Solved Exercise 2.10

Q1. Find two positive integers whose sum is 30 and their product will be maximum.

Solution

Let x and $30 - x$ be two required integers.

So that $f(x) = x(30 - x)$

$f(x) = 1(30 - x) + x(-1)$

$= 30 - x - x = 30 - 2x = 2(15 - x)$

$f'(x) = 0 \quad \Rightarrow 2(15 - x) = 0$

$\Rightarrow \; x = 15$

$f'(x) = 2(0 - 1) = -2$ which is negative.

So $x(30 - x)$ is maximum if $x = 15$ and other positive integers is 30-15=15 thus the required two positive integers are 15 and 15.

Q2. Divide 20 into two parts so that sum of their square will be minimum.

Solution

Let the required two parts of x and $20 - x$.

Let $f(x) = x^2 + (20 - x)^2$, then

$f'(x) = 2x + 2(20 - x)(-1)$

$= 2[x - 20 + x] = 2(2x - 20)$

$= 4(x - 10)$

$f^{`}(x) = 0 \quad \Rightarrow 4(x-10) = 0$

$\Rightarrow \quad x = 10$

$f^{`}(x) = \frac{d}{dx}[4(x-10)] = 2(2x-20)$

$f^{`}(x) = 4(x-10)$

$f^{`}(x) = 0 \quad \Rightarrow \quad 4(x-10) = 0$

$\Rightarrow \quad x = 10$

$f^{`}(x) = \frac{d}{dx}[4(x-10)] = \frac{4d}{dx}(x-10)$

$= 4$ which is positive.

So $x^2(20-x)^2$ is minimum if $x = 10$

Thus, the required two parts of 20 are 10 and 10.

Q3. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Solution

Let x and $12 - x$ be two positive integers

Let $f(x) = x(12-x)^2$then

$f^{`}(x) = 1(12-x)^2 + x[2(12-x)(-1)]$

$= (12-x)^2 - 2x(12-x)$

$= (12-x)\,[12-x-2x]$

$= (12-x)\,[12-3x]$

$= 3(12-x)\,(4-x)$

$= 3(x)^2 - 48x - 144$

$\Rightarrow \quad f^{'}(x) = 0 \quad \Rightarrow = 0$

$\Rightarrow \quad 12 - x = 0 \; or \; x = 4$

$x = 12$ is not +ive

as $12 - x = 12 - 12 = 0$

Now $f^{''}(x) = \frac{d}{dx}[3(12 - x)(4 - x)]$

$= -3[4 - x + 12 - x]$

$= -3(16 - 2x)$

$= -6(8 - x)$

Now $\quad f^{''}(x) = -6(8 - x)$

$f^{''}(x) = -24$ which is -ive

f(x) gives the max value if $x = 4$. The other +ve integers is because 12-8=4

So, first integers $= x = 4$

And second integers $= 8$

Q4. The perimeter of a triangle is $16cm$, if one side of the length of $6cm$ what are the length of the other side for max. area of the triangle?

Solution

Let the length of the unknown side be $x(cm)$, then the length of other unknown side will be

$16 - 6 - x = 10 - x.$

Let y denote the square of the area of the triangle, then we have

$y = S(S - 6)(S - x)[S - (10 - x)]$

Hence $S = \frac{16}{2} = 8$ so,

$$y = 8(8-6)(8-x)[8-10+x]$$

$$y = 8 + 2(8-x)(x-2)$$

$$\frac{dy}{dx} = 16[(-1)(x-2) + (8-x)(1)]$$

$$= 16[2 - x + (8-x)(1)]$$

$$= 16(10-2x)$$

$$= 32(5-x)$$

Put $\frac{dy}{dx} = 0 \Rightarrow 32(5-x) = 0$

$5 - x = 0 \Rightarrow x = 5$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(32(5-x))$$

$$= 32(-1)$$

$$= -32 < 0$$

Which is negative. Thus, y is maximum if $x = 5$, i.e. the area of A is maximum if $x = 5$, and the length of the other side is $10 - 5 = 5$.

Q5. Find the dimension of a rectangle of largest area having perimeter $120cm$.

Solution

Let x be the length of the rectangle and y be the length of the breadth of rectangle in kms, the perimeter is

$\Rightarrow 2x + 2y = 120$ given

$\Rightarrow x + y = 60$ ....(1)

Let area of triangle be A

$\Rightarrow \quad \Delta = xy$ ....(2)

$A = x(60 - x)$ using I

$\frac{dA}{dx} = \frac{d}{dx}[60 - x]$

$= 1(60 - x) + x(-1)$

$= 60 - x - x$

$= 60 - 2x$

$= 2(30 - x)$

Now $\frac{dA}{dx} = 0 \Rightarrow \quad 2(30 - x) = 0$

$\Rightarrow \quad x = 30$

Also $\frac{d^2A}{dx^2} = \frac{d}{dx}[2(30 - x)] = 2\frac{d}{dx}(30 - x)$

$= -2 < 0$ which is negative

Thus $x = 30$ gives max. value of A (The area of rectangle) So y (the breadth of the rectangle) is

$60 - 30 = 30.$

Hence the sides of the rectangle are $30cm$ and $30cm$ for largest area of its perimeter is $120cm.$

Q6. Find the lengths of the sides of the variable rectangle having area $36cm^2$ when its perimeter is minimum.

Solution

Let x and y be the length and width of sides of rectangle in (cms) then xy the area of rectangle (inSq.cm) is 36.

i.e.

$$xy = 36 \quad \Rightarrow \quad y = \frac{36}{x} \text{ ....(1)}$$

Let p be the perimeter of the rectangle then.

$$p = 2x + 2y$$

$$= 2(x + y)$$

$$= 2\left(x + \frac{36}{x}\right) \because y = \frac{36}{x}$$

Now $\frac{dp}{dx} = 2\left[1 + 36\left(\frac{-1}{x^2}\right)\right]$

$$= 2\left(1 - \frac{36}{x^2}\right)$$

$$= 2\left(\frac{x^2 - 36}{x^2}\right)$$

$\frac{dp}{dx} = 0$ gives $2\left(\frac{x^2-36}{x^2}\right) = 0$

$\Rightarrow x^2 - 36 = 0$

$\Rightarrow \quad x = 6 \quad \because x$ is positive

$$\frac{d^2p}{dx^2} = 2\frac{d}{dx}\left(1 - \frac{36}{x^2}\right) = 2\left(\frac{36\times 2}{x^3}\right)$$

$$= \frac{144}{x^3}$$

$\frac{d^2p}{dx^2}$ is positive if $x = 6$ because $\frac{144}{6\times6\times6} = \frac{2}{3}$

Thus p is min. of $x = 6$.

Putting $x = 6$ in (1)

We have $y = \frac{36}{6} = 6$ so we conclude that rectangle having area $36cm^2$has minimum perimeter if the length of its sides are $6cm$ and $6cm$.

Q7. A box with a square base and open top is to have a volume pf $4dm^3$. Find the dimension of the box which will require the least material.

Solution

Let the dimension of the box be x, x and h, then volume of the box x^2h

i.e. $x^2h = 4 \Rightarrow h = \frac{4}{x^2}$ ....(1)

Also $S = x^2 + 4xh$ where s is the total surface area.

$$S = x^2 + 4x\left(\frac{4}{x^2}\right)$$

$$S = x^2 + \frac{16}{x}$$

Now $\frac{ds}{dx} = \frac{d}{dx}\left(x^2 + \frac{16}{x}\right)$

$$= 2x - \frac{16}{x^2}$$

$$= 2\left(\frac{x^3-8}{x^2}\right)$$

Put $\frac{ds}{dx} = 0$

$\Rightarrow \quad x^3 - 8 = 0$

$\Rightarrow \quad x = 2$

$$\frac{d^2s}{dx^2} = \frac{d}{dx}\left(2x - \frac{16}{x^2}\right)$$

$$= 2 + \frac{32}{x^3}$$

$$= 2 + \frac{32}{(2)^3}$$

$$= 2 + \frac{32}{8}$$

$$= 2 + 4 = 6 > 0$$

Hence required

$\Rightarrow \frac{d^2S}{dx^2}$ is positive

Put $x = 2$ in above

$$h = \frac{4}{(2)^2} = \frac{4}{4} = 1$$

Thus, the required dimension for least material are $2dm$. $2dm$ and $1dm$.

Q8. Find the dimension of rectangular garden having perimeter 80 meter if its area is to be maximum.

Solution

Let x and y be length and breadth of rectangular garden in (meters)

$\Rightarrow p = 2x + 2y = 80$ given

$\Rightarrow x + y = 40 \ldots\ldots(1)$

Let A be the area of rectangle

$$A = xy \ldots\ldots.(2)$$

$$A = x(40 - x)$$

$\frac{dA}{dx} = (40 - x) + (0 - 1)$ $\quad\quad\quad \therefore y = 40 - x$

$= 40 - 2x$

$= 2(20 - x)$

$\frac{dA}{dx} = 0$

$2(20 - x) = 0 , x = 20$

Now $\frac{d^2A}{dx^2} = -2$ which is negative

Thus, the garden has max. area if $x = 20$ and $y = 40 - 20 = 20$ i.e. the required dimension are 20 meters and 20 meters.

Q9. An open tank of square base of side x vertical sides is to be constructed to contain a given quantity of water. Find the depth in term of x if the expense of lining the inside of the tank with lead will be least.

Solution

Let the given quantity be in cubic units and h be the depth of tank of square box x.

Then $q = x^2 h$

$$\Rightarrow h = \frac{q}{x^2}$$

And $S = x^2 + 4 \times h$

Where S is the surface area(inside) of the tank

$$= x^2 + 4x\frac{q}{x^2}$$

$$= x^2 + \frac{q}{x}$$

$$\frac{ds}{dx} = 2x + 4q\left(-\frac{1}{x^2}\right)$$

$$= 2x - \frac{4q}{x^2}$$

$$\Rightarrow 2x - \frac{4q}{x^2} = 0$$

$$\Rightarrow 2x^3 - 4q = 0$$

$$2x^3 = 4q$$

$$\Rightarrow x^3 = 2q$$

$$\Rightarrow x = (2q)^{\frac{1}{3}}$$

$$\frac{d^2s}{dx^2} = 2 + \frac{8q}{\left(2q^{\frac{1}{3}}\right)^3}$$

$= 2 + 6$

$= 8$ which is positive

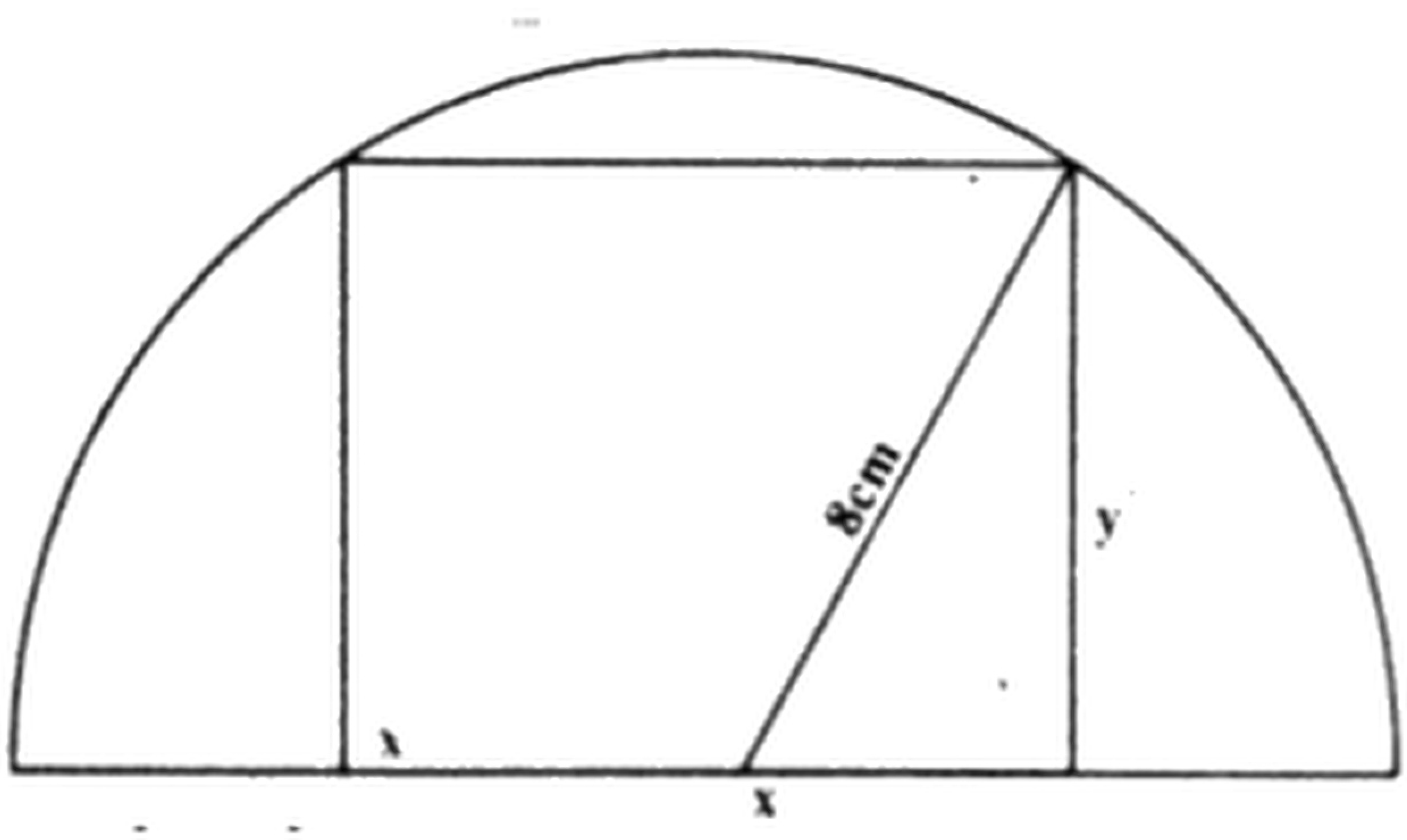
Thus, for the least expense.

$h = \frac{q}{x^2} = \frac{\frac{x^3}{2}}{x^2} = \frac{x}{2}$

Q10. Find the dimension of the rectangle of maximum area which fits inside the semi-circle of radius $8cm$ as shown in figure.

Solution

x be width and $2x$ be length of square then y is the area of the rectangle.



$y = 4x^2(64 - x^2)$

$y = 256x^2 - 4x^4$

$\frac{dy}{dx} = 512x - 16x^3$

$\frac{dy}{dx} = 0$

$\Rightarrow 512 - 16x^3 = 0 \Rightarrow 16x(3x - x^2) = 0$

$x^2 = 32$

$x = 4\sqrt{2} > 0$

Now $\frac{d^2y}{dx^2} = 512 - 48x^2$

$= 512 - 1530$

$= -1024 < 0$ for $x = 4\sqrt{2}$

x =width= $4\sqrt{2}\ cm$

$2x$ =length= $2(4\sqrt{2}) = (8\sqrt{2})cm$

Q11. Find the point on the curve $y = x^2 - 1$ that is closest to the point (3,-1).

Solution

Let l be the distance between two points (x, y) on the curve $y = x^2 - 1$and the point (3,1) then using distance formula

$l = \sqrt{(x-3)^2 + (y-(-1))^2}$

$= \sqrt{(x-3)^2 + (y+1)^2}$

$= \sqrt{(x-3)^2 + (x^2 - x + 1)^2}$ $\quad \therefore y = x^2 - 1$

$= \sqrt{(x-3)^2 + x^4}$

$\frac{dl}{dx} = \frac{1}{2=\sqrt{(x-3)^2+(x)^4}} \times (2(x-3) + 4x^3)$

$= \frac{1\times 2}{2}(x - 3 + 2x^2)$

$= \frac{1}{1}(2x^3 + x - 3)$

$= (x-1)(2x^2 + 2x + 3)$

$\frac{dl}{dx} = 0$

$(x-1)(2x^2+2x+3)=0$

$x = 2x^2+2x+3$ (gives imaginary roots)

1 and $\mathbf{2x^2+2x+3}$ are positive for $\boldsymbol{x=1}-\epsilon$ and $\boldsymbol{x=1}+\epsilon$ where ϵ is very small +ive real number.

The sign of $\frac{dl}{dx}$ depends on the factor $\boldsymbol{x-1}$

$x-1=(1-\epsilon-1)=-\epsilon<0$ if $x=1-\epsilon$.........(1)

And $x-1=(1+\epsilon-1)=\epsilon>0$ if $x=1+\epsilon$...........(2)

$\frac{dl}{dx}$ is −ive before $\boldsymbol{x=1}$ and +ive after $\boldsymbol{x=1}$, so 1 has mini. Value at $\boldsymbol{x=1}$ putting $\boldsymbol{x=1}$in $\boldsymbol{y=x^2-1}$,we have

$y=(1)^2-1=1-1=0$

Hence the required $\boldsymbol{pt}$ on the curve is $(\mathbf{1,0})$.

Q12. Find the point on the curve $y=x^2+1$that is closest to the point $(18,1)$.

Solution

Let $\boldsymbol{l}$ be the positive distance between a point $(\boldsymbol{x,y})$ on the curve $\boldsymbol{y=x^2+1}$ and the point $(\mathbf{18,1})$ then using distance formula.

$l=\sqrt{(x-18)^2+(y-1)^2}$ $\quad\therefore y=x^2+1$

$=\sqrt{(x-18)^2+(x^2+1-1)^2}$

$=\sqrt{(x-18)^2+x^4}$

$=4x^3+2x-36=0$

Let $p(x) = 4x^3 + 2x - 36$

$$p(2) = 4(2)^3 + 2(2) - 36 = 0$$

$x = 2$ is a root

So $x - 2$ is a factor of $4x^3 + 2x - 36$

Now $= 4x^3 + 2x - 36 = (x - 2)(4x^2 + 8x + 18)$

Now $x - 2 = 0 \quad \Rightarrow \quad x = 2$

And $4x^2 + 8x + 18 = 0$

$2(2x^2 + 4x + 9) = 0$ this is imaginary

$$\frac{d^2}{dx^2} = 12x^2 + 2 \qquad \Rightarrow 60 > 0$$

When $x = 2$ then $\Rightarrow \quad y = (2)^2 + 1 = 5$

$\Rightarrow (x, y) = (2,5)$