Solved Exercise 2.9

Q1. Determine the interval in which f is increasing or decreasing for the domain mentioned in each case.

i. $\boldsymbol{f(x) = Sin\ x}$ $\qquad\qquad$ $\boldsymbol{X \in |-\pi, \pi|}$

Solution

$f(x) = Sin\ x$

$f`(x) = Cos\ x$

$As\ f`(x) = Cos\ x - ive \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

So, f is decreasing on intervals $\left[-\pi, \frac{\pi}{2}\right] and \left[\frac{\pi}{2}, \pi\right]$

$But f`(x) = Cos\ x\ is + ive\ for\ \left[\frac{-\pi}{2}, \pi\right]$

So, f is increasing on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **Ans**

ii. $\boldsymbol{f(x) = Cosx}$ $\qquad$ $\boldsymbol{x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$

Solution

$f`(x) = -Sinx$

$As\ Sin\ x\ is - ive\ for -\frac{\pi}{2} < x < 0$

$So\ f\ is\ increasing\ on\ the\ interval\ \left[-\frac{\pi}{2}, 0\right]$

$\Rightarrow \quad f`(x) \ = -Sin\ x\ is\ negative\ for\ 0 < x < \frac{\pi}{2}$

$So\ f\ is\ decreasing\ on\ the\ interval\ \left[0, \frac{\pi}{2}\right]$ **Ans**

iii. $f(x) = 4 - x^2$ $\quad x \in |-2, 2|$

Solution

$$f(x) = 4 - x^2$$

$$f`(x) = -2x$$

As f`(x) is +ve when x is -ive

$\Rightarrow$ $f\ ia\ increasing\ on\ the\ interval\ |-2{,}0|$

As f`(x) is +ve when x is positive

$\Rightarrow$ $f\ is\ decreasing\ on\ the\ interval\ |0{,}2|$ **Ans**

iv. $f(x) = x^2 + 3x + 2$

Solution

$$f`(x) = 2x + 3$$

$As\ f`(x) = 2x + 3\ is\ negative\ for -4 < x < \frac{-3}{2}$

$So\ f\ is\ decreasing\ on\ the\ interval\ \left[-4, \frac{-3}{2}\right]$

$As\ f`(x) = 2x + 3\ is + ive\ for\ \frac{-3}{2} < x < 1$

Thus, f is increasing on the interval $\left[-\frac{3}{2}, 1\right]$ **Ans**

Q2. Find the extreme value for the following function defined as

i. $f(x) = 1 - x^3$

Solution

$$f(x) = 1 - x^3$$

$$\Rightarrow \quad f`(x) = -3x^2$$

$$Put\ f`(x) = 0$$

$$-3x^2 = 0$$

$$\Rightarrow \quad x = 0$$

$$f``(x) = -6x$$

$$f``(0) = 0$$

The 2nd derivation does not help in determining the extreme values

$$y \quad = f`(x) = 1 + x^3$$

$$= 1 - 0 = 1$$

As 1st derivative does not change at x=0, or x∴ $(0,1)$ is the pt. of inflexion

ii. $\mathbf{f(x) = x^2 - x - 2}$

Solution

$$f`(x) = 2x - 1$$

$$f`(x) \ = 0$$

$$\Rightarrow \ 2x - 1 = 1$$

$$\Rightarrow \quad x = \frac{1}{2}$$

$$Let\ x = \frac{1}{2} - t\ on\ t \ \to 0$$

$$f\left(\frac{1}{2} - t\right) = 2\left(\frac{1}{2} - t\right) - 1$$

$$= 1 - 2t - 1 = -2t < u$$

$$f\left(\frac{1}{2} + t\right) = 2\left(\frac{1}{2} + t\right) - 1$$

$$= 1 + 2t - 1 = 2t > 0$$

$$f'(x) < 0 \; before \; x = \frac{1}{2} \; and$$

$$f'(x) > 0 \; after \; x = \frac{1}{2}$$

Hence f has relative minima at $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right) - 2 = \frac{1}{4} - \frac{1}{4} - 2$$

$$= \frac{1-2-8}{4} = -\frac{9}{4}$$

$Note \; that \; f''(x) = 2 > 0$ **Ans**

iii. $\mathbf{f(x) = 5x^2 - 6x + 2}$

Solution

$$\Rightarrow \quad f'(x) = 10x - 6$$

$$f''(x) = 10 > 0$$

$$f'(x) = 10x - 6 = 0$$

$$10x = 6$$

$$x = \frac{6}{10}$$

$$x = \frac{3}{5}$$

$$Let \; x = \frac{3}{5} - \epsilon \; as \; \epsilon \; 0$$

$$f'\left(\frac{3}{5} - \epsilon\right) = 10\left(\frac{3}{5} - \epsilon\right) - 6$$

$$= 6 - 10\epsilon - 6$$

$$= -10\epsilon < 0$$

$$Put \; x = \frac{3}{5} + \epsilon$$

$$f`\left(\frac{3}{5}+\epsilon\right) = 10\left(\frac{3}{5}+\epsilon\right) - 6$$

$$= 6 + 10\epsilon - 6$$

$$= 10\epsilon > 0$$

$f`(x) < 0$ *before* $x = \frac{3}{5}$ *and* $f`(x) > 0$ *after* $x = \frac{3}{5}$

Hence f has realtive minima at $x = \frac{3}{5}$

$$f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2$$

$$= 5 \times \frac{9}{25} - \frac{18}{5} + 2 \qquad = \frac{9}{5} - \frac{18}{5} + 2$$

$$= \frac{9-18+10}{5} \qquad = \frac{19-18}{5}$$

$= \frac{1}{5}$ **Ans**

iv. $f(x) = 3x^2$

Solution

$$f(x) = 3x^2$$

$$\Rightarrow \quad f`(x) = 6x$$

Put $f`(x) = 0$

$$\Rightarrow \quad 6x = 0$$

$$x = 0$$

where ϵ *is very small + ive number*

Now let $x = 0-\epsilon$

Then $f`(0-\epsilon) = 6(0-\epsilon) = -6\epsilon < 0$

$$f`(0+\epsilon) = 6(0+\epsilon) = 6\epsilon > 0$$

$f`(x) < 0$ *before* $x = 0$ *and*

$$f`(x) < 0 \; before \; x = 0$$

$Hence \; f \; has \; relative \; Minima \; at \qquad x = 0$

$$f(0) = 3(0)^2 = 0 \; has \; relative \; minima \; at \; x = 0.$$ **Ans**

v. $\boldsymbol{f(x) = 3x^2 - 4x + 5}$

Solution

$$f(x) = 3x^2 - 4x + 5$$

$$\Rightarrow \quad f`(x) = 6x - 4$$

$$Now \; f`(x) = 0$$

$$\Rightarrow \quad 6x - 4 = 0$$

$$x = \frac{2}{3}$$

$But \; f`` = 6 \; and \; 6 > 0 \; which \; shows \; that \; f \; has \; relative \; minima \; at \; x = \frac{2}{3}$

$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 5$$

$$= 3 \times \frac{4}{3} + 5 = \frac{4}{3} - \frac{8}{3} + 5 = \frac{4-8+15}{3}$$

$$= \frac{11}{3}$$ **Ans**

vi. $\boldsymbol{f(x) = 2x^3 - 2x^3 - 36x + 3}$

Solution

$$\Rightarrow \quad f`(x) = 6x^2 - 4x - 36$$

$$\Rightarrow \quad f`(x) = 0$$

$$6x^2 - 4x - 36 = 0$$

$3x^2 - 2x - 18 = 0$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-18)}}{6}$$

$$x = \frac{2 \pm \sqrt{220}}{6}$$

$$= \frac{1 \pm \sqrt{55}}{3}$$

$$f``(x) = 12x - 4 \quad = \quad 4(3x - 1)$$

$$f``(x) = \frac{1+\sqrt{55}}{3} \quad = \quad 4\left[3\left(\frac{1+\sqrt{55}}{3}\right) - 1\right]$$

$$= 4\left(1 + \sqrt{55} - 1\right)$$

$$= 4\sqrt{55} > 0$$

$\textit{which shows that } f \textit{ has relative minima at } x = \frac{1+\sqrt{55}}{3} i, e$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - 36\left(\frac{1+\sqrt{55}}{3}\right) + 3$$

$$= \frac{2}{27}(1 + 3\sqrt{55} + 3 + 55 + 55\sqrt{55})$$

$$-\frac{2}{9}\left(1 + 2\sqrt{55} + 55\right) - 12\left(1 + \sqrt{55}\right) + 3$$

$$= \frac{2}{27}\left(166 + 58\sqrt{55}\right) - \frac{2}{9}\left(56 + 2\sqrt{55}\right) - 12\left(1 + \sqrt{55}\right) + 3$$

$$= \frac{322}{27} + \frac{116}{27}\sqrt{55} - \frac{4}{9}\sqrt{55} - 12 - 12\sqrt{55} + 3$$

$$= \left(\frac{332}{27} - \frac{112}{9} - 9\right) + \left(\frac{116}{27} - \frac{4}{9} - 12\right)\sqrt{55}$$

$$= -\frac{1}{2}\left(247 + 220\sqrt{55}\right)$$

$$At\ f(x) = \frac{1-\sqrt{55}}{3} \Rightarrow f(x) = 4\left(3\left(\frac{1-\sqrt{55}}{3}\right) - 1\right) = -4\sqrt{55} < 0$$

$\textit{Which show } f \textit{ has relative maxima at } \frac{1-\sqrt{55}}{3}$

$$f``(-2) = 12\left(-\sqrt{2}\right)^2 - 8$$

$= 24 - 8$

$= 16$

$Which\ shows\ that\ f\ has\ relative\ minima\ at\ x = -\sqrt{2}$

$f(-2) = \left(-\sqrt{2}\right)^4 - 4\left(-\sqrt{2}\right)^2 = 4 - 8 = -9$ **Ans**

vii. $\boldsymbol{f(x) = x^4 - 4x^2}$

Solution

$f`(x) = 4x^2 - 8x$

$let\ f``(x) = 0$

$\Rightarrow 4x^3 - 8x = 0$

$4x\ = 0\ or\ x^2 - 2 = 0$

$x\ = 0\ or\ x = \pm\sqrt{2}$

$f``(x)\ = 12x^2 - 8$

$f``(x)\ \ = 12(0) - 8 = -8 < 0$

$f`(x)\ is\ maxmina\ at\ x = 0$

$f(0) = 0 - 0 + 0$

Now

$f`\left(\sqrt{2}\right) = 12\left(\sqrt{2}\right)^2 - 8 + 16 < 0$

$f`\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^4 - 4\left(\sqrt{2}\right)^2 = -4$

$f`\left(-\sqrt{2}\right) = 12\left(-\sqrt{2}\right)^4 - 8\left(\sqrt{2}\right)^2 = 16 > 0$

$Hence\ f(x) has\ a\ minima\ at\ x = -\sqrt{2}\ i.e$

$f\left(\sqrt{2}\right) = \left(-\sqrt{2}\right)^4 - 4\left(-\sqrt{2}\right)^2$

$= 4 - 8 = -4$ **Ans**

viii. $f(x) = (x-2)^2(x-1)$

Solution

$\Rightarrow \quad f`(x) = 2(x-2)(x-1) + (x-2)^2 = (x-2)[2x - 2 + x - 2]$

$f`(x) = (x-2)(3x-4)$

$f`(x) = 0 \Rightarrow (x-2)(3x-4) = 0$

$\Rightarrow \quad x = 2. \qquad x = \frac{4}{3}$

$Also\ f`(x) = 1 + (3x-4) + (x-2)(3) = \quad 3x - 4 + 3x - 6$

$= 6x - 0$

$x = 2$

$f``(x) = 6(2) - 10 = 270$

$\Rightarrow f\ has\ relative\ minima\ at\ n = 2$

$f(2) = (2-2)^2(2-1) = 0 \times 1 = 0$

$f\left(\frac{4}{3}\right) \quad = 6\left(\frac{4}{3}\right) - 10 = 8 - 10 = -2 < 0$

$f``(x) < 0\ where \qquad x = \frac{4}{3}$

$Hence\ f\ has\ relative\ maxima\ at\ x = \frac{4}{3}$

$f\left(\frac{4}{3}\right) = \left(-\frac{2}{3}\right)^2\left(\frac{1}{3}\right)$

$= \left(\frac{4}{9}\right)\left(\frac{1}{3}\right)$

$= \frac{4}{27}$ **Ans**

ix. $f(x) = 5 + 3x - x^3$

Solution

$\Rightarrow \quad f`(x) = 3 - 3x^2$

$f(x) = 0$

$\Rightarrow \quad 3 - 3x^2 = 0$

$x^2 = 1$

$x = \pm 1$

$Now\ f``(x) = -6x$

$at \quad x = 1$

$f``(1) = -6(1) = -6 < 0$

$\Rightarrow \quad f\ has\ relative\ maxima\ at\ x = 1$

$Now\ f``(-1) = -6(-1) = 67°$

$\Rightarrow \quad f\ has\ relative\ minima\ at\ x = 1$

$f(-1) = 5 + 3(-1) - (-1)^3 = 5 - 3 + 1 = 3$ **Ans**

Q3. Find the maximum and minimum value of the function defined by the following equations occurring in the interval $[0, 2\pi]$, $f(x) = sinx + cosx$

Solution

$f(x) = sinx + cosx$

$f^{'}(x) = cosx - sinx$

Now $f(x) = 0$

$\Rightarrow \quad cosx - sinx = 0$

$\Rightarrow \quad cosx = sinx$ it is only the above equation is specified when $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$ for $x \in (0,2\pi)$.

Now $f^{`}(x) = -sinx - cosx$

When $x = \frac{\pi}{4}, f^{`}(x) = -\sin\frac{\pi}{4} - cos\frac{\pi}{4}$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$$

Hence f is maximum value at $x = \frac{\pi}{4}$

$$f\left(\frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}\right) + cos\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \sqrt{2}$$

Now $x = \frac{5\pi}{4}$

$$f^{``}\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{5\pi}{4}\right) - cos\left(\frac{5\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} > 0$$

Thus f has relative minimum at $x = \frac{5\pi}{4} \in (0,2\pi)$.

$$f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) + cos\left(\frac{5\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$$

$$= -\sqrt{2}$$

Q4. Show that $y = \frac{lnx}{n}$ has maximum value at $x = e$

Solution

$$y = \frac{lnx}{n}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x} + lnx\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2} - \frac{lnx}{x^2}$$

$$\frac{dy}{dx} = 0, \quad \Rightarrow \quad \frac{1}{x^2} - \frac{lnx}{x^2} = 0$$

$$\Rightarrow \frac{1-lnx}{x^2} = 0$$

$$\Rightarrow 1 - lnx = 0$$

$$\Rightarrow lnx = 1$$

$$lnx = lne$$

$$\Rightarrow x = e$$

$$y^{\cdot\cdot}(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x^2} - \frac{lnx}{x^2}\right)$$

$$= -\frac{2}{x^3} - \left[\frac{1}{x^3} - \frac{2lnx}{x^3}\right]$$

$$= -\frac{2}{x^3} - \frac{1}{x^3} + \frac{2lnx}{x^3}$$

$$= \frac{-3}{x^3} + \frac{2xlnx}{x^3}$$

$$\frac{d^2y}{dx^2} = x = e = \frac{-3}{e^3} + \frac{2xlnx}{e^3}$$

$$= \frac{-3}{x^3} + \frac{2}{x^3}$$

$$= \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$$

y has maximum value at$x = e$.

Q5. Show that $y = x^x$ has minimum value at $x = \frac{1}{e}$.

Solution

$$y = x^x$$

$$\Rightarrow \quad lny = ln(x)^x = xlnx$$

Diff. w.r.t x we have

$$\frac{d}{dx}[lnx] = \frac{d}{dx}[xlnx]$$

$$\frac{1}{y}\frac{dy}{dx} = (lnx + 1)$$

$$\frac{dy}{dx} = y(lnx + 1)$$

$$\frac{dy}{dx} = x^x(lnx + 1)$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow x[lnx + 1] = 0$$

$$\Rightarrow [lnx + 1] = 0 \quad (fx^n \neq 0)$$

$$\Rightarrow \quad lnx = -1$$

$$\Rightarrow \quad lnx = -lne$$

$$lnx = lne^{-1}$$

$$\Rightarrow \quad x = e^{-1} \quad \Rightarrow \quad x = \frac{1}{e}$$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}[x^x(1 + lnx)]$

$$= x^x(1 + lnx)^2 + x^x \times \frac{1}{x}$$

$$= x^x\left[(1 + lnx)^2 + \frac{1}{x}\right]$$

$$\frac{d^2y}{dx^2} x = \frac{1}{e} = \left(\frac{1}{e}\right)^{\frac{1}{2}}\left[\left(1 + ln\frac{1^2}{e}\right) + \frac{1}{\frac{1}{e}}\right]$$

$$\frac{d^2y}{dx^2}x = \left(\frac{1}{e}\right)^{\frac{1}{e}}[(1-1)^2 + e] = \left(\frac{1}{e}\right)^{\frac{1}{e}}(e)$$

Thus, y has minimum value at $x = \frac{1}{e}$