Exercise 6.4

Q1 Find the focus, vertex and direction of the parabola, sketch the graph.

i. $\boldsymbol{y^2 = 8x}$

Solution:

Given $y^2 = 8x$

Comparing with $y^2 = 4ax$, we have

$4a = 8 \Rightarrow a = 2$

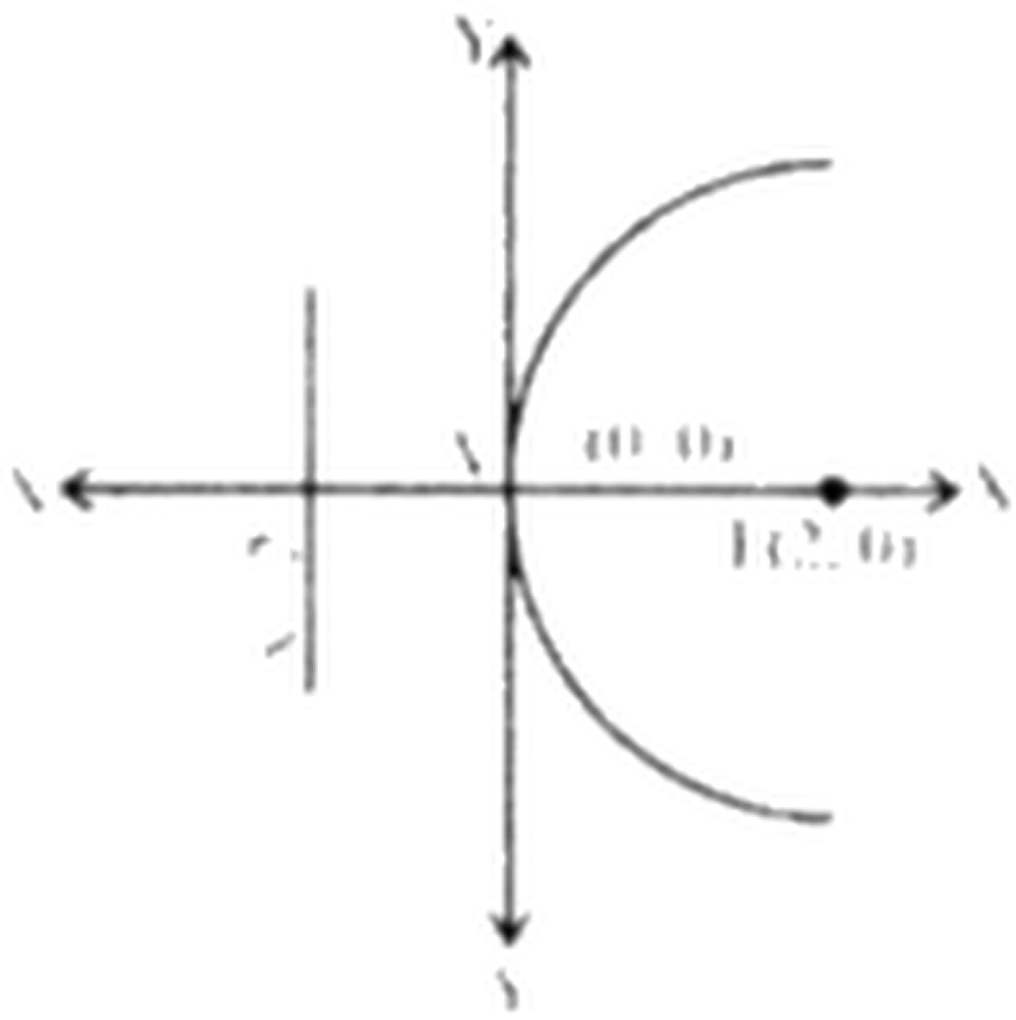
Hence focus $S = (2,0)$ vertex $V = (0,0)$ and axis f

Parabola

Equation of direction $= x = -a = -2$

Equation of axis $= y = 0$

x-axis sketch is



ii. $\boldsymbol{x^2 = -16y}$

Solution:

Comparing with $x^2 = -4ay$, we have

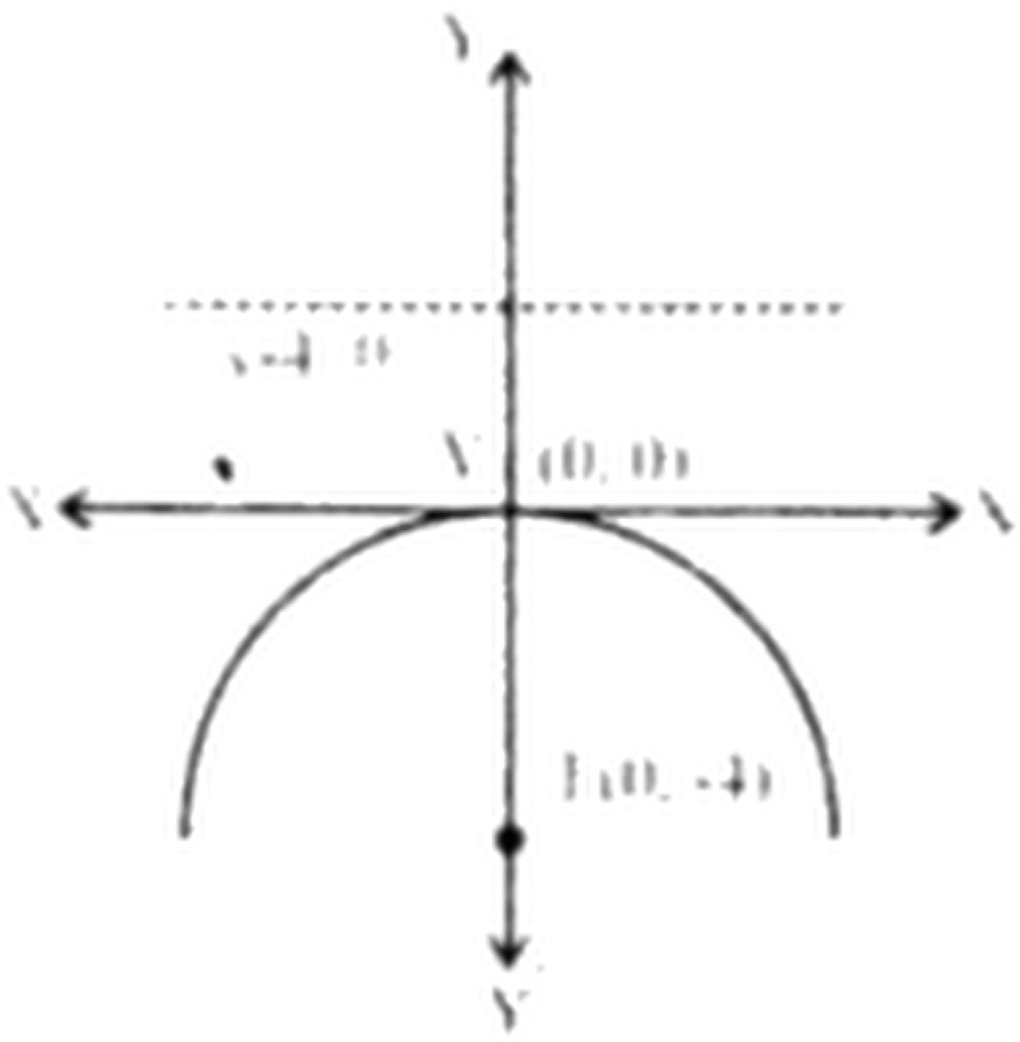
$4a = 16 \Rightarrow a = 4$

Hence focus $F = (0, -4)$ vertex $V = (0,0)$ and axis of

Parabola is y axis i.e. $x = 0$

Equation of direction $= y = a = 4$

$y - 4 = 0$



iii. $\boldsymbol{x^2 = 5y}$

Solution:

Given $\boldsymbol{x^2 = 5y}$

Comparing with $x^2 = 4ay$, we have

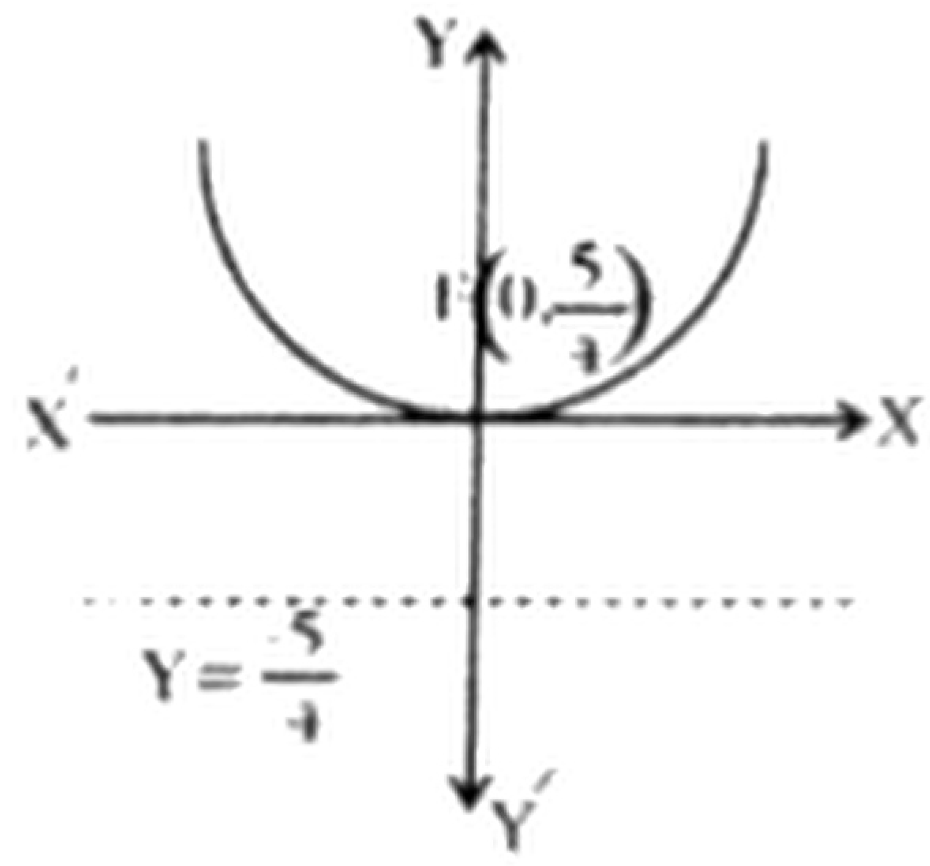
$4a = 5 \Rightarrow a = \frac{5}{4}$

Hence focus $F = \left(0, \frac{5}{4}\right)$ vertex $V = (0,0)$ and axis of

Parabola is y axis i.e. $x = 0$

Equation of direction = $y = -\frac{5}{4}$

The sketch is



iv. $\boldsymbol{y^2 = -12x}$

Solution:

Comparing with $y^2 = -4ax$, we have

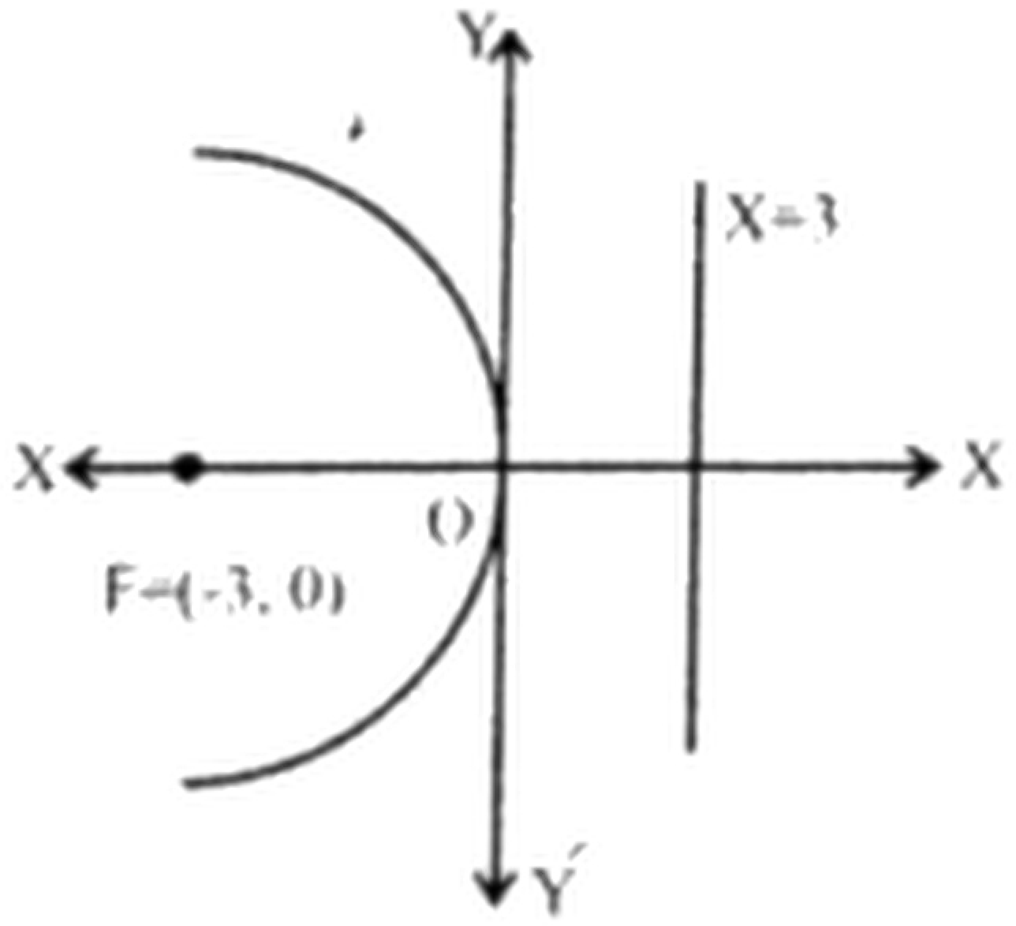
$4a = 12 \Rightarrow a = \frac{12}{4} = 3$

Hence focus $F = (-3,0)$ vertex $V = (0,0)$ and axis f

Parabola is x-axis

Equation of directrix = $x = a = 3$

Equation of axis $y = 0$



v. $x^2 = 4(y-1)$

Solution:

Shift the origin to $(0,1)$ so, that

$X = x + 0$ and $Y = y + 1$ referred to xy-axis with $(0,0)$ as origin

i.e. $x = x,$ $y - 1 = 1$

the equation (1) becomes

$x^2 = 4y$

This is parabola whose focus lies on

$x = 0$, coordinates of the focus of (2) are

$x = 0,$ $y = 1$

$x = 0,$ $y - 1 = 1$

$x = 0,$ $y = 2$

Thus coordinate of the focus of the

Parabola (1) are $(0,2)$

Axis of (2) is $x = 0$

i.e. $x = 0$ is the axis of

vertex of (2) has has coordinates $x = 0$, $y = 0$

$x = 0,$ $y - 1 = 1$

$x = 0,$ $y = 1$

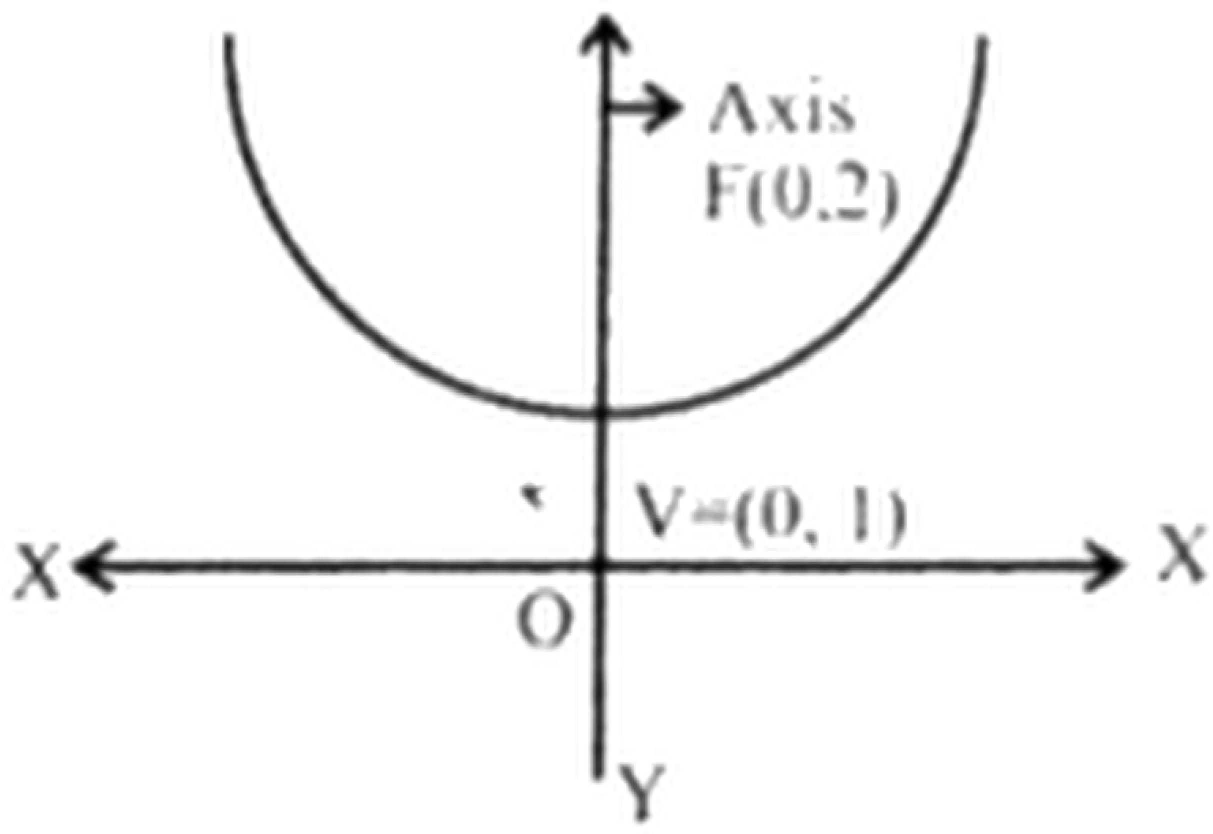
Are the coordinates of the vertex of (1)

Directrix of (2) has equation $y = -1$

i.e. $y - 1 = -1$

or , $y = 0$ is an equation of directrix if 1

the graph of (1) can easily be sketched, and is shown as



vi. $y^2 = -8(x-3)$

Solution:

Shift the origin to (3,0) so, that

$X = x + 3$ and $Y = y + 0$ referred to xy-axis with (3,0) as origin

i.e. $x - 3 = x,$ $y = y$

the equation (1) becomes

$y^2 = -8x$

This is parabola whose focus lies on

$y = 0$, coordinates of the focus of (2) are

$x = -2,$ $y = 0$

$x - 3 = -2,$ $y = 0$

$x = 1,$ $y = 0$

Thus coordinate of the focus of the

Parabola (1) are (1,0)

Axis of (2) is $y = 0$

i.e. $y = 0$ is the axis of (1)

vertex of (2) has has coordinates $x = 0,\ y = 0$

$x - 3 = 0, \qquad y = 0$

$x = 3, \qquad y = 1$

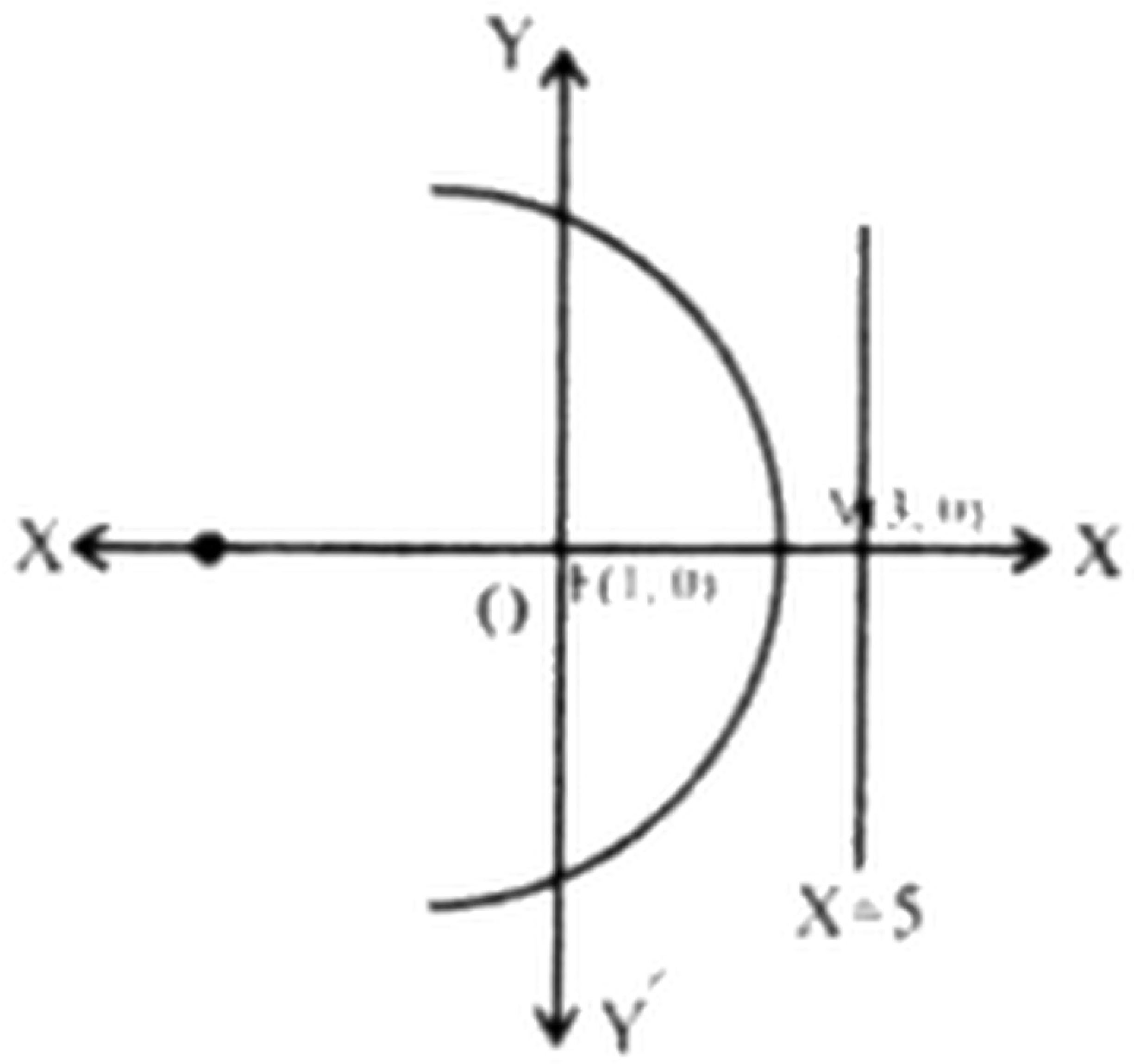
Are the coordinates of the vertex of (1)

Directrix of (2) has equation $x = 2$

i.e. $x - 3 = 2$

or , $x = 5$ is an equation of directrix (1)

sketch is



vii. $(x - 1)^2 = 8(y + 2)$

Solution:

Shift the origin to $(1, -2)$ so, that

$X = x + 1$ and $Y = y - 2$ referred to xy-axis with $(1, -2)$ as origin

i.e. $x - 1 = x, \qquad y + 2 = y$

the equation (1) becomes

$x^2 = 8y$______________(2)

Which is parabola whose focus lies on

$x = 0$, coordinates of the focus of (2) are

$x = 0,$ $y = 2$

$x - 1 = 0,$ $y + z = 2$

$x = 1,$ $y = 2$

Thus coordinate of the focus of the

Parabola (1) are (1,0)

Axis of (2) is $x = 0$

i.e. $x - 1 = 0$ $y + 2 = 0$

$x = 1,$ $y = 2$

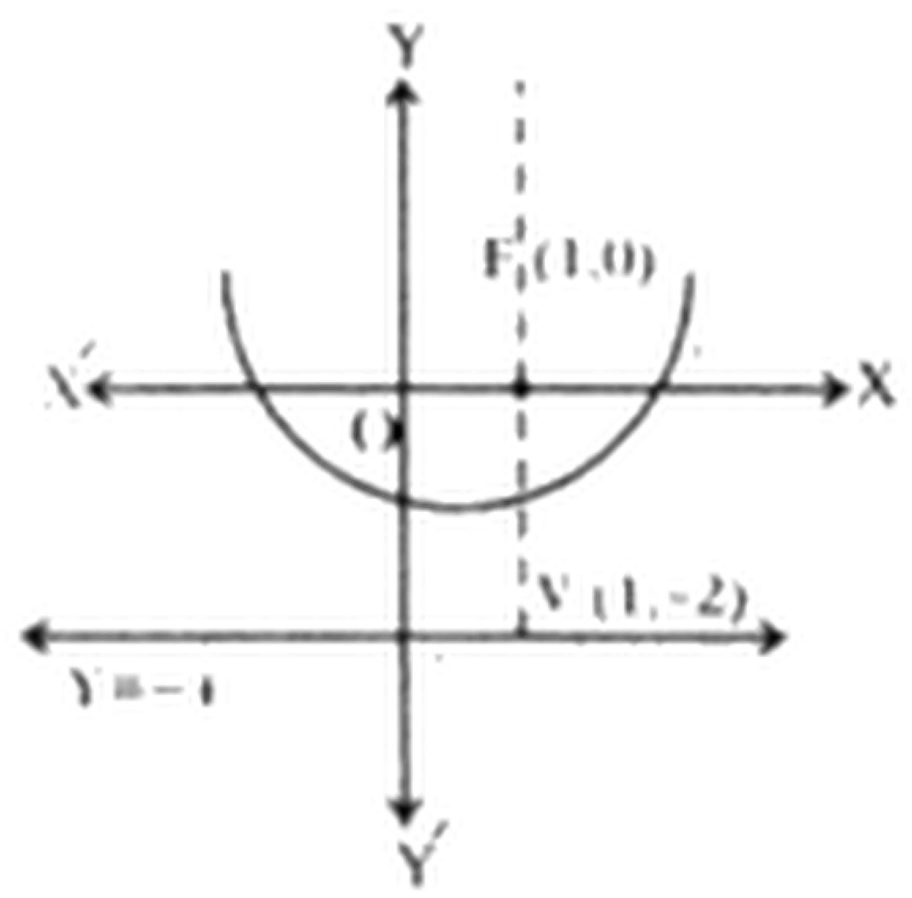
Thus the coordinates of the vertex of parabola of (1) are $(1, -2)$

Directrix of (2) has equation $y = -2$

i.e. $y + 2 = -2$

or , $y = -4$ is an equation of directrix of (1)

sketch is



viii. $\boldsymbol{y = 6x^2 - 1}$

Solution:

$6x^2 = y + 1$

$x^2 = \frac{1}{6}(y + 1)$

Shift the origin to $(0, -1)$ so, that

$X = x + 0$ and $Y = y - 1$ referred to xy-axis with $(0, -1)$ as origin

i.e. $x = x,$ $y + 1 = y$

the equation (1) becomes

$x^2 = \frac{1}{6}y$ ______________(2)

Which is parabola whose focus lies on

$x = 0$, coordinates of the focus of (2) are

$x = 0,$ $y = \frac{1}{24}$

$x = 0,$ $y + 1 = \frac{1}{24}$

$x = 0,$ $y = \frac{1}{24} - 1 = -\frac{23}{24}$

Thus coordinate of the focus of the

Parabola (1) are $\left(0, -\frac{23}{24}\right)$

Axis of (2) is $x = 0$ is the axis of (1) vertex of (2) has coordinated $x = 0$ *and* $y = 0$

i.e. $x = 0$ $y + 1 = 0$

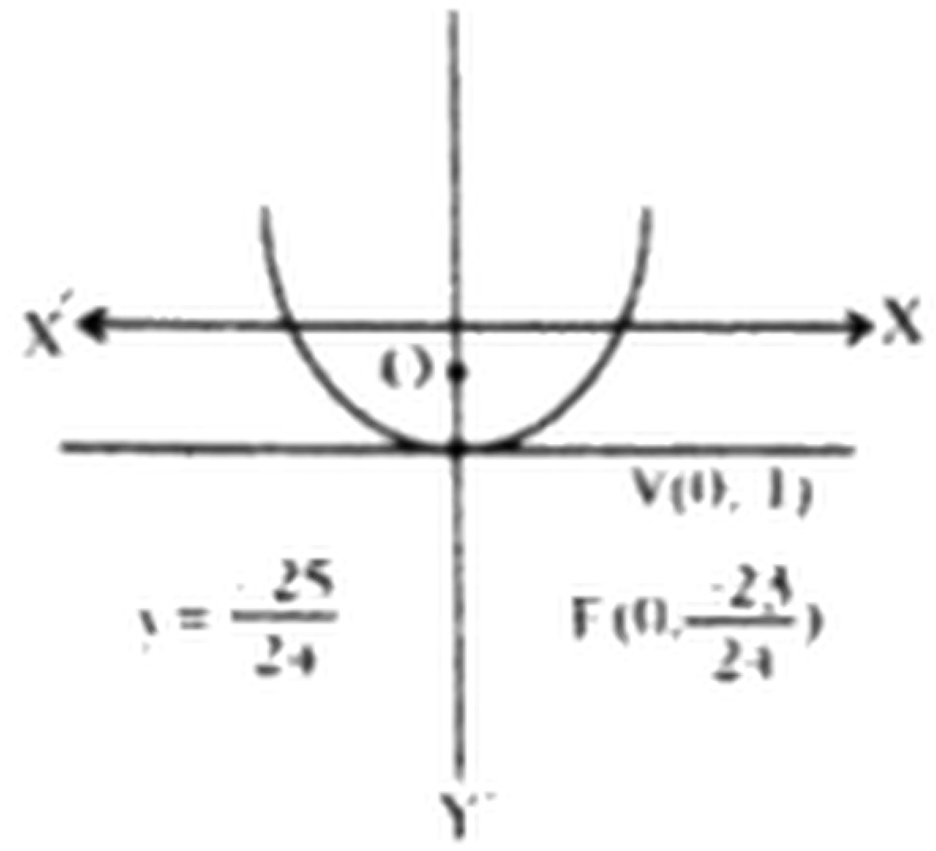
$x = 0,$ $y = -1$

Thus the coordinates of the vertex of parabola of (1) are $(0, -1)$

Directrix of (2) has equation $y = -\frac{1}{24}$

i.e. $y = -\frac{1}{24} - 1 = -\frac{25}{24}$ is an equation of directrix of (1)

sketch is



ix. $\boldsymbol{x + 8 - y^2 + 2y = 0}$

Solution:

$y^2 - 2y = x + 8$

$y^2 - 2y + 1 = x + 8 + 1$

$(y - 1)^2 = x + 9$

Shift the origin to $(-9,1)$ so, that

$X = x - 9$ and $Y = y + 1$ referred to xy-axis with $(-9,1)$ as origin

i.e. $X + 9 = x,$ $Y - 1 = y$

the equation (1) becomes

$y^2 = x$______________(2)

Which is parabola whose focus lies on

$y = 0$, coordinates of the focus of (2) are

i.e. $x = \frac{1}{4},$ $y = 0$

i.e. $x + 9 = \frac{1}{4},$ $y - 1 = 0$

i.e. $x = \frac{1}{4} - 9,$ $y = 1$

i.e. $x = -\frac{35}{4},$ $y = 1$

Thus coordinate of the focus of the

Parabola (1) are $\left(-\frac{35}{4}, 1\right)$

Axis of (2) is $y = 0$

i.e. $x + 9 = 0$ $y - 1 = 0$

$x = -9,$ $y = 1$

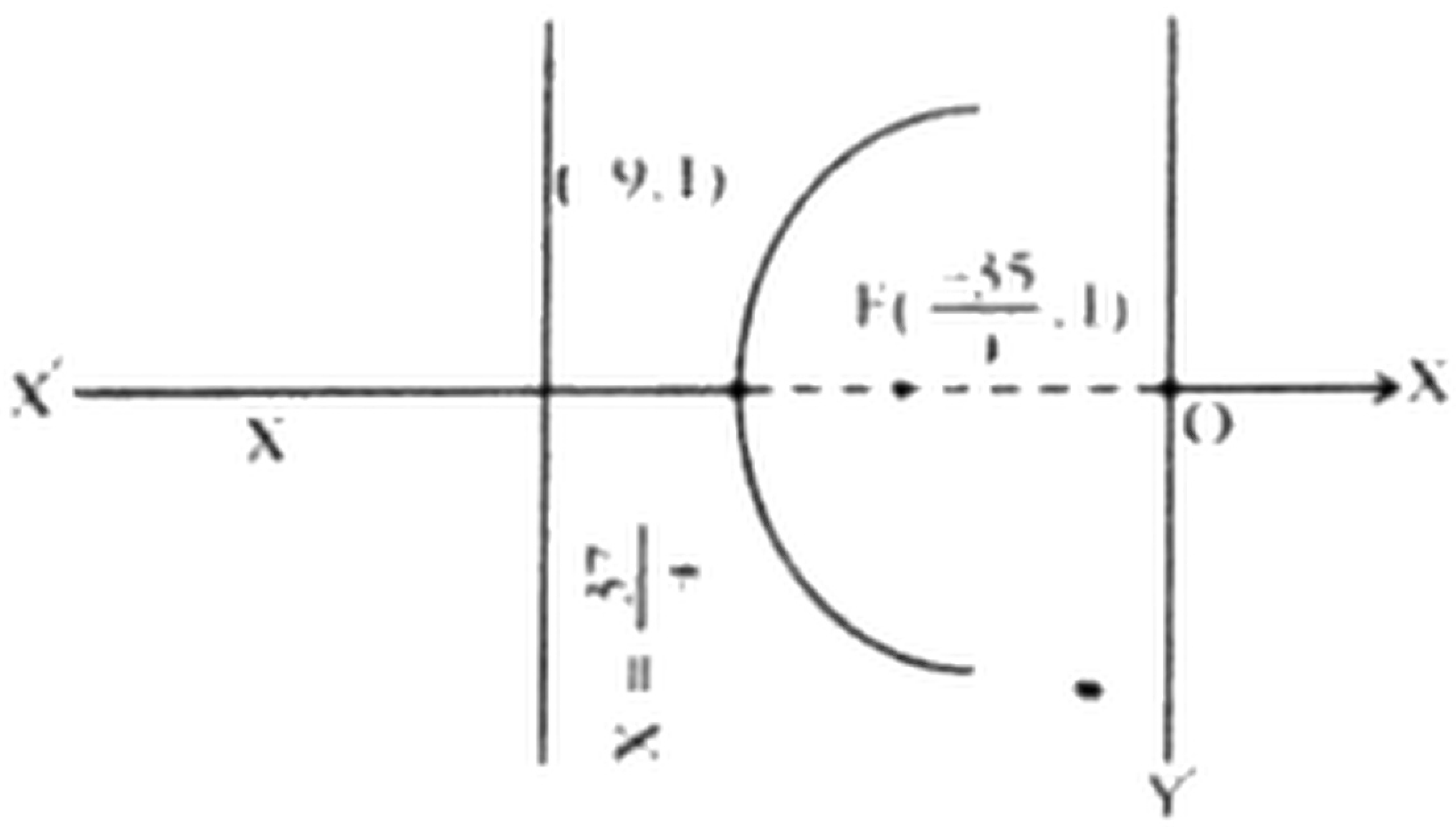
Thus the coordinates of the vertex of parabola of (1) are $(-9,1)$

Directrix of (2) has equation $X = -\frac{1}{4}$

i.e. $x + 9 = -\frac{1}{4}$

$x = -\frac{1}{4} - 9 = -\frac{37}{4}$ is an equation of directrix of (1)

sketch is



x. $\mathbf{x^2 - 4x - 8y + 4 = 0}$

Solution:

$x^2 - 4x + 4 = 8y$

$(x-2)^2 = 8y$

Shift the origin to (2,0) so, that

$X = x + 2$ and $Y = y + 0$ referred to xy-axis with (2,0) as origin

i.e. $x - 2 = x,$ $y = y$

the equation (1) becomes

$x^2 = 8y$

which is parabola whose focus lies on

$x = 0$, coordinates of the focus of (2) are

i.e. $x = 0,$ $y = 2$

i.e. $x - 2 = 0,$ $y = 2$

i.e. $x = 2,$ $y = 2$

Thus coordinate of the focus of the Parabola (1) are (2,2)

Axis of (2) is $x = 0$

i.e. $x - 2 = 0$

$x = 2$ is the axis of (1)

vertex of (2) has coordinates $x = 0, y = 0$

$x - 2 = 0,$ $y = 0$

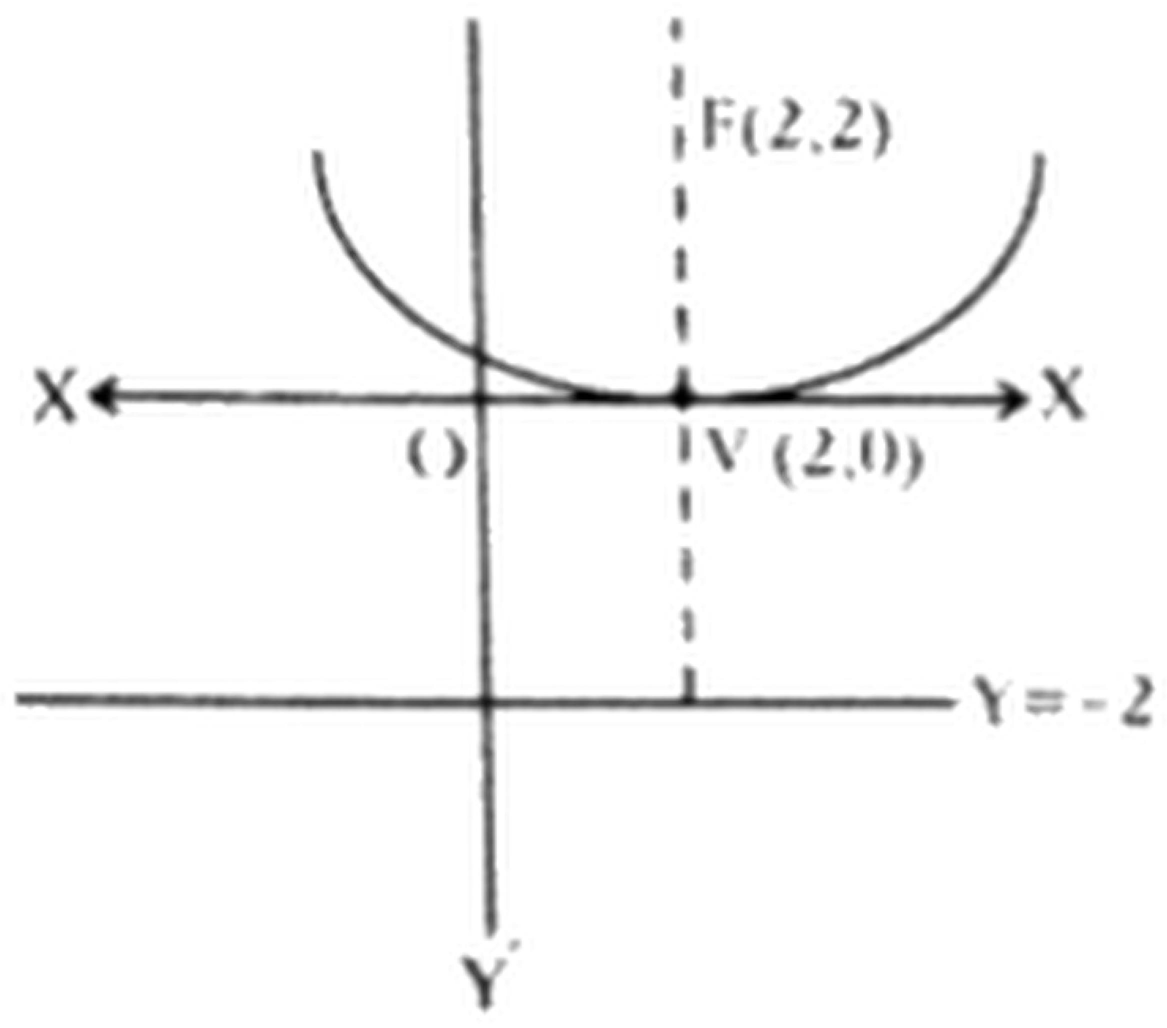
$x = 2,$ $y = 0$

Are the coordinates of the vertex of parabola (1) are (2,0)

Directrix of (2) has equation $y = -2$

or $, y = -2$ is an equation of directrix of (1)

Sketch is



Q2 Write an equation for the parabola with given elements.

i. **Focus $(-3, 0)$, directrix $x = 3$**

Solution:

Let $P(x, y)$ be any point on the parabola

Let F be the focus (−3,0) and $\overline{AB}$ the directrix $x - 3 = 0$

Then $|PF| = |PM|$ (by def)

$$\sqrt{(x+3)^2 + (y-0)^2} = \frac{|x-3|}{\sqrt{1^2+0^2}}$$

$$x + 3 + (y)^2 = \frac{|x-3|^2}{1} \text{ (squaring both sides)}$$

$$x^2 + 6x + 9 + y^2 = x^2 - 6x + 9$$

$\Rightarrow \quad y^2 = -6x - 6x$

$\Rightarrow \quad y^2 = -12x$ is the required equation of parabola

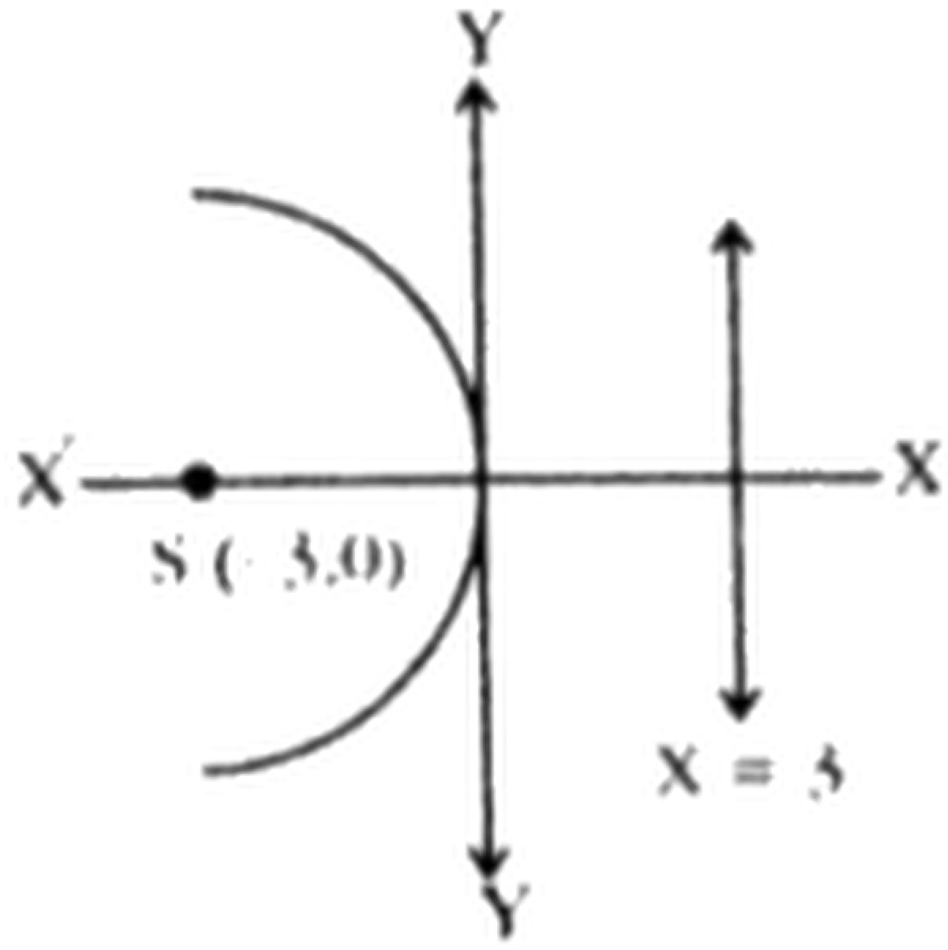
Axis $= y = 0$ (x axis)

Focus $= x = -3, \quad y = 0$

Vertex $= x = 0, \quad y = 0$

Directrix $= x = 3$

Sketch is



ii. Focus $(2, 5)$, directrix $y = 1$

Solution:

Let $P(x, y)$ be any point on the parabola

Let F be the focus (2,5) and $\overline{AB}$ the directrix $y = 1$

Then $|PF| = |PM|$ (by def)

$\sqrt{(x-2)^2 + (y-5)^2} = |y - 1|$

$(x-2)^2 + (y-5)^2 = (y-1)^2$ (squaring both sides)

$x^2 + 4 - 4x + y^2 - 10y + 25 = y^2 - 2y + 1$

$\Rightarrow$ $x^2 + 4 - 4x = 10y - 2y + 1$

$\Rightarrow$ $(x-2)^2 = 8y - 24$

$(x-2)^2 = 8(y-3)$ is the required equation of parabola

Axis $= x - 2 = 0$ i.e. $x = 2$

Focus $= x - 2 = 0, \quad y - 3 = 2$

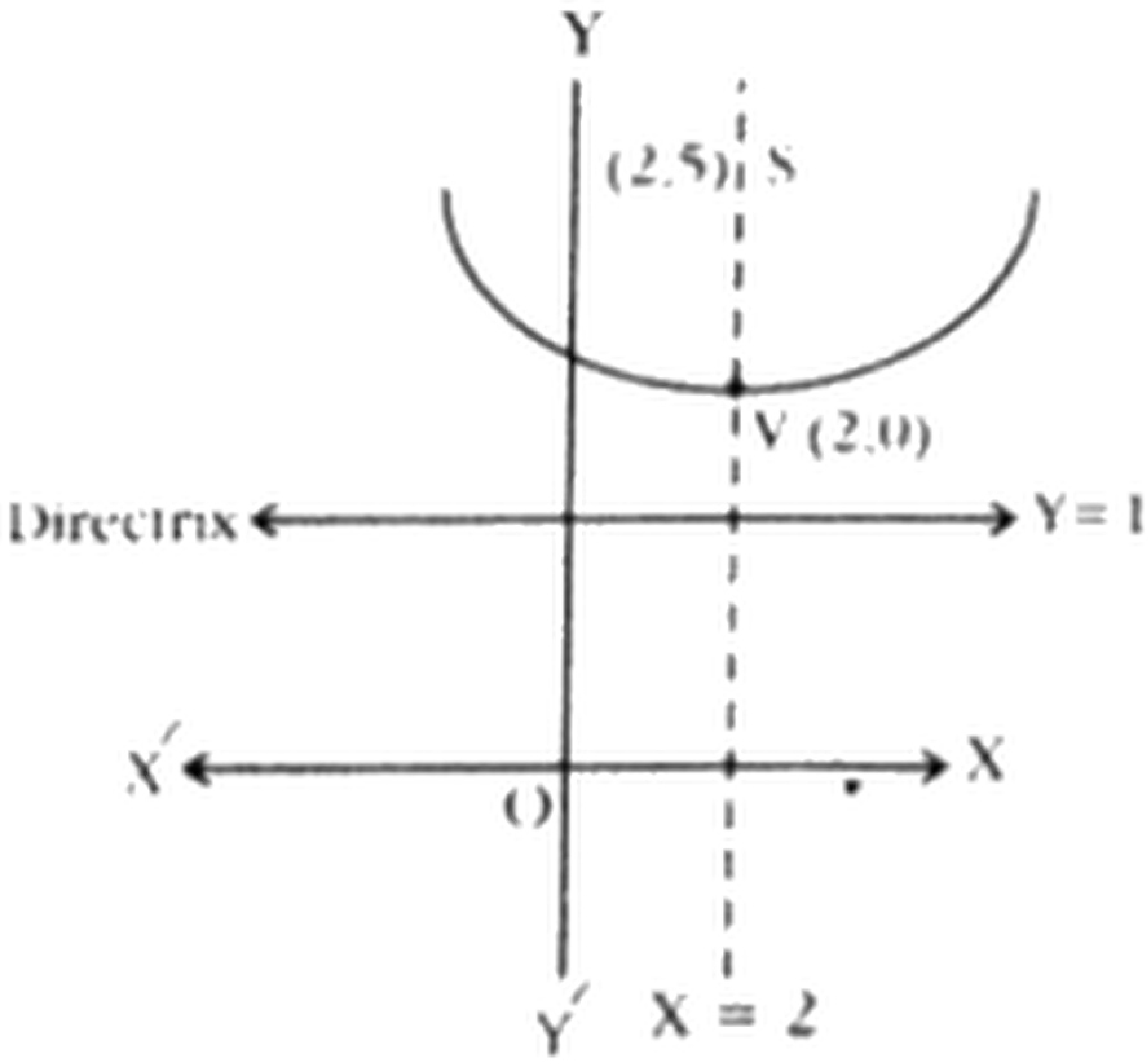
i.e. . $x = 2, \quad y = 5 \quad i.e. (2,5)$

Vertex $= x - 2 = 0, \quad y - 3 = 0$

i.e. . $x = 2, \quad y = 3 \quad i.e. (2,3)$

Directrix $= y - 3 = -2 \quad i.e. y = 1$

Sketch is



iii. **Focus $(-3, 1)$, directrix $= x - 2y - 3 = 0$**

Solution:

By definition of parabola

$$\frac{|PF|}{|PM|} = e = 1 \Rightarrow |PF| = |PM|$$

$$|PF| = \sqrt{(x+3)^2 + (y-1)^2}$$

$$|PM| = \frac{|(1)(x) + (-2)(y) + (-3)|}{\sqrt{(1)^2 + (-2)^2}} = \frac{|x - 2y - 3|}{\sqrt{5}}$$

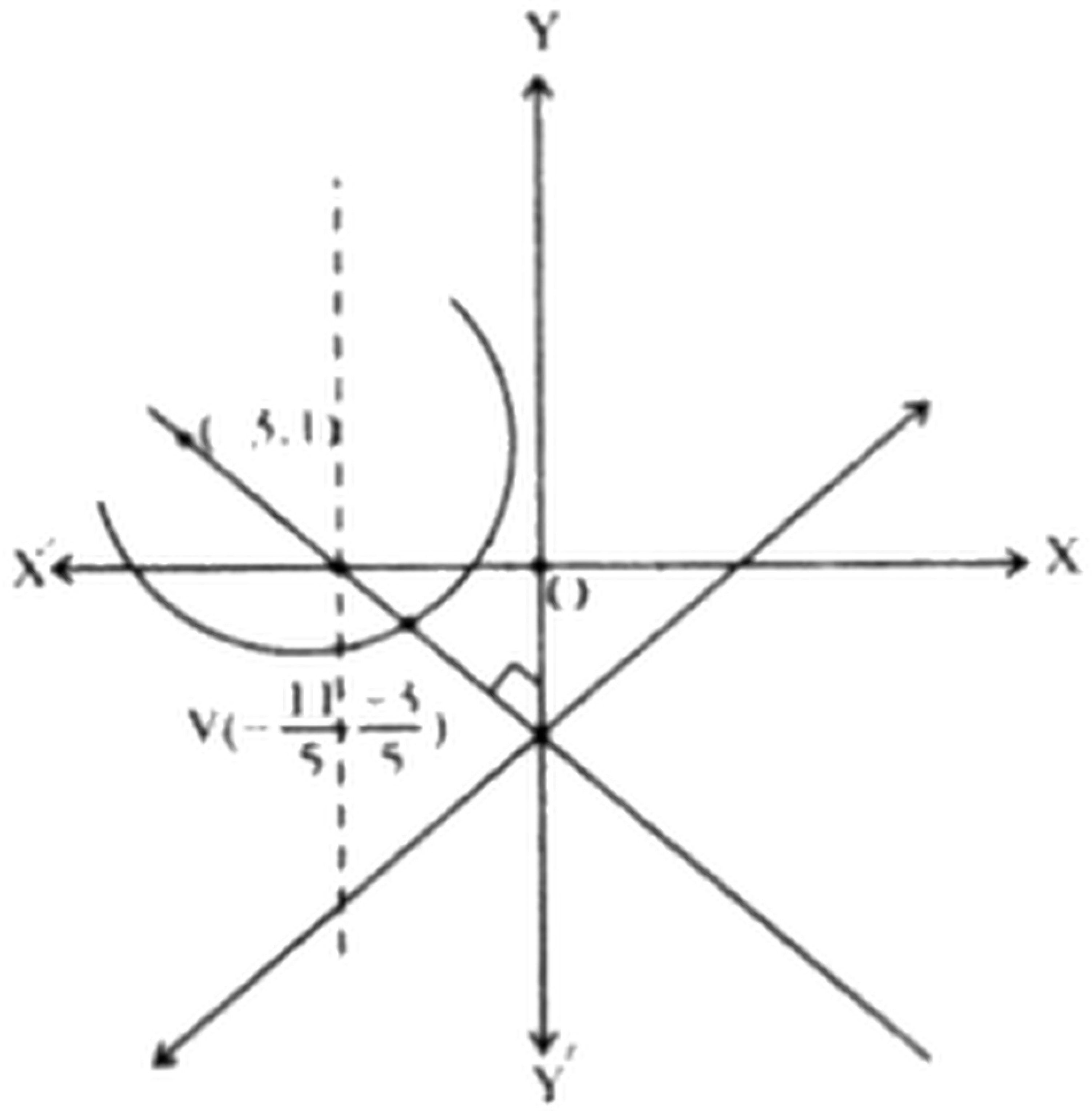
Putting (2) and (3) in (1)

$$\sqrt{(x+3)^2 + (y-1)^2} = \frac{|x - 2y - 3|}{\sqrt{5}}$$

$$x^2 + 6x + 9 + y^2 + 1 = \frac{x^2 + (-2y)^2 + (-3)^2 + 2(x)(-2y) + 2(-2y)(-3) + 2(x)(-3)}{5}$$

$$5x^2 + 30x + 5y^2 - 10y + 50 = x^2 + 4y^2 + 9 - 4xy + 12y - 6x$$

$$4x^2 + 36x + y^2 - 22y + 4xy + 41 = 0$$



iv. **Focus** $(1, 2)$, **vertex** $(3, 2)$

Solution:

Here Focus (1,2), vertex (3,2) implies the axis of parabola is x axis

Let equation of parabola is $(y-2)^2 = 4a(x-3)$ ____________(1)

$$a = -\sqrt{(3-1)^2 + (2-2)^2}$$

(because parabola open left side)

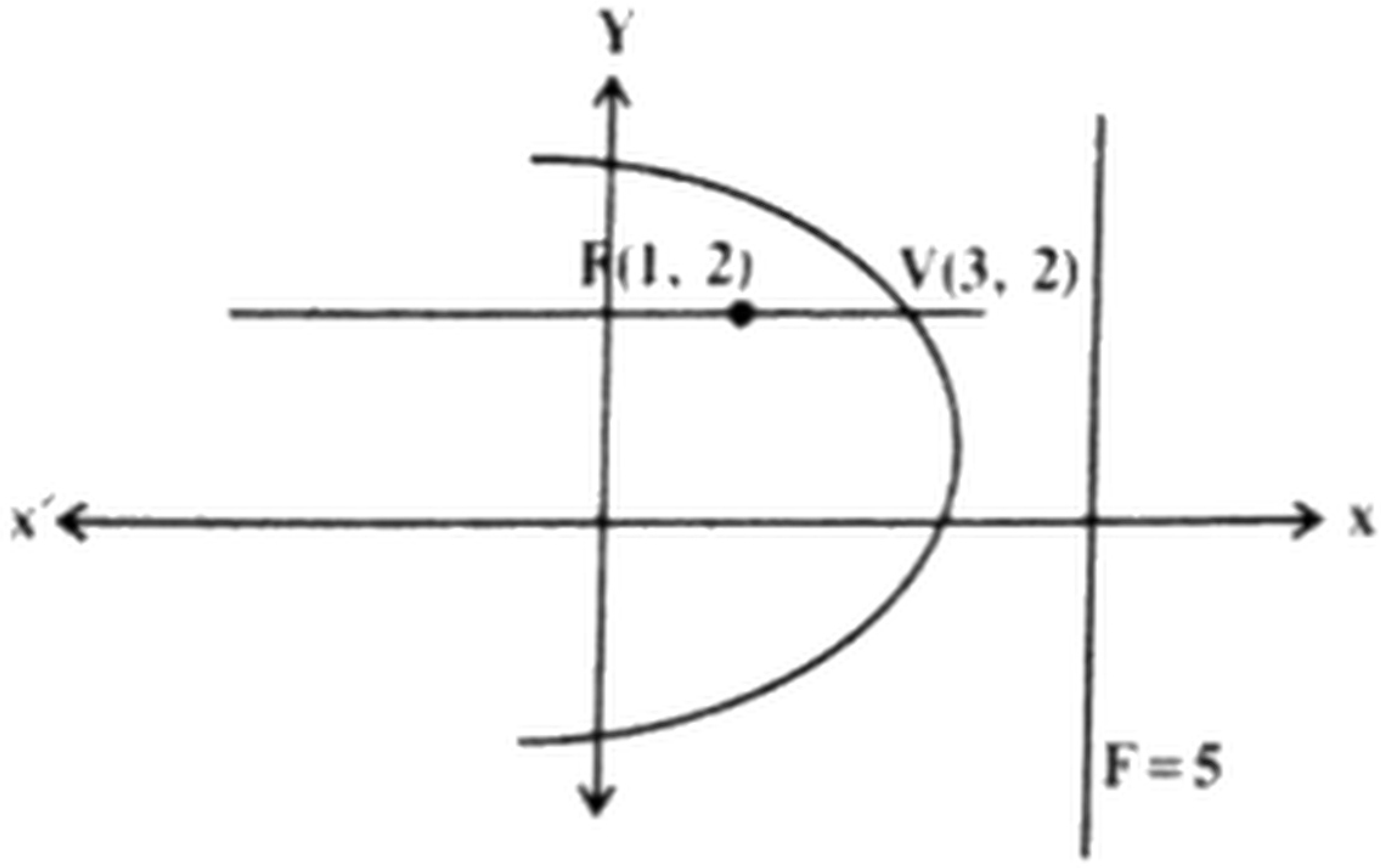
Substitute the value of $a = -2$ in equation (1) we have

$$(y-2)^2 = 4(-2)(x-3)$$

$$(y-2)^2 = -8(x-3)$$

$$y^2 - 4y + 4 = -8x + 24$$

$$y^2 - 4y + 8x - 20 = 0$$



v. **Focus** $(-1,0)$, **vertex** $(-1,2)$

Solution:

Here Focus $(-1,0)$, vertex $(-1,2)$ implies the axis of parabola is y axis

Let equation of parabola is $(x+1)^2 = 4a(y-3)$____________(1)

$$a = -\sqrt{(-1+1)^2 + (2-0)^2}$$

(because parabola open downward)= $\sqrt{0+4} = 2$

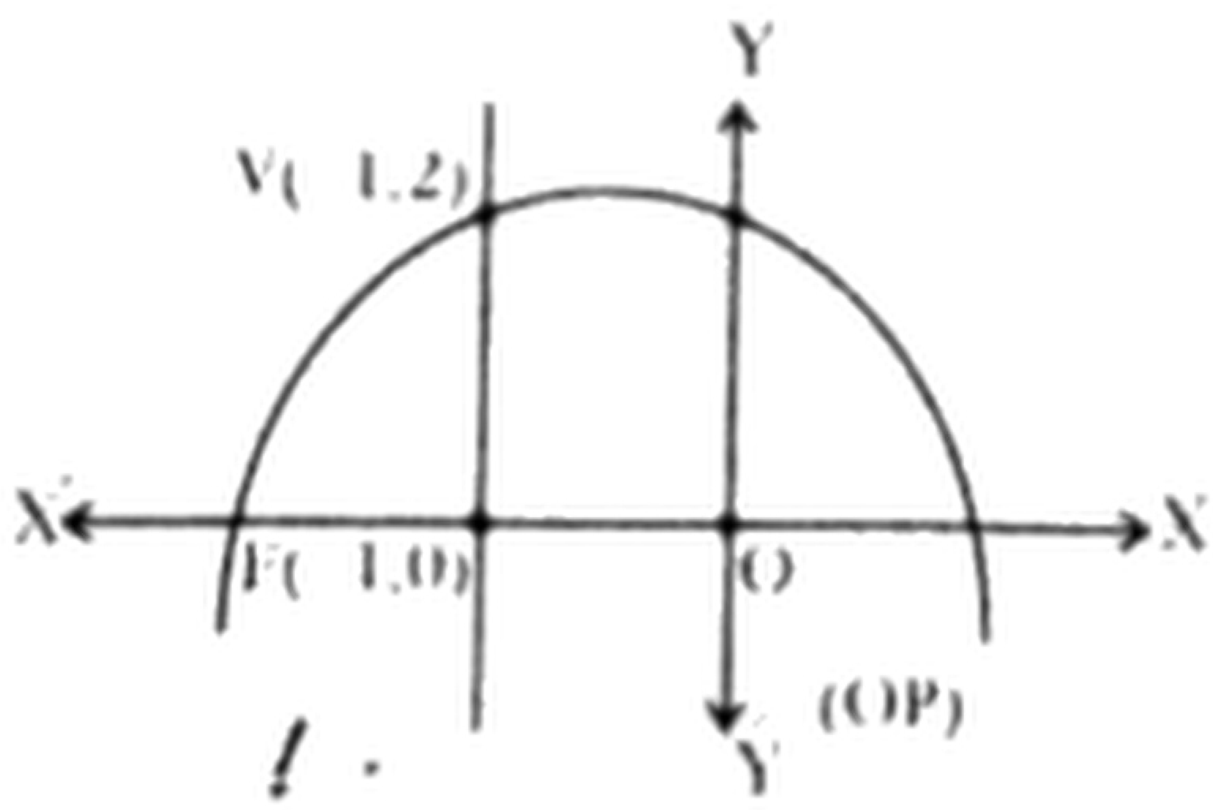
Substitute the value of $a = -2$ in equation (1) we have

$$(x+1)^2 = 4(-2)(y-3)$$

$$(x+1)^2 = -8(y-3)$$

$$x^2 + 2x + 1 = -8y + 24$$

$$x^2 + 2x + 8y - 15 = 0$$



vi. **Directrix $x = -2$, Focus $(2, 2)$**

Solution:

Let $P(x, y)$ be any point on the parabola

Let F be the focus (2,2) and $\overline{AB}$ the directrix $x = -2$

Then $|PF| = |PM|$ (by def)

$$\sqrt{(x-2)^2 + (y-2)^2} = |x+2|$$

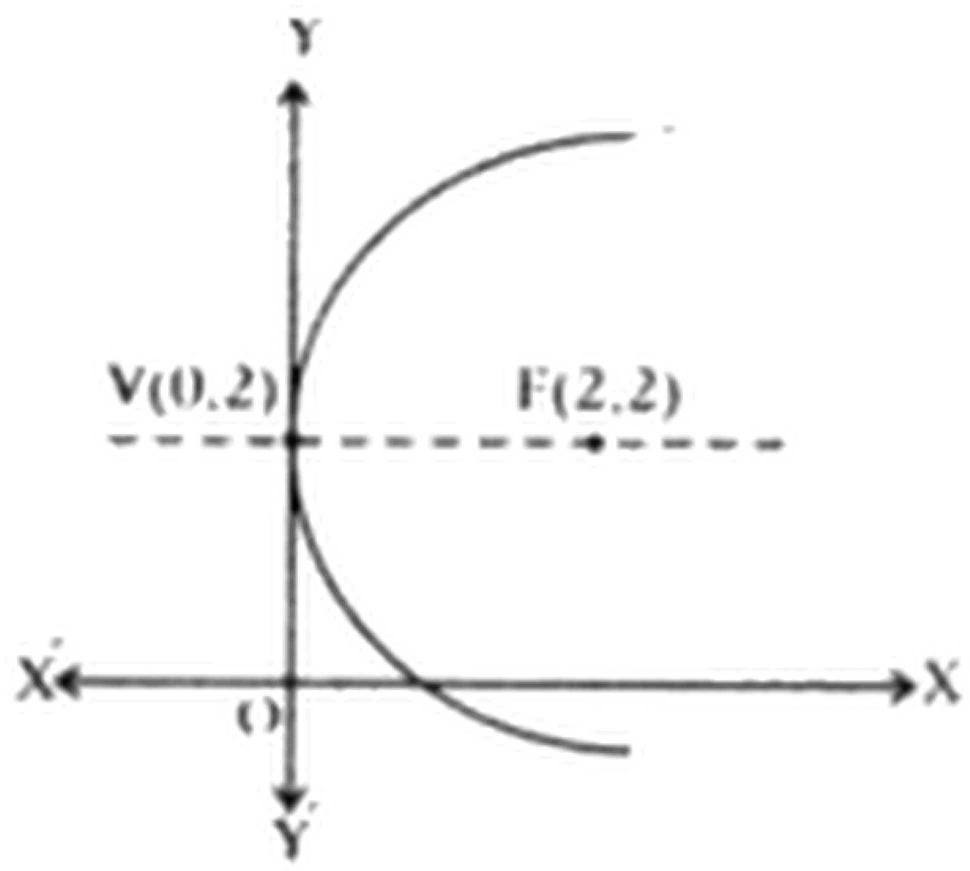
$$(x-2)^2 + (y-5)^2 = (x+2)^2 \text{ (squaring both sides)}$$

$$x^2 + 4 - 4x + y^2 - 4y + 4 = x^2 + 4x + 4$$

$$y^2 - 4y - 8x + 4 = 0$$

or $(y^2 - 4y + 4) = 8x$

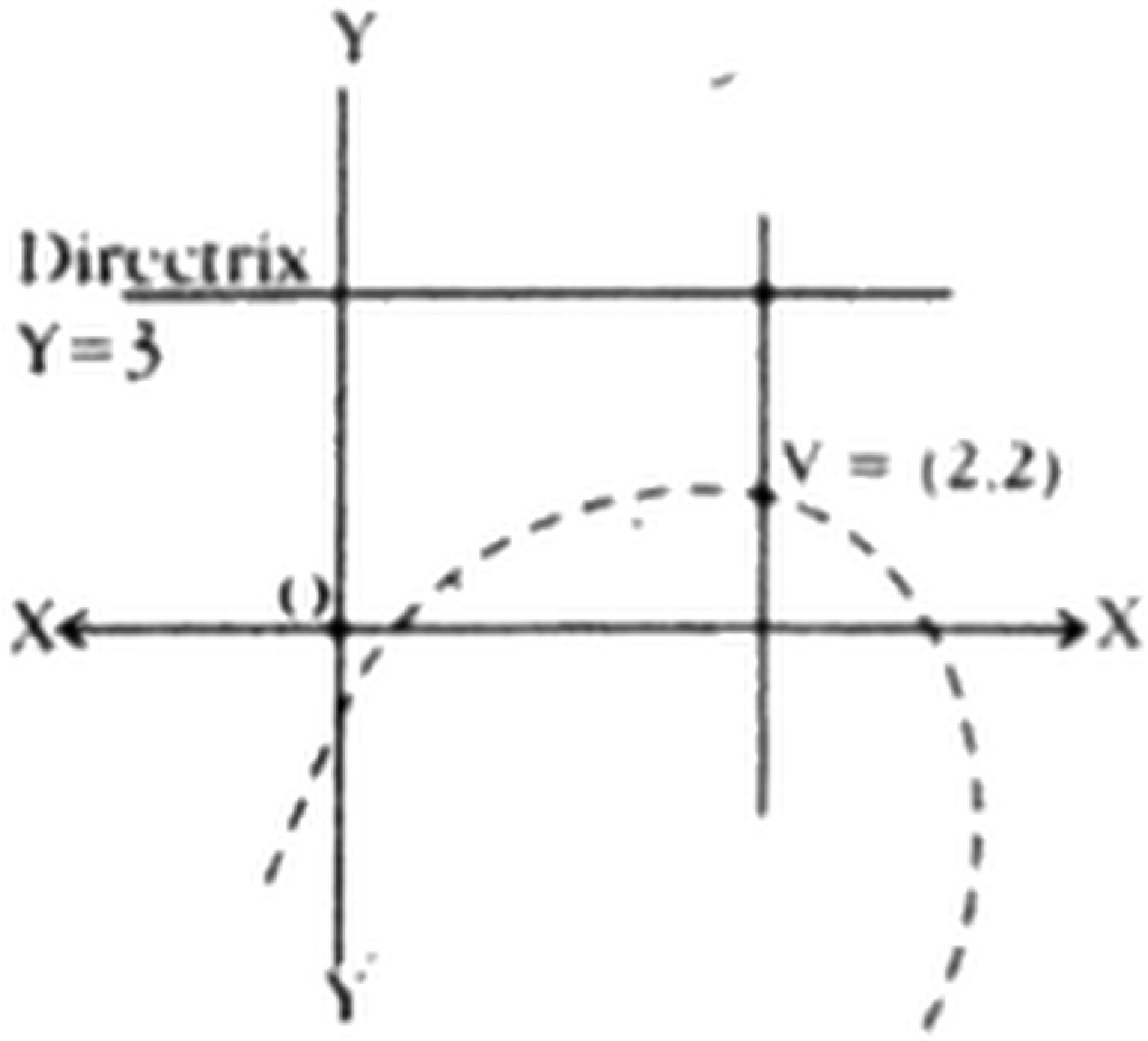
or $y^2 - 4y - 8x + 4 = 0$



vii. Directrix $y = 3$, Vertex $(2, 2)$

Solution:

Clearly distance $\frac{b}{w}$ vertex and directrix ie equal to one so focus $= F(2,1)$



Let $P(x, y)$ be any point on the parabola

Let F be the focus (2,1) and $\overline{AB}$ the directrix $y = 3$

Then $e = 1 \Rightarrow |PF| = |PM|$ (by def)

$\sqrt{(x-2)^2 + (y-1)^2} = |y - 3|$

$(x-2)^2 + (y-1)^2 = (y-3)^2$ (squaring both sides)

$x^2 + 4 - 4x + y^2 - 2y + 1 = y^2 - 6y + 9$

or $y^2 - 4y - 4x - 4 = 0$

or $(x^2 - 4x + 4) + 4y - 8 = 0$

or $(x-2)^2 = -(4y - 8)$

$(x-2)^2 = -4(y-2)$ is the equation of parabola

viii. Directrix $y = 1$, length of latus rectum is 8 opens downward.

Solution:

Here $L = 8 = 4a$

As parabola opens downward so equation of parabola is

$(x-h)^2 = -4a(y-k)$ ________ (1)

But $k = -1$, because distance vertex and directrix is equal to distance between vertex and focus i.e. one and sign is negative because of downward.

Thus equation (1) becomes $(x-h)^2 = -8(y+1)$

$$x^2 - 2xh + h^2 = -8y - 8$$

or $\quad x^2 - 2xh + 8y + 8 + h^2 = 0$

ix. Axis $y = 0$, through $(2, 1)$ and $(11, -2)$

Solution:

The equation of parabola having axis $y = 0$ is

$(y-k)^2 = 4a(x-h)$ ________ (1)

Because the axis of parabola is x-axis, so $k = 0$

Thus equation (1) becomes

$y^2 = 4a(x-h)$ ___________ (2)

As (2,1) lies on equation (2) thus it must satisfy the equation (2) i.e.

$$1^2 = 4a(2-h)$$

$$1 = 8a - 4ah \text{ ______________ (3)}$$

Also $(11, -2)$ lies on equation (2) thus it must satisfy the equation (2) i.e.

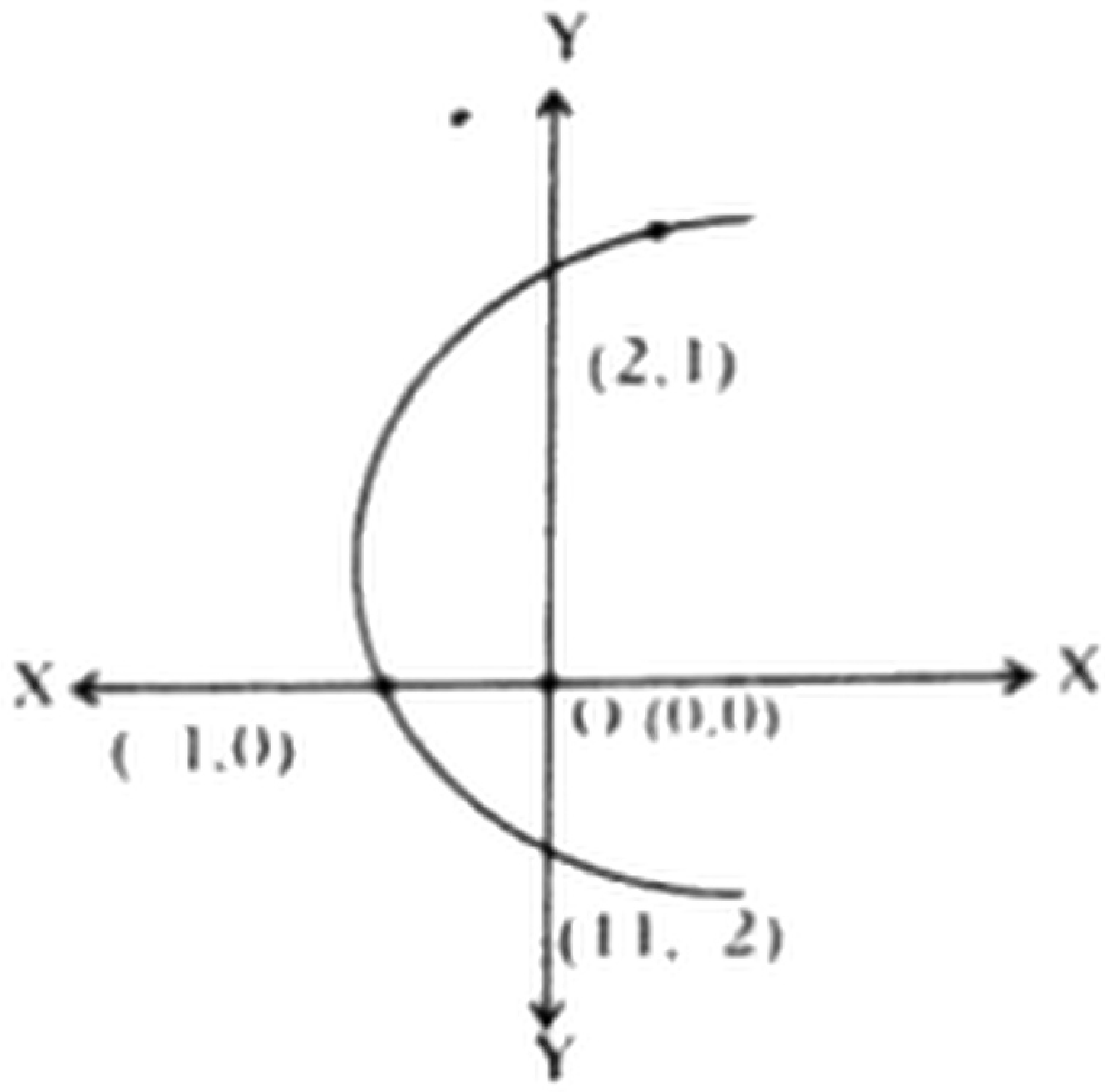
$$(-2)^2 = 4a(11-h)$$

$$4 = 44a - 4ah \text{ ________________ (4)}$$

Subtracting (3) and (4), we have

$$36a = 3$$

$a = \frac{1}{12}$



Subtracting the value of a in equation (3), we have

$1 = 8\left(\frac{1}{12}\right) - 4\left(\frac{1}{12}\right)h$ or $1 = \frac{2}{3} - \frac{1}{3}h$

$3 = 2 - h$ or $h = -1$

Subtracting the value of a and h in equation (2), we have

$y^2 = 4(\frac{1}{12})(x - (-1))$

Or $y^2 = \frac{1}{3}(x + 1)$

Or $3y^2 = (x + 1)$

x. Axis parallel to y axis, the points $(0, 3)$, $(3, 4)$ and $(4, 11)$ lie on the graph

Solution:

A axis of parabola is to y axis, so equation of the parabola is

$(x - h)^2 = 4a(y - k)$_________(1)

As (0,3) lies on equation (1) becomes

$(0 - h)^2 = 4a(3 - k)$

$\Rightarrow h^2 = 12a - 4ak$ ________________ (2)

As (3,4) lies on equation (1) then

$$(3-h)^2 = 4a(4-k)$$

$$9 - 6h + h^2 = 16a - 4ak \text{ ______________ } (3)$$

Also (4,11) lies on equation (1) then

$$(4-h)^2 = 4a(4-k)$$

$$16 - 8h + h^2 = 16a - 4ak \text{ ________________ } (4)$$

Subtracting (3) from (4), we have

$$7 - 2h = 28a$$

$$28a - 2h = 7 \text{ ___________ } (5)$$

Subtracting the equation (2) from (3), we have

$$9 - 6h + 4a = 0$$

$$\Rightarrow \quad 6h + 4a = 9 \text{ ___________ } (6)$$

Multiplying equation (6) by "7" we get

$$28a + 42h = 63$$

$$\underline{-8a + 2h = 7}$$

$$40h = 56$$

$$\Rightarrow \quad h = \frac{56}{40} = \frac{7}{5}$$

Putting the value of h in equation (6), we have

$$4a + 6\left(\frac{7}{5}\right) = a = 7\ 4a\ \frac{42}{5} = 9$$

$$\Rightarrow \quad 4a = a - \frac{42}{5} \Rightarrow 4\text{a}\frac{45-42}{5}$$

$$\Rightarrow \quad 4a = \frac{3}{5} \Rightarrow \text{a} = \frac{3}{20}$$

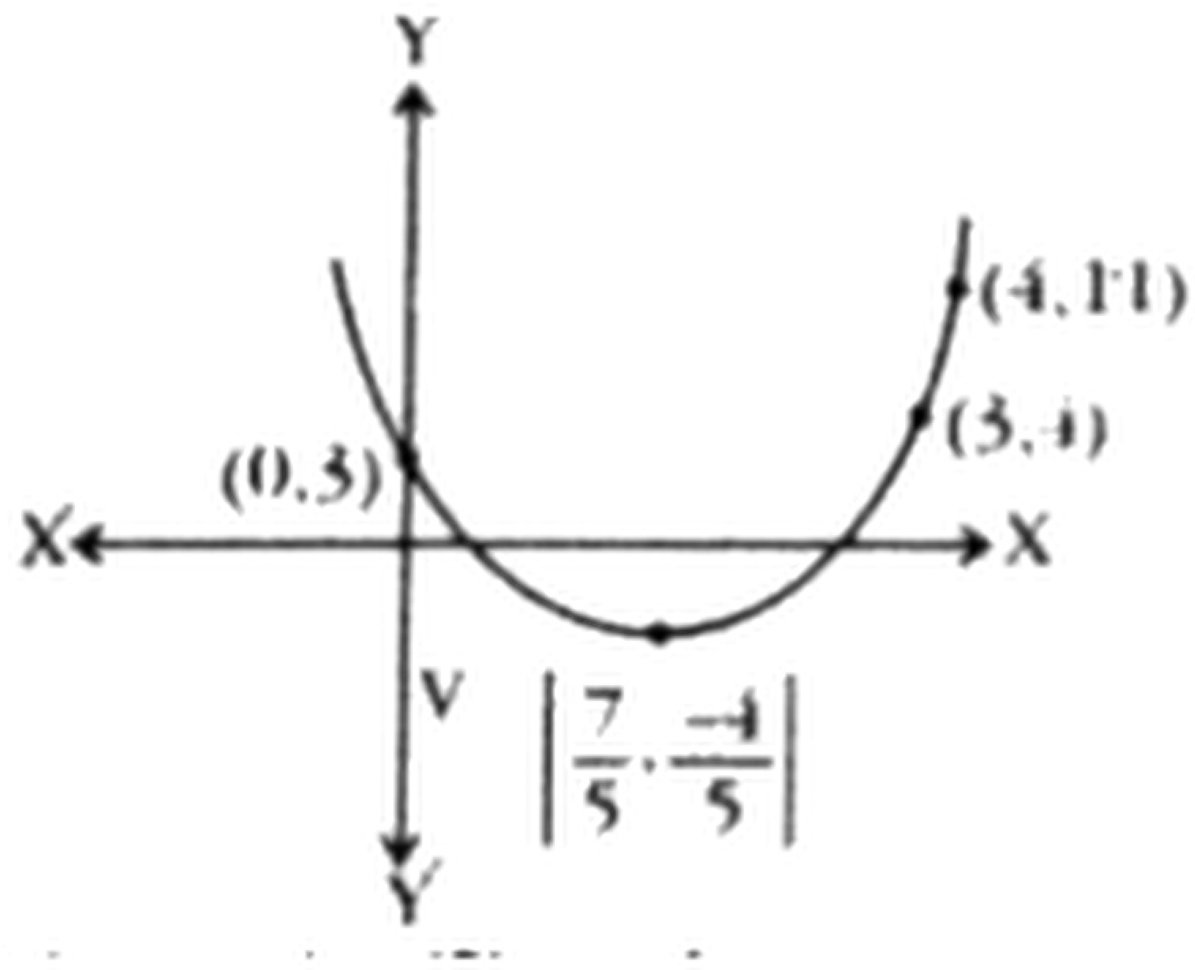
Substitute the values of a and h in equation (2), we have

$$\left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right) - 4\left(\frac{3}{20}\right)k$$

$\Rightarrow \quad \frac{49}{25} = \frac{9}{5} - \frac{3}{5}k \quad \Rightarrow \quad -\frac{49}{25} + \frac{9}{5} = \frac{3}{5}k$

$\Rightarrow \quad \frac{3}{5}k = \frac{45-49}{25} \quad \Rightarrow \quad \frac{3}{5}k = \frac{-4}{25}$

$\Rightarrow \quad k = \frac{-4}{25}$



Substitute the values of a and h in equation (2), we have

$$\left(x - \frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\{y - \left(\frac{-4}{15}\right)\}$$

$\Rightarrow \quad \left(x - \frac{7}{5}\right)^2 = \left(\frac{3}{5}\right)\{y + \left(\frac{4}{15}\right)\}$

Is the required equation of parabola

Q3 Find the equation of parabola having its focus at the origin and directrix is parallel to

i. **x-axis**

ii. **y-axis**

Solution:

i. **Directrix parallel to x-axis**

Since directrix is parallel to x-axis

Its equation is $y = a \Rightarrow y - a = 0$

Focus is at $(0,0)$

Now let $P(x, y)$ be any point on the parabola

Then $|PF| = |PM|$ (by definition)

$$\sqrt{(x-2)^2 + (y-0)^2} = \frac{|y-a|}{\sqrt{0^2+1^2}}$$

$x + y^2 = \frac{|y-3|^2}{1}$ (squaring both sides)

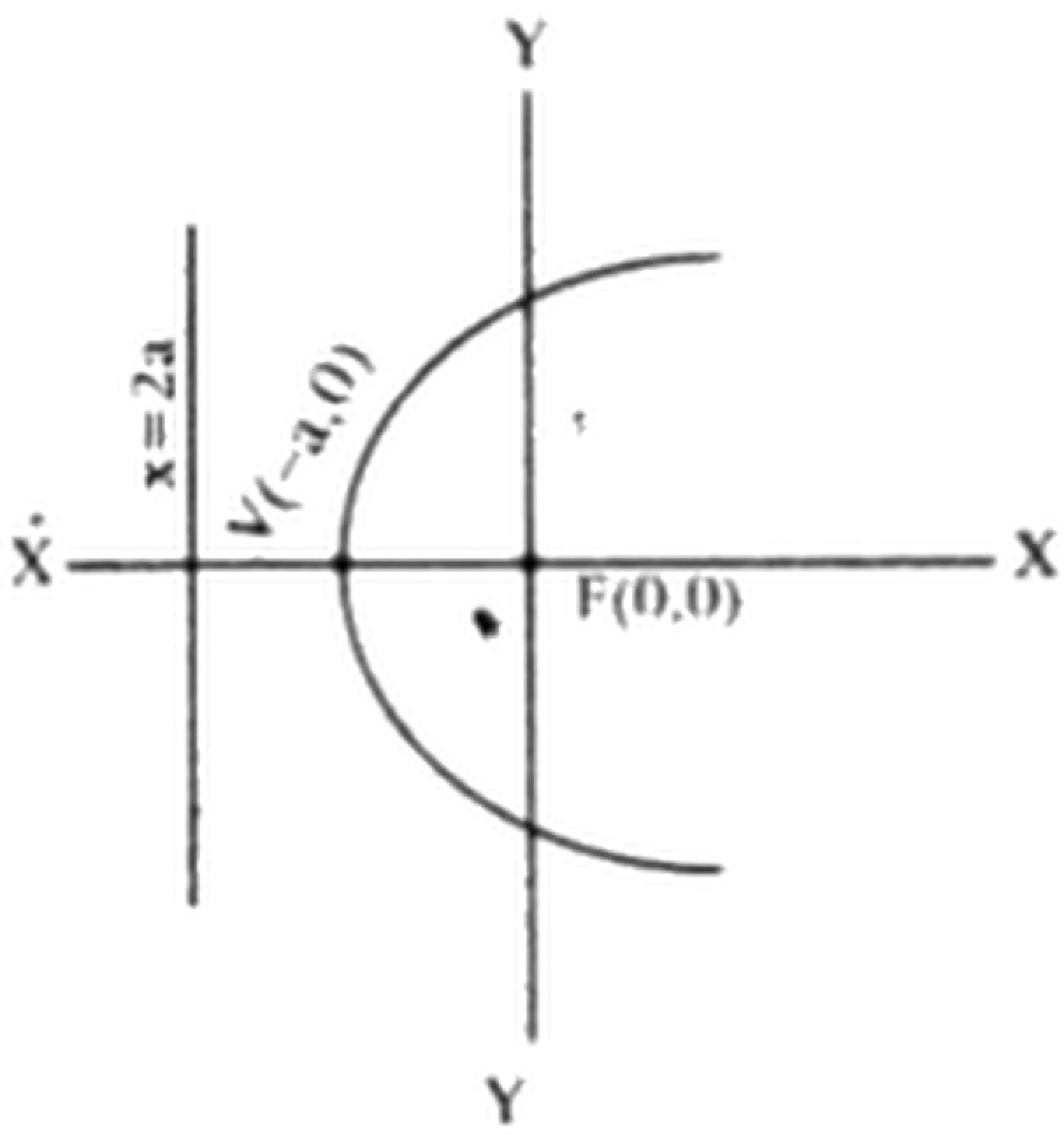
$x^2 + y^2 = y^2 - 2ay + a^2$

$x^2 = -2ay + a^2$

$x^2 = -2a(y - \frac{a}{2})$

$\Rightarrow \quad y^2 \pm 4ay - 4a^2 = 0$

Equation of the parabola



ii. Directrix parallel to y-axis

Since directrix is parallel to y-axis

Its equation is $x = a \Rightarrow x - a = 0$

Focus is at $(0,0)$

Now let $P(x, y)$ be any point on the parabola

Then $|PF| = |PM|$ (by definition)

$$\sqrt{(x-0)^2 + (y-0)^2} = \frac{|x-a|}{\sqrt{1^2 + 0^2}}$$

$$x + y^2 = \frac{|x-3|^2}{1} \text{ (squaring both sides)}$$

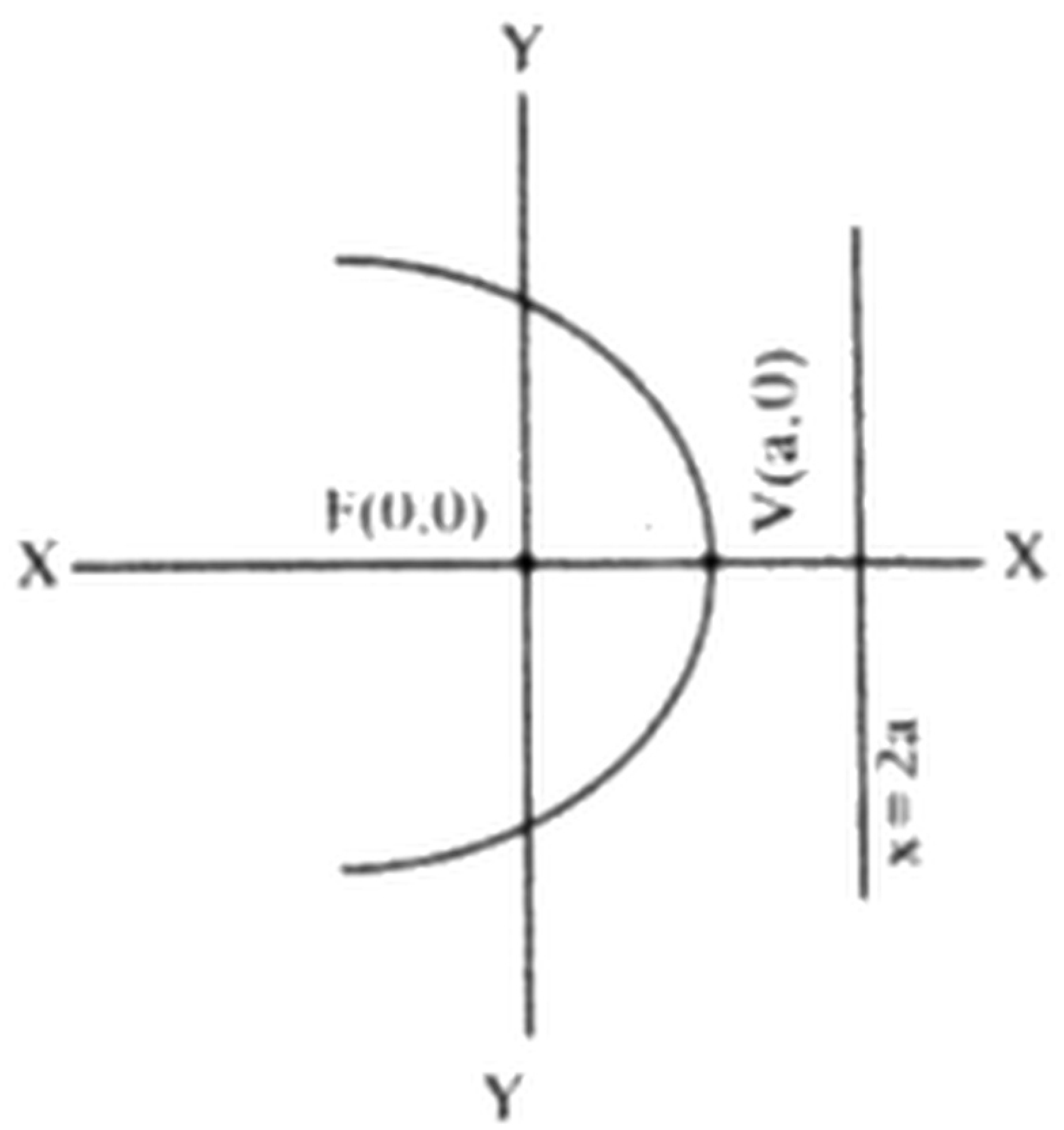
$$x^2 + y^2 = x^2 - 2ax + a^2$$

$$y^2 = -2ax + a^2$$

$$y^2 = -2a(x - \tfrac{a}{2})$$

$\Rightarrow \quad y^2 \pm 4ax - 4a = 0$

Equation of the parabola



Q4 Show that the parabola having

$(x \sin\alpha - y\cos\alpha)^2 = 4a(x\cos\alpha - y\sin\alpha)^2\alpha$

Has focus at $(a\cos\alpha, a\sin\alpha)$ and its directrix is

$x\cos\alpha + y\sin\alpha + a = 0$

Solution:

Here focus $= F(a\cos\alpha, a\sin\alpha)$ and equation of directrix is $x\cos\alpha + y\sin\alpha + a = 0$

Now let $P(x, y)$ be any point on the parabola

Then $|PF| = |PM|$ (by definition)

$\sqrt{(x - a\cos\alpha)^2 + (y - a\sin\alpha)^2} = \frac{|x\cos\alpha + y\sin\alpha + a|}{\sqrt{\cos\alpha^2 + \sin\alpha^2}}$

$(x - a\cos\alpha)^2 + (y - a\sin\alpha)^2 = (x\cos\alpha + y\sin\alpha + a)^2$

$x^2 + a^2cos^2\alpha - 2a\cos\alpha\, x + y^2 + a^2sin^2\alpha - 2a\sin\alpha\, y$

$x^2cos^2\alpha + y^2sin^2\alpha a^2 + 2\cos\alpha\sin\alpha\, xy$

$a^2sin^2\alpha - a^2 + 2a\sin\alpha\, y + 2a\cos\alpha\, x + 2a\cos\alpha\, x + 2a\sin\alpha\, y - a^2 - a^2sin^2\alpha - a^2cos^2\alpha$

$x^2(1 - cos^2\alpha) + y^2(1 - sin^2\alpha) - 2\cos\alpha\sin\alpha\, xy$

$a^2 + 4a\cos\alpha\, x + 4a\sin\alpha\, y - a^2(sin^2\alpha + cos^2\alpha)$

$x^2sin^2\alpha + y^2cos^2\alpha - 2xy\cos\alpha\sin\alpha$

$4ax\cos\alpha + 4ay\sin\alpha$

$(x\sin\alpha - y\cos\alpha)^2 = 4a(x\cos\alpha - y\sin\alpha)$

Q5 Show that the ordinate at any point P of the parabola is the mean proportional between the length of the latus rectum and the abscissa of P.

Solution:

Let $P(x, y)$ be any point on the parabola

$y^2 = 4ax$

$y = \pm\sqrt{(4a)(x)}$

(ordinate of P)

$$= \pm\sqrt{(length\ of\ latusrectum)(abscissa\ of\ P)}$$

Which shows ordinates of P is mean P proportional between the length of the latus rectum and the abscissa of P

Q6 A comet has a parabola orbit with the earth at the focus. When the comet is 150,000 km from the earth, the line joining the comet and the earth makes an angle of $30^o$with the axis of the parabola. How close will the comet come to the earth?

Solution:

Let the earth E be the origin. If vertex A has coordinates (-a,0) them directrix of the parabola is$x = -2a$. If the is at $P(x, y)$ then

$\sqrt{(x-0)^2 + (y-0)^2} = |x + 2a|$

$x^2 + y^2 = (x + 2a)^2$_______________(1)

Now from right angle triangle EQP

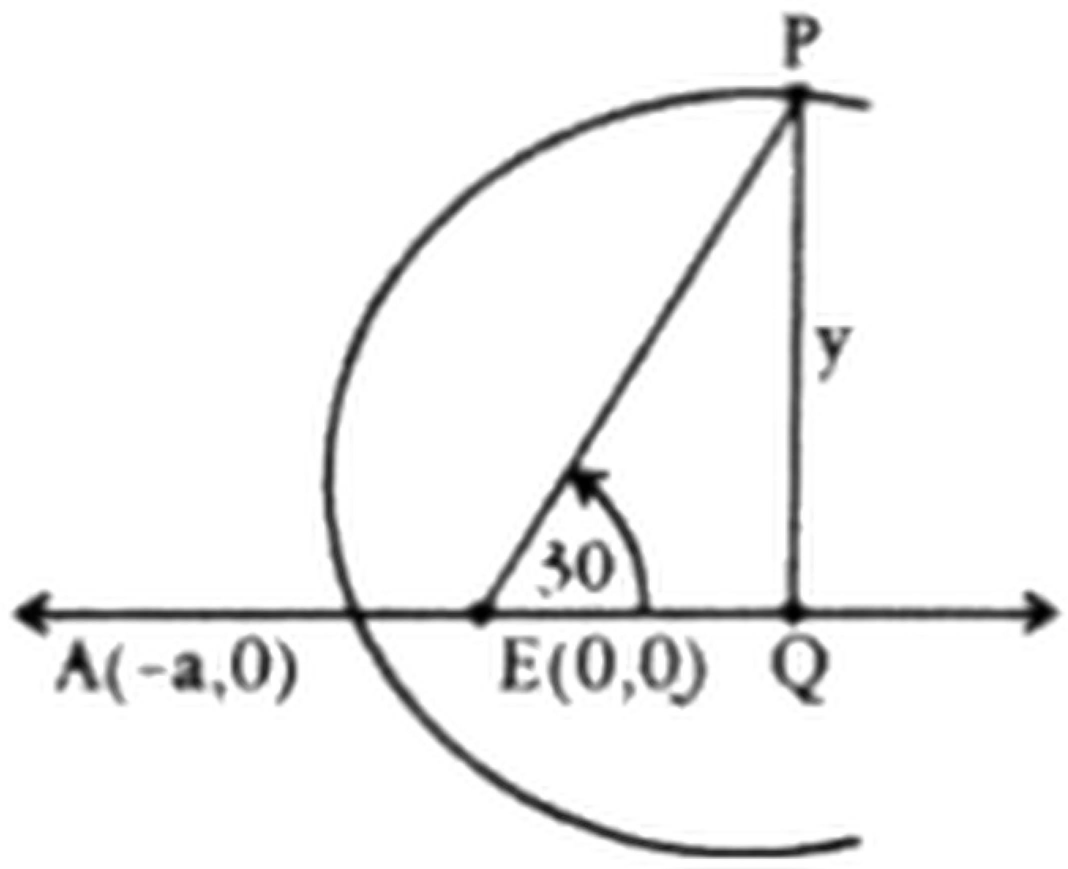
$x^2 + y^2 = (150{,}000)^2$__________________(2)

From (1) and (2)

$(x + 2a)^2 = (150{,}000)^2$

$(x + 2a)^2 = 150{,}000$

Now from $\Delta\, EQP$



$\cos 30^o = \frac{x}{150{,}000} \quad \Rightarrow 150{,}000 * \cos 30^o$

$= 150{,}000 * \frac{\sqrt{3}}{2}$

$= 150{,}000 * \sqrt{3}$

Put $x = 75{,}000 * \sqrt{3}$ in above

$75{,}000 * \sqrt{3} + 2a = 150{,}000$

$2a = 150{,}000 - 75{,}000 * \sqrt{3}$

$2a = 75{,}000(2 - \sqrt{3})$

$a = 75{,}000\left(\frac{2-\sqrt{3}}{2}\right) = 10048.095km$

Thus the comet is closest to the earth when it is$75{,}000\left(\frac{2-\sqrt{3}}{2}\right)$km from the earth.

Q7 Find an equation of the parabola formed by the cables of a suspension bridge whose span is "a" meters and vertical height of the supporting towers is "b" metres.

Solution:

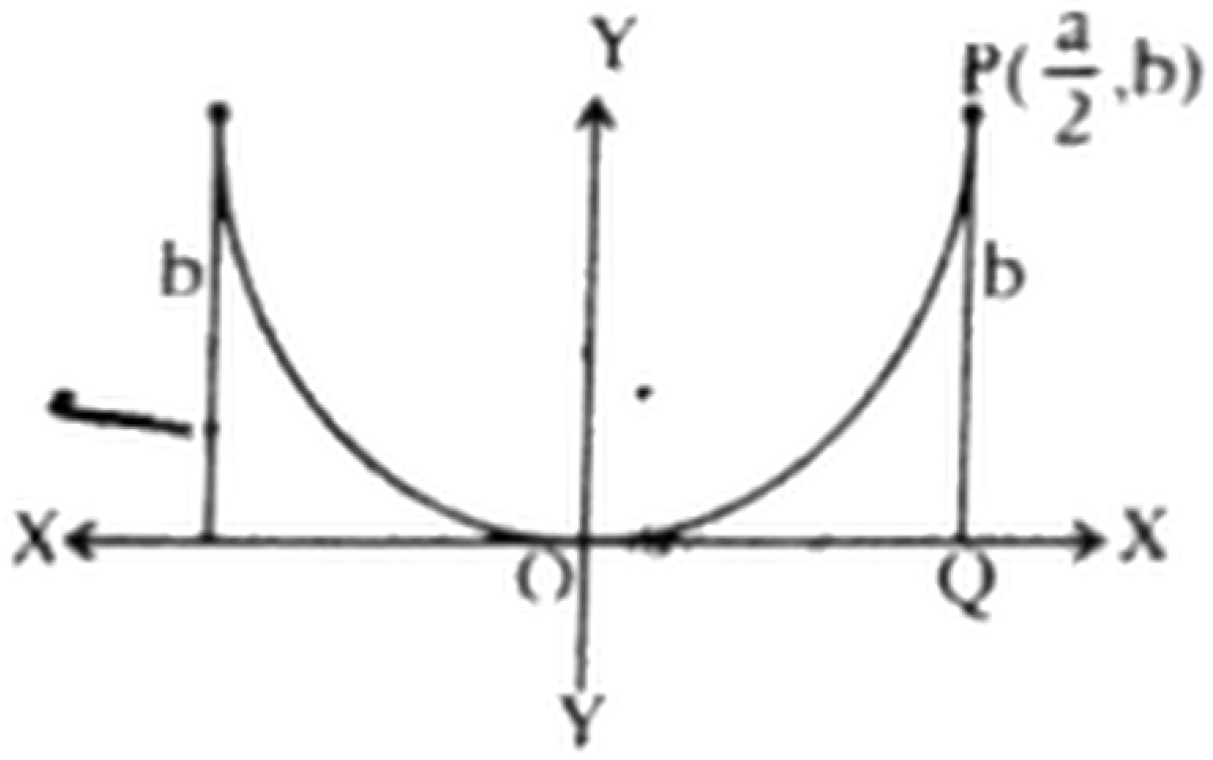
An equation of the parabola is

$x^2 = 4Py$____________________(1)

The point $P(\frac{a}{2}, b)$lies on the parabola and so

$\left(\frac{a}{2}\right) = 4Pb$

$\left(\frac{a^2}{4}\right) = 4Pb$



$\Rightarrow \quad P = \left(\frac{a^2}{16b}\right)$ (put in (1))

$x^2 = 4\left(\frac{a^2}{16b}\right)y$

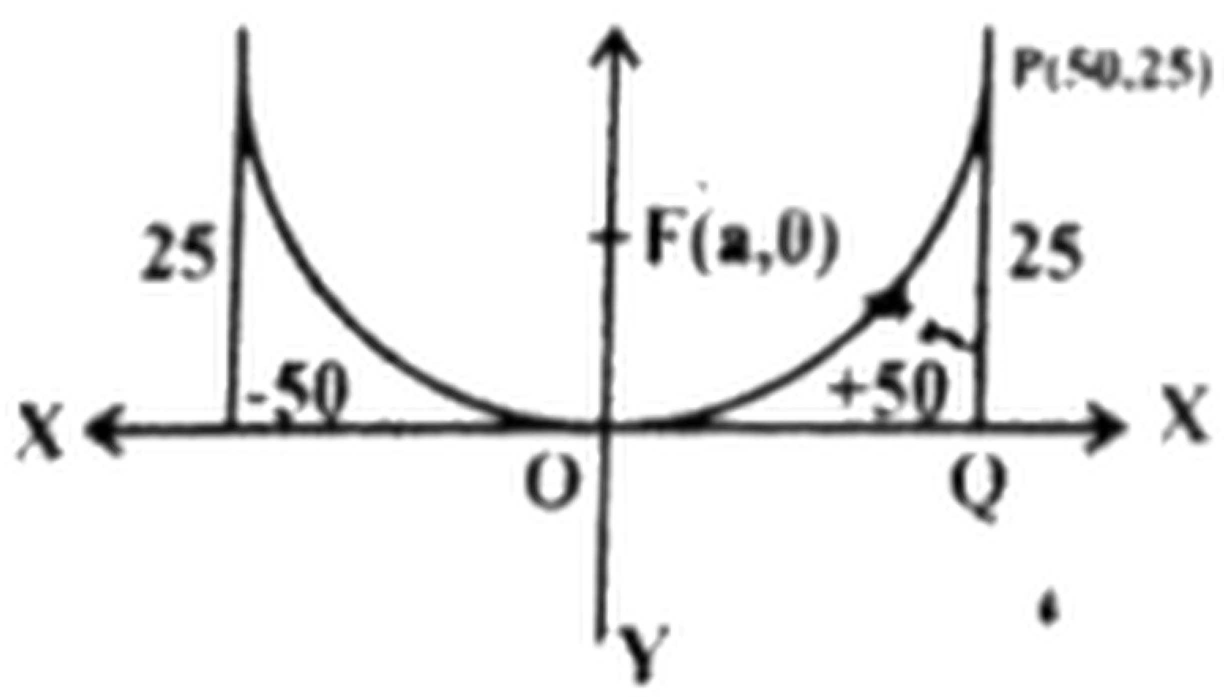
$x^2 = \left(\frac{a^2}{4b}\right)y$

Q8 A parabola arch has 100m base and height 25m. find the height of the arch at a point 30m from the centre of the base.

Solution:

An equation of this parabola is

$x^2 = 4ay$



The point $P(50,25)$lies on the parabola and so

$(50)^2 = 4a(25)$

$2500 = 100a$

$\Rightarrow \quad a = 25$

$x^2 = 4(25)y$

$x^2 = (100)y$

To find the height of the arch when $x = 30$we have

$(30)^2 = (100)y$

$900 = (100)y$

$y = 9$m

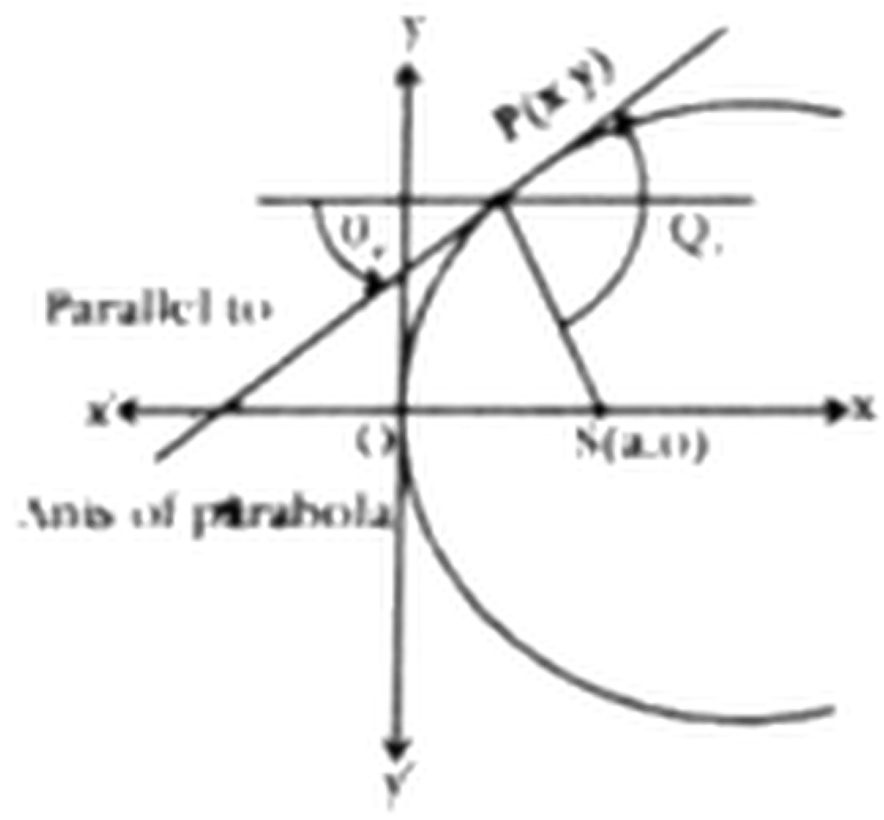
Is required height.

Q9 Show that tangent at any point P of a parabola makes equal angles with the line PF and the line through P parallel to the axis of the parabola being focus (these angles are called respectively angle of incidence and angle of reflection)

Solution:

Let the $P(x_1, y_1)$ be any point on the parabola.

Let the angle between the tangent line point P and $\overline{PF}$ be θ_1 .let the angle between the tangent line at point P and a line passing through point P which is parallel to the axis of parabola be θ_1



To prove $\theta_1 = \theta_2$

Let the equation of parabola be $y^2 = 4ax$____________(1)

Diff (1) w.r.t "x", we have

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{\frac{dy}{dx}}{(x_1,y_1)} . (x_1x_1) = \frac{2a}{y}$$

Thus the slope of tangent line at point $P(x_1, y_1)$ is equal to $\frac{2a}{y_1}$

$\Rightarrow \quad m_1 = \frac{2a}{y_1}$, (say)

The slope of line parallel to axis of parabola is equal to zero

$\Rightarrow \quad m_2 = 0$, (say)

The slope of

$$\overline{PF} = \frac{y_1 - 0}{x_1 - a} = \frac{y_1}{x_1 - a}$$

$\Rightarrow \quad m_3 = \frac{y_1}{x_1 - a}$, (say)

Now $\quad \tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\left(\frac{2a}{y_1}\right) - \left(\frac{y_1}{x_1 \ \ a}\right)}{1 + \left(\frac{2a}{y_1}\right)\left(\frac{y_1}{x_1 \ \ a}\right)}$

$$= \frac{\frac{(2a(x_1 \quad a)) \quad (y_1)^2}{[y_1(x_1 \quad a)]+2ay_1}}{(x_1-a)\,y_1} = \frac{2ax_1-2a^2-(y_1)^2}{x_1y_1-ay_1+2ay_1}$$

$$= \frac{2ax_1-2a^2-4ax_1}{x_1y_1+ay_1}$$

On the parabola it must satisfied the equation of the parabola

$$= \frac{-2ax_1-2a^2}{x_1y_1+ay_1} = \frac{-2a(x_1+a)}{y_1(x_1+a)}$$

$$\tan\theta_1 = \frac{-2a}{y_1} \underline{\qquad\qquad\qquad} (2)$$

$$\Rightarrow \qquad \theta_1 = \tan\left(\frac{-2a}{y_1}\right)$$

And $\tan\theta_2 = \frac{m_2-m_1}{1+m_2m_1} = \frac{-0.\left(\frac{2a}{y_1}\right)}{1+(0)\left(\frac{2a}{y_1}\right)}$

$$= \frac{\left(-\frac{2a}{y_1}\right)}{1+(0)}$$

$$\tan\theta_2 = \left(\frac{-2a}{y_1}\right) \underline{\qquad\qquad\qquad} (3)$$

From (2) and (3)

$\theta_1 = \theta_2$ as required

Hence Proved