Exercise 4.5

Find the lines represented by each of the following and also find measure of the angle between them. (Problem 1-7)

1. $10x^2 - 23xy - 5y^2 = 0$

Solution

The equation may be written as

$$5\left(\frac{y}{x}\right)^2 + 23\left(\frac{y}{x}\right) - 10 = 0$$

$$\frac{y}{x} = \frac{-(23)\pm\sqrt{(23)^2 - 4(5)(-10)}}{2(5)}$$

$$= \frac{-(23)\pm\sqrt{529+200}}{10}$$

$$= \frac{-(23)\pm\sqrt{729}}{10}$$

$$= \frac{-(23)\pm 27}{10} = \frac{4}{10}, \frac{50}{10}$$

$$\Rightarrow \quad = m_1 = \frac{2}{5} \text{ and } m_2 = -5$$

The two lines have equations

$$y = \frac{2}{5}x \quad , y = -5x$$

or $2x - 5y = 0$, $5x + 4y = 0$

To find measure of angle between the lines represented by (1). We have a = 10, $h = \frac{-23}{2}$, b = -5

If θ is the measure of angle between the given lines then

$$= \frac{2\sqrt{\left(\frac{23}{2}\right)^2 - (10)(-5)}}{10+(-5)}$$

$$= \frac{2\sqrt{\frac{529}{4}+50}}{10-5}$$

$$= \frac{2\sqrt{\frac{529+200}{4}}}{5}$$

$$= \frac{2\sqrt{\frac{729}{4}}}{5}$$

$$= \frac{2\left(\frac{27}{2}\right)}{5}$$

$$= \frac{27}{5}$$

Thus $\theta = \tan^{-1}\left(\frac{27}{5}\right) = 79.51^\circ$

Q2. $\mathbf{3x^2 + 7xy + 2y^2 = 0}$

Solution

The equation may be written as

$$2\left(\frac{y}{x}\right)^2 + 7\left(\frac{y}{x}\right) + 3 = 0$$

Solving it for $\frac{y}{x}$, *we have*

$$\frac{y}{x} = \frac{-7\pm\sqrt{(7)^2-4(2)(3)}}{2(2)}$$

$$= \frac{-7\pm\sqrt{49-24}}{4}$$

$$= \frac{-7\pm\sqrt{25}}{4}$$

$$= \frac{-7\pm 5}{4} = -\frac{2}{4}, -\frac{12}{4}$$

The two lines represented by (1) have equations

$$y = \frac{1}{2}x \quad , y = -3x$$

or $x + 2y = 0,\ 3x + y = 0$

To find measure of angle between the lines represented by (1). We have a = 3, h = $\frac{7}{2}$, b = 2

If θ is the measure of angle between the given lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\left(\frac{7}{2}-\right)^2 - (3)(2)}}{3+2}$$

$$= \frac{2\sqrt{\frac{49}{4} - 6}}{5}$$

$$= \frac{2\sqrt{\frac{49 \quad 24}{4}}}{5}$$

$$= \frac{2\sqrt{\frac{25}{4}}}{5}$$

$$= \frac{2\left(\frac{5}{2}\right)}{5}$$

$$= \frac{5}{5} = 1$$

Thus $\theta = \tan^{-1}(1) = 45^\circ$

3. $\mathbf{9x^2 + 24xy + 16y^2 = 0} \qquad \mathbf{(1)}$

Solution

The equation may be written as

$$16\left(\frac{y}{x}\right)^2 + 24\left(\frac{y}{x}\right) + 9 = 0$$

$$\frac{y}{x} = \frac{-(24) \pm \sqrt{(24)^2 - 4(16)(9)}}{2(16)}$$

$$= \frac{-(24) \pm \sqrt{576 - 576}}{32}$$

$$= \frac{-(24) \pm 0}{32} = \frac{24}{32}, \frac{-24}{32}$$

$$= -\frac{3}{4}, \frac{3}{4} \Rightarrow \; = m_1 = \frac{-3}{4} \text{ and } m_2 = \frac{-3}{4}$$

The two lines represented by (1) have equations

$$y = -\frac{3}{4}x \quad , y = \frac{3}{4}x$$

or 3x+4y = 0, 3x+4y = 0

Since the two lines are parallel the measure of the acute angle between them is zero.

Q4 $2x^2 + 3xy - 5y^2 = 0$ **(1)**

Solution

The equation may be written as

$$5\left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right) - 2 = 0$$

$$\Rightarrow \quad \frac{x}{y} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(5)(-2)}}{2(5)}$$

$$= \frac{3 \pm \sqrt{9 + 40}}{10}$$

$$= \frac{3 \pm \sqrt{49}}{10}$$

$$= \frac{3 \pm 7}{10}$$

$$\Rightarrow \quad \frac{x}{y} = \frac{3-7}{10} \text{ and } \frac{x}{y} = \frac{3+7}{10}$$

$\Rightarrow \quad \frac{x}{y} = -\frac{4}{10}$ and $\frac{x}{y} = \frac{10}{10}$

$\Rightarrow \quad \frac{x}{y} = -\frac{2}{5}$ and $\frac{x}{y} = 1 \qquad (2)$

$\Rightarrow \quad$ 5y = -2x and y = x

$\Rightarrow$ 2x+5y = 0 and x – y = 0

Are the required lines

$m_1 = Slope$ of first line $= -\frac{2}{5}$

$m_2 = Slope$ of second line $= 1$

Angle between the lines is

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2.m_1} = \frac{1-\left(-\frac{2}{5}\right)}{1+(1)\left(-\frac{2}{5}\right)}$$

$$= \frac{1+\frac{2}{5}}{1-\frac{2}{5}} = \frac{\frac{7}{5}}{\frac{3}{5}}$$

$$= \frac{7}{3}$$

Thus $\theta = \tan^{-1}\left(\frac{7}{3}\right) = 66.8^{\circ}$

Q5. $6x^2 - 19xy + 15y^2 = 0$

Solution

The equation may be written as

$15\left(\frac{y}{x}\right)^2 - 19\left(\frac{y}{x}\right) + 6 = 0$

Solving it for $\frac{y}{x}$, *we have*

$$\frac{y}{x} = \frac{-(-19) \pm \sqrt{(-19)^2 - 4(15)(6)}}{2(15)}$$

$$= \frac{19 \pm \sqrt{1}}{30}$$

$$= \frac{19 \pm 1}{30} = \frac{2}{3}, -\frac{3}{5}$$

The two lines have equations

$$y = \frac{2}{3}x \quad , y = \frac{3}{5}x$$

or $2x - 3y = 0, 3x - 5y = 0$

To find measure of angle between the lines represented by (1). We have $a = 6, h = -\frac{19}{2}, b = 15$

If θ is the measure of angle between the given lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\left(-\frac{19}{2}\right)^2 - (6)(15)}}{6+15}$$

$$= \frac{2\sqrt{\frac{361}{4} - 90}}{5}$$

$$= \frac{2\sqrt{\frac{361 \quad 360}{4}}}{5}$$

$$= \frac{2\sqrt{\frac{1}{4}}}{5}$$

$$= \frac{2\left(\frac{1}{2}\right)}{5}$$

$$= \frac{1}{21}$$

Thus $\theta = \tan^{-1}\left(\frac{1}{21}\right) = 2.73^\circ$

Solution

$$. \quad \mathbf{x^2} + 2xy\ secx + \mathbf{y^2} = 0$$

$$\mathbf{x^2} + 2xy\ secx + \mathbf{x^2} = 0$$

$$\frac{y^2}{x^2} + \frac{2xy\ sec}{x^2} + \frac{x^2}{x^2} = 0$$

Which is a quadratic equation in $\frac{y}{x}$

$$\frac{y}{x} = \frac{-2sec \pm \sqrt{(2sec)^2 - 4(1)(1)}}{2(1)} m$$

$$= \frac{-2sec \pm \sqrt{2\ sec^2 - 4}}{2}$$

$$= \frac{-2sec\ a \pm 2\sqrt{tan^2 a}}{2}$$

$$= sec \pm tan$$

$$= \frac{1}{cos} \pm \frac{Sin}{cos} = \frac{-1 \pm sin}{cos}$$

$$\frac{y}{x} = \frac{-1 \pm sin}{cos}$$

$$y = \left[\frac{-1+sin}{cos}\right] x$$

$$m_1 = \frac{-1+sin}{cos}$$

And (cos)y = -(1-Sin)x

(1-Sin)x +(cos)y = 0

$$\frac{y}{x} = \frac{-1-sin}{cos}$$

$$y = \left[\frac{-1-sin}{cos}\right] x$$

$$m_2 = \frac{-11sin}{cos}$$

(cos)y = -(1+Sin)x

∴ (1+Sin a)x + (cos)y = 0

$$\frac{-1-sin}{cos} \qquad \left[\frac{-1+sin}{cos}\right]$$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2.m_1} = \frac{\left[\frac{1 \quad Sin}{cos}\right] - \left(-\frac{1+Sin}{cos}\right)}{1 + \left[\frac{1 \quad Sin}{cos}\right]\left(-\frac{1+Sin}{cos}\right)}$$

$$\frac{\frac{1 \quad Sin + 1 \quad Sin}{cos}}{cos2 \quad (Sin^2 \quad 1)}$$

$2y = x + 5 \Rightarrow y = -\frac{1}{2}x + \frac{5}{2} \Rightarrow m = -\frac{1}{2}$

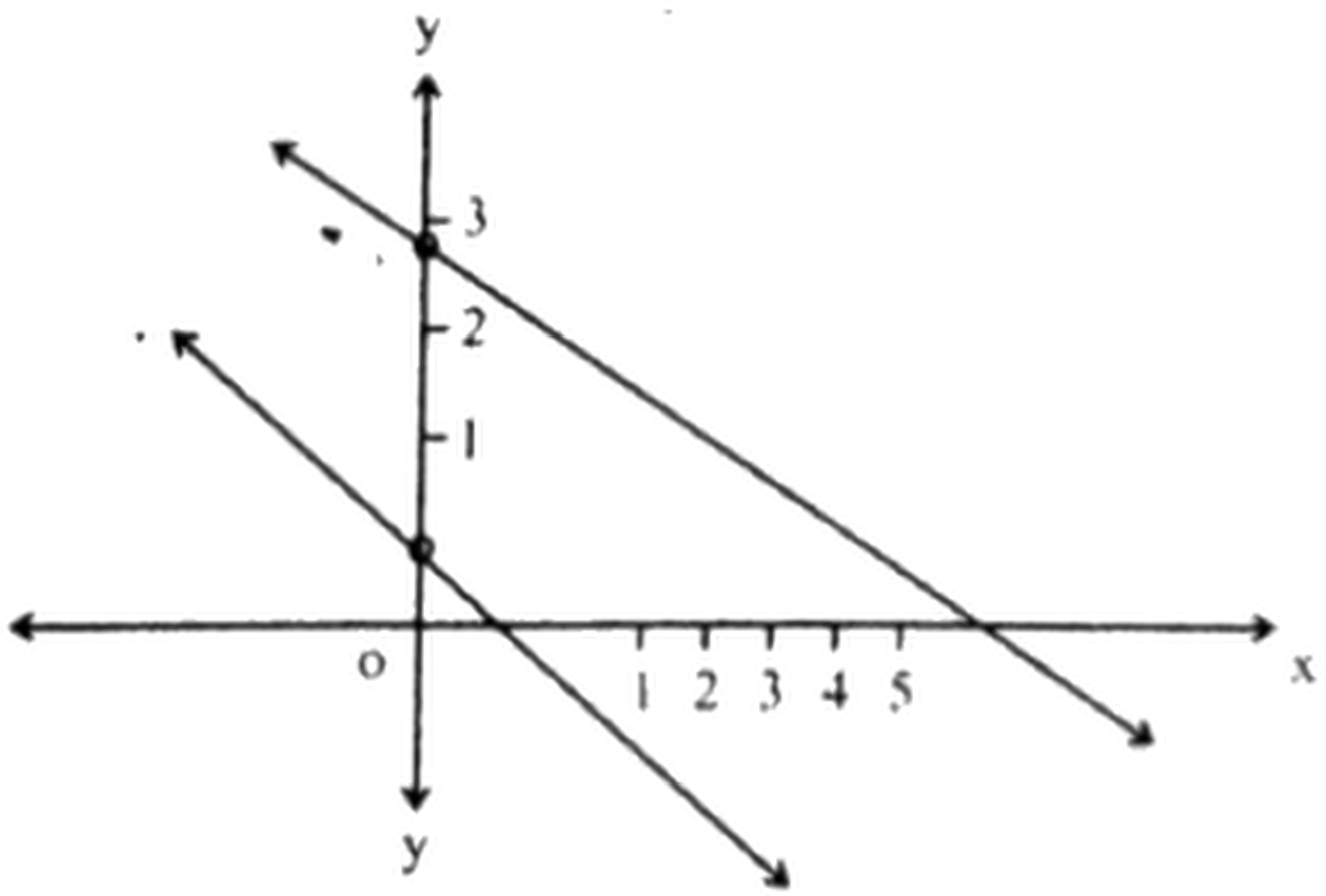
Equation of line through $\left(0, \frac{11}{8}\right)$ *with slope* $= -\frac{1}{2}$

$y - \frac{11}{8} = \frac{11}{8} = 0 \Rightarrow 4x + 8y - 11 = 0$

SKETCH

Put y = (i)	put y = (i)
x + 0 = 5	2x + 0 = 1
x = 5	$x = \frac{1}{2}$

Hence $\left(\frac{1}{2}, 0\right)$ is another point on (ii)



$$\frac{\frac{2Sin}{cos}}{\frac{cos2 + (cos^2)}{cos^2}}$$

$$\frac{\frac{2Sin}{cos}}{\frac{2cos2}{cos^2}} = -tan$$

For an acute angle between the lines

Tanθ = (-tan) = tan

7. Find a joint equation of the lines through the origin and perpendicular to the line

$$x^2 - 2xy\ tan - y^2 = 0$$

$$x^2 - 2xy\ tan - y^2 = 0$$

$$-y^2 - 2xy\ tan - x^2 = 0$$

$$\frac{-y^2}{-x^2} - \frac{2xy\ tan}{-x^2} + \frac{x^2}{-x^2} = 0$$

$$\left[\frac{xy}{x} + 2tan\left(\frac{y}{x}\right)\right] - 1 = 0$$

Which is a quadratic equation in $\frac{y}{x}$

$$\frac{y}{x} = \frac{-2tan \pm \sqrt{(2tan)^2 - +4)}}{2(1)} m$$

$$= \frac{-2tan \pm \sqrt{2\ tan^2 a + 4}}{2}$$

$$= \frac{-2tan \pm 2\sqrt{sec^2}}{2}$$

$$= -tan \pm sec$$

$$= \frac{-Sin}{cos} \pm \frac{1}{cos} = \frac{-Sin \pm 1}{cos}$$

$\frac{y}{x} = \frac{-Sin \pm 1}{cos}$

$y = \left[\frac{-Sin+1}{cos}\right]$

$m_1 = \frac{-Sin-1}{cos}$

$\frac{y}{x} = \frac{-Sin-1}{cos}$

$y = \left[\frac{-Sin-1}{cos}\right]$

$m_2 = \frac{-Sin-1}{cos}$

Slope of required line is slope of required line is

$m_3 = \frac{Cos}{-Sin+1}$

$= \frac{Cos}{Sin-1}$

$m_4 = \frac{Cos}{-Sin-1}$

$= \frac{Cos}{Sin+1}$

Equations of lines through origin and perpendicular to (i)

$y = \left[\frac{Cos}{-Sin - 1}\right] x$

$(Sin - 1)y = (Cos)x$

$(Sin - 1)x - (Sin - 1)y = 0$

$y = \left[\frac{Cos}{-Sin + 1}\right] c$

$(Sin + 1)y = (Cos)x$

$(Cos - 1)x - (Sin + 1)y = 0$

Joining above two equations

$[(\cos)x - (\sin - 1)y][(\cos)x - (\sin - 1)y] = 0$

$(\cos^2)x^2 - (\cos)(Sin\ + 1)\ xy -$

$(cos)(sin\text{-}1)x\ y + (Sin2\ -\ 1)^2$

$(\cos^2)x^2\ -\ (\cos \sin)xy\ -\ (\cos)xy$

$-(cos\ Sin)xy + (cos)xy - (1\text{-}\ Sin^2)$

$(\cos^2)x^2 - 2(cos\ sin)xy\ \text{-}\ (\cos^2)$

Dividing the equation by $\cos^2 a$

$\frac{(\cos^2)x^3}{\cos^2} - \frac{2(cosSin)xy}{cos^2} - \frac{(\cos^2 Sin)y^2}{\cos^2} = 0$

$x^2\ - (2\tan)xy -\ y^2\ =\ y^2\ =\ 0$

Q8. Find a equation of the lines through the origin and perpendicular of the line

$ax^2 + 2hxy\ + by^2\ =\ 0$

$ax^2 + 2hxy\ + by^2\ =\ 0$

$b\ x^2 + 2hxy\ + ay^2\ =\ 0$

$\frac{by^2}{bx^2} - \frac{2hxy}{bx^2} + \frac{ax^2}{b\ x^2}\ =\ 0$

$[y]^2 \quad 2h\ [y] \quad 9$

Which is a quadratic equation in $\frac{y}{x}$

$$\frac{y}{x} = \frac{\frac{24}{b} \pm \sqrt{\left[\frac{2h}{b}\right]^2 - 4(1)\left[\frac{a}{b}\right]}}{2(1)}$$

$$= \frac{\frac{2h}{b} \pm \sqrt{\frac{4h^2}{b} - 4\left[\frac{4a}{b}\right]}}{2(1)}$$

$$= \frac{2h}{b} \pm \sqrt{\frac{4h^2 - 4ab}{2}}$$

$$= \frac{2h}{b} \pm \sqrt{\frac{h^2 - ab}{b}}$$

$$= \frac{-2h \pm 2\sqrt{h^2 - ab}}{2} = \frac{-h \pm 2\sqrt{h^2 - ab}}{b}$$

$$\frac{y}{x} = \frac{-h + 2\sqrt{h^2 - ab}}{b}$$

$$y = \frac{-h - \sqrt{h^2 - ab}}{b}$$

$$m_1 = \frac{-h - \sqrt{h^2 - ab}}{b}$$

Slope of required lined

$$m_3 = \frac{b}{-h + \sqrt{h^2 - ab}}$$

$$\frac{y}{x} = \frac{-h - 2\sqrt{h^2 - ab}}{b}$$

$$y = \left[\frac{-h - \sqrt{h^2 - ab}}{b}\right]$$

$$m_3 = \frac{-h - \sqrt{h^2 - ab}}{b}$$

Slope of required lined

$$m_3 = \frac{b}{h + \sqrt{h^2 - ab}}$$

Equations of lines through Origin and perpendicular to (i)

are

$$y = m_3 x$$

$$y = \left[\frac{b}{h - \sqrt{h^2 - ab}}\right] x$$

$$(h - \sqrt{h^2 - ab})y = bx$$

$$bx - (h - \sqrt{h^2 - ab})y = 0$$

$$y = m_3 x$$

$$y = \left[\frac{b}{h + \sqrt{h^2 - ab}}\right] c$$

$$(h + \sqrt{h^2 - ab})y = bx$$

$$bx - (h + \sqrt{h^2 - ab})y = 0$$

Joining the above two equations.

$b^2x^2 - b(h - \sqrt{h^2 - ab})xy -$

$b\,(h - \sqrt{h^2 - ab})xy + [h^2 - (h^2 - ab)]\,y^2 = 0$

b^2x^2 -bhxy - b$\sqrt{h^2} - abxy - bhxy +$

b$\sqrt{h^2} - abxy + (h^2 - h^2 + ab) = 0$

Dividing the equation by b

$\frac{b^2x^2}{b}$ - $\frac{2bhxy}{b} + \frac{aby^2}{b} = 0$

$b^2x^2 - 2hxy + ay^2 = 0$

Q9. Find the area of the region bounded

$\mathbf{10x^2 - xy - 21y^2 = 0 \text{ and } x + y + 1 = 0}$

Solution

$10x^2 - xy - 21y^2 = 0$

$-21y^2 - xy + 10x^2 = 0$

$\frac{-21y^2}{-21x^2} + \frac{1}{21}\left(\frac{y}{x}\right) - \frac{10}{21} = 0$

Which is a quadratic equation in $\frac{y}{x}$

$\frac{y}{x} = \frac{\frac{1}{21} \pm \sqrt{\left(\frac{1}{21}\right) - 4\,(1)\left[-\frac{10}{21}\right]}}{2(1)}$

$= \frac{\frac{1}{21} \pm \sqrt{\frac{1}{44} + \frac{40}{21}}}{2}$

$= \frac{\frac{1}{21} \pm \sqrt{\frac{841}{441}}}{2}$

$= \frac{\frac{1}{21} \pm \frac{29}{21}}{2} = \frac{-1 \pm 29}{\frac{21}{2}} = \frac{-1 \pm 29}{42}$

$= \frac{28}{42}$

$= \frac{2}{3}$

3y = 2x

∴ 2x – 3y = 0 _____(ii)

Solving (i) and (ii)

From x = y -1 ____(iv)

$= \frac{30}{42}$

$= \frac{5}{7}$

7y = 5x

5x+7y = 0 _______(iii)

Putting (iv) in (ii)

2(-y-1) -3y = 0

-2y-2-3y = 0

5 -y = 0

y =- $\frac{2}{5}$

Putting y = $\frac{2}{-5}$ in _______(iv)

x = $\left(-\frac{2}{5}\right) - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$

A$\left[\frac{3}{-5}, -\frac{2}{5}\right]$

Solving (i) and (iii)

5(-y-1)+7y =

Putting (iv) in (iii) = x $\left(-\frac{2}{5}\right) - 1$

x = $-\frac{7}{2}$

Lines (ii) and (iii) pass through the origin hence their point of intersection as the origin i.e 0(0,0).

Area of triangle ΔOAB as:

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ -3-5 & -2/5 & 1 \\ -7/2 & 5/2 & 1 \end{vmatrix}$$

$= \frac{1}{2}\left[0 - 0 + 1\left(-\frac{3}{5} - \frac{7}{2}\right)\right]$