Exercise 3.2

Q1. Evaluate the following indefinite integrals.

i. $\int(3x^2 - 2x + 1)dx$

Solution

$= \int(3x^2 - 2x + 1)dx$

$= \int(3x^2)dx - \int(2xdx + \int 1dx$

$= 3\int(x^2)dx - 2\int x\, dx + \int dx$

$= 3\left(\frac{x^{2+1}}{2+1}\right) - 2\left(\frac{x^{1+1}}{1+1}\right) + x + c$

$= 3\left(\frac{x^3}{2}\right) - 2\left(\frac{x^2}{2}\right) + x + c$

$= x^3 - x^2 + x + c$

ii. $\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx, x > 0$

Solution

$\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$

$= \int\left(x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{2}}}\right)dx$

$= \int\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$

$= \int\left(x^{\frac{1}{2}}\right)dx + \int \left(x^{-\frac{1}{2}}\right)dx \qquad \therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

iii. $\int \mathbf{x}(\sqrt{x+1})\, dx, x > 0$

Solution

$$\int x(\sqrt{x+1})\, dx$$

$$= \int x(x+1)^{\frac{1}{2}} dx$$

$$= \int (x^{\frac{3}{2}} + x) dx$$

$$= \int x^{\frac{3}{2}} dx + \int x\, dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{\frac{1}{2}+1} + C$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c$$

$$= \frac{2}{5}x^{\frac{3}{2}} + \frac{1}{2}x^2 + c$$

iv. $\int (\mathbf{2x+3})^{\frac{1}{2}} dx$

Solution

$$\int (2x+3)^{\frac{1}{2}} dx$$

$$= \frac{1}{2}\int (2x+3)^{\frac{1}{2}}\, 2\, dx$$

$$= \frac{1}{2}\frac{(2x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

As $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

$$= \frac{1}{2}\frac{(2x+3)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{3}(2x + 1)^{\frac{3}{2}} + C$$

v. $\int(\sqrt{x} + 1)^2\, dx$

Solution

$$\int(\sqrt{x} + 1)^2\, dx$$

$$= \int((\sqrt{x})^2 + 2(\sqrt{x})(1) + (1)^2)dx$$

$$= \int(x + 2\sqrt{x} + 1)dx$$

$$= \int(x + 2x^{\frac{1}{2}} + 1)dx$$

$$= \int(x)dx + \int 2x^{\frac{1}{2}}dx + \int dx$$

$$= \frac{x^{1+1}}{1+1} + 2\int \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + x + C$$

$$= \frac{x^2}{2} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + x + C$$

$$= \frac{x^2}{2} + \frac{2.2}{3}x^{\frac{3}{2}} + x + C$$

$$= \frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}} + x + C$$

i. $\int(\sqrt{x} - \frac{1}{\sqrt{x}})^2\, dx$

Solution

$$\int(\sqrt{x} - \frac{1}{\sqrt{x}})^2\, dx \qquad \therefore (a - b)^2 = a^2 - 2ab + b^2$$

$$= \int((x)^2 - 2(\sqrt{x})(\frac{1}{\sqrt{x}}) + (\frac{1}{\sqrt{x}})^2)dx$$

$$= \int(x - 2 + \frac{1}{x})\, dx$$

$$= \int x\, dx - 2\int dx + \int \frac{1}{x} dx$$

$= \frac{x^2}{2} - 2x + \ln x + C$

vii. $\int \frac{3x+2}{\sqrt{x}} dx, x > 0$

Solution

$\int \frac{3x+2}{\sqrt{x}} dx$

$= \int (\frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}}) dx$

$= \int (3\sqrt{x} + \frac{2}{\sqrt{x}}) dx$

$= \int (3\, x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}) dx$

$= \int 3x^{\frac{1}{2}}\, dx + \int 2x^{-\frac{1}{2}} dx$

$= 3\int x^{\frac{1}{2}}\, dx + 2\int x^{-\frac{1}{2}} dx$

$= 3\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + 2\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + C$

$= 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{6}{3} x^{\frac{3}{2}} + 2.2x^{\frac{1}{2}} + C$

$= 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + C$

viii. $\int \frac{\sqrt{y}\,(y+1)}{y} dy, y > 0$

Solution

$\int \frac{\sqrt{y}\,(y+1)}{y} dy$

$= \int \frac{y^{\frac{1}{2}}(y+1)}{y} dy$

$= \int (y^{\frac{1}{2}} + y^{-\frac{1}{2}})dy$

$= \int y^{\frac{1}{2}}dy + \int y^{-\frac{1}{2}}dy$

$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$+C

$= \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^{\frac{1}{2}}}{\frac{1}{2}}$+C

$= \frac{2}{3}\, y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + C$

ix $\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}}\mathbf{d}\boldsymbol{\theta}\,, \boldsymbol{\theta} > \mathbf{0}$

Solution

$\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}}\,\mathrm{d}\theta$

$= \int (\frac{\theta - 2\sqrt{\theta}+1}{\sqrt{\theta}})\mathrm{d}\theta$

$= \int (\frac{\theta}{\sqrt{\theta}} - \frac{2\sqrt{\theta}}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}})\mathrm{d}\theta$

$= \int (\sqrt{\theta} - 2 + \frac{1}{\sqrt{\theta}})\mathrm{d}\theta$

$= \int (\theta^{\frac{1}{2}} - 2 + \theta^{-\frac{1}{2}})\mathrm{d}\theta$

$= \int (\theta^{\frac{1}{2}})\mathrm{d}\theta - \int 2\mathrm{d}\theta + \int (\theta^{-\frac{1}{2}})\mathrm{d}\theta$

$= \int (\theta^{\frac{1}{2}})\mathrm{d}\theta - 2\int \mathrm{d}\theta + \int (\theta^{-\frac{1}{2}})\mathrm{d}\theta$

$= \frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 2\theta + \frac{\theta^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$+C

$= \frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} - 2\theta + \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}}$+C

$= \frac{2}{3}\,\theta^{\frac{3}{2}} - 2\theta + 2\theta^{\frac{1}{2}} + C$

x. $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}} \mathbf{d}x, x > 0$

Solution

$= \int \frac{(1-\sqrt{x})^2}{\sqrt{x}} dx$

$= \int (\frac{1+x-2\sqrt{x}}{\sqrt{x}}) dx$

$= \int (\frac{dx}{\sqrt{x}} + \int \frac{x}{\sqrt{x}} dx - 2\int \frac{\sqrt{x}}{\sqrt{x}} dx$

$= \int x \frac{-1}{2} dx + \int x \frac{1-1}{2} dx - 2\int dx$

$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \int x^{\frac{1}{2}} dx - 2.x + C$

$= 2x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2x + C$

$= 2x^{\frac{1}{2}} + 2x^{\frac{3}{2}} - 2x + C$

xi. $\int \frac{e^{2x} + e^x}{e^x} \mathbf{dx}$

Solution

$= \int \frac{e^{2x}}{e^x} + \frac{e^x}{e^x} dx$

$= \int (e^x + 1) dx$

$= \int e^x dx \int 1 dx$

$= e^x + x + C$

Q2 Evaluate

i $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$ $\{x + a > 0\}$ **{x+ b>0}**

Solution

Rationalize

$$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})dx}{(\sqrt{x+a})^2-(\sqrt{x+b})^2}$$

(a+b)(a-b) = $a^2 - b^2$

$$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{(x+a)-(x-b)}\,dx$$

$$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{x+a-x-b}\,dx$$

$$= \int \frac{(\sqrt{x+a}-\sqrt{x})}{a-b}\,dx$$

$$= \frac{1}{a-b}\int (\sqrt{x+a} - \sqrt{x+b})dx$$

$$= \frac{1}{a-b}[\int (x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}]\,dx$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x+b)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$$

$$= \frac{1}{a-b}\left[\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x+b)^{\frac{3}{2}}\right] + C$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$$

ii. $\int \frac{1-x^2}{1+x^2}\,dx$

Solution

$$\int \frac{1-x^2}{1+x^2}\,dx$$

It can be written as

$$= \int \frac{-1-x^2+2}{\ }\,dx$$

$= \int (\frac{2}{1+x^2} - \frac{1+x^2}{1+x^2})\, dx$

$= 2\int \frac{1}{1+x^2}\, dx - \int 1\, dx$

$= 2\tan^{-1} x - x + C$

iii $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}\ \mathbf{x > 0, a > 0}$

$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

Rationalize it

$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x}} \times \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}$

$= \int \frac{(\sqrt{x+a}-\sqrt{x})dx}{(\sqrt{x+a})^2-(\sqrt{x})^2}$

$= \int \frac{(\sqrt{x+a}-\sqrt{x})}{x+a-a}\, dx$

$= \int \frac{(\sqrt{x+a}-\sqrt{x})}{x}\, dx$

$= \frac{1}{a}\int (\sqrt{x+a} - \sqrt{x})dx$

$= \frac{1}{a}[\int (x+a)^{\frac{1}{2}} - x^{\frac{1}{2}}]\, dx$

$= \frac{1}{a}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]$

$= \frac{1}{a}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$= \frac{1}{a}\left[\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right] + C$

$= \frac{1}{3a}\left[(x+a)^{\frac{3}{2}} - x^{\frac{3}{2}}\right] + C$

iv $\int (a-2x)^{\frac{3}{2}}\, \mathbf{dx}$

$$\int (a-2x)^{\frac{3}{2}}\,dx$$

Multiply and divide by -2

$$= -\frac{1}{2}\int (a-2x)^{\frac{3}{2}}(-2)\,dx$$

$$= -\frac{1}{2}\int (a-2x)^{\frac{3}{2}}(-2\,dx)$$

$$= -\frac{1}{2}\frac{(a-2x)^{\frac{3}{2}+1}}{\frac{3}{2}+1}+C$$

$$= -\frac{1}{2}\frac{(a-2x)^{\frac{5}{2}}}{\frac{5}{2}}+C$$

$$= -\frac{1}{5}(a-2x)^{\frac{5}{2}}+C$$

v $\int \frac{(1+e^x)^3}{e^x}\,dx$

Solution

$$\int \frac{(1+e^x)^3}{e^x}\,dx$$

$$= \int \frac{1+e^{3x}+3e^x(1+e^x)}{e^x}\,dx$$

$$(a+b)^3 = a^3+b^3+3ab(a+b)$$

$$= \int \frac{1+3e^x+3e^{2x}+e^{3x}}{e^x}\,dx$$

$$= \int \left(\frac{1}{e^x}+\frac{3e^x}{e^x}+\frac{3e^{2x}}{e^x}+\frac{e^{3x}}{e^x}\right)dx$$

$$= \int (e^{-x}+3+3e^x+e^{2x})\,dx$$

$$= \int e^{-x}\,dx + 3\int dx + 3\int e^x dx + \int e^{2x}\,dx$$

$$= \frac{e^{-x}}{-1}+3x+3e^x+\frac{e^{2x}}{2}+C \qquad \therefore \int e^{2x}\,dx = \frac{e^{ax}}{a}$$

$$= -e^{-x}+3x+3e^x+\frac{1}{2}e^{2x}+C$$

vi. $\int \sin(a+b)x\,dx$

Solution

$\int \sin(a+b)x\,dx$

Multiply divide by a +b

$= \frac{1}{a+b}\int \sin(a+b)x\,(a+b)\,dx$

$= -\frac{1}{a+b}\int -\sin(a+b)x\,(a+b)\,dx$

$= -\frac{1}{a+b}\cos(a+b)x + C$

vii $\int \sqrt{1-Cos\,2x}\;dx \quad (1 - Cos2x > 0)$

Solution

$\int \sqrt{1-Cos\,2x}\;dx$

$= \int \sqrt{2sin^2x}\;dx \qquad 1 - \cos 2x = 2\,sin^2x$

$= \sqrt{2}\int \sqrt{sin^2x}\;dx$

$= \sqrt{2}\int \sin x\;dx$

$= -\sqrt{2}\,\cos x + C \qquad \int \sin x\;dx = -\cos x$

viii. $\int (\ln x)\times \frac{1}{x}dx$

Solution

$\int (\ln x)\times \frac{1}{x}dx$

$= \frac{(\ln x)^{1+1}}{1+1} + C$

$= \frac{(\ln x)^2}{2} + C$

Solution

$\int sin^2x$ dx $\quad\therefore\quad sin^2x = \frac{1-Cos\ 2x}{2}$

$= \int (\frac{1-Cos\ 2x}{2})dx$

$= \int (\frac{1}{2} - \frac{1}{2}Cos\ 2x)dx$

$= \int \frac{1}{2}\ dx - \int \frac{1}{2}Cos\ 2xdx$

$= \frac{1}{2}\int\ dx - \frac{1}{2}\int Cos\ 2xdx$

$= \frac{1}{2}x - \frac{1}{2}\frac{sin\ 2x}{2}$ +C

$\int$ Cos ax dx $= \frac{sin\ ax}{a}$

$= \frac{1}{2}x - \frac{1}{4}sin\ 2x$ +C

$= \frac{1}{2}(x - \frac{sin\ 2x}{2}$ +C

x. $\int \frac{1}{1+Cos\ x}$ **dx** $(\frac{\pi}{2} < x < \frac{\pi}{2})$

Solution

$\int \frac{1}{1+Cos\ x}$ dx

$\Rightarrow$ 1 +Cos x = $2Cos^2\frac{x}{2}$

$= \int \frac{1}{\frac{2Cos\ x^2}{2}}$ dx

$= \frac{1}{2}\int \frac{1}{2Cos^2\frac{x}{2}}$ dx

$= \frac{1}{2}\int sec^2\frac{x}{2}$ dx $\quad\therefore \int sec^2x$ dx = tan x

$= \frac{1}{2}\frac{tan\frac{x}{2}}{\frac{1}{2}}$ +C

$= tan\frac{x}{2} + C$

xi. $\int \frac{ax+b}{ax^2+2bx+C}dx$

Solution

$$\int \frac{ax+b}{ax^2+2bx+C}dx$$

$$= \int (ax^2 + 2bx + C)^{-1}(ax+b)dx$$

Multiply and divide by 2

$$= \frac{1}{2}\int (ax^2 + 2bx + C)^{-1}\, 2(ax+b)dx$$

$$= \frac{1}{2}\int (ax^2 + 2bx + C)^{-1}\,(2ax+2b)dx$$

$$= \frac{1}{2}\int \left(\frac{2ax+2b}{ax^2+2bx+C}\right)dx$$

$$= \frac{1}{2}\ln \quad ax^2 + 2bx + C \;+ C_1$$

xii. $\int$ **Cos3xsin2x dx**

Solution

$$\int \text{Cos3xsin2x dx}$$

Multiply and divide by 2

$$= \frac{1}{2}\int 2\text{Cos3xsin2x dx} \quad \therefore 2\text{Cos ax sin bx} = \sin(a+b)+\sin+(a-b)$$

$$= \frac{1}{2}\int [\sin(3x + 2x) + \sin(3x - 2x)]dx$$

$$= \frac{1}{2}\int [\sin 5x + \sin x]\, dx$$

$$= \frac{1}{2}\int \sin 5x\, dx + \frac{1}{2}\int \sin x\, dx \quad \therefore \quad \int \sin ax\, dx = \frac{-Cos\, ax}{a}$$

$$= \frac{1}{2}\left(\frac{-Cos\, 5x}{5}\right) + \frac{1}{2}(-Cos\, x) + C$$

$$= -\frac{1}{2}\left[\frac{Cos\, 5x}{5} + Cos\, x\right] + C$$

Solution

$\int \frac{Cos2x-1}{1+Cos2x}\,dx$

$= -\int \frac{1-\text{Cos } 2x}{1+\text{ Cos } 2x}\,dx$

$= -\int \frac{2+sin^{2x}}{2\,Cos^{2x}}\,dx$

$= -\int \tan^2 x\,dx$

$= -\int (\sec^2 x - 1)\,dx$

$= -\int \sec^2 x\,dx + \int dx$

$= -\tan x + x + C$

xiv. $\int \tan^2 x\,dx$

Solution

$\int \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\,dx$

$= \int \sec^2 x\,dx - \int dx$

$= \tan x - x + C$