Exercise 3.8

Q1. Find that each of the following equations written against the differential equation is its solution.

i. $x\frac{dy}{dx} = 1+y$, $y = Cx - 1$

Solution

$$y = Cx-1$$

$$y + 1 = Cx \quad \ldots\ldots\ldots..(a)$$

taking differentiation on both sides

$$\frac{dy}{dx} = C$$

Now

$$x\,\frac{dy}{dx} = X.\,C \quad \ldots\ldots\ldots..(b)$$

putting the values of $\frac{dy}{dx}$ in above eq.(b)

$$x.\,\frac{1+y}{x} = y+1$$

$$1+y = 1+y$$

Hence the required result

ii. $x^2(2y+1)\frac{dy}{dx} - 1 = 0 \ldots..(a)$, $y^2+y = c - \frac{1}{x}$

Solution

$$y^2+y = C - \frac{1}{x}$$

Differentiating both sides by 'x' we get

$$2y\frac{dy}{dx} + \frac{dy}{dx} = 0+\frac{1}{x^2}$$

$$x^2(2y+1)\frac{dy}{dx} = 1$$

$$x^2(2y+1)\frac{dy}{dx} - 1 = 0$$

Hence the result putting above this line in L.H.S of(a) we get,

$$1 - 1 = 0$$

iii. $y\frac{dy}{dx} - e^{2x} = 1, \quad y^2 = e^{2x}+2x+1$

Solution

$$y\frac{dy}{dx} - e^{2x} = 1 \qquad (a)$$

$$y^2 = e^{2x}+2x+1 \qquad (b)$$

Differentiating w.r.t of eq (b) both sides by we get

$$\frac{dy}{dx}(y^2) = \frac{d}{dx}[e^{2x}+2x+1]$$

$$2y\frac{dy}{dx} = e^{2x} \times 2 + 2$$

$$y\frac{dy}{dx} = e^{2x} + 1$$

$$y\frac{dy}{dx} - e^{2x} = e^{2x} + 1 - e^{2x} = 1$$

Hence the result.

iv. $\frac{1}{x}\frac{dy}{dx} - 2y = 0, \qquad y = C\,e^{x^2}$

Solution

$$\frac{1}{x}\frac{dy}{dx} - 2y = 0 \quad \ldots\ldots\ldots\ldots\text{(i)}$$

$$y = C\,e^{x^2} \quad \ldots\ldots\ldots\text{(ii)}$$

Differentiating (ii) w.r.t 'x' we get

$\frac{dy}{dx}$ = $2.\ C\ e^{x^2}$ putting the value of 'e'

$= (\frac{y}{e^{x^2}}) \times e^{x^2}$ from eq.(ii)

$= 2xy$

$\frac{1}{x}\frac{dy}{dx}$ -2y $= \frac{1}{x}(2xy)$-2y

$= 2y - 2y = 0$

Hence the required result

v. $\frac{dy}{dx} = \frac{y^2+1}{e^{\ x}}$ **, y = tan(e^x +C)**

Solution

$\frac{dy}{dx} = \frac{y^2+1}{e^{\ x}}$ …………(i)

y = tan(e^x +C)……..(ii)

from (ii) Diff. w.r.t 'x' gives

$\frac{dy}{dx} = Sec^2((e^x$ +C)

$\frac{dy}{dx}$ = 1+tan2((e^x +C) × e^x

$= (1+y^2)\ e^x$ by(ii)

$\frac{dy}{dx} = \frac{(1+y^2)}{e^{\ x}}$

Hence the result

Q2. Solve the following differential equation.

$\frac{dy}{dx} = -y$

Solution

$$\frac{dy}{y} = -dx$$

Integrating we have

$$\int \frac{1}{y}\, dy = -\int 1\, dx$$

$$\ln y = -x + C_1$$

$$y = e^{-x + C_1} = e^{-x} . e^{C_1}$$

$$y = Ce^{-x}$$

Q3. ydy + xdy = 0

Solution

$$y\, dx + x\, dy = 0$$

$$y\, dx = -x\, dy$$

$$\frac{1}{x}\, dx = -\frac{1}{y}\, dy$$

Integrating on both sides we get

$$\int \frac{1}{x}\, dx = -\int \frac{1}{y}\, dy$$

$$\ln(x) = -\ln(y) + \ln C$$

$$\ln(x) + \ln(y) = \ln C$$

$$\ln(xy) = \ln C$$

$$xy = C$$

Q4. $\frac{dy}{dx} = \frac{1-x}{y}$

Solution

Separating variable we get

$$\int y\,dy = \int (1-x)\,dx$$

$$\frac{y^2}{2} = \int 1\,dx - \int x\,dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + C_1$$

$$y^2 = 2x - x^2 + 2C_1$$

$$y^2 = x(2-x) + C$$

Q5. $\frac{dy}{dx} = \frac{y}{x^2}$ **(y>0)**

Solution

Separating variable we have

$$\frac{1}{y}dy = \frac{1}{x^2}\,dx$$

Integrating variable we have

$$\int \frac{1}{y}\,dy = \frac{1}{x^2}\,dx$$

$$\text{Ln } y = \int x^{-2}\,(1)\,dx$$

$$\text{Ln } y = \frac{x^{\;2+1}}{-2+1} + C_1$$

$$\text{Ln } y = -\frac{1}{x} + C_1$$

$$y = e^{-\frac{1}{x} + C_1}$$

$$y = e^{-\frac{1}{x}} \cdot e^{C_1}$$

$$y = C\,e^{-\frac{1}{x}}$$

Q6. **Sin y . Cosec. x** $\frac{dy}{dx}$ **= 1**

Solution

$\text{Sin y}\frac{1}{\text{Sin x}}\ \frac{dy}{dx} = 1$ $\qquad$ $\text{Cosec x} = \frac{1}{\text{Sin x}}$

Separating variables gives

$$\int \text{Sin y}\, dy \ = \int \text{Sin x}\, dx$$

$$-\text{Cos y} = \ -\text{Cosx} + C_1$$

$$\text{Cos y} = \text{Cos x} + C \quad \text{When } C = -C_1$$

Q7. x dy +y(x-1) dx = 0

Solution

$$x\, dy + y(x-1)\, dx = 0$$

$$x\, dy = \ -y(x-1)\, dx$$

Separating variables gives

$$\frac{1}{y}dy \ = \ -(\frac{x-1}{x})dx$$

$$\frac{1}{y}dy \ = \ -(\ 1 \ - \frac{1}{x})dx$$

Integrating on both sides, we have

$$\int \frac{1}{y}\ dy \ = \ \frac{1}{y}\ (\ 1 \ - \frac{1}{x})dx$$

$$\int \frac{1}{y}\ dy \ = \ - \int 1\ dx \ + \int \frac{1}{x}\ dx$$

$$\text{Ln } y = \ln|x| \ -x + \ln|C|$$

$$\text{Ln } y = \ln|Cx| \ -x + \ln|C|$$

$$\ln y - \ln Cx = \ln|e^{-x}|$$

$$\ln(\frac{y}{Cx}) = \ln|e^{-x}|$$

$$(\frac{y}{Cx}) = e^{-x}$$

$$y = \ Cx\, e^{-x}$$

Q8. $\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$

Solution

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$$

Separating variables we have

$$\frac{x^2+1}{y+1} = \frac{y+1}{y}dy$$

$$(x+\frac{1}{x})dx = (x+\frac{1}{y})dy$$

$$(x+\frac{1}{y})dy = (x+\frac{1}{x})dx$$

Integrating both sides we have,

$$\int(1+\frac{1}{y})\,dy = \int(x+\frac{1}{x})\,dx$$

$$y+\ln y = \frac{x^2}{2}+\ln(x)+\ln(C)$$

$$\ln ye^y - \ln e\,x = \frac{x^2}{2}$$

$$\ln(\frac{ye^y}{Cx}) = \frac{x^2}{2}$$

$$\frac{ye^y}{Cx} = e^{x^2/2}$$

$$ye^y = Cxe^{x^2/2}$$

Q9. $\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}(1+y^2)$

Solution

Separating variables we have

$$\frac{1}{1+y^2}dy = \frac{x}{2}dx$$

Integrating both sides gives

$$\tan^{-1}(y) = \frac{1}{2}\frac{x^2}{2} + C_1$$

$$\tan^{-1}(y) = \frac{x^2}{4} + C_1$$

Q10 $2x^2 y\frac{dy}{dx} = x^2 - 1$

Solution

$$2x^2 y\frac{dy}{dx} = x^2 - 1$$

Separating variables, we have

$$2y\, dy = \frac{x^2 - 1}{x^2}$$

Integrating both sides, we have

$$\int 2y\, dy = \frac{1}{2}\int(1 - \frac{1}{x^2})\, dx$$

$$y^2 = \int 1\, dx + \int \frac{1}{x^2}\, dx$$

$$y^2 = x + \frac{1}{x} + C$$

Q11. $\frac{dy}{dx} + \frac{2xy}{2y+1} = x$

Solution

$$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$$

$$\frac{dy}{dx} = x - \frac{2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{2xy + x - 2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{x}{2y+1}$$

Separating variables we get

$$\int (2y + 1)\, dy = \int x\, dx$$

$$\int 2y\, dy + \int 1\, dx = \int 1\, dx$$

$$y^2 + y = \frac{x^2}{2} + C_1$$

$$y(y+1) = \frac{x^2}{2} + C_1$$

Q12. $(x^2 - x^2 y)\frac{dy}{dx} + y^2 + xy^2 = 0$

Solution

$$x^2(1-y)\frac{dy}{dx} + y^2(1+x) = 0$$

$$x^2(1-y)\frac{dy}{dx} = -\,y^2(1+x)$$

$$\left(\frac{1-y}{-y^2}\right)dy = \left(\frac{1+x}{x^2}\right)dx$$

$$\left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \left(\frac{1}{x^2} - \frac{1}{x}\right)dx$$

$$\int\left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \int\left(\frac{1}{x^2} - \frac{1}{x}\right)dx$$

$$\int \frac{1}{y}\,dy + \int -\frac{1}{y^2}\,dy = \int \frac{1}{x^2}\,dx + \int\left(\frac{1}{x}\right)dx$$

$$\text{Ln}y + \frac{1}{y} = -\frac{1}{x} + \ln x + C$$

$$= \ln x - \frac{1}{x} + C$$

Q13. $\text{Sec}^2x \tan y\, dx + \text{Sec}^2y \tan x\, dy = 0$

Solution

$$\text{Sec}^2x \tan y\, dx + \text{Sec}^2y \tan x\, dy = 0$$

$$\text{Sec}^2y \tan x\, dy = -\text{Sec}^2x \tan y\, dx$$

$$\frac{\text{Sec}^2y}{}\, dy = -\,\frac{\text{Sec}^2x}{}\, dx$$

$$\int \frac{\text{Sec}^2 y}{\tan y} dy = -\int \frac{\text{Sec}^2 x}{\tan x} dx$$

$$\ln \tan y = -\ln \tan x + \ln |C|$$

$$\ln |\tan y| = \ln \tan x = \ln |C|$$

$$\ln |\tan y \tan x| = \ln |C|$$

$$\tan x \tan y = C$$

Q14. $y\text{-}x\frac{dy}{dx} = 2(y^2 + \frac{dy}{dx})$

Solution

$$-2\frac{dy}{dx} - x\frac{dy}{dx} = 2y^2 - y$$

Separating variables , we have

$$(-2+x)\frac{dy}{dx} = 2y^2 - y$$

$$\frac{1}{y-2y^2} dy = \frac{1}{2+x} dx$$

$$\frac{1}{y(1-2y)} = \frac{1}{1+2x}$$

$$[\frac{1}{y} + \frac{1}{(1-2y)}]dy = \frac{1}{1+2x} dx$$

Integrating both sides we get

$$\int [\frac{1}{y} + \frac{1}{(1-2y)}]dy = \int \frac{1}{1+2x} dx$$

$$\int \frac{1}{y} dy - \int \frac{2}{2y-1}]dy = \int \frac{1}{1+2x} dx$$

$$\ln y - \ln(2y-1) = \ln(2+x) + \ln C$$

$$\ln (\frac{y}{2y-1}) = \ln C(x+2)$$

$$\frac{y}{2y-1} = C(x+2)$$

Solution

$$1+\text{Cos } x \tan y \frac{dy}{dx} = 0$$

$$\text{Cos } x \tan y \frac{dy}{dx} = -1$$

Separating variables we have

$$\tan y\ dy = -\frac{1}{\text{Cos } x} dx$$

$$\frac{-\,Sin\ y}{\text{Cos } y}\ .\ dy = \text{Sec } x\ .\ dx$$

Integrating both sides, we get

$$\int (\text{Cos } y)^{-1}(-Sin\ y)\ dy = \int Sec\ x\ dx$$

$$\text{Ln } |\text{Cos } y| = \ln|\text{Sec } x + \tan x| + \ln|C|$$

$$\text{Ln } |\text{Cos } y| = \ln|\text{Sec } x + \tan x\, C|$$

$$\text{Cos } y = C(\text{Sec } x + \tan x)$$

Q16. $y - x\frac{dy}{dx} = 3(1+x\frac{dy}{dx})$

Solution

$$y - x\frac{dy}{dx} = 3(1+x\frac{dy}{dx})$$

$$y - x\frac{dy}{dx} = 3+3x\frac{dy}{dx}$$

$$y - x\frac{dy}{dx} = 3$$

$$-4x\frac{dy}{dx} = 3 - y$$

Separating the variables

$$\frac{-4}{3-y} dy = \frac{1}{x}\ dx$$

$$\frac{4}{y-3} dy = \frac{1}{x}\ dx$$

$\int \frac{4}{y-3} dy = \int \frac{1}{x}$ dx

4ln(y-3) = ln x+ ln C

$\ln(y-3)^4 = \ln xC_1$

$(y-3)^4 = xC_1$

y-3 = $(C_1\, x)^{1/4} = C\, x^{1/4}$

y-3 = $(C_1\, x)^{1/4} = x^{1/4} C_1{}^{1/4} = C\, x^{1/4}$

y = 3+$C\, x^{1/4}$

Q17. Sec x +tan y $\frac{dy}{dx}$ = 0

Solution

Sex x +tan y $\frac{dy}{dx}$ = 0

Separating variables we have

tan y. Sec x +tan y $\frac{dy}{dx}$ = -Sex x

Integrating both sides we give

$$\int -\frac{\text{Sin y}}{\text{Cos y}} dy = \int Sec\ x\ dx$$

ln Cos y = ln (Sec x + tan x) + ln C

ln Cos y = ln C (Sec x + tan x)

Cos y = C(Sec x + tan x)

Q18. $(e^x + e^{-x})\ \frac{dy}{dx} = e^x - e^{-x}$

Solution

dy = $(\frac{e^x - e^{-x}}{e^x + e^{-x}})$dx

Q19. Find the general solution of the equation $\frac{dy}{dx}$ -x = xy^2. Also find the particular solution if y=1 when x =0.

Solution

$$\frac{dy}{dx} - x = xy^2$$

$$\frac{dy}{dx} = x + xy^2$$

$$\frac{dy}{dx} = x(1+y^2)$$

$$\frac{1}{1+y^2}dy = x\,dx \qquad \text{(i)}$$

Integrating both sides of (i) gives

$$\int \frac{1}{1+y^2}dy = \int x\,dx$$

$$\tan^{-1} y = \frac{x^2}{2} + C \qquad \text{(a)}$$

Put y=1 and x=0 in ,we get

$$\tan^{-1}(1) = \frac{0}{2} + C \qquad \Rightarrow \frac{\pi}{4} = C$$

Thus $\tan^{-1}(y) = \frac{x^2}{2} + \frac{\pi}{4}$

Q20. Solve the differential of $\frac{dy}{dt} = 2x$ given that x=4 when t=0.

Solution

$$\frac{dy}{dt} = 2x$$

$$\frac{1}{x}\,dx = 2dt \text{ ....(i)}$$

Integrating both sides of(i),give

$$\int \frac{1}{x}\,dx = 2\int dt$$

$$\ln x = 2t + \ln C$$

$x = Ce^{2t}$

using condition x =4 when t=0 in ii.

$4 = Ce^{2.0} = 1 \quad \Rightarrow C = 4$

$x = 4e^{2t}$

Q21. Solve the differential equation $\frac{dS}{dt} + 2St = 0$. Also find the particular solution If S =4e when t = 0.

Solution

$$\frac{dS}{dt} + 2St = 0$$

$$\frac{dS}{dt} = -2St$$

$$\frac{1}{S}\,dS = -2t\,dt \text{............(i)}$$

Integrating both sides of (i) gives

$$\int \frac{1}{S}\,dS = -2\int dt$$

$$\ln S = -2\frac{t^2}{2} + \ln C$$

$$\ln S - \ln C = -t^2$$

$$\ln\frac{S}{C} = -t^2$$

$$\frac{S}{C} = e^{-t^2} \quad \Rightarrow \quad S = Ce^{-t^2}$$

S = 4.e when t =0

$$4e = C.\,e^0$$

$$S = 4e\,e^{-t^2} \quad \Rightarrow \quad S = 4C$$

$$S = 4e^{1-t^2}$$

Q22. In a culture, bacteria increase at the rate proportion to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours. Find the number of bacteria present four hours later.

Solution

Let P be any number of bacteria present at time 't'. then

$$\frac{dP}{dt} = KP \qquad (K>0)$$

OR $\frac{1}{P}dP = kdt$

Integrating we get,

$$\int \frac{1}{P}\ dP = \int Kdt$$

$$\ln(P) = Kt + C_1$$

$\Rightarrow$ $P = e^{Kt+C_1}$

$$P = e^{Kt}.e^{C_1}$$

$$P = e^{Kt}.C \quad ....(i) \qquad e^{C_1} = e$$

Applying given condition

That is P = 200 WHEN T=0,We get

$$200 = Ce^{K(0)}C$$

$\Rightarrow$ $200 = Ce^{(0)} = C$

$$200 = C$$

Putting $C = 200$ in (i)

$$P = 200\ e^{Kt} ....(2)$$

P will be 400 when t = 2 (hours)

So 2 becomes

$$400 = 200\ e^{K(2)}$$

$\Rightarrow$ $2 = e^{2K}$ $\Rightarrow \ln 2 = 2K$

Putting $K = \frac{1}{2}\ln 2$ in 2, we get

$P = 200\, e^{Kt}$

$= 200\, e^{(\frac{1}{2}\ln 2)t} = 200\, e^{(\frac{1}{2}\ln 2)}$

$= 200\, e^{\ln(2^{\frac{1}{2}})} = (200)\ 2^{\frac{1}{2}}$

If t = 4 (hours), then

$P = 200 \times 2^{\frac{1}{2}}$

P = 200 × 4 = 800

Q23. A ball s thrown vertically upward with a velocity of 2450 cm/Sec neglecting air resistance find

(i) Velocity of ball at any time t.

(ii) Distance traveled in any time t.

(iii) Maximum height attained by the ball.

Solution

(i) Let v be velocity of ball at any time t. then by Newton's law of motion.

$\frac{dv}{dt} = -g$

⇒ $dv = -g dt$ ....(i)

Integrating both sides of (i) gives

$\int dv = \int -g\, dt$

$v = -gt + C_1$ ....(ii)

apply initial condition t = 0 and v =2450 cm\sec. when t=0

$2450 = -g(0) + C_1$

⇒ $C_1 = 2450$

v = -980t +2450 or

v = 2450 = -g(0) +C_1

C_1 = 2450

Hence (ii) becomes

v = -980t +2450 or

v = 2450 - 980 t ....(iii)

(ii)h be height of ball at any time "t" we have

v = $\frac{dh}{dt} = 2450 - 980\,t$

⇒ dh = (2450-980 t) dt ....(9)

Integrating both sides of (9) we get

$\int dh = \int (2450 - 980t)t$

h = 2450 t – 980 $\frac{t^2}{2}$ +C_2

h = 2450 t- 490 $t^2 + C_2$ ....(iv)

when h = 0, when t =0 ⇒ $C_2 = 0$

hence (iv) can be written as

h = 2450 t - 490 t^2

iii For maximum height of ball v = 0

Then from (iii) we have

0 = 2450 – 980 t

⇒ 980 t = 2450

t = $\frac{2450}{980}$ ⇒= $\frac{5}{2}$(sec)

hence maximum height attained (in cm)

h(max.) = 2540 × $\frac{5}{2}$ -490 × $(\frac{5}{2})^2$

h(max.) = 3062.5 Cm = 30. 625