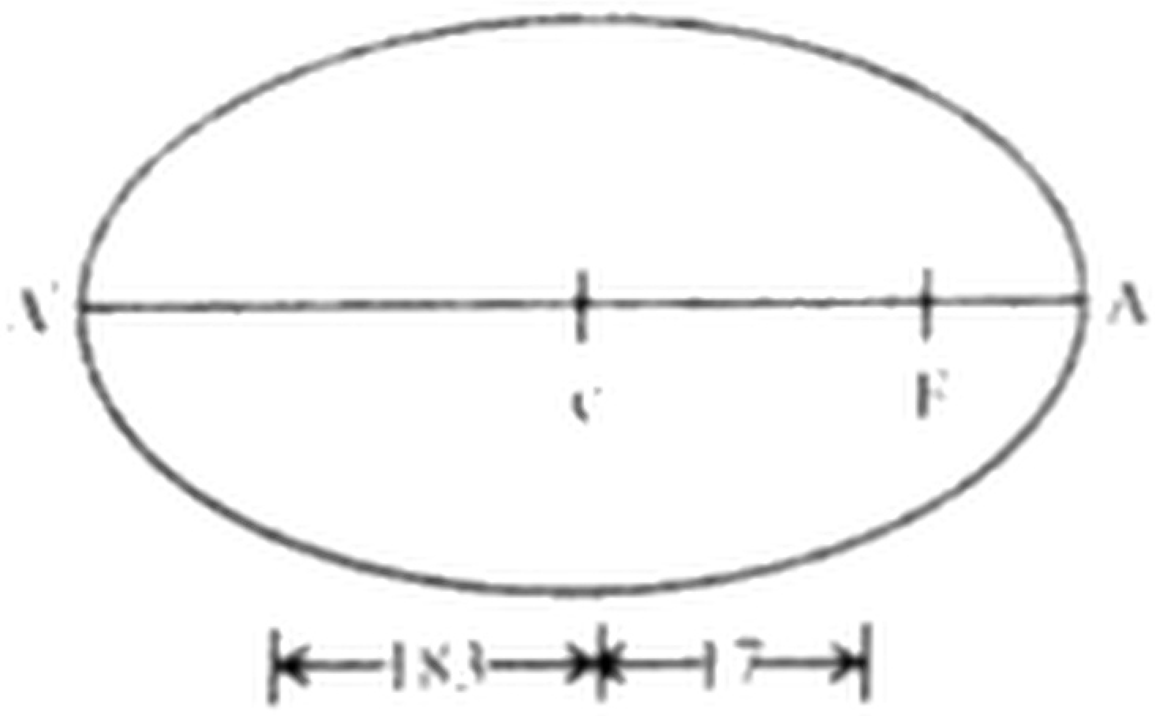
$$\frac{X^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{X^2}{8} + \frac{y^2}{4} = 1$$

Q7. An asteroid has elliptic orbit with the sun at one focus. Its distance from the sun ranges from 17 million to 183 million miles. Write an equation of the orbit of the asteroid.

Solution

As



$$2a = 200$$

$$\Rightarrow \quad a = 100$$

$$ae = a - 17 = 100 - 17 = 83$$

$$e = \frac{83}{100} = 0.83$$

$$e^2 = \frac{a^2 - b^2}{a^2} \qquad (0.83)^2 = \frac{10000 - b^2}{10000}$$

$$\Rightarrow \quad b^2 = 3111$$

Equation of orbit of astoid is

$$\Rightarrow \quad \frac{X^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{X^2}{10000} + \frac{y^2}{3111} = 1$$

Q8. An arc in the shape of a semi ellipse in 90m wide at the base and 30m high. At what distance from the centre of earth $20\sqrt{2}$m high?

Solution

$$2a = 90$$

$$\Rightarrow \quad a = 45 \qquad b = 30$$

So equation of ellipse is

$\frac{x^2}{45^2}+\frac{y^2}{30^2}=1$ ------------------- (1)

As height is $20\sqrt{2}$

i.e. $y = 20\sqrt{2}$

putting the value of y in equation (1)

$\frac{X^2}{45^2}+\frac{(20\sqrt{2})^2}{30^2}=1$

$\frac{X^2}{45^2}+\frac{800}{900}=1$

$\Rightarrow \frac{X^2}{45^2}+\frac{8}{9}=1$

$\frac{X^2}{45^2}=1-\frac{8}{9}$

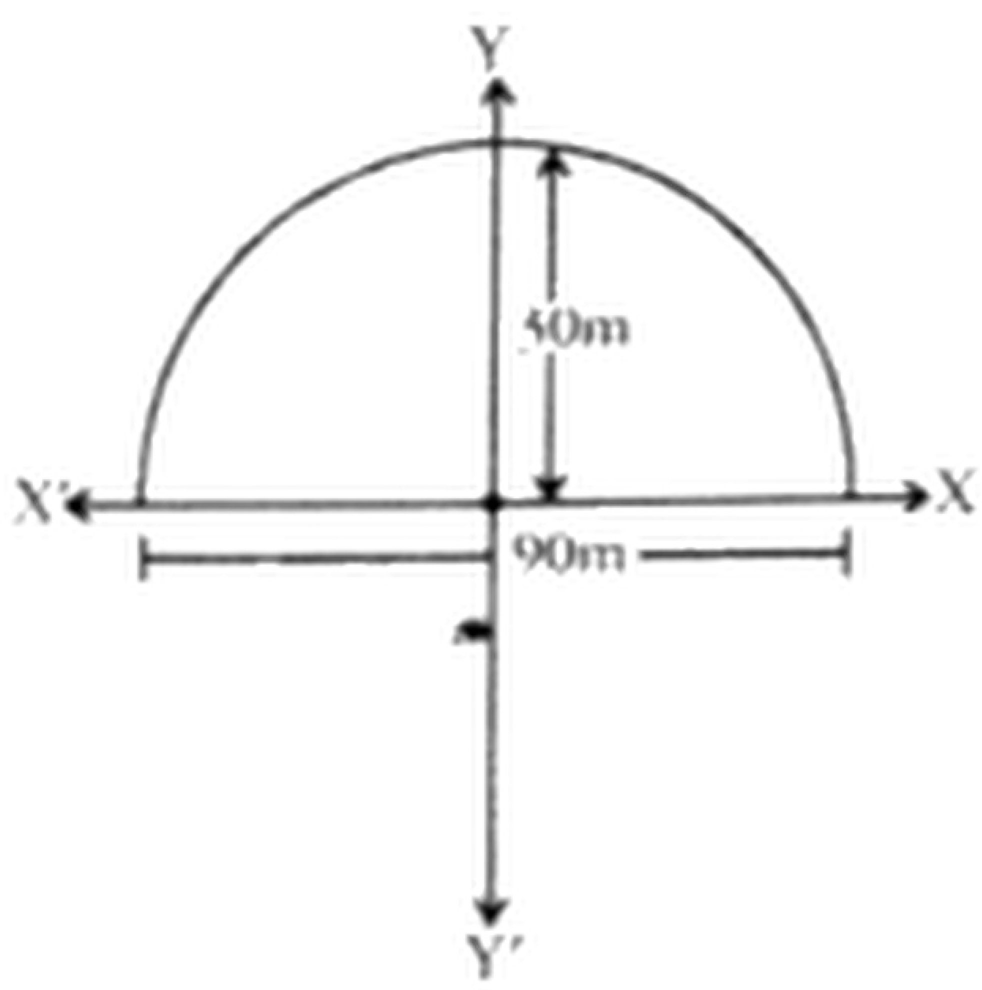
$\frac{X^2}{45^2}=\frac{9-8}{9}$

$\frac{X^2}{45^2}=\frac{1}{9}$

$x^2=\frac{45^2}{9}$

$x=\sqrt{\frac{45^2}{9}}$

x = 15m



Q9. The moon orbits the earth in an elliptic path with earth at one focus. The major and minor ases of earth are 786,806 km and 767,746 km respectively. Find the greatest and distance (in astronomy called apogee and perigee) of the moon from earth?

Solution

The greatest distance is V_2F_1 or V_1F_2

The least distance is F_1V_1 or F_2V_1

Given

As $c = ae$

$2a = 786{,}806$ $a = 384403$ km

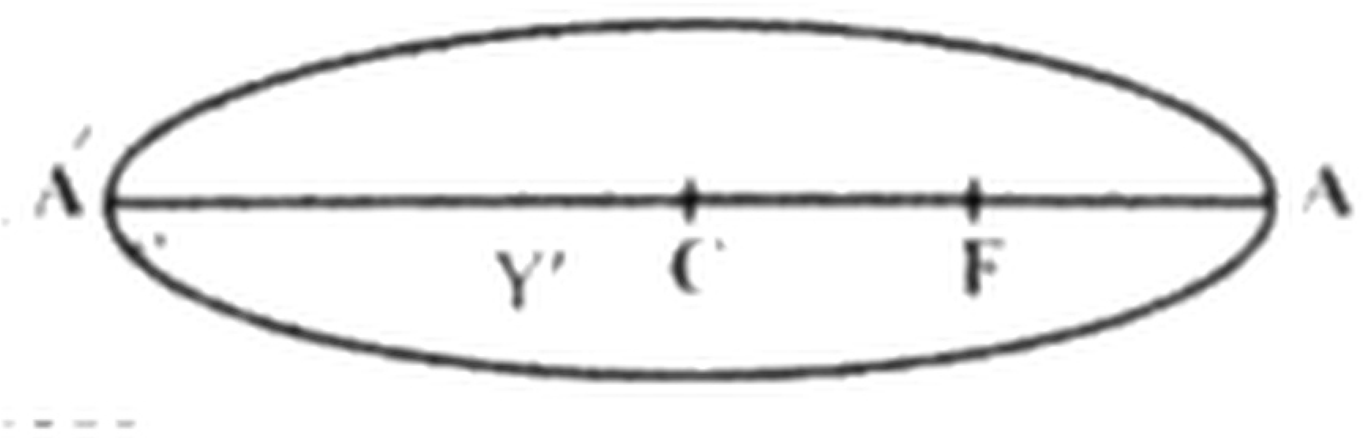
$2b = 767{,}746$ $b = 383873$ km

$a^2 = 147765666409$

$b^2 = 147358480129$

$c^2 = a^2 - b^2$

$c^2 = 147765666409 - 147358480129$



$c^2 = 407186280$

$c = 20178.85725$ km

Thus greatest distance

$a + c = 384403 + 20178.85725$

$= 404581.85725 = 404582$ km (about)

And least distance

$a - c = 384403 - 20178.85725$

$= 364224.14275 = 364224$ km (about)