Solved Exercise 2.7

Q1. ***Find*** y^2 ***if***

i. $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Solution

Diff w.r.t x

$$\frac{dy}{dx} = 10x^4 - 12x^3 + 12x^2 + 1 - 0$$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

Diff. again w.r.t x

$$y_1 = 40x^3 - 36x^2 + 24x$$ **Ans**

ii. $y = (2x + 5)^{3/2}$

Solution

$$y = (2x + 5)^{3/2}$$

Diff. w.r.t

$$y_1 = \frac{3}{2}(2x + 5) \frac{d}{dx}(2x + 5)$$

$$= \frac{3}{2}(2x + 5)^{1/2}(2)$$

$$y_1 = 3(2x + 5)^{1/2} \ldots\ldots\ldots\ldots\ldots(1)$$

Diff. again w.r.t x

$$y_2 = 3\frac{1}{2}(2x + 5)^{1/2-1} \frac{d}{dx}(2x + 5)$$

$$= \frac{3}{2}(2x + 5)^{1/2} \ldots\ldots\ldots..(2)$$

$$y_2 = \frac{3}{(\sqrt{2x+5})} \quad \textbf{Ans}$$

iii. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}} + x^{\frac{1}{2}}\right)$$

$$y_1 = \frac{1}{2}x^{\frac{1}{2}\ 1} - \frac{1}{2}x^{-\frac{1}{2}\ 1}$$

$$= \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{\ \frac{3}{2}}$$

$$y_1 = \frac{1}{2}\left(x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}}\right)$$

Diff. again w.r.t x

$$y_2 = \frac{1}{2}\left(-\frac{1}{2}x^{\frac{1}{2}1} - \left(-\frac{3}{2}x^{\frac{3}{2}\ 1}\right)\right)$$

$$= \frac{1}{2}\left(-\frac{1}{2}x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{3}{2}}\right)$$

$$= -\frac{1}{4x^{\frac{3}{2}}} + \frac{3}{4x^{\frac{3}{2}}}$$

$$= \frac{-x+3}{4x^{\frac{3}{2}}} \qquad \textbf{Ans}$$

Q2. $\boldsymbol{Find\ y_2\ if(1)y = x^2e^{-x}}$

Solution

$$y = x^2e^{-x}$$

Diff. w.r.t x

$$y = e^{-x}(2x) + (x^2)(-e^{-x})$$

$$= e^{-x}(2x - x^2)$$

Diff. w.r.t x

$$y_2 = e^{-x}(2 - 2x) + (2x - x^2)(-e^{-x})$$

$$= e^{-x}(2 - 2x - 2x + x^2)$$

$$= e^{-x}(x^2 - 4x + 2) \qquad \textbf{Ans}$$

ii. $\boldsymbol{y = ln\left(\frac{2x+3}{3x+2}\right)}$

Solution

$$y = ln\left(\frac{2x+3}{3x+2}\right)$$

$$y = \ln(2x + 3) - \ln(3x + 2)$$

Diff. w.r.t x

$$y = \frac{1}{2x+3}2 - \frac{1}{3x+2}$$

$$= \frac{2}{2x+3} - \frac{3}{3x+2}$$

Diff. again w.r.t x

$$y_2 = 2\left[\frac{d}{dx}(2x + 3)^{-1}\right] - 3\left[\frac{d}{dx}(3x + 2)^{-1}\right]$$

$$= 2(-1)(2x + 3)^{-1-1} \times 2 - 3((-1)(3x + 2)^{-2}(3))$$

$$y_2 = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2}$$

$$y_2 = -\frac{[4(3x + 2)^2 - 9(2x + 3)^2]}{(2x + 3)^2(3x + 2)^2}$$

$$y_2 = \frac{[4(9x^2+12x+4)-9(4x^2+12x+9)}{(2x+3)^2(3x+2)^2}$$

$$= \frac{[36^2+48x+16-36^2-108x-81}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{60x+65}{(2x+3)^2(3x+2)^2}$$ **Ans**

Q3. $\boldsymbol{Find\ y_2\ if}$

i. $\boldsymbol{x^2 + y^2 = a^2}$

Solution

Diff. both sides w.r.t x

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$y\frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx}y_1 = -\frac{x}{y} \ ....(1)$$

Now diff.(1) w.r.t x

$$y_2 = \frac{y(-1)-(-x)\frac{dy}{dx}}{(y)^2}$$

$$= \frac{-y+x\frac{dy}{dx}}{y^2}$$

$$= \frac{-y+x\left(-\frac{x}{y}\right)}{y^2}$$

$$= \frac{-y^2-x^2}{y^2}$$

$$y_2 = \frac{(x^2+y^2)}{y^2} = \frac{a^2}{y^2}$$ **Ans**

ii. $x^3 - y^3 = a^3$

Solution

Diff. both sides of above equation w.r.t x

$$3x^2 - 3y^2\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} = 3x^2$$

$$y^2\frac{dy}{dx} = x^2$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\Rightarrow y_1\frac{dy}{dx} = \frac{x^2}{y^2} \text{ ....(1)}$$

Now diff.(1) w.r.t x

$$y_2 = \frac{2xy^2 - \left(2y\frac{dy}{dx}\right)x^2}{y^4}$$

$$= \frac{2xy^2 - x^2(2y)\left(\frac{x^2}{y^2}\right)}{y^4} \text{ (using 1)}$$

$$= \frac{2xy^2 - \frac{2x^4}{y}}{y^4}$$

$$= \frac{2xy^3 - 2x^4}{y^5}$$

$$= \frac{2x(y^3 - x^3)}{y^5}$$

$$= \frac{2x(x^3 - y^3)}{y^5}$$

$$= \frac{2a^3x}{y^5} \quad \textbf{Ans}$$

iii. $\boldsymbol{x = a\cos\emptyset,\ \ y = a\sin\emptyset}$

Solution

$$x = a\cos\emptyset, y = a\sin\emptyset\ s$$

$$\frac{dx}{d\emptyset} = -a\sin\emptyset, \frac{dy}{d\emptyset} = a\cos\emptyset$$

$$\frac{dy}{dx} = \frac{dy}{d\emptyset} \times \frac{d\emptyset}{dx}$$

$$= a\cos\emptyset \times \frac{1}{-a\sin\emptyset}$$

$$y_1 = \frac{dy}{dx} = -\cot\emptyset$$

Diff. again

$$y_2 = -(-cosec^2\emptyset)\frac{d\emptyset}{dx}$$

$$= cosec^2\emptyset \times \left(-\frac{1}{a\sin\emptyset}\right)$$

$$= -\frac{1}{asin^3\emptyset} \quad \textbf{Ans}$$

iv. $\boldsymbol{x = at^2, \qquad y = bt^4}$

Solution

$$x = at^2, \frac{dy}{dt} = 4bt^3$$

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{at}{dx}$$

$$= 4bt^3 \times \frac{1}{2at}$$

$$y_1 = \frac{dy}{dx} = \frac{2bt^2}{a}$$

Diff. again

$$y_2 = \frac{2b}{a}\left(2t.\frac{dt}{dx}\right)$$

$$= \frac{2b}{a}\left(2t.\frac{1}{2at}\right)$$

$$= \frac{2b}{a^2}$$ **Ans**

v. $\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$

Solution

Diff. w.r.t x

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$2(y + f)\frac{dy}{dx} = -2(x + g)$$

$$(y + f)y_1 = -(x + g)$$

$$y_1 = -\frac{(x+g)}{y+f}$$

Diff. again

$$\frac{d}{dx}y_1 = \frac{d}{dx}\left(\frac{x+g}{y+f}\right)$$

$$y_2 = \frac{(y+f)(1)-(x+g)}{(y+f)^2}\frac{dy}{dx}(y + f)$$

$$y_2 = \frac{(y+f)-(x+g)\left(-\frac{x+g}{y+f}\right)}{(y+f)^2}$$

$$y_2 = \frac{(y+f)^2+(x+g)^2}{(y+f)^2}$$

$$= \left[\frac{y^2+f^2+2fy+x^2+g^2+2xg}{(y+f)^3}\right]$$

$$= \frac{x^2+y^2+2gx+2fy\pm c+f^2+g^2}{(y+f)^2}$$

$$= \frac{c-f^2-g^2}{(y+f)^3} \quad \textbf{Ans}$$

Q4. $Find\ y^4\ if$

i. $y = \sin 3x$

Solution

$$y_1 = \cos 3x.3 \quad = 3cos.3x$$

$$y_2 = 2(-sin.3x)3 = -9sin.3x$$

$$y_3 = -27c0s.3x$$

$$y_4 = -27(-sin.3x).3$$

$$= 81\ sin.3x \quad \textbf{Ans}$$

ii. $y = \ cos^3x$

Solution

$$As\ cos^3x = 3cosx - 4cos^3x$$

$$\Rightarrow \quad y = 3cosx - 4cos^3x$$

$$y_1 = -3sinx + 12cos^2 \times sin$$

$$= -3sin + 3sin^2x.cosx$$

$$y_2 = -3cosx + 9sin^2x.cosx$$

$= -3cosx + 9(1 - cos^2x)cos^4$

$= 6cosx - 9cos^3x$

$y_3 = -6sinx.27cos^2x(-sinx)$

$= -6sin + 27(1 - sin^2x)sinx$

$= 21sin - 27sinx$

$y_4 = 21cosx - 81sin^2x\cos x$

$= 21\ cosx - 81(1 - cos^2x)cos^4$

$= 21\ cosx - 81\ cosx + 81\ cos^3x$

$= -60\ cosx + 81cos^3x$ **Ans**

iii. $\mathbf{y = \ln(x^2 - 9)}$

Solution

$= ln[(x + 3)(x - 3)]$

$= \ln(x + 3) + \ln(x - 3)$

$\frac{dy}{dx} = \frac{d}{dx}[\ln(x + 3)] + \frac{d}{dx}[\ln(x - 3)]$

$= \frac{1}{x+3})(1 + 0) + \frac{1}{x-3}(1 - 0)$

$= (x + 3)^{-1}(x - 3)^{-1}$

$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}[(x + 3)^{-1}] + \frac{d}{dx}[(x - 3)^{-1}]$

$= -(x + 3)^{-2}(1 + 0) - (x - 3)^{-2}(1 - 0)$

$= -(x + 3)^{-2} - (x - 3)^{-2}$

$y_3 = \frac{d^3y}{dx^3} = \frac{d}{dx}[-(x + 3)^{-2}] - \frac{d}{dx}[(x - 3)^{-2}]$

$= 2(x + 3)^{-2}(1 + 0) + 2(x - 3)^{-3}(1 - 0)$

$$= 2(x+3)^{-3} + 2(x-3)^{-3}$$

$$y_4 = \frac{d^4y}{dx^4} = \frac{d}{dx}[2(x+3)^{-3}] + \frac{d}{dx}[2(x-3)^{-3}]$$

$$= -6(x+3)^{-4}(1+0) - 6(x-3)^{-4}(1-0)$$

$$= \frac{-6}{(x+3)^4} - \frac{6}{(x-3)}$$

$$= -6\left[\frac{1}{(x+3)^4} + \frac{1}{(x-3)^4}\right]$$ **Ans**

Q5. if $x = Sin\emptyset,\ y = sinm\emptyset\ show\ that\ (1-x^2)\ y_2 - xy_1 + m^2y = 0$

Solution

$x = Sin\emptyset \quad \Rightarrow sin^{-1}x = \emptyset$

$y = sinm\emptyset \quad \Rightarrow \sin(msin^{-1}x)$

Diff. both sides w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(msin^{-1}x)]$$

$$= \frac{mcos(msin^{\ \ 1}x)}{\sqrt{1-x^2}}$$

$$(\sqrt{1-x^2})y_1 = mcos(msin^{-1}x)$$

$\Rightarrow \quad (1-x^2){y_1}^2 = m^2cos^2(msin^{-1}x)$

$$(1-x^2){y_1}^2 = m_2[1 - sin^2(msin^{-2}x)]$$

$$(1-x^2){y_1}^2 = m^2(1-y^2)$$

Diff. again w.r.t x

$$(1-x^2)2y_1y_2 - (2x)({y_1}^2) = m^2(-2yy_1)$$

$$(1-x^2)y_2 - xy + m^2y = 0$$ **Ans**

Q6. If $y = e^x \, Sin\, x \; show\; that$

$$\frac{d^2y}{dx^2} = \frac{2dy}{dx} + 2y = 0$$

Solution

$$y = e^x \, Sin\, x \; ....(1)$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{d}{dx}(e^x sin.\, x)$$

$$\frac{dy}{dx} = e^x \sin x + e^x \cos x \; ....(2)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = e^x \sin x + e^x cos^x + e^x cos^x - e^x \sin x$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = 2e^x cos^x$$

$$= 2\left(\frac{dy}{dx} - e^x \sin x\right) using\; 2$$

$$= 2\frac{dy}{dx} - 2y \; using\; (1)$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{2dy}{d^x} + 2y = 0$$ **Ans**

Q7. If $y = e^{ax}\, Sin\, bx$ show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

Solution

$$y = e^{ax}\, Sin\, bx \qquad (1)$$

$$\frac{dy}{dx} = Sin\, bx . e^{ax} . a + e^{ax} Cos\, bx \times b$$

$$\frac{dy}{dx} = e^{ax}(a\, Sin\, bx + bCosbx) \qquad (2)$$

$$\frac{dy}{dx} = e^{ax}(a \sin bx + bCosbx) \qquad (2)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[e^{ax}(0Sinx\; bx + bCosbx)]$$

$$\frac{d^2y}{dx^2} = \left(\frac{d}{dx}e^{ax}\right)[aSin\ bx + bCos\ bx)]$$

$$= e^{ax}\frac{d}{dx}[(a\ Sin\ bx + bCosbx)]$$

$$\frac{d^2y}{dx^2} = ae^{ax}[aSin\ bx + bCosbx]$$

$$= e^{ax}[ab\ Cos\ bx - b^2 Sin\ bx]$$

$$= e^{ax}[a^2 Sin\ bx + abCosbx + abCosbx - b^2\ Sin\ bx]$$

$$= e^{ax}[2abCosbx + 2a^2 Sinbx - 2a^2 Sinbx + a^2 Sinbx - b^2 Sinbx]$$

$$= 2ae^{ax}[aSinbx + bCosbx] - e^{ax}(a^2 + b^2)Sinbx$$

$$\frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - (a^2 + b^2)y$$

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0 \qquad \textbf{Ans}$$

Q8. If $y = (Cos^{-1}x)^2$ prove that $(1 - x^2)y_2 - xy_1 - 2 = 0$

Solution

$$y = (Cos^{-1}x)^2 \qquad \ldots\ldots(1)$$

$$\frac{dy}{dx} = 2Cos^{-1}x\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$-\sqrt{1-x^2}\,y_1 = 2Cos^{-1}x$$

Diff. again w.r.t x

$$\frac{2x}{2\sqrt{1-x^2}}y_1 + \sqrt{1-x^2}\,y_2 = \frac{-2x}{\sqrt{1-x^2}}$$

$$\frac{-xy_1}{\sqrt{1-x^2}} + \sqrt{1-x^2}\,y_2 = \frac{2x}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad -xy + (1-x^2)y_2 = 2$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 - 2 = 0 \qquad \textbf{Ans}$$

Q9. if $y = a\, Cos(lnx) + bSin(lnx)$

$$prove\ that\ x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Solution

$$y = a\ Cos\ (lnx) + b\ Sin(lnx) \ \ldots\ldots\ldots..(1)$$

$$\frac{dy}{dx} = -\frac{a\ Sin(lnx)}{n} + \frac{b\ Cos(lnx)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -a\ Sin(lnx) + b\ Cos(lnx)$$

Diff. again w.r.t we have

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dx}(-a\ Sin(lnx) + bCos(lnx))$$

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} = -\frac{a\ Coslnx}{x} + \frac{-bSinlnx}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = -[a\ Cos(lnx) + b\ Sin(lnx)]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \qquad by(i)$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$ **Ans**