Solved Exercise 2.8

Q1. Apply the Maclaurin series expansion to prove that

i. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots\dots\dots\dots$

Solution

$$Let\ f(x) = \ln(1+x)$$

$$f`(x) = \frac{1}{1+x}$$

$$f``(x) = -\frac{1}{(1+x)^2}$$

$$f```(x) = \frac{(-1)(-2)}{(1+x)^3} = \frac{2}{(1+x)^3}$$

$$f````(x) = \frac{-6}{(1+x)^4}$$

$$put\ x = 0\ we\ have$$

$$f(0) = 1 \times (1+0) = \ln(1) = 0$$

$$f`(0) = \frac{1}{1+0} = 1$$

$$f``(0) = \frac{1}{(1+0)^2} = -1$$

$$f```(0) = \frac{1}{(1+0)^3} = 2$$

$$f````(0) = \frac{(-1)(-2)(-3)}{(1+4)^4} = -6$$

$$f`````(0) = \frac{(-1)(-2)(-3)}{(1+4)^4} = -6$$

By Maclaurin Expansion we have

$$f(x) = f`(0) + \frac{(0)}{1!}x + \frac{f``(0)}{2!}x^2 + \frac{f```(0)}{3!}x^3 + \frac{f````(0)}{4!}x^4 + \cdots \dots\dots\dots$$

$$\ln(1+x) = 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \quad \text{Ans}$$

ii. $\boldsymbol{Cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots}$

Solution

$Let \quad f(x) = cosx \qquad put\ x = 0$

$f(x) \quad = Cosx \qquad f(0) = Cos0 = 1$

$f(x) \quad = -Sinx \qquad f(0) = -Sin0 = 0$

$f`(x) = -Cosx \qquad f`(0) = -Cos0 = -1$

$f``(x) = Sin \qquad f```(0) = Sin = 1$

$f````(x) = Cosx \qquad f````(0) = Cos0 = 1$

$f`````(x) = -Sin \qquad f`````(0) = 0$

Using Maclaurin Expansion we have

$$f(x) = (0) + \frac{f^{i}}{1!} + \frac{f^{ii}}{2!} + \frac{f^{iii}}{3!} + \frac{f^{iv}}{4!} + \frac{f^{v}}{5!} + \ldots\ldots\ldots\ldots$$

$$Cosx \quad = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \ldots\ldots.$$

$$Cosx \quad = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots\ldots \quad \text{Ans}$$

iii. Prove that $\boldsymbol{\sqrt{1-x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{10} + \ldots\ldots\ldots}$

Solution

$$f(x) = (1+x)^{\frac{1}{2}}$$

$$f`(x) = \frac{1}{2\sqrt{1+x}}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{\frac{3}{2}} = \frac{-1}{4(1+x)^{\frac{3}{2}}}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-\frac{5}{2}} = \frac{3}{8(1+x)^{\frac{5}{2}}}$$

$$f^{iv}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1+x)^{\frac{7}{2}} = \frac{-15}{8(1+x)^{\frac{7}{2}}}$$

Putting x = 0 we have

$$f(0) = (1+0)^{\frac{1}{2}} = 1$$

$$f'(0) = \frac{1}{2\sqrt{1+0}} \quad = \frac{1}{2}$$

$$f''(0) = \frac{-1}{4(1+0)^{\frac{3}{2}}} \quad = -\frac{1}{4}$$

$$f'''(0) = \frac{3}{8(1+0)^{\frac{5}{2}}} \quad = \frac{3}{8}$$

$$f^{iv}(0) = \frac{-15}{16(1+0)^{\frac{7}{2}}} \quad = -\frac{15}{16}$$

Using Maclaurin expansion, we have

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \dots\dots\dots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} + \frac{-1}{(4)(2!)}x^2 + \frac{3}{(8)3!}x^3 - \frac{15}{(16)(4!)}x^4 \dots\dots\dots\dots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots\dots\dots\dots \quad \textbf{Ans}$$

iv. $\boldsymbol{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^{23}}{3!} + \dots\dots\dots\dots}$

Solution

$\boldsymbol{Let\ f(x) = e^x}$ $\qquad\qquad$ $\boldsymbol{f(0) = e^0 = 1}$

$\boldsymbol{f'(0) = e^x}$ $\qquad\qquad$ $\boldsymbol{f'(0) = e^0 = 1}$

$$f``(0) = e^x \qquad f``(0) = e^0 = 1$$

$$f```(0) = e^x \qquad f```(0) = e^0 = 1$$

By Maclaurin expansion we have

$$f(x) = f(0) + f`(0)x + \frac{f``(0)}{2!}x^2 + \frac{f```(0)}{3!}x^3 + \ldots\ldots\ldots$$

$$e^x \quad = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots$$ **Ans**

v. $e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$

Solution

$$f(x) = e^{2x} \qquad f(0) = e^0 = 1$$

$$f`(x) = 2e^{2x} \qquad f`(0) = 2e^0 = 2$$

$$f``(x) = 2^2e^{2x} \qquad f``(0) = 4e^0 = 4$$

$$f```(x) = 2^3e^{2x} \qquad f```(0) = 8e^0 = 8$$

$$f^{iv}(x) = 2^4e^{2x} \qquad f^{iv}(0) = 16e^0 = 16$$

By Maclurins expansion we have

$$f(x) \quad = f(0)x + f`(0)x + \frac{f`(0)}{2!}x^2 + \frac{f```(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!} + \ldots\ldots\ldots\ldots$$

$$e^{2x} \quad = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \ldots\ldots\ldots\ldots$$ **Ans**

Q2. Show that is

$Cos(x+h) = C0sx - h\,Sin\,x - \frac{h^2}{2!}Cosx + \frac{h^3}{3!}Sinx + \ldots\ldots$ **and Evaluate Cos 61^0.**

Solution

$$Let f(x+h) = f(x) + \frac{f`(h)}{1!} + \frac{f``(x)}{2!}h^2 + \frac{f```(x)}{3!}h^3 + \ldots$$

$$f(x) = Cos\ x$$

$$f`(x) = -Sinx$$

$$f``(x) = -Cosx$$

$$f```(x) = Sinx$$

Using values, we have

$$\Rightarrow \quad Cos(x+h) = Cosx - (sinx)h - \frac{Cosx}{2!}h^2 + \frac{Sin}{3!}h^3 + \ldots\ldots B$$

$$= Cosx - h\ sinx - \frac{h^2}{2!}Cosx + \ldots\ldots\ldots$$

Putting x+h=$61^0 + 1^0 = \frac{\pi}{3} + \frac{\pi}{180^0}. So\ that\ x = 60^0 = \frac{\pi}{3}$

$$Cos61^0 = Cos\frac{\pi}{3} - \left(Sin\frac{\pi}{3}\right)\left(\frac{\pi}{180}\right) - \frac{Cos\frac{\pi}{2}}{2!}\left(\frac{\pi}{180}\right)^2 + \frac{Sin\frac{\pi}{3}}{3!}\left(\frac{\pi}{180}\right)^3$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{\frac{1}{2}}{2!}\left(\frac{\pi}{180}\right)^2 + \frac{\frac{\sqrt{3}}{2}}{3!}\left(\frac{\pi}{180}\right)^3 +$$

$$Cos61^0 = \frac{1}{2} - \frac{\sqrt{3}}{2}(0.017485) - \frac{\frac{1}{2}}{2!}(0.017485)^2 +$$

$$= .5 - (.866)(.017485) - \frac{1}{4}(.0003045)$$

$$= .5 - .015116 - .00076$$

$= .4848$ **Ans**

Q3. Show that $2^{x+h} = 2^x\{1 + \ln 2)h + (\ln 2)^2 + (\ln 2)^3\ h^3 + \ldots.\}$

Solution

$$So. Let\ (x+h) = 2^{x+h}$$

$$\Rightarrow \quad f(x) = 2^x$$

$$Diff. w.r.t\ x\ we\ have$$

$$f`''(x) = 2^x \ln 2$$

$$f`(x) = 2^x \ln 2^2$$

$$f```(x) = 2^x \ln 2^3$$

$$f^{iv}(x) = 2^x \ln 2^4$$

By using Taylors series, we have

$$f(x+h) = f(x) + \frac{f`(x)}{1!}h + \frac{f``(x)}{2!}h^2 + \frac{f```(x)}{3!}h^3 + \frac{f^{iv}(x)}{4!}h^4 + \ldots\ldots$$

$$2^{x+h} = 2^x + (2`\ln 2)h + \frac{2`(\ln 2)^2}{2!}(h)^2 + \frac{2`(\ln 2)^3}{3!} + \ldots\ldots$$

$$2^{x+h} = 2^x\left[1 + (\ln 2)h + \frac{(\ln 2)^2}{2!}h^2 + \frac{(\ln 2)^3}{3!}h^3 + \cdots\right]$$ **Ans**