Solved Exercise 2.4

Q1. Find $\frac{dy}{dx}$ making suitable substitutions in the following function defined as:

i. $y = \sqrt{\frac{1-x}{1+x}}$

Solution

$$y = \sqrt{\frac{1-x}{1+x}}$$

Let $u = \frac{1-x}{1+x}$ ....(1) then

$$y = u^{\frac{1}{2}} \text{ ....(2)}$$

Diff. w.r.t x, we name

$$\frac{du}{dx} = \frac{(-1)(1+x)-(1)(1-x)}{(1+u)^2}$$

$$= \frac{-1-x-1+x}{(1+x)^2}$$

$$= \frac{-2}{(1+x)^2}$$

Diff. w.r.t u

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} \Rightarrow \frac{1}{2}u^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{u}}$$

By Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{x}} \cdot \frac{-2}{(1+x)^2}$$

But $u = \frac{1-x}{1+x} \Rightarrow \sqrt{u} = \sqrt{\frac{1+x}{1-x}}$

$= \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \times \left(\frac{-2}{(1+x)^2}\right)$

$= \frac{-\sqrt{1+x}}{(\sqrt{1-x})(1+x)^2}$

$= \frac{-1}{\sqrt{1-x}\ (1+x)^{1/2}}$

ii. $\boldsymbol{y = \sqrt{x + \sqrt{x}}}$

Solution

Let $u = \sqrt{x + \sqrt{x}}$ ....(1)

Then $y = \sqrt{u} = u^{\frac{1}{2}}$ ....(2)

From(1) $\frac{du}{dx} = 1 + \frac{1}{2\sqrt{u}}$

$= \frac{2\sqrt{x+1}}{2\sqrt{x}}$

Also from (2)

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$

By Chain rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \cdot \frac{2\sqrt{x}+1}{2\sqrt{x}}$

But $u = x + \sqrt{x}$

$= \frac{1}{2\sqrt{x+\sqrt{x}}} \times \frac{2\sqrt{x}+1}{2\sqrt{x}}$

$= \frac{2\sqrt{x}+1}{4\sqrt{x}\ \sqrt{x+\sqrt{x}}}$

iii. $Y = x\sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{x^2(a+x)}{a-x}} = \sqrt{\frac{(ax^2+x^2)}{a-x}}$

Solution

Let $t = \frac{ax^2+x^3}{a-x}$ $therefore\ y = \sqrt{t} = t^{1/2}$

$$\frac{dt}{dx} = \frac{d}{dx}\left(\frac{ax^2+x^3}{a-x}\right)$$

$$= \frac{(a-x)d/dx(ax^2+x^3)-(ax^2+x^2)d/dx(a-x)}{(a-x)^2}$$

$$= \frac{(a-x)(2ax+3x^2)-(ax^2+x^2)(0-1)}{(a-x)^2}$$

$$= \frac{2a^2x+3ax^2-2ax^2-3x^3+ax^2+x^3}{(a-x)^2}$$

$$= \frac{2a^2x+2ax^2-2x^3}{(a-x)^2}$$

$$= \frac{2x(a^2+ax-x^2)}{(a-x)^2}$$

$$and\ \frac{dy}{dx} = \frac{d}{dt}(t)^{1/2}$$

$$= \frac{1}{2}t^{-1/2}$$

$$= \frac{1}{2\sqrt{t}}$$

$$= \frac{1}{2\sqrt{\frac{ax^2+x^3}{a\ x}}}$$

$$= \frac{\sqrt{a-x}}{2\times\sqrt{a+x}}$$

By chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= \frac{a-x}{2x\sqrt{a+x}}\times\frac{2x(a^2+ax-x^2)}{(a-x)^2}$$

$$= \frac{a^2+ax-x^2}{\sqrt{a+x}(a-x)^2(a-x)^{1/2}}$$

$$= \frac{a^2+ax-x^2}{\sqrt{a+x}(a-x)^{1/2}} \quad \text{Ans}$$

iv. $\boldsymbol{y = (3x^2 - 2x + 7)^6}$

Solution

Let $u = 3x^2 - 2x+7$..........(1)

Then $y = u^2$ ... ...(2)

From (1)

$$\frac{du}{dx} = \frac{d}{du}(u)^6 = 6u^5$$

Thus

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Chain rule

$$= 6u^5(6x-2) = 6(3x^2-2x+7)^5 - 2(3x-1)$$

$$= 12(3x-1)(3x^2-2x=7)^5$$

v. $\boldsymbol{y = \sqrt{\frac{a^2+x^2}{a^2-x^2}}}$

Solution

$$y = \sqrt{\frac{a^2+x^2}{a^2-x^2}}$$

Let $y = u^{1/2}$ ....(1)

Where $u = \frac{a^2+x^2}{a^2-x^2}$........(2)

From (1) $\frac{dy}{du} = \frac{d}{du}\left(u^{1/2}\right) = \frac{1}{2}u^{-1/2}$

$$= \frac{1}{2}\left[\frac{a^2+x^2}{a^2-x^2}\right]^{-1/2} = \frac{1}{2}\left[\frac{a^2-x^2}{a^2-x^2}\right]^{-1/2}$$

From (2) $\frac{dy}{dx} = \frac{(a^2-x^2)(2x)-(a^2+x^2)(-2x)}{(a^2-x^2)^2}$

$$= \frac{4a^2x}{(a^2-x^2)^2}$$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2}\frac{(a^2-x^2)^{1/2}}{(a^2+x^2)^{1/2}} \qquad \frac{4a^2x}{(a^2-x^2)^2}$$

$$= \frac{2a^2x}{\sqrt{(a^2+x^2)}\,(a^2-x^2)^{3/2}}$$ **Ans**

Q2. Find $\frac{dy}{dx}$ if

(i)$3x + 4y + 7 = 0$

Solution

$$3x + 4y + 7 = 0$$

Diff.w.r.t. x

$$3 + 4\frac{dy}{dx} = 0$$

$$4\frac{dy}{dx} = -3$$

$$\frac{dy}{dx} = \frac{-3}{4}$$ Ans

ii. Find $\frac{dy}{dx}$ $if\ xy + y^2 = 2$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}[xy] + \frac{d}{dx}y^2 = \frac{d}{dx}(2)$$

$$x\frac{dy}{dx} + y(1) + 2y\frac{dy}{dx} = 0$$

$$x\frac{d}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x+2y}$$ **Ans**

iii. $\boldsymbol{x^2 - 4xy - 5y = 0}$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}(x^2) - 4\frac{d}{dx}(xy) - 5\frac{d}{dx}(y) = \frac{d}{dx}(0)$$

$$2x - 4x\left[\frac{dy}{dx} + (1)y\right] - 5\frac{dy}{dx} = 0$$

$$(4x + 5)\frac{dy}{dx} = \frac{2x-4y}{4x+5} = \frac{2(x-2y)}{4x+5}$$ **Ans**

iv. Find = if $\boldsymbol{4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}(4x^2 + 2hxy + by^2 + 2gx + 2fy + c) = \frac{d}{dx}(0)$$

$$8x + 2h\left(\left[x\frac{ay}{ax} + y\right] + b\left[2y\frac{dy}{dx}\right] + 2g + 2f\frac{dy}{dx}\right) = 0$$

$$2(hx + by + f)\frac{dy}{dx} = -2(4x + hy + g)$$

$$\frac{dy}{dx} = \frac{-2(4x+hy+g)}{2(hx"+by+f)}$$

$$\frac{dy}{dx} = \frac{4x+hy+g}{hx+by+f}$$ **Ans**

v. if x $\sqrt{1+y} + y\sqrt{1+y} = 0$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}(x.\sqrt{1+y}.1 + \frac{d}{dx}\left(y\sqrt{1+x}\right) = 0$$

$$\left(\sqrt{1+y}\right)(1) + \frac{x}{2\sqrt{1+y}}\frac{dy}{dx} + \sqrt{1+x}\frac{dy}{dx} + \frac{1}{2\sqrt{1+h}} = 0$$

$$\left[\frac{x}{2\sqrt{1+y}} + \sqrt{1+x}\right]\frac{dy}{dx} = \frac{-y}{2\sqrt{x+1}} - \left[\frac{y+2\sqrt{1}+h(\sqrt{1+y})}{2\sqrt{x+1}}\right]$$

$$\frac{dy}{dx} = \frac{(y+2\sqrt{1}+x\sqrt{1+y}}{2\sqrt{1+x}} \cdot \frac{2\sqrt{1+y}}{x+2\sqrt{1+x.\sqrt{1+y}}}$$

$$\frac{dy}{dx} = \frac{\sqrt{1+y}(y+2\sqrt{(1+x)(1+y)}}{\sqrt{1+x}(x+2\sqrt{(1+x)(1+y)}}$$ **Ans**

vi. If $y(x^2-1) = x.\sqrt{x^2}+4$

Solution

Diff.w.r.t. x

$$\frac{d}{dx}(y?\,(x^2-1))\frac{d}{dx}\left(x.\sqrt{x^2+4}\right)$$

$$\frac{dy}{dx} \quad (x^2-1) + y(2x) = 1.\sqrt{x^2}+4+x.\frac{2x}{2\sqrt{x^2+4}}$$

$$(x^2-1)\frac{dy}{dx} = \sqrt{x^2}+4+\frac{x^2}{\sqrt{x^2+4}} - 2xy$$

$$\frac{dy}{dx} = \frac{x^2+4+x^2}{\sqrt{x^2+4}} - 2xy$$

$$= \frac{2(x^2+2)-2xy\sqrt{x^2+4}}{\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{2(x^2+2)-2-xy\sqrt{x^2+4}}{(x^2-1)\sqrt{x^2+4}}$$

$$= \frac{-2(3x^2+2)}{\sqrt{x^2+4}(x^2+1)^2} \quad \textbf{Ans}$$

Q3. Find $\frac{dy}{dx}$ of the following parametric functions.

i. $\boldsymbol{\theta + \frac{1}{\theta}, y = \theta + 1}$ **ii.** $\boldsymbol{x = \frac{a(1-t)}{1+t^2}, y = \frac{2bt}{1+t^2}}$

i. $\boldsymbol{\theta + \frac{1}{\theta}, y = \theta + 1}$

Solution

Take $x = \theta + \theta^{-1}$

Diff. w.r.t θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\theta + \theta^{-1})$$

$$= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\theta^{-1})$$

$$= 1 - \theta^{-2}$$

$$= 1 - \frac{1}{\theta^2}$$

$$\frac{dx}{d\theta} = \frac{\theta^2-1}{\theta^2}$$

$$y = \theta + 1$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta + 1)$$

$$= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(1)$$

$$= 1 + 0$$

$$\frac{dy}{d\theta} = 1$$

By chain rule.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 1 \times \frac{\theta^2}{\theta^2 \ 1} = \frac{\theta^2}{\theta^2 - 1}$$ **Ans**

ii. $x = \frac{a(1-t)}{1+t^2}, y = \frac{2bt}{1+t^2}$

Solution

$$x = \frac{a(1-t^2)}{1+t^2}, y = \frac{2bt}{1+t^2}$$

Take $\frac{dx}{dt} = \frac{d}{dt}\left[\frac{a(1-t^2)}{1+t^2}\right]$

And $\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2bt}{1+t^2}\right)$

$$= \frac{(1+t^2)^d/_{dt}(2bt) - 2bt^d/_{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(2b) - 2bt(0+2t)}{(1+t^2)^2}$$

$$= \frac{2b + 2bt^2 4bt^2}{(1+t^2)^2}$$

$$= \frac{2b - 2bt^2}{(1+t^2)^2}$$

$$= \frac{2b(1-t^2)}{(1+t^2)^2}$$

By chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at}$$

$$= -\frac{b(1-t^2)}{2at}$$ **Ans**

Q4. Prove that $y\frac{dy}{dx} + x = 0\ if$

If x= $\frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

Solution

Take $y = \frac{2t}{1+t^2}$

Diff.w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{(1+t)2-2t(2t)}{(1+t^2)^2}$$

$$= \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$= \frac{2(1-t^2)}{(1+t^2)^2}$$

Also, x $= \frac{1-t^2}{1+t^2}$

Diff.w.r.t. x

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}$$

$$= \frac{-2t-2t^2-2t+2t^2}{(1+t^2)^2}$$

$$= \frac{-4t}{(1+t^2)^2}$$

By chain rule

$$\Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4t}$$

$$\frac{dy}{dx} = \frac{t^2-1}{2t}$$

Now we prove

$$y\frac{dy}{dx}+x=0$$

$$=\left(\frac{2t}{1+t^2}\right)\left(\frac{t^2-1}{2t}\right)+\frac{1-t^2}{1+t^2}$$

$$=\frac{t^2-1}{1+t^2}+\frac{1-t^2}{1+t^2}$$

$$=\frac{t^2-1+1-t^2}{1+t^2}$$

$$=\frac{0}{1+t^2}$$

$$=0$$

$= R.H.S$ **Ans**

Q5. Differentiate

i. $x^2-\frac{1}{x^2}\quad w.r.t.\, x^4$

Solution

Let $y=x^2-\frac{1}{x^2}$ ....(1)

and $u=x^4$ .... (2)

1. $\Rightarrow \frac{dy}{dx}=2x+\frac{2}{x^3}$

2. $\Rightarrow \frac{du}{dx}=4x^3$

By chain rule

$$\frac{dy}{du}=\left(2x+\frac{2}{x^3}\right)\left(\frac{1}{4x^3}\right)$$

$=\frac{x^4+1}{2x^4}$ **Ans**

ii. $(1+x^2)^n\quad w.r.t.\; x^2$

Solution

Let $y = (1 + x^2)^n$.....(1)

and $u = x^2$ .....(2)

(1) $\Rightarrow$ $\frac{dy}{dx} = n(1 + x^2)^{n-1}(2x)$

$= 2nx(1 + x^2)^{n-1}$

(2) $\Rightarrow$ $\frac{dn}{du} = \frac{dy}{dx} . \frac{dn}{du}$

$= \frac{2nx(1+x^2)^{n-1}}{2x}$

$= n(1 + x^2)^{n-1}$ **Ans**

iii. $\frac{x^2+1}{x^2-1}$ $w.r.t$ $\frac{x-1}{x+1}$

Solution

Let $y = \frac{x^2+1}{x^2-1}$.......(1)

and $u = \frac{x-1}{x+1}$ ....(2)

(1) $\Rightarrow$ $\frac{dy}{dx} = \frac{2x(x^2-1)-(2x)(x^2+1)}{(x^2-1)^2}$

$= \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$

$= \frac{-4x}{(x^2-1)^2}$

And (2) $\Rightarrow$ $\frac{du}{dx} = \frac{1(x+1)-1(x-1)}{(x+1)^2}$

$= \frac{x+1-x+1}{(x+1)^2}$

$= \frac{2}{(x+1)^2}$

Now $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$

$$= \frac{-4x}{(x^2-1)^2} \frac{(x-1)^2}{2}$$

$$= \frac{-2x(x+1)^2}{(x-1)^2(x+1)^2}$$

$\frac{dy}{du} = \frac{-2X}{(X-1)^2}$ **Ans**

iv. Let $y = \frac{ax+b}{cx+d}$. w.r.t $\frac{ax^2+b}{ax^2+d}$

Solution

Let $y = \frac{ax+b}{cx+d}$ (1)

and $u = \frac{ax^2+b}{cx^2+d}$ (2)

(1) $\Rightarrow$ $\frac{dy}{dx} = \frac{a(cx+d)-c(ax+b)}{(cx+d)^2}$

$$= \frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2}$$

(2) $\Rightarrow$ $\frac{du}{dx} = \frac{2ax(cx^2+d)-2cx(ax^2+b)}{(cx^2+d)^2}$

$$= \frac{2acx^3+2adx-2acx^3-2bcx}{(cx^2+d)^2}$$

$$= \frac{2adx-2bcx}{(cx^2+d)^2}$$

$$= \frac{2ax(d-b)}{(ax^2+d)^2}$$

Now $\frac{dy}{du} = \frac{dy}{dx} + \frac{dn}{du}$

$$\frac{dy}{du} = \frac{ad-bc}{(cx+d)^2} \times \frac{(cx^2+d)^2}{2x(ad-bc)}$$

$$= \frac{(ad-bc)}{(cx^2+d)^2} \times \frac{(ax^2+d)^2}{2ax(d-b)}$$

$$= \frac{(ad-bc)(ax^2+d)^2}{2ax(d-b)^2(cx+d)^2}$$ **Ans**

v. $\frac{x^2+1}{x^2-1}$ $w.r.t.$ x^3

Solution

Let $y = \frac{x^2+1}{x^2-1}$.........(1)

$u = x^3$ ....(2)

Diff.(1)Both Sides

$$\frac{dy}{dx} = \frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}$$

$$= \frac{2x^3-2x+2x^3(x^2+1)}{(x^2-1)^2}$$

$$= \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

and $\frac{dy}{dx} = 3x^3$

By chain rule

Now $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$

$$= \frac{-4x}{(x^2-1)^2} \times \frac{1}{3x^3}$$

$$= \frac{-4}{3x(x^2-1)^2}$$ **Ans**