Solved Exercise 2.5

Q.1.Differentiate the following trigonometric functions from the first Principle.

i. $\sin 2x$

Solution

Let $y = \sin 2x$

$$y + \delta y = \sin 2(x + \delta x)$$

$$\Rightarrow \quad \delta y = \sin 2(x + \delta x) - \sin 2x$$

$$= 2\cos\left[\frac{2x+2\delta x+2x}{2}\right]\sin\left(\frac{2x+2\delta x-2x}{2}\right)$$

$$= 2\cos\left(\frac{4x+2\delta x}{2}\right)\sin\left(\frac{2\delta x}{2}\right)$$

$$= 2\sin\delta x.\cos(2x+\delta x) \qquad (1)$$

Dividing both sides by δx and take limit as $\delta x \to 0$

$$\Rightarrow \quad \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}$$

$$= 2\lim_{\delta x\to 0}.\cos(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\lim\delta y}{\delta x} = 2\lim_{\delta x\to 0}\cos(2x+\delta x)$$

$$\frac{dy}{dx} = 2(1)\cos 2x$$

$$= 2\cos 2x \qquad \textbf{Ans}$$

ii. Tan 3x

Solution

Let $y = Tan\,3x$

$$y + \delta y = Tan\ x\ (x + \delta x)$$

$$\Rightarrow \quad \delta y = Tan\ 3(x + \delta x) - Tan\ 3x$$

$$\Rightarrow \quad = \frac{\sin 3(x+\delta x)}{\cos 3(x+\delta x)} - \frac{\sin 3x}{\cos 3x}$$

$$= \frac{\sin 3(x+\delta x).\cos 3x - \cos 39x+\delta x),\sin 3x}{\cos 3(x+\delta x).\cos 3x}$$

$$\Rightarrow \frac{\sin \delta x}{\delta x} \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{3}{\cos 3(x+\delta x).\cos 3x} \times \lim_{\delta x \to 0} \frac{3\delta x}{3\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{3}{\cos 3(x+\delta x).\cos 3x}\right] \times \lim_{\delta x \to 0} \frac{3\delta x}{3\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{\cos 3x.\cos 3x}$$

$$= 3\ Sec^2\ 3x$$ **Ans**

iii. $\sin 2x + \cos 2x$

Solution

Let $y = \sin 2x + \cos 2x\ \ then$

$$\Rightarrow \quad \delta y = (\sin 2(x + \delta x) - \sin 2x) + \cos 2(x + \delta x) - \cos 2x$$

$$= 2\cos\left(\frac{2(x+\delta x)}{2}\right)\sin\left(\frac{2(x+\delta x)-2x}{2}\right)$$

$$= 2\sin\left(\frac{2(x+\delta x)+2}{2}\right).\sin\left(\frac{2(x+\delta x)-2x}{2}\right)$$

$$= 2\left[(\cos(2x + \delta x).\sin \delta x - \sin(2x + \delta x)\cos \delta x)\right]$$

$$\frac{\delta y}{\delta x} = 2[(\cos(2x + \delta x).\sin \delta x - \sin(2x + \delta x)\cos \delta x)]$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 2\cos(2x + \delta x).\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= \lim_{\delta x \to 0} 2\cos(2x + \delta x0).\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= \left[2\left\{\cos(2x + \delta x) - \lim(2x + \delta x)\right\}\right] \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$\frac{dy}{dx} = 2(\cos 2x - \sin 2x) \times 1 = 2(\cos 2x - \sin 2x)$$

$$= 2\cos 2x - 2\sin 2x$$

iv. $\cos x^2$

Solution

Let $y = \cos x^2$

$$\delta x = \cos(x + \delta x)^2 - \cos x^2$$

$$= -2\sin\frac{(x+\delta x)^2 + x^2}{2}\sin\frac{(x+\delta x)^2 - x}{2}$$

$$= -2\sin\left(\frac{x+\delta x}{2x}\right).\sin\frac{(x+\delta x)^2 + \delta x}{2}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} 2 \times \sin(x^2 + x\delta x).\lim_{\delta x \to 0}\frac{\sin x\delta x}{x\delta x}$$

$$\frac{dy}{dx} = -2x\sin\frac{2x^2}{2}$$

$$= -2x\sin x^2$$ **Ans**

v. $Tan^2 x$

Solution

Let $y = tan^2 x$

$$y + \delta y = tan^2(x + \delta x)$$

$$\Rightarrow \quad \delta y = tan^2(x + \delta x) - tan^2 x$$

$$\delta y = [\tan(x + \delta x) - tanx][\tan(x + \delta x) + tanx]$$

$$= \left[\frac{\sin(x+\delta x) - \sin x}{\cos(x-\delta x)\cos x}\right]\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} + \frac{-\sin x}{\cos x}\right]$$

$$\delta y = \left[\frac{\sin(x-\delta x)\cos x - \sin x\cos(x+\delta x)}{\cos(x+\delta x)\cos x}\right]$$

$$\times\left[\frac{\sin(x+\delta x)\cos x+\cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}\right]$$

$$=\frac{\sin(x+\delta x-x)}{\cos(x+\delta x)\cos x}\,\frac{\sin(x+\delta x+x)}{\cos(x+\delta x)\cos x}$$

$$\delta y=\frac{\sin\delta x\sin(2x+\delta x)}{\cos^2(x+\delta x)\cos^2 x}$$

Dividing by δx and taking limit $\delta x\to 0$,we have

$$\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{\sin(2x+\delta x)}{\cos^2(x+\delta x)\cos^2 x}\right]$$

$$=\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}\lim_{\delta x\to 0}\left[\frac{\sin(2x+\delta x)}{\cos^2(x+\delta x)\cos^2 x}\right]$$

$$\frac{dy}{dx}=1.\frac{\sin 2x}{\cos^2 x\cos^2 x}$$

$$=\frac{2\sin x\cos x}{\cos^2 x\cos^2 x}-\frac{2\sin x}{\cos x}$$

$$=\frac{2\sin x}{\cos x}\cdot\frac{1}{\cos^2 x}$$

$$=2tan\times sec^2x$$

$$Thus\ \frac{dy}{dx}[tan^2x]=2tan\times sec^2$$ **Ans**

vi. $\sqrt{tanx}$

Solution

Let $y=\sqrt{tanx}$

$$y+\delta y=\sqrt{\tan(x+\delta x)}$$

$$\Rightarrow\quad \delta y=\sqrt{\tan(x+\delta x)}-\sqrt{tanx}$$

$$=\frac{\sqrt{\sin(x+\delta x)}}{\sqrt{\cos(x+\delta x)}}-\frac{\sqrt{sinx}}{\sqrt{cosx}}$$

$$=\frac{\sqrt{\sin(x+\delta x)cosx-\sqrt{cosx(x+\delta x)}sinx}}{\sqrt{\cos(x+\delta x)}\sqrt{cosx}}$$

By rationalizing we get

$$\delta y = \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\left(\sqrt{\sin(x+\delta x)}\sqrt{\cos x}\sqrt{\cos(x+\delta x)\sin x}\sqrt{\cos(x+\delta x)\cos x}\right)}$$

$$= \frac{\sin(x+\delta x - x)}{\left[\sqrt{\sin(x+\delta x)\cos x}+\sqrt{\cos(x+\delta x)\sin x}\right]\sqrt{\cos(\delta x)\cos x}}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{\sin(\delta x)}{|\sin(x+\delta x)| + \sqrt{\cos(x+\delta x)\sin x}\sqrt{\cos(x+\delta x)\cos x}}$$

$$= \frac{1}{\left[\sqrt{\sin x\cos x}+\sqrt{\cos x\sin x}\right]\sqrt{\cos x\sin x}\sqrt{\cos x\cos x}}$$

$$= \frac{1}{2\sqrt{\sin x\cos x\sqrt{\cos x}}}$$

$$\frac{dy}{dx} = \frac{\sqrt{\cos x}}{2\sqrt{\sin x}}\, x\, \sec x \frac{d}{dx}\left[\sqrt{\tan x}\right]$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}$$

vii. $\cos\sqrt{x}$

solution

let $y = \cos\sqrt{x}$

$$y + \delta y = \cos\sqrt{x+\delta x}$$

$$\Rightarrow \quad \delta y = \cos\sqrt{x+\delta x} - \cos\sqrt{x}$$

$$= -2\sin\left[\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right] \sin\left[\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right]$$

$$\frac{\delta y}{\delta x} = \frac{\sin\left[\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right]\sin\left[\frac{\sqrt{x+\delta x}\ \ \sqrt{x}}{2}\right]}{2x\left(\sqrt{x+\delta x}+\sqrt{x}\right)\frac{(x+\delta x)}{2}}$$

$$\lim\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{-\sin\left[\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right]}{\sqrt{x+\delta x}+\sqrt{x}}\sin x\, \frac{\left[\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right]}{\frac{\sqrt{x+\delta x}\ \ \sqrt{x}}{2}}$$

$$(-1)\frac{\sin\frac{2\sqrt{x}}{2}}{2\sqrt{x}}\left[\frac{\sqrt{x+\delta x}-\sqrt{x}}{2} \to 0 \text{ when } \delta x \to 0\right]$$

$$= \frac{-1}{2}\frac{\sin\sqrt{x}}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Q2. Differentiate the following w.r.t the variables involved

i. $x^2 sec4x$

Solution

Let $y = x^2 sec4x$

$$\frac{dy}{dx} = \frac{d}{dx}[x^2 sec4x]$$

$$= x^2 \frac{d}{dx}[sec4x] + sec4x \frac{d}{dx}(x^2)$$

$$= x^2(sec4x + tan4x) \times 4 + (sec4x) \times 2x$$

$$\frac{d}{dx}[x^2 sec4x] = 2xsec4 \times [2xtan4x + 1]$$ **Ans**

ii. $tan^2 \emptyset sec^2 \emptyset$

Solution

Differentiate w.r.t '∅' we have

$$\frac{dy}{dx} = \frac{d}{d\emptyset}[tan^2\emptyset sec^2\emptyset]$$

$$= sec^2\emptyset \left[\frac{d}{d\emptyset}(tan^2\emptyset)\right] + tan^2\emptyset \frac{d}{d\emptyset}(sec^2\emptyset)$$

$$= sec^2\emptyset \left[3tan^2\emptyset \frac{d}{d\emptyset}(tan\emptyset)\right] + tan^2\emptyset \left[2sec\emptyset \frac{d}{d\emptyset}(sec\emptyset)\right]$$

$$= sec^2\emptyset \left[3tan^2\emptyset \frac{d}{d\emptyset}(tan\emptyset)\right] + tan^2\emptyset[2sec\emptyset sec\emptyset tan\emptyset]$$

$$= 3tan^2\emptyset sec^4\emptyset + 2sec^2\emptyset tan^4\emptyset$$

$$\frac{d}{d\emptyset}(tan^3\emptyset sec^2\emptyset) = sec^2\emptyset tan^2\emptyset tan[3sec^2\emptyset + 2tan^2\emptyset]$$ **Ans**

iii. $(sin2\emptyset - cos3\emptyset)^2$

Solution

Let $y = (sin2\emptyset - cos3\emptyset)^2$

Differentiating w.r.t '∅' we have,

$$\frac{dy}{d\emptyset} = \frac{d}{d\emptyset}[sin2\emptyset - cos3\emptyset]^2$$

$$= 2[sin2\emptyset - cos3\emptyset]\frac{d}{d\emptyset}(sin2\emptyset - cos3\emptyset)$$

$$= s[sin2\emptyset - cos3\emptyset]\left[cos2\emptyset(-sin3x)\frac{d}{d\emptyset}(3\emptyset)\right]\frac{d}{d\emptyset}$$

$$(sin2\emptyset - cos3\emptyset)$$

$$= 2(sin2\emptyset - cos3\emptyset)[(cos2\emptyset).2 + (sin3\emptyset)3]$$

$$\frac{dy}{d\emptyset} = 2(sin2\emptyset - cos3\emptyset)[2cos2\emptyset + 3sin3\emptyset]$$ **Ans**

iv. $Cos\sqrt{x} + \sqrt{sinx}$

Solution

Let $y = cos\sqrt{x} + \sqrt{sinx}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[cos^{1/2} + sin^{1/2}x\right]$$

$$= \frac{d}{dx}\left(cosx^{1/2}\right) + \frac{d}{dx}(sin^{1/2}x)$$

$$= -sinx^{1/2}x^{-1/2} + \frac{1}{2}sin^{-1/2}x\ cosx$$

$$= -\frac{sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}$$

$$= \frac{1}{2}\left(\frac{sin\sqrt{x}}{\sqrt{x}} + \frac{\cos x}{\sqrt{\sin x}}\right)$$ **Ans**

Q3. $\boldsymbol{Find}$ $\frac{dy}{dx}$ **if**

i. $\boldsymbol{y = x\cos y}$ **ii.** $\boldsymbol{x = y\sin y}$

i. $\boldsymbol{y = xcosy}$

Solution

Differentiating w.r.t. "x" we have

$$\frac{dy}{dx} = \frac{d}{dx}[x\cos y]$$

$$= x\frac{d}{dx}(\cos y) + \cos y\frac{d}{dx}(x)$$

$$= x\left[-\sin y\frac{d}{dx}\right] + \cos y$$

$$\frac{dy}{dx} = -x\sin y\frac{dy}{dx} + \cos y$$

$$\frac{dy}{dx} + x\sin y.\frac{dy}{dx} = cosy$$

$$(1 + x\sin y)\frac{dy}{dx} = cosy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos y}{1+x\sin y}$$ **Ans**

ii. $\boldsymbol{x = y\sin y}$

Solution

Differentiating w.r.t 'x' we have

$$\frac{dy}{dx}(x) = \frac{d}{dx}(y\sin y)$$

$$1 = y\frac{d}{dx}(\sin y) + \sin y\frac{dy}{dx}$$

$$1 = y\cos y \times \frac{dy}{dx} + \sin y\frac{dy}{dx}$$

$$1 = (y\cos y + \sin y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{y\cos y + \sin y}$$ **Ans**

Q4. Find the derivative w.r.t 'x'

i. $\sqrt{\frac{1+x}{1+2x}}$ **ii.** $sin\sqrt{\frac{1+2x}{1+x}}$

i. $\sqrt{\frac{1+x}{1+2x}}$

Solution

$$Let\ y \quad = \cos u\ where\ u\ = \sqrt{\frac{1+x}{1+2x}}$$

$$\frac{dy}{dx} = \sin u \qquad (1)$$

$$= \sin u \qquad (1)$$

$$\frac{d}{dx}\,(u) = \frac{d}{dx}\left[\frac{1+x}{1+2x}\right]$$

$$= \frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{-1/2} \frac{(1+2x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1+2x)}{(1+2x)^2}$$

$$= -\frac{1}{2}\sqrt{\frac{1+2x}{1+x}} \qquad \times \frac{(1+2x)(1)-(1+x)-(2)}{(1+2x)^2}$$

$$\frac{du}{dx} = \frac{1}{2}\sqrt{\frac{1+2x}{1+2x}} \qquad \times \frac{-1}{(1+2x)^2}$$

$$= x\,\frac{-1}{\sqrt{1+x}(1+2x)^{3/2}}$$

$$Now\,\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= \sin u\ x\ \frac{-1}{2\sqrt{1+x}(1+2x)^{3/2}}$$

$$= \frac{\sin u}{2\sqrt{1}+x(1+2x)^{3/2}}$$

$$\frac{dy}{dx} = \left(cos\sqrt{\frac{1+x}{1+2x}}\right)$$

$$= \frac{sin\sqrt{\frac{1+x}{1+2x}}}{2\sqrt{1+x}(1+2x)^{3/2}} \quad \textbf{Ans}$$

ii. $sin\sqrt{\frac{1+2x}{1+x}}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(sin\sqrt{\frac{1+2x}{1+x}}\right)$$

$$= cos\sqrt{\frac{1+2x}{1+x}} \times \frac{d}{dx}\left[\sqrt{\frac{1+2x}{1+x}}\right]$$

$$= cos\sqrt{\frac{1+2x}{1+x}} \times \frac{1}{2}\left[\frac{1+2x}{1+x}\right]$$

$$= \frac{(1+x)\frac{d}{dx}(1+2x)-(1+2x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{1}{2}\left[cos\sqrt{\frac{1+2x}{1+x}}\right]\sqrt{\frac{1+2x}{1+x}}$$

$$\frac{(1+x)(2)-(1+2x)(1)}{(-x)}$$

$$= \frac{1}{2}cos\sqrt{\frac{1+2x}{1+x}} \times \frac{1}{\sqrt{1+2x}(1+x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{cos\sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x}(1+x)^{3/2}} \quad \textbf{Ans}$$

Q5. Differentiate

i. sin x w.r.t. cot x

Solution

Let $y = \sin x$ and $u = \cot x$

$\frac{dy}{dx} = \frac{d}{dx}(\sin\ x),$ $\frac{du}{dx} = \frac{d}{dx}(\cos x)$

$\frac{dy}{dx} = \cos x,$ $\frac{du}{dx} = -cosec^2\ x$

$\Rightarrow$ $\frac{dx}{du} = \frac{1}{cosec^2 x}$ $\Rightarrow \frac{dx}{du} = sin^2 x$

Now $\frac{dy}{du}\ \frac{dx}{du}$ *by chain rule*

$\frac{dy}{du} = (\cos x)(-sin^2 x)$

$= -(\cos x\ sin^2\ x)$ **Ans**

ii. $sin^2\ x\ w.r.t.\ cos^4$ **x**

Solution

Let $y = sin^2\ x\ and\ u =\ cos^4\ x$

$we\ shall\ find\ \frac{d}{dx}\ (u) = \frac{d}{dx}\ cos^4$ x

$= 4cos^3\ x\ \frac{d}{dx}\ (c0s)x$

and $y = sin^2 x$

and $\frac{dy}{dx} = \frac{d}{dx}(\sin x)^2 . 4\cos x^2\ x(-\sin x)$

$\frac{du}{dx} = -4cos^3 \times \sin x$

$\frac{du}{dx} = -4cos^3 \times \sin x$

$= 2\left(\sin x\ \frac{d}{dx}\right)(\sin x)$

$\frac{dy}{dx}\ = 2sin \times cos \times \frac{dx}{du} = \frac{-1}{4cos^3 \times \sin x}$

$\Rightarrow\ \frac{dx}{du} = \frac{dy}{dx}\ \frac{dx}{du}$

$$= 2sin \times \cos x \quad \times \frac{-1}{4cos^3 \times \sin x}$$

$$= \frac{-1}{2cos^2 x}$$

$$\frac{d(y)}{du} = \frac{-1}{2} sec^2 x$$ **Ans**

Q6. $\boldsymbol{If \tan y(1 + \tan x) = 1 - \tan x \, , show\ that\ \frac{dy}{dx} = -1}$

$\boldsymbol{\frac{dy}{dx} = -1}$

Solution

$$\tan y(1 + tan\ x) = \ 1 - \tan x$$

Diff. w.r.t x

$$sec^2 y \frac{dy}{dx}(1 + \tan x) + \tan y(sec^2\ x) = sec^2\ x$$

$$(1 + tan^2 y)(1 + \tan x)\frac{dy}{dx} \ = sec^2\ x(a + tany\ y)$$

$$\left[1 + \left(\frac{1-\tan x}{1+\tan x}\right)^2\right](1 + tan\ x)\frac{dy}{dx} = sec^2 x$$

$$\left[1 + \frac{1-\tan x}{1+\tan x}\right]\left[\therefore\ tany \frac{1-\tan x}{1-\tan x}\right] vc$$

$$\left[\frac{1+tan^2 x+2\tan x+1-2\tan x+tan^2 x}{(1+\tan x)^2}\right](1 + \tan x)$$

$$\frac{dy}{dx} = -sec^2 x\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]$$

$$\frac{2(1+tan^2\ x)dy}{1+\tan x \quad dx} = -sec^2\ x\left[\frac{2}{1+\tan x}\right]$$

$$\left[\frac{2sec^2\ x}{1+\tan x}\right]\frac{dy}{dx} = \frac{.2sec^2\ x}{1+\tan x}$$

$\Rightarrow$ $$\frac{dy}{dx} = -1$$ **Ans**

Q7. $y = \sqrt{\sqrt{\tan x} + \sqrt{\tan x} + \sqrt{\tan x + \ldots . \infty}}$

Prove that $(2y-1)\frac{dy}{dx} = sec^2 x$

Solution

$$y = \sqrt{\tan x + \left(\sqrt{\tan x + \sqrt{\tan x + \ldots \ldots \infty}}\right)}$$

$$y^2 = \tan x + \sqrt{\tan x + \cdots \cdots \propto}$$

$$y^2 = \tan x + y$$

Differentiating w.r.t. 'x'

$$2y\frac{dy}{dx} = sec^2 x + \frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = sec^2 x$$

$\Rightarrow (2y-1)\frac{dy}{dx} = sec^2 x$ **Ans**

Q8. $\boldsymbol{if\ x = acos^3 \emptyset y = b\ sin^3 \emptyset}$

show that $a\frac{dy}{dx} + b\tan\emptyset = \emptyset$

Solution

$$x = acos^2\emptyset$$

Differentiating w.r.t. 'Ø' we have

$$\frac{dx}{d\emptyset} = \frac{d}{dx}(a\ cos^3\emptyset)$$

$$= 3acos^2\emptyset(-sin\emptyset)$$

$$\Rightarrow \quad \frac{dx}{d\emptyset} = 3acos^2\emptyset\ sin\emptyset$$

$$\frac{d\emptyset}{dx} = -\frac{1}{3a\ cos^2\ \emptyset\ sin\emptyset}$$

$Now \quad y = bsin^3\ \emptyset$

$diff.w.r.t.\emptyset\ we\ have$

$$\frac{dy}{d\emptyset} = \frac{d}{d\emptyset}(b\ sin^3\emptyset)$$

$$= b(3sin^2\ \emptyset.\cos\emptyset)$$

$$= 3b\ sin^2\ \emptyset.\cos\emptyset$$

$$Now\ \frac{dy}{dx} = \frac{dy}{d\emptyset} \times \frac{d\emptyset}{dx}$$

$$= 3b\ sin^2\ \emptyset.\cos\emptyset \times \frac{-1}{3a\ cos^2\emptyset\ sin\emptyset}$$

$$\frac{dy}{dx} = -\frac{b}{a}\tan\emptyset$$

$$a\frac{dy}{dx} = -b\tan\emptyset$$

$\Rightarrow a\frac{dy}{dx} + btan\emptyset = 0$ Ans

Q9. $\boldsymbol{Find\ \frac{dy}{dx}\ if\ x = a(\cos t + \sin t)}$

$$y = a(\sin t - t\cos t)$$

Solution

$x = a(\cos t - \sin t)$ ....(1)

$y = a(\sin t - tcost)$..........(2)

Diff.(1) w.r.t t both sides

$$\frac{dy}{dt} = a[\cos t - (\cos t - t\sin t)]$$

$$= a[\cos t - \cos t + t\sin t]$$

$$= at\sin t.$$

Diff.(1) w.r.t

$$\frac{dx}{dt} = a(-\sin t + \cos t)$$

$and\ \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= at \sin t \times \frac{1}{a(\cos t - \sin t)}$

$Now\ \frac{dy}{dx} = \frac{t \sin t}{\cos t - \sin t}$ **Ans**

Q10.Differentiate w.r.t 'x'

i. $cos^{-1} \frac{x}{a}$

Solution

$Let \quad y = cos^{-1} \frac{x}{a}$

Diff. w.r.t x

$\frac{dy}{dx} = \frac{d}{dx} cos^{-1} \frac{x}{a}$

$= -\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \frac{d}{dx}\left(\frac{x}{a}\right)$

$= -\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \frac{1}{a}$

$= -\frac{1}{\sqrt{\frac{a^2\ \ x^2}{a^2}}} \frac{1}{a}$

$= -\frac{a}{\sqrt{a^2-x^2}} \frac{1}{a}$

$= -\frac{1}{\sqrt{a^2-ax^2}}$ **Ans**

ii. $cot^{-1} \frac{x}{a}$

Solution

$Let \quad y = cot^{-1} \frac{x}{a}$

Diff.w.r.t. x

$$\frac{dy}{dx} = -\frac{1}{1+\left(\frac{x}{a}\right)^2}\,\frac{1}{a} = \frac{1}{1+\frac{x^2}{a^2}}\,\frac{1}{a}$$

$$= -\frac{1}{\frac{a^2+x^2}{a^2}}\,\frac{1}{a}$$

$$= -\frac{a^2}{a^2+x^2}\,\frac{1}{a}$$

$$= -\frac{a}{a^2+x^2}$$ **Ans**

iii. $\frac{1}{a}\, sin^{-1}\left(\frac{a}{x}\right)$

Solution

$$Let \quad = \frac{1}{a}\, sin^{-1}\frac{a}{x}$$

Diff.w.r.t. x

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{a}\, sin^{-1}\frac{a}{x}\right)$$

$$= \frac{1}{a}\left(\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\,\frac{d}{dx}\left(\frac{a}{x}\right)\right)$$

$$= \frac{1}{a}\left(\frac{1}{\sqrt{1-\frac{a^2}{x^2}}}\,\frac{-a}{x^2}\right)$$

$$= \frac{1}{a}\left(\frac{x}{\sqrt{x^2-a^2}} \times \frac{-a}{x^2}\right)$$

$$= \frac{-1}{x\sqrt{x^2-a^2}}$$ **Ans**

iv. $sin^{-1}\sqrt{1-x^2}$

Solution

$$Let \quad y = sin^{-1}\sqrt{1+x^2}$$

Diff.w.r.t. x

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{1-x^2})}} \frac{d}{dx}\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-1+x^2}} \qquad \frac{1}{2\sqrt{1-x^2}}(-2x)$$

$$= \frac{1}{\sqrt{x^2}} \qquad \frac{(-2x)}{2(\sqrt{1-x^2})}$$

$$= \frac{-2}{2(x)\sqrt{1-x^2}}$$

$$= \frac{-1}{\sqrt{1-x^2}} \qquad \textbf{Ans}$$

v. $sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Solution

$$Let \qquad y = sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$$

Diff.w.r.t. x

$$\Rightarrow = \frac{1}{\frac{x^2+1}{x^2-1}\left(\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}\right)} \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{x^2-1}{x^2+1\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}} \left[\frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}\right]$$

$$\Rightarrow \qquad \frac{(x^2-1)^2}{(x^2+1)\sqrt{(x^2+1)^2-(x^2-1)^2}} \left[\frac{2x^3-2x-2x^2-2x}{(x^2-1)^2}\right]$$

$$= \frac{-4x}{(x^2+1)\sqrt{x^4+2x^3+1-x^4+2x^2-1}}$$

$$= \frac{-4x}{(x^2-1)\sqrt{4x^2}}$$

$$= \frac{-4x}{(x^2+1)} \qquad \textbf{Ans}$$

vi. $\boldsymbol{cot^{-1}\left(\frac{2x}{1-x^2}\right)}$

Solution

$Let \qquad y = cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Diff.w.r.t. x

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2}\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$= \frac{1}{1+\frac{4x^2}{(1-x^2)^2}}\left[\frac{2(1-x^2)-(-2x)(2x)}{(1-x^2)^2}\right]$$

$$= -\frac{(1-x^2)^2}{(1-x^2)^2+4x^2}\left[\frac{2-2x^2+4x^2}{(1-x^2)^2}\right]$$

$$= \frac{-(2x^2+2)}{1+x^4-2x^2+4x^2}$$

$$= \frac{-\ (2x^2+2)}{1+2x^2+x^4}$$

$$= \frac{-2(x^2+1)}{(x^2+1)^2}$$

$$= \frac{-2}{1+x^2} \qquad \textbf{Ans}$$

vii. $\boldsymbol{cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)}$

Solution

$Let \qquad y = cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Diff.w.r.t. x

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}}\left[\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}\right]$$

$$= \frac{(1+x^2)}{\sqrt{(1+x^2)^2-(1-x^2)^2}}\left[\frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}\right]$$

$$= \frac{4x}{\sqrt{1+x^4+2x^2-1-x^4+2x^4}} \quad \left[\frac{-4x}{(1+x^2)^2}\right]$$

$$= \frac{4x}{\sqrt{4x^2(1+x^2)}}$$

$$= \frac{2}{(1+x^2)}$$ **Ans**

Q11. $\boldsymbol{if\ \frac{y}{x} = \ tan^{-1}\left(\frac{y}{x}\right)\ show\ that\ \frac{dy}{dx} = \frac{y}{x}}$

Solution

$$\frac{y}{x} = x\,tan^{-1}\left(\frac{y}{x}\right)$$

Diff.w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[xtan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\frac{dy}{dx} = 1 \times tan^{-1}\left(\frac{y}{x}\right) + \left[\frac{1}{1+\left(\frac{x}{y}\right)^2}\right]\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= tan^{-1}\frac{y}{x} + \frac{y^2}{x^2+y^2} \times \frac{y-x\frac{dy}{dx}}{y^2}$$

$$= \frac{y}{x} + \frac{x}{x^2+y^2}\left[y - x\frac{dy}{yx}\right] from\ above$$

$$\frac{dy}{dx} \quad = \frac{y}{x} + \frac{xy}{x^2+y^2} - \frac{x^2}{x^2+y^2}\ \frac{dy}{dx}$$

$$\left[1+\frac{x^2}{x^2+y^2}\right]\frac{dy}{dx} = y\,\frac{(x^2+y^2)+x^2y}{x(x^2+y^2)}$$

$$\left[1+\frac{x^2+y^2+x^2}{x^2+y^2}\right]\frac{dy}{dx} = y\left[\frac{x^2+y^2+x^2}{x(x^2+y^2)}\right]$$

$$\left(\frac{2x^2+y^2}{x^2+y^2}\right)\frac{dy}{dx} = y\left[\frac{x^2+y^2+x^2}{x(x^2+y^2)}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$ **Ans**

Q12. If $y = \tan(ptan^{-1}x)$ show that

$$(1 + x^2)y_1 - p(1 + y^2) = 0$$

Solution

$$y = \tan(ptan^{-1}x)$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}[\tan(ptan^{-1}x)] \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$y_1 = sec^2(ptan^{-1}).\frac{d}{dx}(ptan^{-1}x)$$

$$= 1 + tan^2(ptan^{-1}x).p\frac{1}{1+x^2}$$

From(1)

$$(1 + x^2)y_1 = p(1 + y^2)$$

$$(1 + x^2)y_1 = -p(1 + y^2) = 0$$

Hence proved.