Exercise 3.4

Q1. Evaluate the following integral by parts add a word representing all the functions are defined

i. $\int$ **x Sinxdx**

Solution

$I = \int x \sin x\, dx$

Integral by parts taking x as first function

$= x(-\cos x) - \int \cos x \frac{d}{dx} x(dx)$

$= -x \cos x + \int \cos x dx$

$= -x \cos x + \sin x + C$

$= \sin x - x \cos x + C$

ii. $\int$ **ln x dx**

Solution

$I = \int \ln x .1dx$

Integral by parts taking lnx as first function

$= \ln x . x \int x \frac{d}{dx}(lnx)dx$

$= x \ln x - \int x \frac{1}{x} dx$

$= x\ln - \int dx$

$= x\ln x - x + C$

Solution

$$I = \int \ln x\, .x dx$$

Integral by parts taking lnx as first function

$$= \ln x \frac{x^2}{2} - \int \frac{x^2}{2} \frac{d}{dx}(\ln x)dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x^2 . \frac{1}{x} dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x\, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \left(\frac{x^{1+1}}{1+1}\right) + C$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} . \frac{x^2}{2} + C$$

$$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C$$

$$= \frac{x^2}{2} [\ln x - \frac{1}{2}] + C$$

iv. $\int \mathbf{x^2 \ln x dx}$

Solution

$$I = \int x^2 ln\, x\, dx$$

Integral by parts taking lnx as first function

$$= \ln x \frac{x^3}{2} - \int \frac{x^3}{2} \frac{d}{dx}(\ln x)dx$$

$$= \frac{x^3}{2} \ln x - \frac{1}{3} \int x^3 . \frac{1}{x} dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2\, dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} . \frac{x^3}{3} + C$$

$$= \frac{x^3}{2} \ln x - \frac{1}{9} x^3 + C$$

v. $\int x^3 \ln x dx$

Solution

$$I = \int x^3 ln\, x\, dx$$

Integral by parts taking lnx as first function

$$= \frac{x^4}{4} \ln x - \int \frac{x^4}{4} . \frac{1}{x} dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} . \frac{x^4}{4} + C$$

$$= \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + C$$

$$= \frac{x^4}{4} [\ln x - \frac{1}{4}] + C$$

vi. $\int x^4 \ln x dx$

Solution

$$I = \int x^4 ln\, x\, dx$$

Integral by parts taking lnx as first function

$$= \frac{x^5}{5} \ln x - \int \frac{x^5}{5} . \frac{1}{x} dx$$

$$= \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4 \, dx$$

$$= \frac{x^5}{5} \ln x - \frac{1}{5} . \frac{x^5}{5} + C$$

$$= \frac{x^5}{5} \ln x - \frac{1}{25} x^5 + C$$

$$= \frac{x^5}{5} [\ln x - \frac{1}{5}] + C$$

vii. $\int \tan^{-1} x \, dx$

$I = \int \tan^{-1} x \ dx$

Integral by parts taking $\tan^{-1} x$ as first function

$= \tan^{-1} x . x - \int x \frac{d}{dx}(\tan^{-1} x)dx$

$= x \tan^{-1} x - \int \frac{x}{x^2+1} dx$

$= x \tan^{-1} x - \frac{1}{2}\int \frac{2x}{x^2+1} dx$

$= x \tan^{-1} x - \frac{1}{2}\ln|x^2 + 1| + C$

viii. $\int x^2$ **Sin x dx**

Solution

$I = \int x^2 \text{Sin x dx}$

Integral by parts taking x^2 as first function

$= x^2 (-\text{Cos x}) - \int - \text{Cos x} \frac{d}{dx}(x^2)dx$

$= -x^2 \text{Cos x} + \int \text{Cos x}. 2x dx$

$= -x^2 \text{Cos x} + 2\int x \text{Cos x dx}$

Again integrating by parts x as first function

$= -x^2 \text{Cos x} + 2[x\ Sin\ x - \int \text{Sin x dx}$

$= -x^2 \text{Cos x} + 2[x\ Sin\ x - (-\text{Cos x})] + C$

$= -x^2 \text{Cos x} + 2x \sin x + 2 \text{Cos x} + C$

ix. $\int x^2 \tan^{-1} x \ dx$

Solution

$I = \int x^2 \tan^{-1} x \ dx$

$$= \tan^{-1} x \,.\frac{x^3}{3} - \int \frac{x^3}{3}\frac{d}{dx}(\tan^{-1} x)dx$$

$$=\frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{x^2+1}dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{2}\int (x - \frac{x}{x^2+1})dx \text{ by actual division}$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int \frac{x}{x^2+1}dx$$

$$=\frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int x\,dx + \frac{1}{3.2}\int \frac{2x}{x^2+1}dx$$

$$=\frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\frac{x^2}{2} + \frac{1}{6}\ln|x^2 + 1| + C$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\ln|x^2 + 1| + C$$

x. $\int x \tan^{-1} x \; dx$

Solution

$$I = \int x \tan^{-1} x \; dx$$

Integral by parts taking $\tan^{-1} x$ as first function

$$= \tan^{-1} x \,.\frac{x^2}{2} - \int \frac{x^2}{2}\frac{d}{dx}(\tan^{-1} x)dx$$

$$=\frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int (\frac{x^2+1-1}{x^2+1})dx \text{ by actual division}$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int (\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1})dx$$

$$=\frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int 1\,dx + \frac{1}{32}\int \frac{1}{x^2+1}dx$$

$$=\frac{x^2}{2}\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C$$

$$= \frac{1}{2}\tan^{-1} x\ [x^2 + 1] - \frac{1}{2}x + C$$

Solution

$$I = \int x^3 \tan^{-1} x \; dx$$

Integral by parts taking $\tan^{-1} x$ as first function

$$= \tan^{-1} x . \frac{x^4}{4} - \int \frac{x^4}{4}\frac{d}{dx}(\tan^{-1} x)dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4}\int \frac{x^4}{x^2+1}dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4}\int x^2dx + \frac{1}{4}\int (\frac{x^2+1-1}{x^2+1})dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4}\int x^2dx + \frac{1}{4}\int 1 \; dx - \frac{1}{4}\int \frac{1}{x^2+1}dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4}\frac{x^3}{3} + \frac{1}{4}x - \frac{1}{4}\tan^{-1} x + C$$

$$= \frac{1}{4}[x^4 \tan^{-1} x - \tan^{-1} x - \frac{x^3}{3} + x] + C$$

xii. $\int x^3$ **Cos x dx**

Solution

$$I = \int x^3 \text{ Cos x dx}$$

Integral by parts taking x as first function

$$= x^3 \text{ Sin x} - \int \text{Sin x } 3x^2dx$$

$$= x^3 \text{ sin x} - 3\int x^2 \text{Sin x dx}$$

Again integrating by parts

$$= x^3 \text{ sin x} - 3[x^2(-Cosx) - \int (-\text{Cos x}).\, 2x dx]$$

$$= x^3 \text{ sin } 3x^2 Cos\, x - 6\int \text{x Cos x dx}$$

$$= x^3 \text{ sin } 3x^2 Cos\, x - 6[x\, Sin\, x - \int \text{Sin x .1 dx}]$$

$$= x^3 \text{ sin } 3x^2 Cos\, x - 6x\, Sin\, x + 6\int \text{Sin x dx}$$

$$= x^3 \text{ sin } 3x^2 Cos\, x - 6x\, Sin\, x - 6\text{Cos x} + C$$

xiii. $\int \sin^{-1} x \, dx$

Solution

$$I = \int \sin^{-1} x \; dx \; dx$$

Integral by parts taking $\sin^{-1} x$ as first function

$$= \sin^{-1} x \, . \, x - \int x \frac{d}{dx}(\sin^{-1} x) dx$$

$$= x\sin^{-1} x - \int x \, . \frac{1}{\sqrt{1-x^2}} dx$$

$$= x\sin^{-1} x - \int x(1-x^2)^{-\frac{1}{2}} dx$$

$$= x\sin^{-1} x + \frac{1}{2}\int -2x(1-x^2)^{-\frac{1}{2}} dx$$

$$= x\sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= x\sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= x\sin^{-1} x + \sqrt{1-x^2} + C$$

xiv. $\int x \sin^{-1} x \, dx$

Solution

$$I = \int x \sin^{-1} x \; dx \; dx$$

Integral by parts taking $\sin^{-1} x$ as first function

$$= \sin^{-1} x \, . \frac{x^2}{2} - \int \frac{x^2}{2} \frac{d}{dx}(\sin^{-1} x) dx$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x] - \frac{1}{2}\sin^{-1}x$+C

$=\frac{x^2}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + \frac{1}{4}\sin^{-1}x - \frac{1}{2}\sin^{-1}x$+C

$=\frac{x^2}{2}\sin^{-1}x + \frac{1}{4}\sin^{-1}x - \frac{1}{2}\sin^{-1}x+\frac{x\sqrt{1-x^2}}{4}$+C

$=\frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x+\frac{x\sqrt{1-x^2}}{4}$+C

xv. $\int e^x$ **Sin x Cos x dx**

Solution

$=\int e^x$ Sin x Cos x dx

Integrating by parts taking e^x as first function

$= e^x.\frac{sin^2x}{2} - \int \frac{sin^2x}{2}.e^x\,dx$

$= e^x.\frac{sin^2x}{2} - \frac{1}{2}.\frac{1}{2}\int (2sin^2x).e^x\,dx$

$=e^x.\frac{sin^2x}{2} - \frac{1}{4}\int (1-Cos2x).e^x\,dx$

$=e^x.\frac{sin^2x}{2} - \frac{1}{4}\int e^x\,dx + \frac{1}{4}\int Cos2x.e^x\,dx$

$=e^x.\frac{sin^2x}{2} - \frac{1}{4}e^x + \frac{1}{4}[e^x.\frac{Sin\,2x}{2} - \int \frac{sin\,2x}{2}e^x\,dx$

$=e^x.\frac{sin^2x}{2} - \frac{1}{4}e^x + \frac{e^xSin\,2x}{8} - \frac{1}{8}\int 2Sin\,x\,Cos\,xe^x\,dx$

$=e^x.\frac{sin^2x}{2} - \frac{1}{4}e^x + \frac{e^xSin\,2x}{8} - \frac{1}{4}\int e^xSin\,x\,Cos\,x\,dx$

$=e^x.\frac{sin^2x}{2} - \frac{1}{4}e^x + \frac{e^xSin\,2x}{8} - \frac{1}{4}$ (I)

I+$\frac{1}{4}I = e^x.\frac{sin^2x}{2} - \frac{1}{4}e^x + \frac{e^xSin\,2x}{8}$

$\frac{5}{4}I = e^x.\frac{sin^2x}{2} - \frac{1}{4}e^x + \frac{e^xSin\,2x}{8}$

$I = \frac{2}{5} e^x . \frac{sin^2 x}{2} - \frac{1}{5} e^x + \frac{1}{10} e^x sin\ 2x + c$

xvi. $\int x$ **Sin x Cos x dx**

Solution

$I = \int x \text{ Sin x Cos x dx}$

$= \frac{1}{2} \int x \text{ 2 Sin x Cos x dx}$

$= \frac{1}{2} \int x \text{ Sin 2x dx}$

Integrating by parts taking x as first function

$= \frac{1}{2} [x . \frac{-Cos\ 2x}{2} - \int \frac{-Cos\ 2x}{2} \text{ dx}]$

$= \frac{1}{2} [- \frac{xCos\ 2x}{2} + \frac{1}{2} \int Cos\ 2x \text{ dx}]$

$= [- \frac{xCos\ 2x}{4} + \frac{1}{4} \frac{Sin\ 2x}{2}] + C$

$= \frac{1}{4} [-x\ Cos\ 2x + \frac{2Sin\ xCos\ x}{2}] + C$

$= \frac{1}{4} [-x\ Cos\ 2x + Sin\ x\ Cos\ x] + C$

xvii. $\int$ **x Cos ^{2}x dx**

Solution

$I = \int \text{x Cos 2x dx}$

$= \int x\ (\frac{1 + Cos\ 2x}{2}) \text{ dx}$

$= \frac{1}{2} \int (x + xCos2x) dx$

$= \frac{1}{2} \int xdx + \frac{1}{2} \int xCos2x\ dx$

Integrating by parts

$= \frac{x^2}{4} + \frac{1}{2}[x.\frac{sin\ 2x}{2} - \frac{1}{2}\int sin\ 2x.dx]$

$= \frac{x^2}{4} + \frac{xsin\ 2x}{4} - \frac{1}{4}\int \sin 2x.\ dx$

$= \frac{x^2}{4} + \frac{xsin\ 2x}{4} - \frac{1}{4}(-\ \frac{Cos\ 2x}{2})$ +C

$= \frac{x^2}{4} + \frac{xsin\ 2x}{4} + \frac{1}{8}Cos\ 2x$ +C

$= \frac{1}{4}(x^2 + xsin\ 2x + \frac{1}{2}Cos\ 2x)$ +C

xviii. $\int$ **x Sin ^{2}x dx**

Solution

$I = \int x\ Sin^2 x\ dx$

$= \int x\ (\frac{1-Cos\ 2x}{2})\ dx$

$= \frac{1}{2}\int x(x - xCos2x\ )dx$

$= \int (\frac{1}{2}x\ - \frac{1}{2}xCos2x\ )dx$

$= \frac{1}{2}\int xdx - \frac{1}{2}\int xCos2x\ dx$

Integrating by parts

$= \frac{1}{2}.\frac{x^2}{2} - \frac{1}{2}[x.\frac{sin\ 2x}{2} - \int\ \frac{sin\ 2x}{2}.dx]$

$= \frac{x^2}{4} - \frac{1}{4}x\ Sin\ 2x\ + \frac{1}{4}\int sin\ 2x.dx$

$= \frac{x^2}{4} - \frac{xsin\ 2x}{4} - \frac{1}{4}.\frac{Cos2x}{2} + C$

$= \frac{1}{4}(x^2 - xsin\ 2x - \frac{cos2x}{2})$ +C

xix. $\int$ **(lnx)^{2}dx**

Integrating by parts taking $(\ln x)^2$ as first function

$$=(\ln x)^2.x - \int x\frac{d}{dx}(lnx)\frac{1}{x}\,dx$$

$$=x(\ln x)^2 - 2\int lnx.1\,dx$$

Again integrating by parts

$$=x(\ln x)^2 - 2[lnx.x - \int x.\frac{d}{dx}(lnx)dx]$$

$$=x(\ln x)^2 - 2[lnx.x - \int x.\frac{1}{x}dx]$$

$$=x(\ln x)^2 - 2[xlnx - \int dx]$$

$$=x(\ln x)^2 - 2[xlnx - x]+C$$

$$=x\ln x(\ln x-2)+2x+C$$

xx. $\int \ln(\tan x)\mathrm{Sec}^2x\mathrm{d}x$

Solution

$$I = \int \ln(\tan x)\mathrm{Sec}^2x\mathrm{d}x$$

Integrating by parts taking ln(tan x) as first function

$$=\ln(\tan x).\tan x - \int \tan x\frac{d}{dx}(ln\ tan)dx$$

$$= \tan x \ln \tan x - \int \tan x.\frac{1}{tan\ x}.\mathrm{Sec}^2x\,dx$$

$$= \tan x \ln \tan x - \int \mathrm{Sec}^2x\,dx$$

$$= \tan x \ln \tan x - \tan x + C$$

xxi. $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,\mathrm{d}x$

Solution

$$I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

$=-[\sqrt{1-x^2}\sin^{-1}x - \int \sqrt{1-x^2}\frac{d}{dx}(\sin^{-1}x)dx]$

$=-[\sqrt{1-x^2}\sin^{-1}x - \int \sqrt{1-x^2}.\frac{1}{\sqrt{1-x^2}}dx]$

$=x-\sqrt{1-x^2}\sin^{-1}x + \int 1\, dx$

$= x-\sqrt{1-x^2}\sin^{-1}x + C$

Q2. Evaluate the following integrals

i. $\int \tan^4 x\, dx$

Solution

$I = \int \tan^4 x\, dx$

$=\int \tan^2 x\, tan^2 x\ dx$

$=\int\ tan^2 x.\sec^2 x - tan^2 x\ )\ dx$

$=\int\ (tan^2 x \sec^2 x)dx - \int\ tan^2 x\, dx$

$=\frac{tan^3 x}{3} - \int (\sec^2 x - 1)dx$

$=\frac{tan^3 x}{3} - \int \sec^2 x dx + \int dx$

$=\frac{tan^3 x}{3} - tan\, x + x + C$

ii. $\int Sec^4 x\, dx$

Solution

$I = \int Sec^4 x\, dx$

$=\int \sec^2 x\, sec^2 x\ dx$

$=\int (1 + tan^2 x).\sec^2 x\, dx$

$=\int\ (tan^2 x \sec^2 x)dx - \int\ sec^2 x\, dx$

$=\frac{tan^3 x}{3} + tan\, x + C$

iii. $\int e^x Sin\, 2x\, Cos\, 2x\, dx$

$=\frac{1}{2}\int$ [e^x 2Sin 2x Cos 2x] dx

$= \frac{1}{2}\int e^x[\ \text{Sin}(\ 2x + x) + \text{Sin}(2x - x)\ ]\ dx$

$=\frac{1}{2}\int e^x \text{ Sin } 3x dx + \frac{1}{2}\int e^x \text{ Sinx } dx$

$=\frac{1}{2}I_1 + \frac{1}{2}I_2$..........(A)

$I_1 \quad = \int e^x \text{ Sin } 3x dx$

$= \frac{-Cos\ 3x}{3} e^x + \frac{1}{3}\int e^x \text{ Cos } 3x\ dx$

$= \frac{-Cos\ 3x}{3} e^x + \frac{1}{3}[\frac{\boldsymbol{Sin\ 3x}}{\boldsymbol{3}} e^x - \frac{1}{3} \text{ Sin } 3xe^x\ dx$

$=\frac{-Cos\ 3x}{3} e^x + \frac{1}{3}\text{Sin } 3xe^x \ - \frac{1}{9}\ I_1 + C_1$

$(\ 1+\frac{1}{9})\ I_1 \ = \frac{-Cos\ 3x}{3} e^x + \frac{1}{9}\text{Sin } 3xe^x \ + C_1$

$\frac{10}{9}\ I_1 \ = \frac{-Cos\ 3x}{3} e^x + \frac{1}{9}\text{Sin } 3xe^x \ + C_1$

$I_1 \ = \frac{9}{10}[\ \frac{-Cos\ 3x}{3} e^x + \frac{1}{9}\text{Sin } 3xe^x\ ] + \frac{9}{10}C_1$

$I_1 \ = \frac{1}{10}[-3\text{Cos } 3xe^x + \text{Sin } 3xe^x\ ] + \frac{9}{10}C_1$

$I_1 \ = \frac{e^x}{10}[-3\text{Cos } 3x + \text{Sin } 3x\ ] + \frac{9}{10}C_1$

$I_2 \ = \int e^x \text{ Sin } x dx$ integrating by parts

$= -\text{ Cos } xe^x - \int (\mathbf{-Cosx}) \frac{\boldsymbol{d}}{\boldsymbol{dx}}(e^x)dx$

$=-\text{ Cos } xe^x + \int \mathbf{Cosx}\ e^x dx$

$= -\text{ Cos } xe^x + e^x\ Sin\ x\ \ - + \int \mathbf{Sin\ x}\ e^x dx + C_2$

$I_2 \ \ =-\text{ Cos } xe^x + e^x\ Sin\ x\ \ - I_2 + C_2$

$2I_2 \ = e^x[-\ Cos\ x + \ Sin\ x] + C_2$

$I_2 \ = \ \frac{1}{2}e^x[\ Sin\ x - Cos\ x] + \frac{1}{2}C_2$

By putting values I_2 and I_1 in (A) we get

Hence $\int e^x \text{Sin } 2x \text{ Cos } 2x \, dx = \frac{e^x}{4}[\frac{1}{5} Sin\ 3x - \frac{3}{5}\ Cos 3x\ + Sin\ x\ - Cos\ x] + C$

iv. $\int \tan^3 x \text{ Sec } x \, dx$

Solution

$$I = \int \tan^2 x(\tan x \text{ Sec } x)\, dx$$

$$= \text{Sec } x \tan^2 x - \int \sec x\ 2\tan x \sec^2 x\, dx$$

$$= \text{Sec } x \tan^2 x - 2[\sec x \tan x\ (1 + \tan^2 x)\, dx]$$

$$= \text{Sec } x \tan^2 x - 2\int \sec x \tan x dx - 2\int \tan^3 x\ sec\ x\, dx$$

$$I = \text{Sec } x \tan^2 x - 2\sec x\ - 2]$$

$$I + 2I = \text{Sec } x \tan^2 x - 2\sec x$$

$$3I = \text{Sec } x \tan^2 x - 2\sec x$$

$$I = \frac{1}{3}[\text{ Sec } x \tan^2 x - 2\sec x] + C$$

Hence

$$\int \tan^3 x \text{ Sec } x\, dx = \frac{1}{3}[\text{ Sec } x\ (Sec2x\ - 1) - 2\sec x] + C$$

$$= \frac{1}{3}[\text{ Sec } x \tan^2 x - 2\sec x] + C$$

v. $\int x^3 e^{5x}\, dx$

Solution

$$\int x^3 e^{5x}\, dx$$

$$= \frac{e^{5x}}{5} x^3 - \frac{1}{5}\int e^{5x} \frac{d}{dx}(x^3)dx$$

$$= \frac{1}{5} e^{5x} x^3 - \frac{1}{5}\int e^{5x} 3x^2 dx$$

$$= \frac{1}{5}[e^{5x} x^3 - 3\int e^{5x} x^2 dx]$$

$$= \frac{1}{5}[e^{5x} x^3 - 3(\frac{e^{5x} x^2}{5} - \frac{1}{5}\int e^{5x} 2x dx)]$$

$$=\frac{1}{5}[e^{5x}x^3 - \frac{3e^{5x}x^2}{5} + \frac{6}{5}(\frac{e^{5x}x}{5} - \frac{1}{5}\int e^{5x}dx)]$$

$$= \frac{1}{5}[e^{5x}x^3 - \frac{3e^{5x}x^2}{5} + \frac{6}{25}e^{5x}x - \frac{6}{52} \times \frac{e^{5x}}{5}]+C$$

$$= \frac{1}{5}[e^{5x}x^3 - \frac{3e^{5x}x^2}{5} + \frac{6}{25}e^{5x}x - \frac{6}{125}e^{5x}]+C$$

$$=\frac{1}{5}e^{5x}[x^3 - \frac{3x^2}{5}+\frac{6}{25} - \frac{6}{125}]+C$$

vi. $\int e^{-x} \text{Sin } 2x \, dx$

Solution

$$I = \int e^{-x}\text{Sin}^2 x\, dx = -\frac{Cos^2 x}{2}e^{-x} + \frac{1}{2}\int \text{Cos } 2x(-e^{-x})dx]$$

$$=\frac{Cos^2 x}{2}e^{-x} - \frac{1}{2}[\frac{Sin\ x\ 2x}{2}e^{-x}\int \frac{Sin\ 2x}{2}(-e^{-x})dx]$$

$$=\frac{Cos^2 x}{2}e^{-x} - \frac{Sin\ \ 2x}{4}e^{-x} - \frac{1}{4}\int Sin\ 2x(-e^{-x})dx]$$

$$I = \frac{Cos^2 x}{2}e^{-x} - \frac{Sin\ \ 2x}{4}e^{-x} + C_1 + \frac{1}{4}I$$

$$I + \frac{1}{4}I = \frac{Cos^2 x}{2}e^{-x} - \frac{Sin\ \ 2x}{4}e^{-x} + C_1$$

$$\frac{1}{5}I = -\frac{1}{2}(Cos2x)e^{-x} - \frac{1}{4}(Sin\ 2x)e^{-x} + C_1$$

$$I = \frac{4}{5} \times (-\frac{1}{2}e^{-x})[Cos\ 2x - \frac{1}{2}Sin\ 2x] + C$$

$$= -\frac{2}{5}e^{-x}[Cos\ 2x - \frac{1}{2}\ Sin\ 2x\ ] + C$$

vii. $\int e^{2x} \text{Cos } 3x \, dx$

Solution

$$I = \int e^{2x}\text{ Cos } 3x\ dx$$

$$= \frac{Sin\ 3x}{3}e^{2x} - \frac{1}{3}\int \text{Sin } 3x\ \frac{d}{dx}(e^{2x})dx$$

$= \frac{Sin\ 3xe^{2x}}{3} - \frac{2}{3}[- \frac{Cos\ 3xe^{2x}}{3} + \frac{1}{3}\int Cos\ 3x \times (e^{2x} \times 2)dx$

$,= \frac{Sin\ 3xe^{2x}}{3} + \frac{2Cos\ 3xe^{2x}}{9}\ \frac{4}{9}I + C$

$I + \frac{4}{9}I = \frac{e^{2x}}{3}[Sin\ 3x + \frac{2}{3}Cos\ 3x] + C$

$\frac{13}{9}I = \frac{e^{2x}}{3}[Sin\ 3x + \frac{2}{3}Cos\ 3x] + C$

$I = \frac{9}{13} \times \frac{e^{2x}}{3}[Sin\ 3x + \frac{2}{3}Cos\ 3x] + C$

$\int e^{2x}\ Cos\ 3x\ dx = \frac{3\ e^{2x}}{13}[Sin\ 3x + \frac{2}{3}Cos\ 3x] + C$

viii. $\int Cosec^3 x\ dx$

Solution

$I = \int Cosec^3 x\ dx = \int Cosec^2 x \times Cosec\ x\ dx$

$= -\ Cot\ x\ Cosec\ x + \int Cot\ x \frac{d}{dx}(Cosec\ x)\ dx$

$= -\ Cot\ x\ Cosec\ x + \int Cot\ x \times Cosec\ x\ Cot\ x\ dx$

$= -\ Cot\ x\ Cosec\ x - \int Cot^2 x\ Cosec\ x\ dx$

$= -\ Cot\ x\ Cosec\ x - \int (Cosec^2 x - 1)(Cosec\ x)\ dx$

$= -\ Cot\ x\ Cosec\ x - \int Cosec^3 x\ dx + \int Cosec\ x\ dx$

$I = -\ Cot\ x\ Cosec\ x - I + \ln|Cosec\ x - Cot\ x| + C_1$

$2I = -\ Cot\ x\ Cosec\ x - 1 + \ln|Cosec\ x - Cot\ x| + C_1$

$I = -\frac{1}{2}[\ Cot\ x\ Cosec\ x - \ln\ |Cosec\ x\ Cot\ x| + C$

$\int Cosec^3 x\ dx = -\frac{1}{2}[\ Cot\ x\ Cosec\ x - \ln\ |Cosec\ x\ Cot\ x| + C$

Q3. Show that

Solution

$$I = \int e^{ax} \text{Sin bx dx}$$

$$= \frac{-Cos\ bx}{b} e^{ax} - \int \frac{-Cos\ bx}{b} e^{ax} \times adx$$

$$= \frac{-Cos\ bx}{b} e^{ax} + \frac{a}{b}\int \text{Cosbx } e^{ax} dx$$

$$= \frac{-Cos\ bx}{b} e^{ax} + \frac{a}{b}[\frac{Sin\ bx\ e^{ax}}{b} - \int \frac{Sin\ bx}{b} e^{ax} \times adx]$$

$$= \frac{-Cos\ bx}{b} e^{ax} + \frac{a}{b^2} Sin\ bx\ e^{ax} - \frac{a}{b^2}\int \text{Sin bx } e^{ax}\ dx$$

$$I = \frac{-Cos\ bx}{b} e^{ax} + \frac{a}{b^2} Sin\ bx\ e^{ax} - \frac{a}{b^2} I + C$$

$$I + \frac{a^2}{b^2} I = \frac{a}{b^2} Sin\ bx\ e^{ax} - \frac{Cos\ bx}{b} e^{ax} + C$$

$$(1 + \frac{a^2}{b^2}) I = \frac{1}{b^2}[a\ Sin\ bx - b\ Cos\ b\ x] e^{ax} + C$$

$$(\frac{a^2+b^2}{b^2}) I = \frac{1}{b^2}[a\ Sin\ bx - b\ Cos\ b\ x] e^{ax} + C$$

$$I = \frac{b^2}{a^2+b^2} \times \frac{1}{b^2}[a\ Sin\ bx - b\ Cos\ b\ x] e^{ax} + C$$

$$= \frac{1}{a^2+b^2}[a\ Sin\ bx - b\ Cos\ b\ x] e^{ax} + C$$

$$= \frac{1}{a^2+b^2} e^{ax} sin(bx - \tan^{-1}(\frac{b}{a})) + C$$

Q4. Evaluate the following integrals.

i. $\int \sqrt{a^2 - x^2}\,dx$

Solution

$$I = \int \sqrt{a^2 - x^2}\,dx \quad = \int (1).\sqrt{a^2 - x^2}\,dx$$

$$= \sqrt{a^2 - x^2}\ x - \int x \times \frac{1}{2}(a^2 - x^2)^{\frac{-1}{2}}(-2x)\ dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}}\ dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}}\ dx$$

$$I = x\sqrt{a^2 - x^2} + a^2\int \frac{1}{\sqrt{a^2-x^2}}\, dx - I$$

$$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}(\frac{x}{a}) + C_1$$

$$I = \frac{1}{2}[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}(\frac{x}{a})] + C$$

Thus $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}(\frac{x}{a}) + C$

ii. $\int \sqrt{a^2 - x^2}\, dx$

Solution

$$I = \int (1)\sqrt{a^2 - x^2}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int x \frac{d}{dx}[(x^2 - a^2)^{\frac{1}{2}}]\, dx$$

$$= x\sqrt{a^2 - x^2} - \int x \times \frac{1}{2}(x^2 - a^2)^{-\frac{1}{2}}(2x)\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{x^2-a^2+a^2)}{\sqrt{x^2-a^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{x^2-a^2}{\sqrt{x^2-a^2}}\, dx - \int \frac{a^2}{\sqrt{x^2-a^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - I - a^2\int \frac{1}{\sqrt{x^2-a^2}}\, dx$$

Let $I_1 = \int \frac{1}{\sqrt{x^2-a^2}}\, dx$

Put $x = a \operatorname{Sec}\theta$, then $dx = a \operatorname{Sec}\theta \tan\theta\, d\theta$

$$I_1 = \int \frac{1}{\sqrt{(a \operatorname{Sec}\theta)^2-a^2}} \times a \operatorname{Sec}\theta \tan\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{a^2(\operatorname{Sec}^2\theta-1)}} \times a \operatorname{Sec}\theta \tan\theta\, d\theta$$

$$= \int \frac{1}{a\sqrt{\tan^2\theta}} \times a \operatorname{Sec}\theta \tan\theta\, d\theta = \int \frac{\operatorname{Sec}\theta \tan\theta}{\tan\theta}\, d\theta$$

$$= \int \operatorname{Sec}\theta\, d\theta$$

But x = $a\ Sec\ \theta \quad \Rightarrow Sec\theta \frac{x}{a}$

$\tan\theta = \sqrt{(\text{Sec}^2\theta - 1)}$

$= \sqrt{(\frac{x}{a})^2 - 1} \quad = \frac{1}{a}\sqrt{x^2 - a^2}$

$= \ln\ |\text{Sec}\ \theta + tan\theta| + C_1$

$= \ln|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}| + C_1$

$= \ln\ |x + \sqrt{x^2 - a^2}| - ln\ a + C_1$

$= \ln|x + \sqrt{x^2 - a^2}| + C_2$

(a)Becomes

$2I = x\sqrt{a^2 - x^2} - a^2 ln|x + \sqrt{x^2 - a^2}| + C_2$

$I = \frac{1}{2}x\sqrt{a^2 - x^2} - \frac{a^2}{2}\ ln|x + \sqrt{x^2 - a^2}| + C$

Hence = $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} - \frac{a^2}{2}\ ln|x + \frac{1}{2}\sqrt{x^2 - a^2}| + C$

iii. $\int \sqrt{\mathbf{4 - 5x^2}}\ \mathbf{dx}$

Solution

$I = \sqrt{5}\int \sqrt{(\frac{4}{5} - x^2)}\ dx$

$= \sqrt{5}\int \sqrt{((\frac{2}{\sqrt{5}})^2 - x^2)}\ dx$

$= \sqrt{5}\ [\frac{x}{2}\sqrt{((\frac{2}{\sqrt{5}})^2 - x^2)}\ dx + \frac{1}{2}(\frac{2}{\sqrt{5}})^2 \sin^{-1}(\frac{x}{\frac{2}{\sqrt{5}}})] + C$

$= \sqrt{5}\frac{x}{2}\sqrt{\frac{4}{5} - x^2} + \sqrt{5}\ (\frac{1}{2} \times \frac{4}{5}\sin^{-1}(\frac{\sqrt{5}\,x}{2}) + C$

$= \frac{\sqrt{5}}{2} \times \frac{\sqrt{(4-x^2)}}{\sqrt{5}} + \frac{2}{\sqrt{5}}\sin^{-1}(\frac{\sqrt{5}\,x}{2}) + C$

iv. $\int \sqrt{3 - 4x^2}\, dx$

Solution

$= \int \sqrt{(\sqrt{3})^2 - (2x)^2}$, let $a = \sqrt{3} \Rightarrow a^2 = 3$

$= \int \sqrt{a^2 - y^2}\, \frac{1}{2}\, dy$ $y = 2x \Rightarrow dy = 2dx$

$= \frac{1}{2} \sqrt{a^2 - y^2}\, dy$ $\Rightarrow dx = \frac{1}{2} dy$

$= \frac{1}{2}\left[\frac{y}{2}(2x)\sqrt{(\sqrt{3})^2 - (2x)^2} + \frac{(\sqrt{3})^2}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right] + C$

$= \frac{1}{2}\left[x\sqrt{3 - 4x^2} + \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right] + C$

v. $\int \sqrt{x^2 + 4}\, dx$

Solution

Let $I = \int (1)\sqrt{x^2 + 4}\, dx$

$= x\sqrt{x^2 + 4} - \int x \times \frac{1}{\sqrt{x^2+4}}(2x)\, dx$

$= x\sqrt{x^2 + 4} - \int \frac{x^2}{\sqrt{x^2+4}}\, dx$

$= x\sqrt{x^2 + 4} - \int \frac{x^2+4-4)}{\sqrt{x^2+4}}\, dx$

$= x\sqrt{x^2 + 4} - \int \frac{x^2+4}{\sqrt{x^2+4}}\, dx + \int \frac{4}{\sqrt{x^2+4}}\, dx$

$= x\sqrt{x^2 + 4} - I + 4\int \frac{1}{\sqrt{x^2+4}}\, dx$

Let $I_1 = \int \frac{1}{\sqrt{x^2+4}}\, dx$

Put $x = 2\tan\theta$, $then\ dx = 2\,Sec^2\theta\, d\theta$

$I_1 = \int \frac{1}{\sqrt{(2\tan\theta)^2+4}} \times 2\,Sec^2\theta\, d\theta$

$=\int \frac{1}{2\sqrt{Sec^2\theta}} \times 2\,Sec^2\theta\, d\theta = \int \frac{Sec^2\theta}{Sec\,\theta}\, d\theta$

$=\int Sec\,\theta\, d\theta$

$= \ln\,|Sec\,\theta\ + tan\theta| + C_1$

$= \sqrt{(\frac{x}{2})^2\ + 1}\ + \frac{x}{2}\ ) + C_1$

$=\ln|\frac{x^2+4}{2} + \frac{x}{2}|\ + C_1$

$=\ln\ |\frac{\sqrt{x^2+4}+x}{2}|\ + C_1$

$= \ln\ |\sqrt{x^2+4} + x| - \ ln\,2 + C_1$

$= \ln\ |x + \sqrt{x^2+4}| + C_2$

(a)Becomes

$2I = x\sqrt{a^2+4}\ +\ 4ln|x + \sqrt{x^2+4}| + C_2$

$I = \frac{1}{2}x\sqrt{a^2+4}\ + \frac{4}{2}\ ln|x + \sqrt{x^2+4}| + C$

Hence $= \int \sqrt{x^2+4}\,dx\ = \frac{x}{2}\sqrt{a^2+4}\ + 2\ ln|x + \sqrt{x^2+4}| +$

vi. $\int \mathbf{x^2 e^{ax}\, dx}$

Solution

$=\frac{e^{ax}}{a}\ .x^2\ - \int \frac{e^{ax}}{a}.(2x)\,dx$

$=\frac{e^{ax}}{a}\ .x^2\ \ - \frac{2}{a}(\frac{e^{ax}}{a} - x\int \frac{e^{ax}}{a}.1\,dx)$

$=\frac{e^{ax}}{a}\ .x^2\ \ - \frac{2}{a^2}e^{ax}.x + \frac{2}{a^2}.\frac{e^{ax}}{a} + C$

$= \frac{e^{ax}}{a}[x^2\ - \frac{2}{a}x + \frac{2}{a^2}] + C$

Q5. Evaluate the following integrals

i. $\int e^x(\frac{1}{x} + \ln x)\, dx$

Solution

Let $\int (x) = \ln x$, then $\int (x)\, x = \frac{1}{x}$

$\int e^x(\frac{1}{x} + \ln x)\, dx = \int e^x\, [f(x)+f'(x)]dx$

$= \int \frac{d}{dx}(e^x f(x))dx$

$= e^x f(x)+C$

$= e^x \ln x + C$

ii. $\int e^x(\text{Cos x} + \text{Sin x})dx = \int e^x(\text{Sin} + \text{Cos x})dx$

Solution

$f(x)$ = Sin x then $f'(x)$ = Cos x. So

$=\int e^x(Cos\ x + Sin\ x)dx = \int e^x[f(x)+ f'(x)]dx$

$=\int \frac{d}{dx}(e^x f(x))dx$

$=e^x f(x)+C$

$= e^x Sin\ x + C$

iii. $\int e^{ax}[a\ \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}}]dx$

Solution

Let $\sec^{-1} x = f(x)$ then $f'(x) = \frac{1}{x\sqrt{x^2-1}}$

Thus $\int e^{ax}[a\ \sec^{-1} x + \frac{1}{\ldots}]dx$

$=\int \frac{d}{dx}(e^{ax} f(x)+ f'(x))dx$

$= e^{ax} \sec^{-1} x + C$

iv. $\int e^{3x}(\frac{3 \text{ Sin x} - \text{Cos x}}{\text{Sin}^2 x})dx$

Solution

$= \int e^{3x}(\frac{3 \text{ Sin x}}{\text{Sin}^2 x} - \frac{\cos x}{\sin^2 x})dx$

$=\int e^{3x}[3\ Cosec\ x - Cot\ x\ Cosec\ x\ ]$

$=\int e^{3x} Cosec\ x - \int e^{3x}\ Cot\ x\ Cosec$

Integrate by first integral by taking Cosec x as first function

$=3 \text{ Cosec x}(\frac{1}{3} e^{3x}) - 3\int (\frac{1}{3} e^{3x})(-Cot\ x\ Cosec\ x)\text{-} \int e^{3x}\ Cot\ x\ Cosec\ x$

$=e^{3x}\ Cosec\ x + \int e^{3x}\ Cot\ x\ Cosec\ x - \int e^{3x}\ Cot\ x\ Cosec\ x$

$= e^{3x}\ Cot\ x\ Cosec\ x + C$

v. $\int e^{2x}(-\text{Sin x} + 2 \text{ Cos x})\ dx$

Solution

$= \int e^{2x}(-\text{Sin x} + 2 \text{ Cos x})\ dx$

$= \int e^{2x}(2 \text{ Cos x} - \text{Sin x}))\ dx$

$= 2\int e^{2x}\ Cos\ x\ dx - \int e^{2x}\ Sin\ x\ dx$

Integrate by parts first integral by taking Cos x as first function

$= 2[\text{Cos} (\frac{1}{2} e^{2x}) - \int (\frac{1}{2} e^{2x})(-Sin\ x)dx] - \int e^{2x}\ Sin\ x\ dx$

$=e^{2x}\ Cos\ x + \int e^{2x}\ Sin\ x\ dx - \int e^{2x}\ Sin\ x\ dx$

$=e^{2x}\ Cos\ x + C$

vi. $\int \frac{xe^{x}}{(1+x)^{2}}$ **dx**

Solution

$$= \int e^{x}[\frac{1}{1+x} - \frac{1}{(1+x)^{2}}]dx$$

Let $f(x) = \frac{1}{1+x}$ then $f'(x) = -\frac{1}{(1+x)^{2}}$

So $\int \frac{xe^{x}}{(1+x)^{2}}dx$ $\int e^{x}[f(x)+ f'(x)]$

$$= \int \frac{d}{dx}(e^{x}f(x))dx$$

$$=e^{x}f(x)+C$$

$$= e^{x}\frac{1}{1+x} + C$$

vii. $\int e^{-x}[\text{Cos } x - \text{Sin } x]$ **dx**

Solution

$$= \int e^{-x}[\text{Cos } x - \text{Sin } x]\, dx$$

$$=\int e^{-x}\, Cos\, x\, dx - \int e^{-x}\, Sin\, x\, dx$$

Integrate by parts first integral by taking e^{-x} *as first function*

$$=e^{-x}(Sin\, x) - \int \text{Sin } x \,.\, e^{-x}(-1)dx - \int e^{-x}\, Sin\, x\, dx$$

$$=e^{-x}\, Sin\, x + \int e^{-x}\, Sin\, x\, dx - \int e^{-x}\, Sin\, x\, dx$$

$$=e^{-x}\, Sin\, x + C$$

viii. $\int \frac{e^{m}\tan^{-1} x}{1+x^{2}}$ **dx**

Solution

Let $y = \tan^{-1} x$ then

dy 1

dy $= (\frac{1}{1+x^2})dx$

thus $\int \frac{e^m \tan^{-1} x}{1+x^2} dx$

$= \int e^m \tan^{-1} x(\frac{1}{1+x^2})\, dx$

$= \int e^m . dy$

$= \frac{1}{m} . e^m \tan^{-1} x + C$

ix $\int \frac{2x}{1-Sin\, x}\, dx$

Solution

$= \int \frac{2x}{1-Cos\left(\frac{\pi}{2}-x\right)}\, dx \;=\; \int \frac{2x}{1-sin^2\left(\frac{\pi}{2}-\frac{x}{2}\right)}\, dx$

$= \int x\, Cosec\, 2\, \left(\frac{\pi}{2} - \frac{x}{2}\right)\, dx$

$= \left(\frac{\pi}{2} - \frac{x}{2}\right) = u$

- $\frac{1}{2}\, dx = du \;\Rightarrow\; dx = -\, du$

Thus $\int x\, Cosec^2 \left(\frac{\pi}{2} - \frac{x}{2}\right)\, dx$

$= \int \left(\frac{\pi}{2} - 2u\right)\; Cosec^2\, ux \,-\, 2du$

$= \int \left(2u - \frac{\pi}{2}\right)\; Cosec^2\, u\, dx$

$= \int\; (-Cot\, u\,) \times \left(2u - \frac{\pi}{2}\right)$

$= 2[\;(\text{-Cot } u\,)\left(2u - \frac{\pi}{2}\right) + 2\, ln\, |sin\, u\,| + C$

$= 2x\, \text{Cot}\left(\frac{\pi}{4} - \frac{x}{2}\right) + 4\, \ln\, |\,\text{Sin}\left(\frac{\pi}{4} - \frac{x}{2}\right)| + C$

x. $\int \frac{e^x(1+x)}{(2+x)}\, dx$

$= \int \frac{e^x(1+x-1)}{(2+x)^2}\, dx$

$= \int e^x[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2}]\, dx$

$= \int e^x[\frac{1}{(2+x)} - \frac{1}{(2+x)^2}]\, dx$

$f(x) = \frac{1}{2+x}$

thus $f'(x) = \frac{d}{dx}\left[\frac{1}{2+x}\right] = \frac{d}{dx}[(2+x)^{-1}]$

$= (-1)\,(2+x)^{-2} \times 1$

$= -\frac{1}{(2+x)^2}$

Thus $\int e^x[\frac{1}{(2+x)} - \frac{1}{(2+x)^2}]\, dx$

$= \int e^x[\, f(x) + f'(x)]dx$

$= e^x f(x) + C$

$= e^x \frac{1}{(2+x)} + C = \frac{e^x}{(2+x)} + C$

xi. $\int (\frac{1-Sin\, x}{1-Cos\, x})\ e^x$ **dx**

Solution

$= \int (\frac{1-Sin\, x}{2\, Sin^2 \frac{x}{2}})\ e^x\ dx$

$= \int e^x[\frac{1}{2\, Sin^2 \frac{x}{2}} - \frac{sin\, x}{2\, Sin^2 \frac{x}{2}}]\, dx$

$= \int [\frac{1}{2}\, Cosec^2\, \frac{x}{2} - Cot\ \frac{x}{2}]e^x\, dx$

$= \int e^x[-Cot\ \frac{x}{2} - Cosec^2\ \frac{x}{2}]\ e^x\, dx$

$f(x) = -Cot\ \frac{x}{2}$

Thus we have

thus $f'(x) = \frac{1}{2} Cosec^2 \frac{x}{2}$

$= \int e^x[-Cot \frac{x}{2} - Cosec^2 \frac{x}{2}] \; dx$

$= - \int e^x[f(x) + f'(x)] \, dx$

$= -e^x f(x) + C$

$= -e^x \text{Cot} \frac{x}{2} + C$