Exercise 3.7

Q1. Find the area between the x axis and the curve y = x^2+1 From x =1 to x=2

Solution

The required area is

$= \int_1^2 (x^2+1)dx = \int_1^2 x^2 dx + \int_1^2 1\, dx$

$= \left[\frac{x^3}{3}\right]_1^2 + [x]_1^2 = \left(\frac{(2)^3}{3} - \frac{(1)^3}{3}\right) + (2-1)$

$= \left(\frac{8}{3} - \frac{1}{3}\right) + 1 = \frac{7}{3} + 1 = \frac{10}{3}$ Sq units

Q2. Find the area above the x-axis and under the curve y = 5-x^2 from x =-1 to x=2

Solution

The curve cuts axis at $\sqrt{5}$ and -$\sqrt{5}$

x	$-\sqrt{5}$	-2	-1	0	...	2	$\sqrt{5}$
y	0	1	4	5	...	1	0

The required area is

$= \int_1^2 (5-x^2)\, dx$

$= [5x]_{-1}^2 - \left[\frac{x^3}{3}\right]_{-1}^2 = [5(2)-5(-1)] - \left(\frac{(2)^3-(1)^3}{3}\right)$

$= (10+5) - \left(\frac{8+1}{3}\right) = 15 - \frac{9}{3} + 1 = \frac{45-9}{3} = \frac{36}{3} = 12$

Q3. Find the area below the curve y = $3\sqrt{x}$ and above the x-axis between x =1 to x=4

Solution

The curve cuts x- axis and is in the 1st quadrant at x=0.

x	0	1	3	4
y	0	3	5.2	6

The required area is

$= \int_1^4 3\sqrt{x}\ dx = 3\int_1^4 (x)^{1/2}\ dx$

$= \frac{x^{1/2+1}}{1/2+1}\Big|_1^4 = 3\ {}^2/_3\, x^{3/2}\Big|_1^4$

$= 2\ x^{3/2}\Big|_1^4 = 2\left[(4)^{3/2} - (1)^{3/2}\right]$

$=2\left[(2^2)^{3/2} - 1\right] - 2[8\text{-}1]$

=2[7] = 14 (sq. units)

Q4. Find the area bounded by Cos function from x = $-\frac{\pi}{2}$ and x = $\frac{\pi}{2}$

Solution

y = Cos x

The given curve cube the x-axis at x = $-\frac{\pi}{2}$ and x = $\frac{\pi}{2}$

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0

The required area

$= \text{Sin}\,\frac{\pi}{2} - Sin\left(\frac{\pi}{2}\right) = 1 - (-1)$

= 2

The required area

= 2(sq. units)

Q5. Find the area between the x-axis and the curve y =4x - x^2.

Solution

$y = 4x - x^2 = x(4-x)$

Given curve cuts the x-axis at the points (0,0) and (4,0)

x	-2	-1	0	1	2	3	4
y	-12-	-5	0	3	4	3	0

As $4x - x^2 \geq 0$ for $0 \geq 4$ the curve is above the x-axis with interval [0,4] so,

The required area

$= \int_0^4 (4x - x^2)\ dx$

$= 4\ \frac{x^2}{2}\ {}_0^4 - \frac{x^3}{3}\ {}_0^4$

$= 2\ x^2\ {}_0^4 - \frac{x^3}{3}\ {}_0^4 = 2((4)^2 - (0)^2) - \left(\frac{(4)^3 - (0)^3}{3}\right)$

$= 2(16) - \left(\frac{64}{3}\right) = 32 - \frac{64}{3} = \frac{96-64}{3} = \frac{32}{3}$ (Sq. units)

Q6. Determine the area bounded by the parabola y = $x^2 + 2 - 3$ and x-axis.

Solution

For x intercept point y = 0

We have $x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\Rightarrow \quad x = -3 , x = 1$

The parabola cuts the x-axis at x=-3 and 1

x	0	-1	-2	-3	1	2	3	4
y	-3	-4	-3	0	0	5	12	21

As $y \leq 0$ for $-3 \leq x \leq 1$ so the area bounded by the curve is below the x-axis.

Thus

The required area

$= \int_{-3}^{1}(x^2 + 2x - 3)\ dx$

$= \frac{x^3}{3}\ {}_{1}^{-3}\ -\ x^2\ {}_{1}^{-3}\ -\ 3x\ {}_{1}^{-3}$

$=\left(\frac{(-3)^3-(1)^3}{3}\right) + [(-3)^2 - (1)^2]-3[(-3)-1]$

$= \left(\frac{-2-17}{3}\right) + (9-1) - 3(-4)=\frac{-28}{3} + 8 + 12 \ = \frac{-28}{3} +20 =\frac{-28+60}{3} = \frac{32}{3}$ (Sq. units)

Q7. Find the area bounded by the curve y = $x^3 + 1$, the x-axis and line x =2

Solution

$y = x^3 + 1$

$y = (x+1)(x^2 - x + 1)$

Let $\quad y = 0 \Rightarrow (x+1)(x^2 - x + 1) = 0$

$\Rightarrow \quad x = -1$ or $x^2 - x + 1 = 0$

$x = 1 \pm \frac{\sqrt{1-4}}{2}$

$x = \frac{1 \pm \sqrt{3}}{2}$

The curve cuts the x-axis only at the real value x=-1

x	-3	-2	-1	0	1	2	3
y	-26	-7	0	1	2	9	28

The required area

$= \int_{-1}^{2} (x^3 + 1)\ dx$

$= \frac{x^4}{4}\ {}_{-1}^{2} - \ x\ {}_{-1}^{2}$

$= \left(\frac{(2)^4 - (-1)^4}{4} \right) + [(2) - (-1)]$

$= \left(\frac{16-1}{4} \right) + 2 + 1 = \frac{15}{4} + 3\ = \frac{15+12}{4} = \frac{27}{4}$ (Sq. units)

Q8. Find the area bounded by the curve y = $x^3 - 4x$ and the x-axis

Solution

$y = x^3 - 4x \qquad = x(x^2 - 4)$

if $y = 0 \qquad \Rightarrow x(x^2 - 4) = 0 \Rightarrow x = 0$

or $(x^2 - 4) = 0$

i.e $x = 0 \qquad$ or $x^2 = 4 \ \Rightarrow x = \pm 2$

$x = \pm 2$

thus the given curve cuts the x-axis at (0,0) ,(-2,0),(2.0)

$y \geq 0$ *when* $-2 \leq x \leq 0$,therefore the curve is above x-axis	$y \leq 0$ *when* $0 \leq x \leq 2$,therefore the curve is below the x-axis

The required area

$= \int_{-2}^{0} y\ dx - \int_{0}^{2} y\ dx \qquad \int_{-2}^{0} (x^3 - 4x)\ dx - \int_{0}^{2} (x^3 - 4x)\ dx$

$= \int_{-}^{0} x^3\ dx - 4\int_{-}^{0} x\ dx - \int_{-}^{0} x^3\ dx + 4\int_{-}^{0} x\ dx$

$=\left(\frac{(0)^4-(-2)^4}{4}\right)-2[(0)^2-(-2)^2]$

$=\left(\frac{(2)^4-(0)^4}{4}\right)-2[(2)^2-(0)^2]$

$=\left(-\frac{16}{4}\right)-2(-4)-\frac{16}{4}+2(4)$ =-4+8-4+8

The required area = 8 Sq. units

Q9. Find the area between the curve y =x(x-1)(x+1) and the x-axis.

Solution

y = x(x-1)(x+1)

if y = 0 ⇒ x(x-1)(x+1) = 0

⇒ x = 0 x-1 = 0 or x +1 = 0

⇒ x =0 x =1 , x = -1

The given curve cuts the x- axis at (0,0) (-1,0)(1,0)

As , y ≥ 0 for -$\leq x \leq 0$ and y≤0 when 0≤ $x \leq 1$ the curve is above the x-axis for -1<x<0 and the curve is below the x-axis for 0 <x<1

Thus the required area

$= \int_{-1}^{0}(x^3-x)\ dx - \int_{0}^{1}(x^3-x)\ dx$

$= \int_{-1}^{0}x^3\ dx - 4\int_{-1}^{0}x\ dx - \int_{0}^{1}x^3\ dx + 4\int_{0}^{1}x\ dx$

$= \frac{x^4}{4}\ {}^{0}_{-1} - 4\ \frac{x^2}{2}\ {}^{0}_{-1} - \frac{x^4}{4}\ {}^{1}_{0} + 4\ \frac{x^2}{2}\ {}^{1}_{0}$

$=\left(\frac{(0)^4-(-2)^4}{4}\right)-\left(\frac{(0)^2-(-2)^2}{2}\right)-\left(\frac{(1)^4-(0)^4}{2}\right)+\left(\frac{(1)^2-(0)^2}{2}\right)-$

$=-\frac{1}{4}\pm\frac{1}{2}-\frac{1}{4}\pm\frac{1}{2}= -\frac{1}{4}-\frac{1}{4}+\frac{1}{2}+\frac{1}{2} = \frac{-1-1+2+2}{4}$

$=\frac{-2+4}{4} = -\frac{1}{2}$ Sq. units

Q10. Find the area above the x-axis bounded by the curve y^2 = 3-x from x = -1 to x =2.

Solution

y^2 = 3-x $\Rightarrow y = \pm\sqrt{3-x}$

The branch of curve above x-axis is y= $\sqrt{3-x}$ as y ≥ o for -≤ x ≤ 2,so the curve is above the x-axis in the interval (-1,2)

Thus the required area

$= \int_{-1}^{2} \sqrt{3-x}\ \mathrm{dx} = \int_{2}^{-1} \sqrt{3-x}\ \mathrm{dx}$

$= \int_{2}^{-1} (3-x)^{1/2}(-1)\ \mathrm{dx} = \dfrac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\ {}_{2}^{-1}$

$= \dfrac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}}\ {}_{2}^{-1} = \frac{2}{3}\ (3-x)^{3/2}\ {}_{2}^{-1}$

$= \frac{2}{3}[(3-(-1)]^{3/2} - (3-2)^{3/2}]$

$= \frac{2}{3}\left[(4)^{3/2} - (1)^{3/2}\right] = \frac{2}{3}(8-1)$

$= \frac{2}{3} \times 7 = \frac{14}{3}$(Sq. units)

Q11. Find the area between the x-axis and the curvy y = cos $\frac{x}{2}$ from x = -π to π.

Solution

As the graph of Cos $\frac{1}{2}x$ is above the x-axis in the interval -π to π *so*, the required area

$= \int_{-\pi}^{\pi} \mathrm{Cos}\ \frac{x}{2}\ \mathrm{dx} = \int_{-\pi}^{\pi} (\mathrm{Cos}\ \frac{x}{2}\ \mathrm{dx}).\frac{1}{2}\ dx$

$= 2\ \ Sin\ \frac{x}{2}\ {}_{-\pi}^{\pi} = 2[Sin\ \frac{\pi}{2} - Sin(-\ \frac{\pi}{2})]$

=2[1-(-1)] = 2[1+1] = 4

Q12. Find the area between the x-axis and the curve y = Sin 2x from x =0 to x =$\frac{\pi}{3}$

Solution

The required area

$=\int_0^{\pi/3} \text{Sin } 2x \ dx \ = \frac{1}{2}\int_2^{\pi/3} - \text{Sin } 2x(2) \ dx$

$= -\frac{1}{2} \ Cos\ 2x \ _0^{\pi/3} \ = -\frac{1}{2}[Cos \frac{2\pi}{2} - \text{Cos } 0]$

$= -[\frac{-1}{2} - 1] = -\frac{1}{2}[\frac{-1-2}{2}]$

$= (\frac{1}{2})(\frac{-3}{2}) = \frac{3}{4}$ Sq. units

Q13. Find the area between x-axis and the curve y = $\sqrt{2ax - x^2}$ when a>0

Solution

$y = \sqrt{2ax - x^2}$

when y =0 $\Rightarrow 0 = \sqrt{2ax - x^2}$

$\Rightarrow \ 2ax - x^2 = 0(2z\text{-}x)x = 0$

$\Rightarrow \quad x = 0 \ , x = 2a$

Thus the given curve cuts the x-axis at x=0 and x=2a As y ≥ o for $0 \leq x \leq 2a$, so the curve is above the x-axis in the interval (0,2a).

Thus required area.

$= \int_0^{2a} \sqrt{2ax - x^2} \ dx = \int_0^{2a} \sqrt{a^2 - a^2 + 2ax - x^2} \ dx$

$= \int_0^{2a} \sqrt{a^2 - (a^2 - 2ax + x^2)} \ dx = \int_0^{2a} \sqrt{a^2 - (a - x)^2} \ dx$

Put a-x = a Sinθ dx when x = 0

$= a \text{ Cos } \theta \ d\theta$ dx $a = -0 = a \text{ Sin } \theta$

$= a \text{ Cos } \theta \ d\theta$ Sin θ = 1

When x = 2a

a-2a = a Sin θ

-a = Sin θ

Sin θ = -1

Sin θ = - $\pi/2$

Area 2a

$$= \int_0^{2a} \sqrt{a^2 - (a-x)^2}\ \mathbf{dx}$$

$$= \int_{\pi/2}^{-\pi/2} \sqrt{a^2 - a^2\text{Sin}^2\theta} \times (\text{-a}) \text{ Cos}\,\theta\, d\theta$$

$$= \int_{\pi/2}^{-\pi/2} \sqrt{a^2(1 - \text{Sin}^2\theta)}\ (\text{aCos}\,\theta)\, d\theta$$

$$= \int_{\pi/2}^{-\pi/2} \sqrt{a^2\text{Cos}^2\theta}\ (\text{aCos}\,\theta)\, d\theta = \int_{\pi/2}^{-\pi/2} aCos\,\theta \times (\text{aCos}\,\theta)\, d\theta$$

$$= a^2 \int_{\pi/2}^{-\pi/2} \text{Cos}^2\theta\ \ d\theta = \frac{a^2}{2}\int_{\pi/2}^{-\pi/2} 1\ \ d\theta + \frac{a^2}{2}\int_{\pi/2}^{-\pi/2} \text{Cos}\ 2\theta\ \ d\theta$$

$$= \frac{a^2}{2}\ \ \theta\ \ {}^{\pi/2}_{-\pi/2} + \frac{a^2}{2} \times\ \ \frac{Sin\ 2\theta}{2}\ \ {}^{\pi/2}_{-\pi/2}$$

$$= \frac{a^2}{2}[\frac{\pi}{2} - (\frac{\pi}{2})] + \frac{a^2}{4}[Sin\,\pi - Sin(-\pi)]$$

$$= \frac{a^2}{2}[\frac{\pi}{2} + \frac{\pi}{2}] + \frac{a^2}{4}[0 - 0] = \frac{a^2}{2} . \frac{2\pi}{2}$$

The required area = $\frac{\pi}{2}a^2$ (Sq. units)