Unit 7

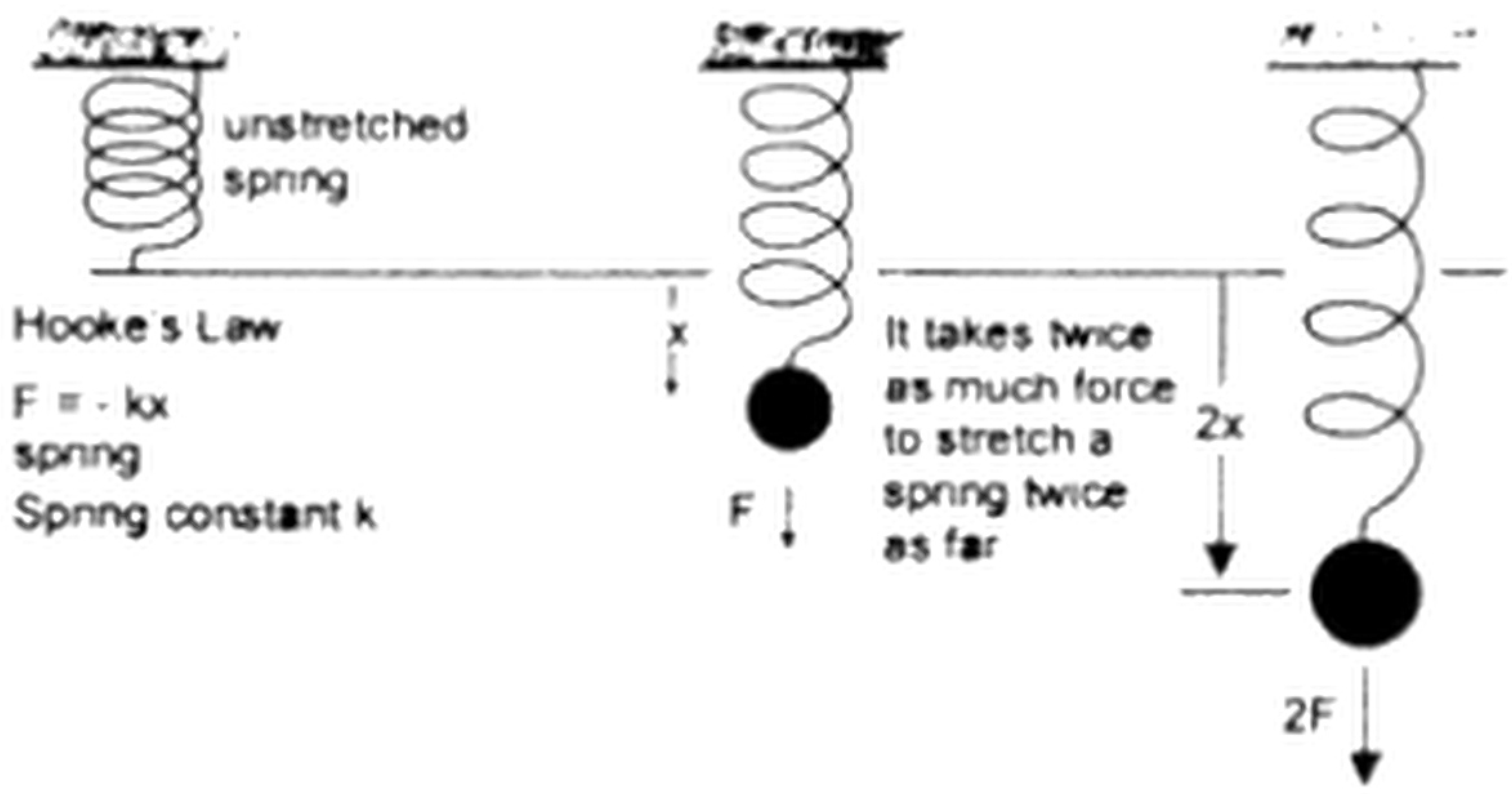
OSCILLATION

Q.1 Define Hook's law and simple harmonic motion? What is restoring force, derive the relation for acceleration of mass attached with a spring?

Answer

Hook's Law

According to Hook's Law, within elastic limit, the applied force is directly proportional to the displacement.



Mathematically

$$\vec{F} \propto \vec{x}$$

Or $$\vec{F} = k\vec{x} \quad ....(1)$$

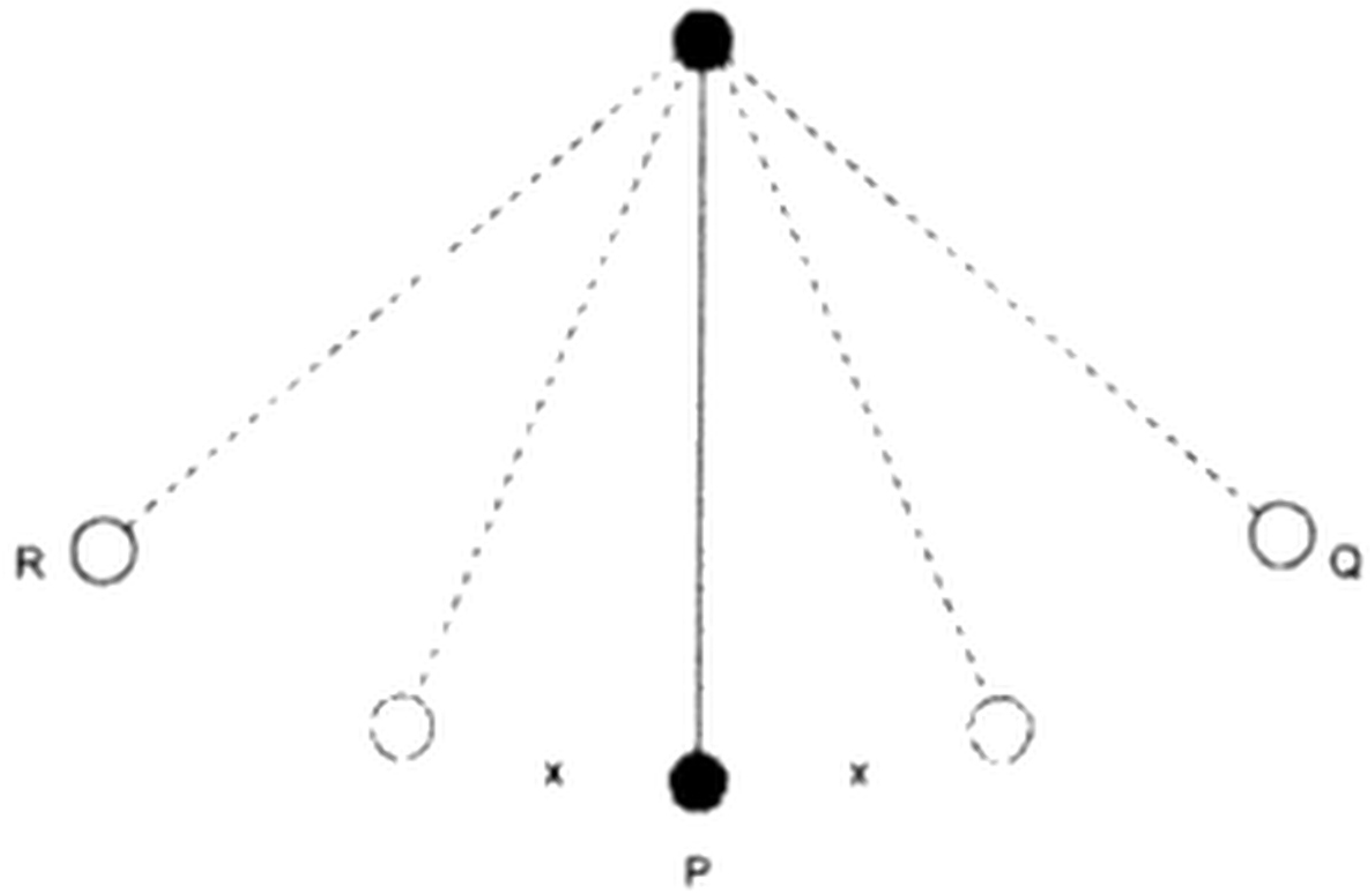
Where k is constant of proportionality, known as spring constant.

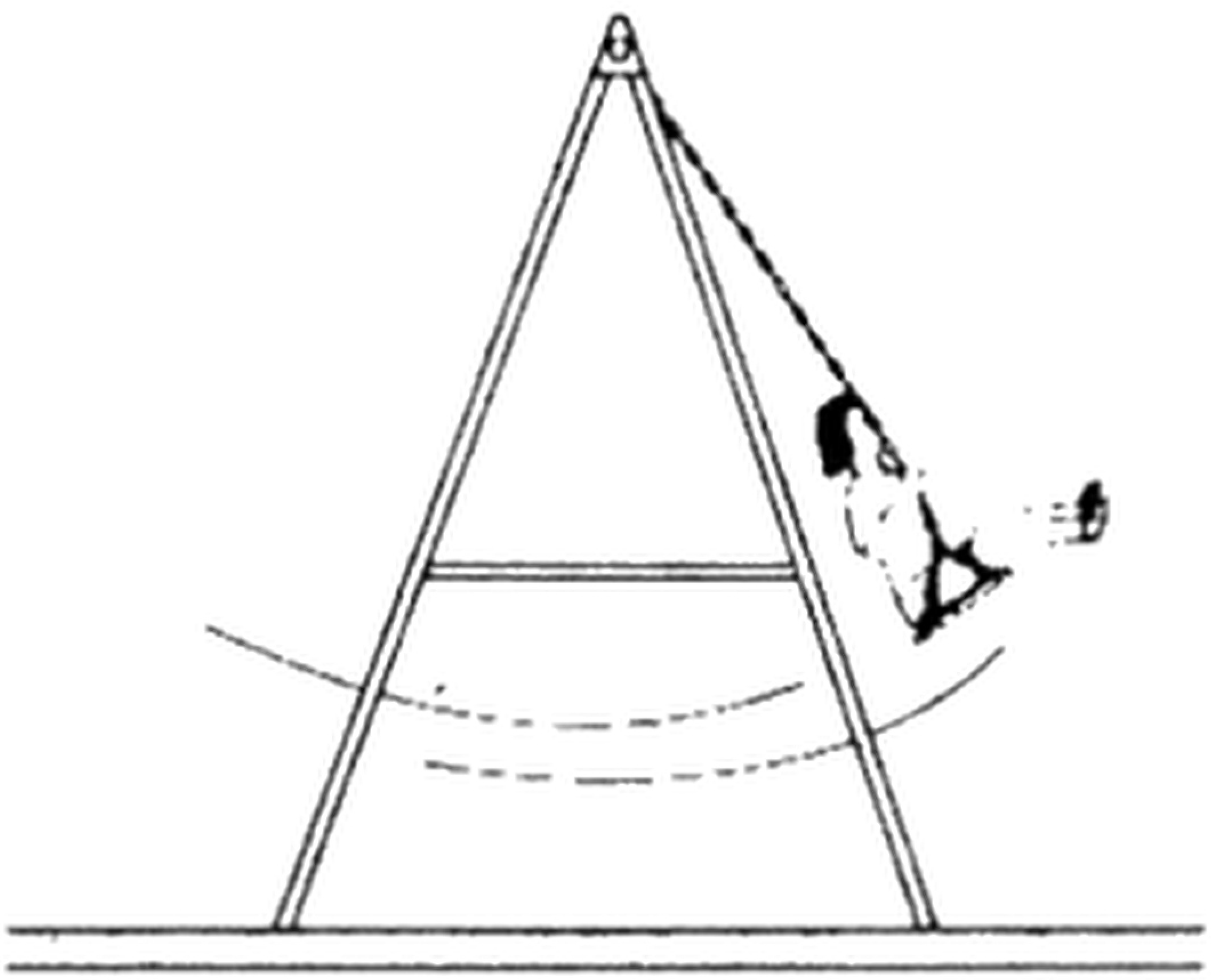
Spring Constant

Spring constant is defined as the force per unit extension. Its SI unit is Nm^{-1} and dimension is $[MT^{-2}]$.

Simple Harmonic Motion

The oscillatory motion, in which acceleration of the body at any instant is directly proportional to displacement from the mean position and directed towards the mean position and fireced towards the mean position, is called simple harmonic motion (SHM).





Examples

- Motion of simple pendulum.
- Motion of mass attached to a spring.
- Motion of a swing

Conditions for SHM

- The system must have inertia.
- The system must obey Hook's law.
- The system should have elastic restoring force.
- The system should be frictionless.

Q.2 Show that motion of mass attached with a spring is SHM.

Answer

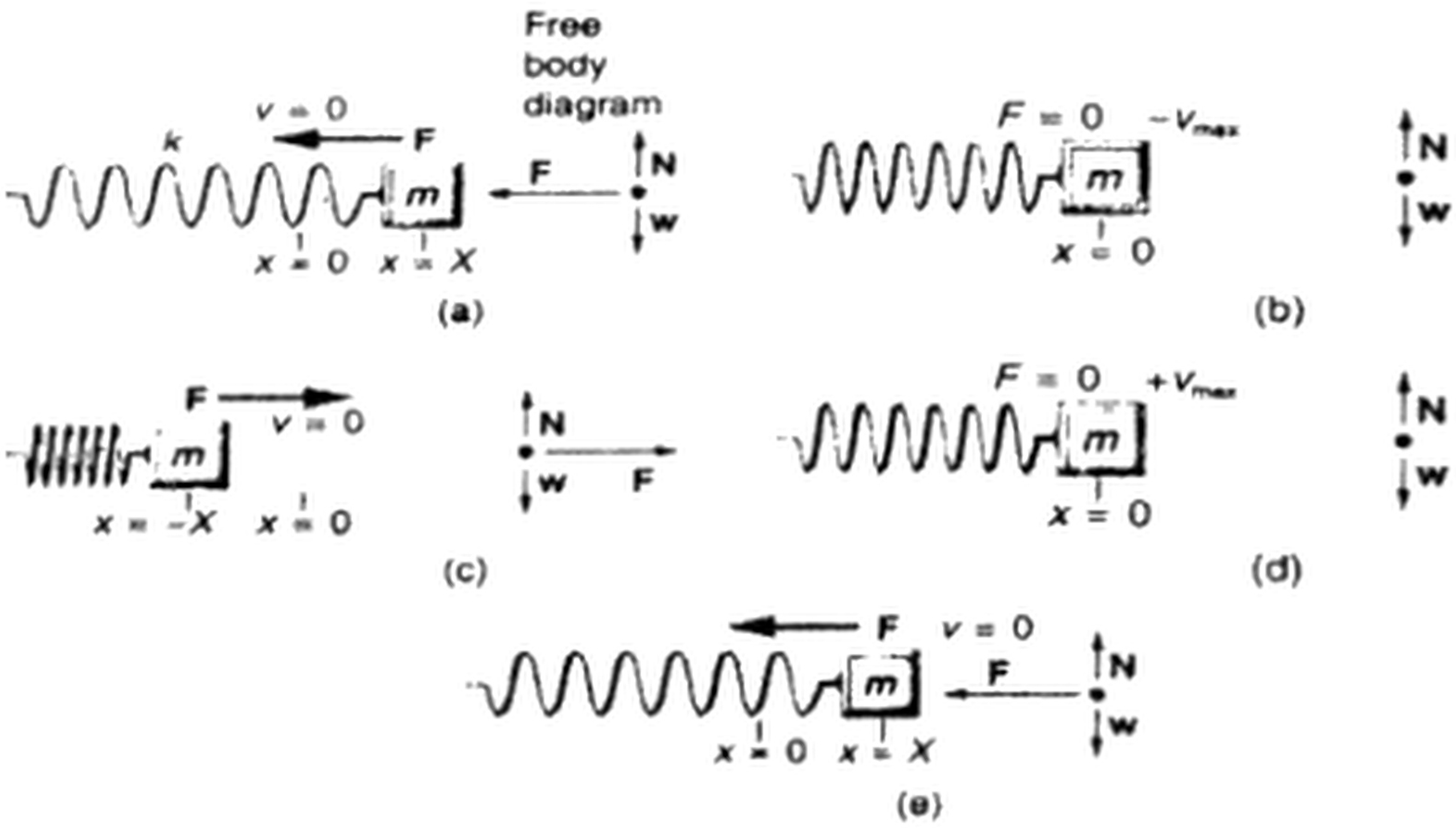
Motion of Mass attached to a spring

Consider a mass m attached with one end of the spring. The mass m can move freely on a frictionless horizontal surface as shown in figure.

When mass m is displaced through a distance x from mean position by a force F then, **According to Hooke's law**

$$\vec{F} = k\vec{x}$$

Due to elasticity, spring opposes the applied force. The opposing force is called restoring force.



Elastic restoring force

The force which is equal but opposite to applied force and brings the body back towards its mean position is called elastic restoring force.

The restoring force is represented by F_r is

$$\vec{F}_r = k\vec{x} \quad \ldots\ldots\ldots\ldots(1)$$

The negative sign shows that F_r is directed opposite to x towards mean position.

When the mass is released, it begins to oscillate about the equilibrium position as shown in figure, such type of oscillations is due to restoring force and inertia. This type of oscillatory motion is called simple harmonic motion.

Expression for acceleration

The acceleration due to restoring force F

$$\vec{F} = m\vec{a} \quad \ldots\ldots\ldots(2)$$

Comparing equations (1) and (2), we get

$$m\vec{a} = -k\vec{x}$$

$$\vec{a} = -\frac{k}{m}\vec{x} \qquad ....(3)$$

$$\vec{a} = -\text{constant}\ \vec{x}$$

$$\vec{a} \propto -\vec{x}$$ (Hence proved)

Q.3 Define the following terms relation to SHM.

- **Wave form of SHM**
- **Instantaneous displacement Amplitude**
- **Vibration**
- **Time period**
- **Frequency**
- **Angular Frequency**

Answer

Wave form of SHM

The curve representing the variation displacement with time is called wave form of SHM.

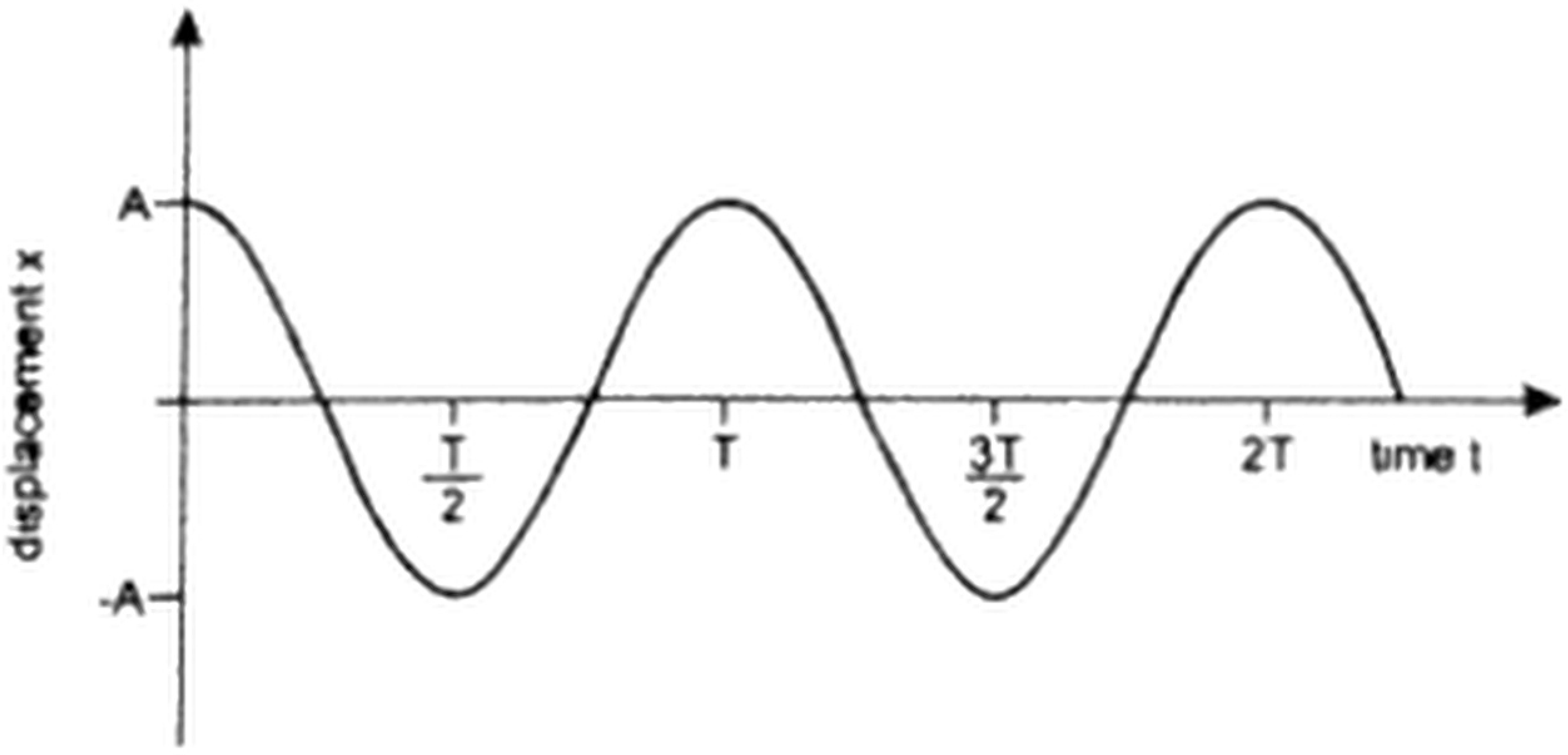
Explanation

Consider a mass spring system with vertical arrangement in such a way that pen attached with mass m from the trace on the strip of paper moving at constant speed from right to left.

So, it provides a time scale on the strip. The sine curve is obtained which shows the variation of displacement with time.

It is called wave form of SHM.

The point A, C and E show its mean position while B and D represent the extreme position.



Instantaneous displacement

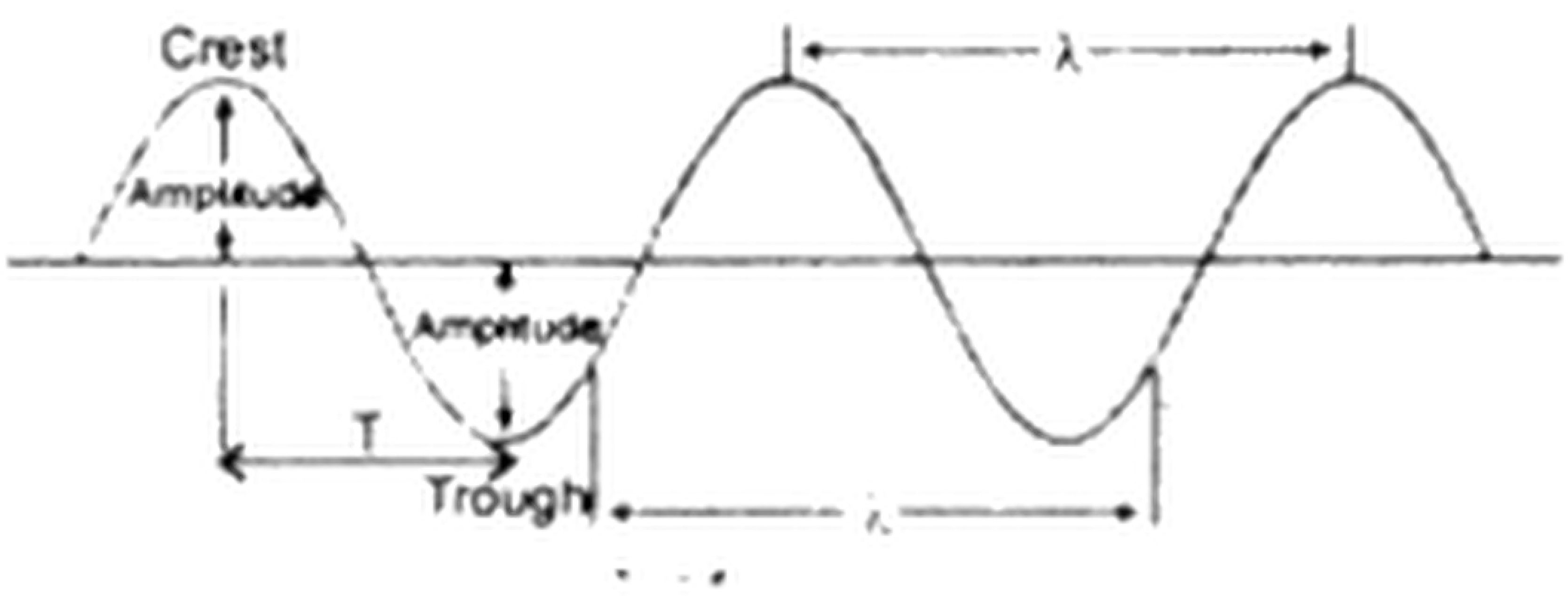
The shortest distance of the vibrating body at any instant from its mean position is called instantaneous displacement.

It is usually denoted by x. The value of instantaneous displacement is zero at mean position while it has maximum value at the extreme position.

15 -1 Characteristics of Wave Motion

Wave characteristics:

- **Amplitude, A**
- **Wavelength, λ**
- **Frequency, f and period, T**
- **Wave velocity, υ = λ f**



Amplitude (x_0)

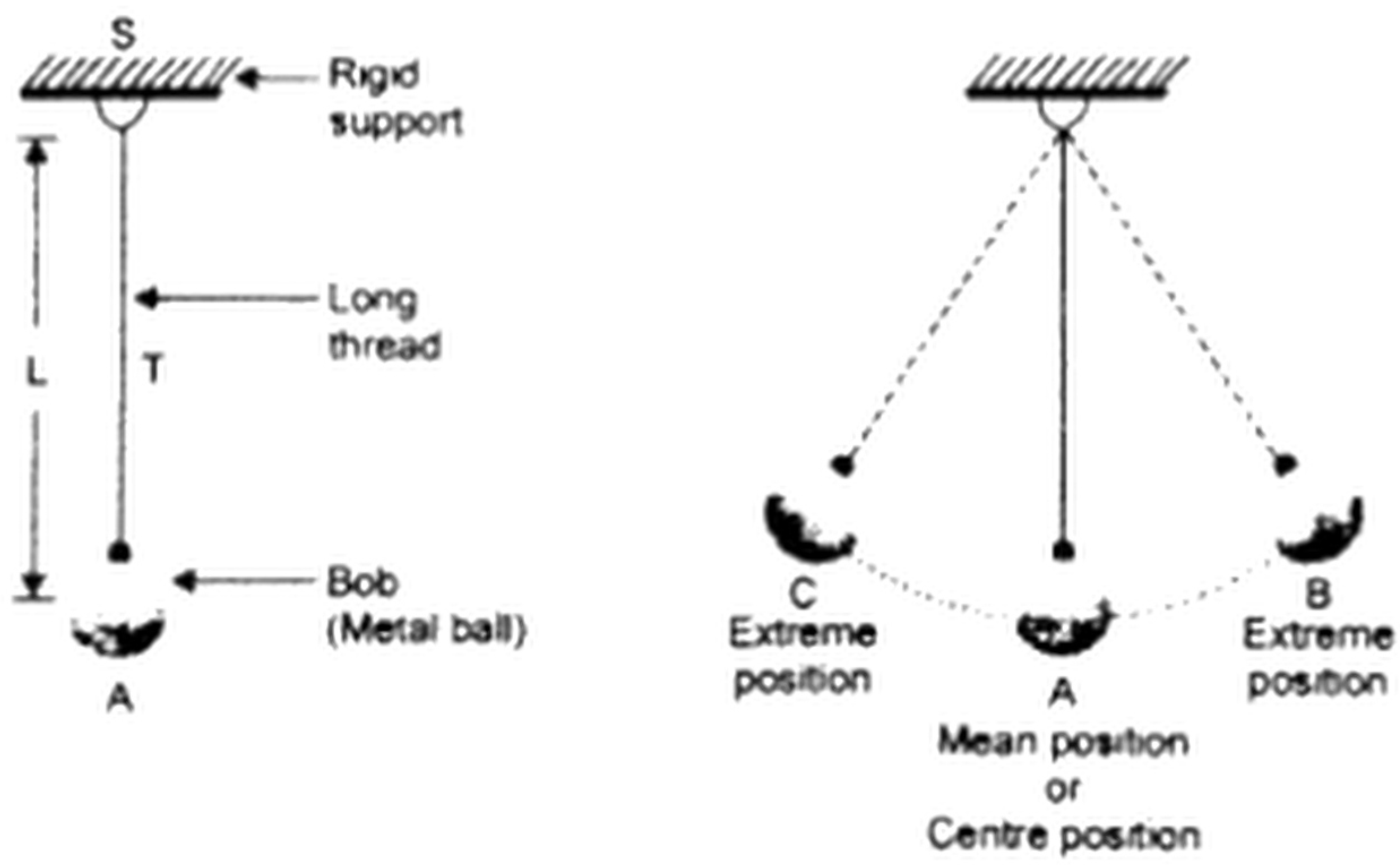
The maximum value of displacement of the vibrating body on either side from its mean position is called amplitude. It is denoted by A.

Vibration

One complete round trip of a body about its mean position is called one vibration.

Explanation

The motion of body from mean position to upper extreme position, from upper extreme position to lower extreme position and back to its mean position is called one vibration. So according to the figure ABCDE shows one vibration of the body.



Time period

The time required to complete one vibration is called time period.

It is represented by T. Its unit is second.

Frequency

The number of vibrations completed in one second by the body is called frequency. It is the reciprocal of the time period. It is represented by f. The unit of frequency is hertz or vibration/sec or cycles/sec.

Hertz

If one vibration is completed in one second then frequency is one hertz.

Mathematically

$$f = \frac{1}{T}$$

Or $$f \times T = 1$$

(i.e. product of frequency and time period equal one hertz)

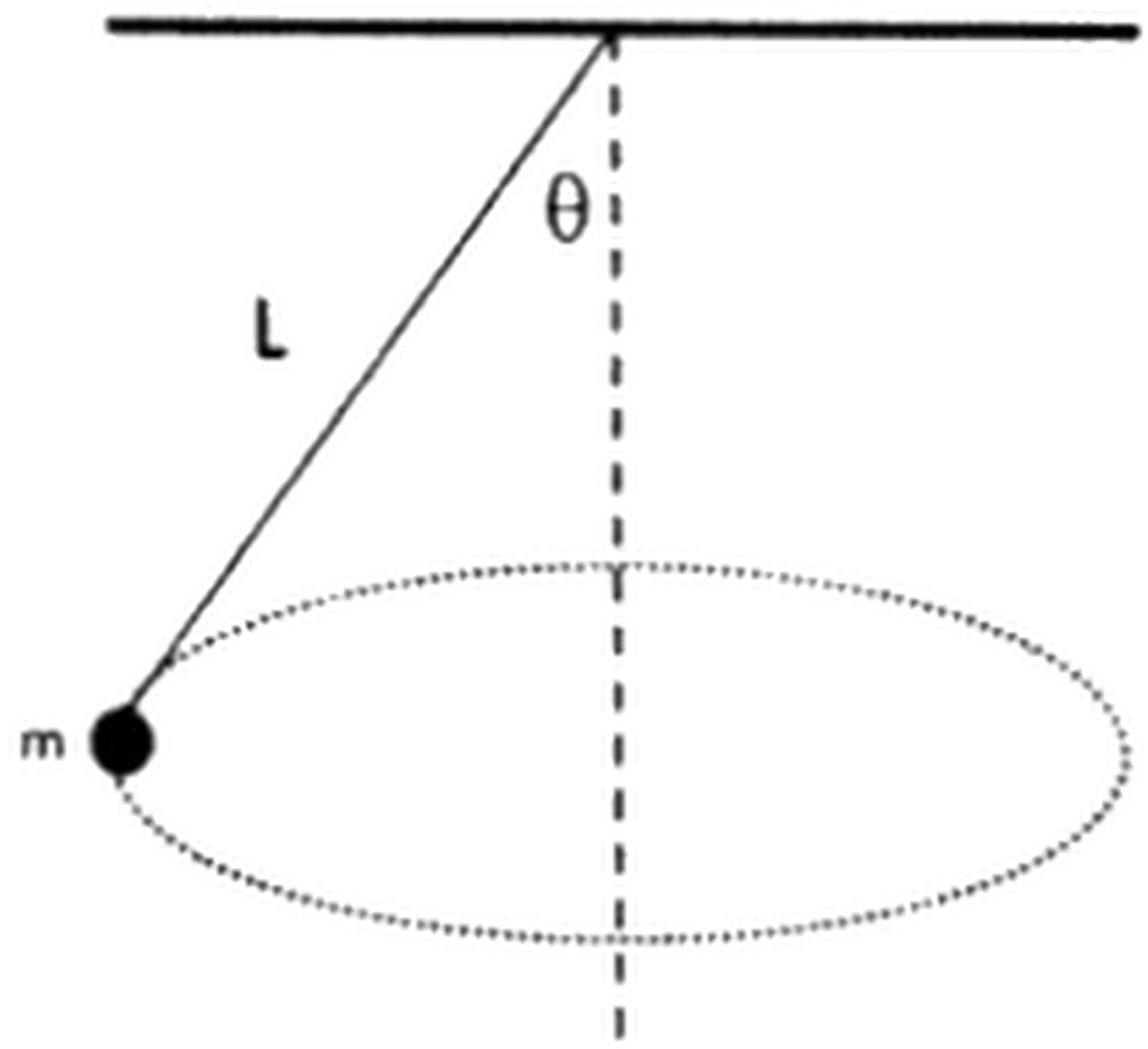
Angular Frequency

If T is the time period of a body executing SHM, its angular frequency (ω) is given as

$$\omega = \frac{2\pi}{T}$$

$$\omega = 2\pi\frac{1}{f}$$

$$\omega = 2\pi f$$



Note

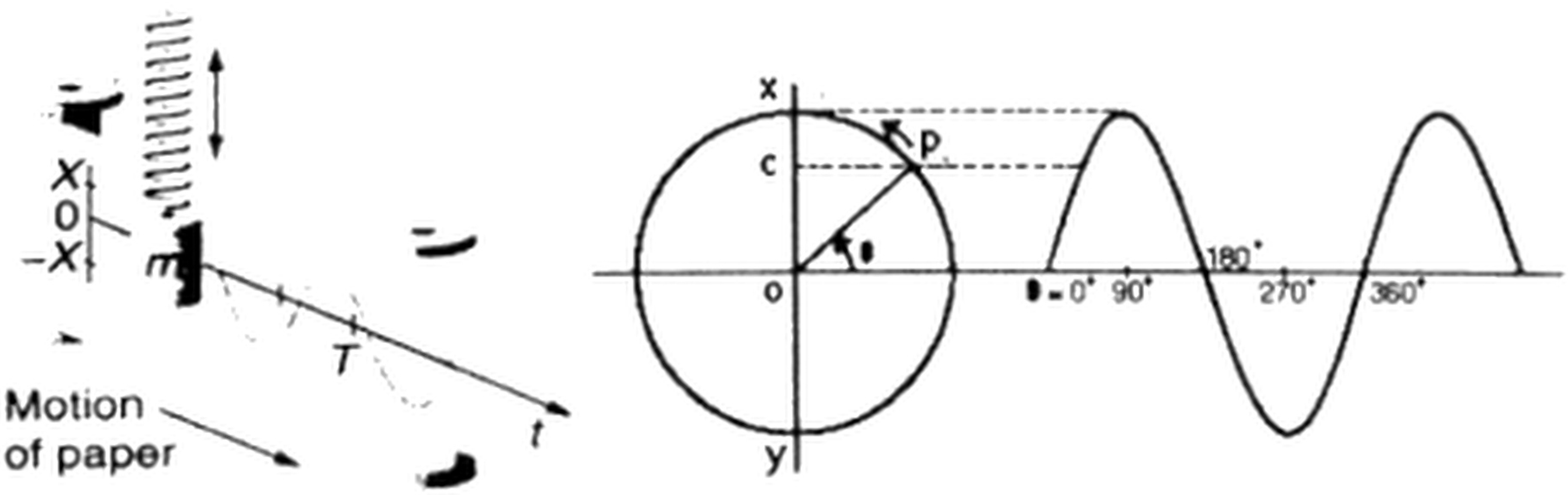
Basically, angular frequency is the property of circular motion. In SHM, it provides an easy method to determine the instantaneous displacement and instantaneous velocity of body executing SHM.

Q.4 Show that the motion of projection of body moving along a circular path is SHM.

Answer

SHM and Uniform Circular Motion

Consider a mass m attached with the end of a vertically suspended spring. It vibrates simple harmonically with period T, frequency f and the amplitude x_0. If motion of the mass is displayed by a pointer P_1.



At t=0 pointer is at position A then at position B, A, C and back to A at instant T/4, T/2, 3T/4 and T respectively.

In circular motion point P is moving in a circle of radius x_0 with uniform angular frequency ω. Now consider the motion of point N, the projection of P on diameter DE. The levels of D and E are similar to points B and C.

With the motion of P on the circle, the point N moves to and fro on DE. Let point P is at O_1 at t=0, the projection N at instants 0-, T/4, T/2, 3T/4 and T will be at O, D, O, E and O respectively.

Result

Hence the comparison of motion of N and P_1 shows that it is a copy of pointer's motion. Hence the motion of projection of particle P moving in a circle is SHM.

Q.5 Derive the expressions for instantaneous displacement, instantaneous velocity and acceleration of the projection of a particle moving in a circle of radius x_0.

Answer

Let N be the projection of a particle P moving in a circle.

Angular frequency of P = ω

The angle subtended by OP at any time t = $\theta = \omega t$

(1) Instantaneous Displacement

From figure (in right angled triangle OPN)

$$\frac{ON}{OP} = \sin\theta$$

Or ON = OP $\sin\theta$ [$\therefore$ < OPN = < $O_1OP = \theta$ (alternate angles)]

But On = x and OP = x_0

So $\boxed{x = x_0 \sin\theta}$

As $\theta = \omega t$

$$X = x_0 \sin \omega t \qquad \text{...........(1)}$$

This equation shows the displacement of pointer N at instant t.

Phase angle (θ)

The angle θ which gives the states of the system during one complete cycle is called phase.

Value of θ

The wave form of SHM is shown in fig 1(c) in which

1. At the **start** of cycle is completed, θ=0.

2. When **1st quarter** of the cycle is completed, $\theta = \pi/2$.

3. When **half** of the cycle is completed, $\theta=\pi$.

4. When **three fourth** of the cycle is completed cycle $\theta=3\pi/2$.

5. For the **complete** cycle, $\theta=2\pi$.

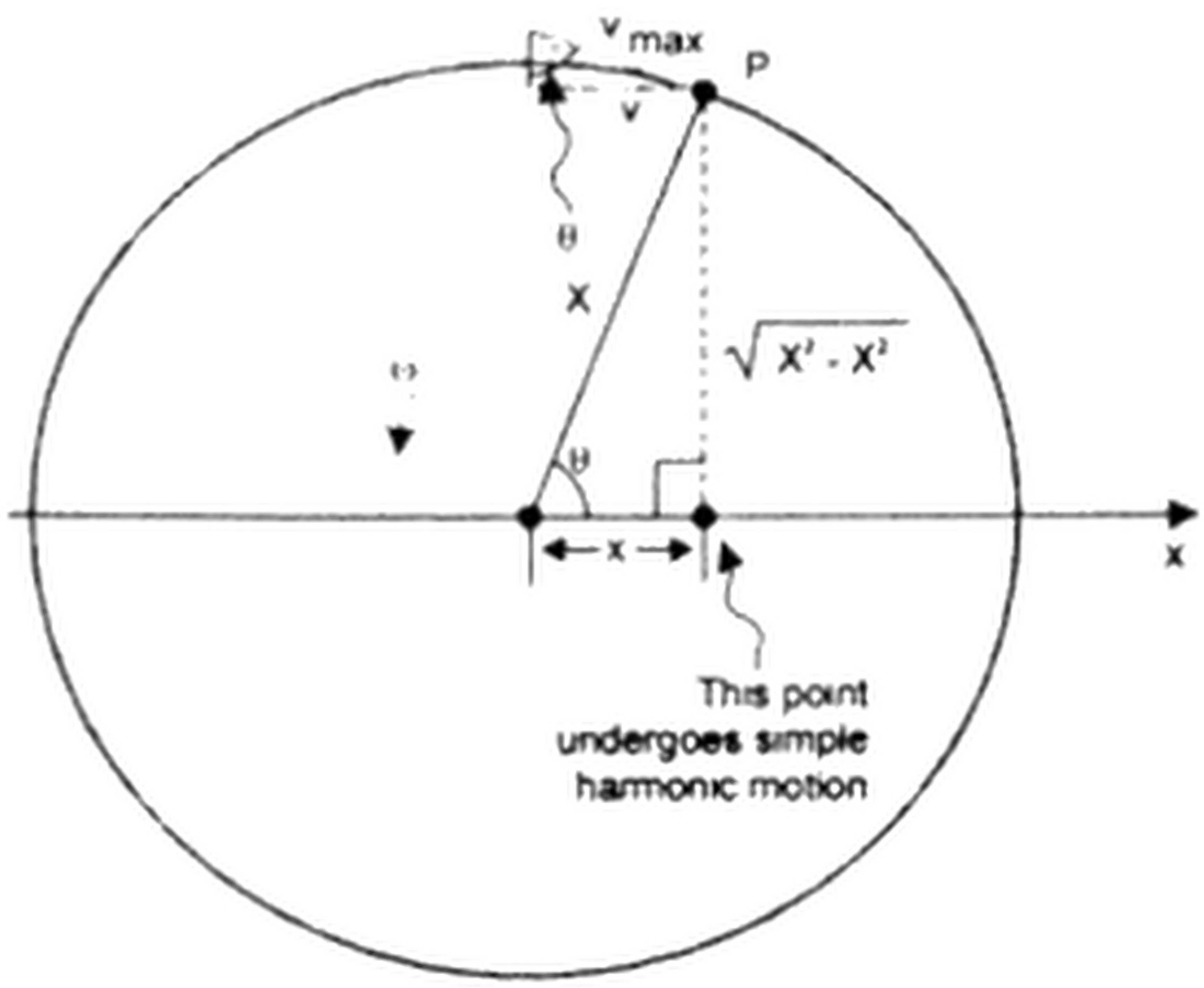
Note

For each quarter of the cycle, the phase of vibration is changed by $\pi/2$ radian.

(2) Instantaneous Velocity (v)

The linear velocity of point P at any instant t = v_p

Then $v_p = x_0\omega$



Since the motion of N on diameter DE is due to the motion of P on the circle. The velocity of N is actually the vertical component of velocity v_p in the direction parallel to De.

The component of velocity parallel to DE is

$$v = v_p \sin(90-\theta)$$

$$v = v_p \cos\theta$$

$$v = x_0\omega\cos\theta$$

$$v = x_0\omega\cos\omega t \qquad (2)$$

From right angled triangle OPN

$$\cos\theta = \frac{PN}{OP} \qquad (3)$$

Applying Pythagorean theorem, for calculating value of PN

$$(OP)^2 = (PN)^2 + (ON)^2$$

$$(PN)^2 = (OP)^2 + (ON)^2$$

Or $$(PN)^2 = x_0^{\ 2} - x^2$$

Or $$PN = \sqrt{x_0^{\ 2} - x^2}$$

Putting values of PN and OP in equation (3), we get

$$\cos\theta = \frac{\sqrt{x_0^{\ 2} - x^2}}{x_0}$$

Putting values of $\cos\theta$ in equation (2), we get

$$v = x_0\omega\frac{\sqrt{x_0^{\ 2} - x^2}}{x_0}$$

Or $$v = \omega\sqrt{x_0^{\ 2} - x^2} \qquad (4)$$

Direction of Velocity of N

The direction of velocity depends upon the value of phase angle.

- When it varies from 0° to 90° then the direction of v is O to D.
- When it varies from 90° to 270° then the direction of v is D to E.
- When it varies from 270° to 360° then the direction of v is E to O.

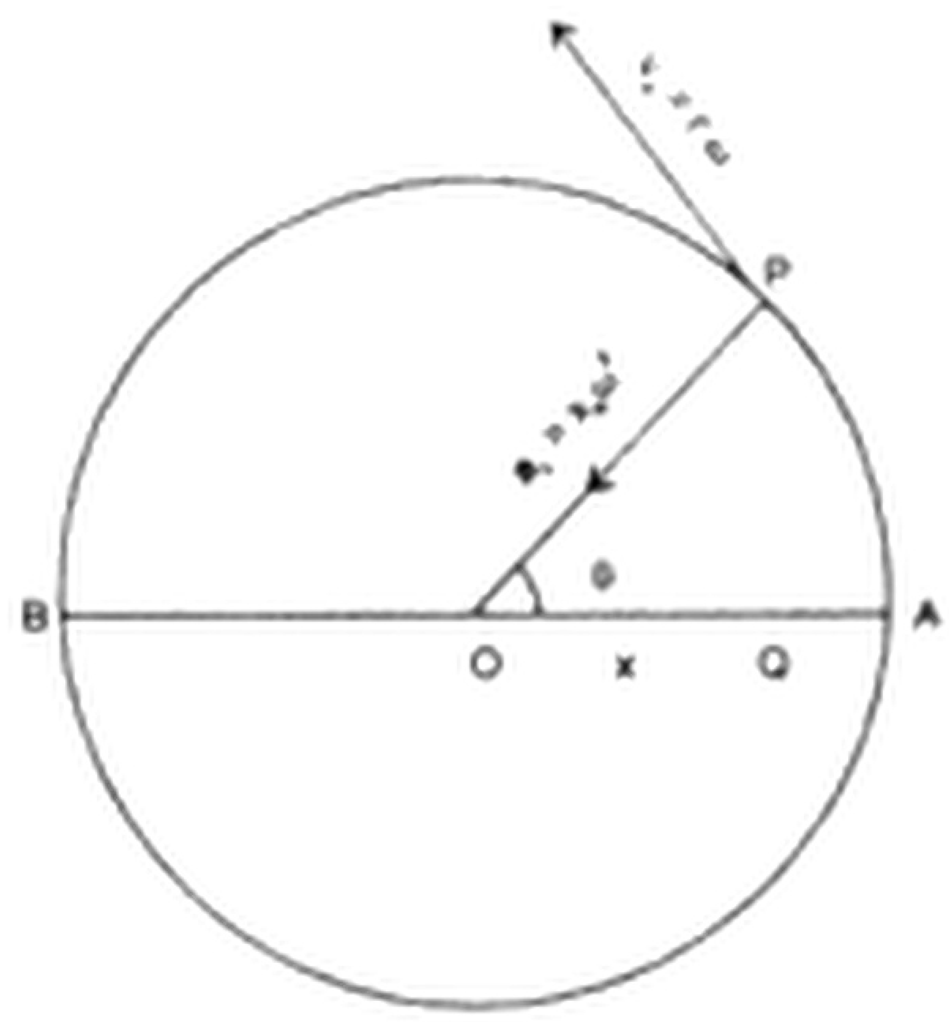
Special cases

- At mean position (i.e. x=0), the velocity is maximum [i.e. $v_{max} = \omega x_0$]
- At extreme position (i.e. $x = x_0$), the velocity is zero [i.e. $v_{min} = 0$]

(3) Instantaneous Acceleration (a)

The acceleration at any point P moving along the circle can be expressed as,

$A_p = x_0\omega^2$



It is always directed towards the centre O.

The acceleration at point N will be component of acceleration a_p along the diameter DE as shown in figure.

So,

$a = a_p \sin\theta$

Or $a = x_0\omega^2 \sin\theta$ (5)

From figure

$\sin\theta = \frac{ON}{OP}$

Or $\sin\theta = \frac{x}{x_0}$ (6)

So equation (5) becomes

$a = x_0\omega^2 \frac{x}{x_0}$

Or $a = \omega^2 x$

The acceleration $\vec{a}$ is directed towards the mean position, so we get that direction of $\vec{a}$ is opposite to $\vec{x}$. So above equation can be written as

$$\vec{a} = -\omega\vec{x}$$

$$\vec{a} = -\text{constant}\ \vec{x} \qquad [\because \omega = \text{constant}]$$

Or $$\vec{a} \propto -\vec{x}$$

This equation shows that the acceleration is directly proportional to displacement and is directed towards the mean position which is the property of SHM.

So, we can say that point N is performing the SHM with the same amplitude, time period and instantaneous displacement of pointer P_1.

Q.6 Define the phase angle.

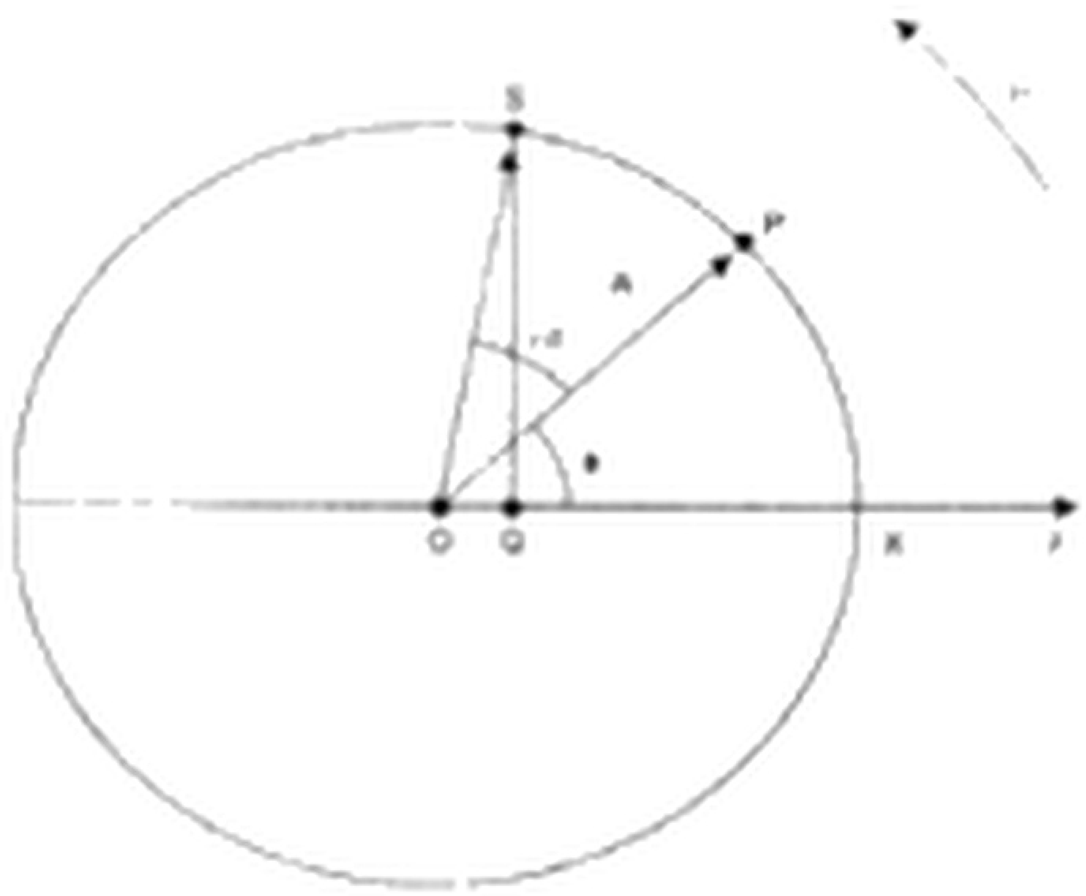
Answer

Phase

The angle ($\theta=\omega t$) which gives the displacement as well as the direction of motion of point executing SHM is known as phase.

OR

The angle ($\theta=\omega t$) which determines the state of motion of the vibrating point is called phase.



Note

This angle is obtained when SHM is related with circular motion.

Displacement in terms of phase

Special case

Lt at t=0, the point P is at O_1 and N is at mean position. Then the displacement of a point having SHM is

$$S = X_0 \sin \omega t$$

Where $\theta=\omega t$ is the angle, which the rotating radius OP makes with the reference line OO_1 at any time t as shown in figure.

General case (concept of initial phase)

Let at t = 0

The angle made by rotating radius OP with the reference line $OO_1 = \phi$

After a time, t,

The radius rotates through angle = ωt

The angle made by rotating radius OP with the reference line OO_1 at time t= $(\omega t+\phi)$

So, the displacement at time t is given by

$$x = x_0 \sin(\omega t+\phi)$$

Initial phase

Now the phase angle is

$$\theta = \omega t+\phi$$

When t = 0, $\theta = \phi$, so ϕ is called initial phase.

Now taking initial phase as 90° or $\left(\frac{\pi}{2}\right)$, then displacement is

$$x = x_0 \sin(\omega t + 90)$$

Or $$x = x_0 \cos \omega t$$

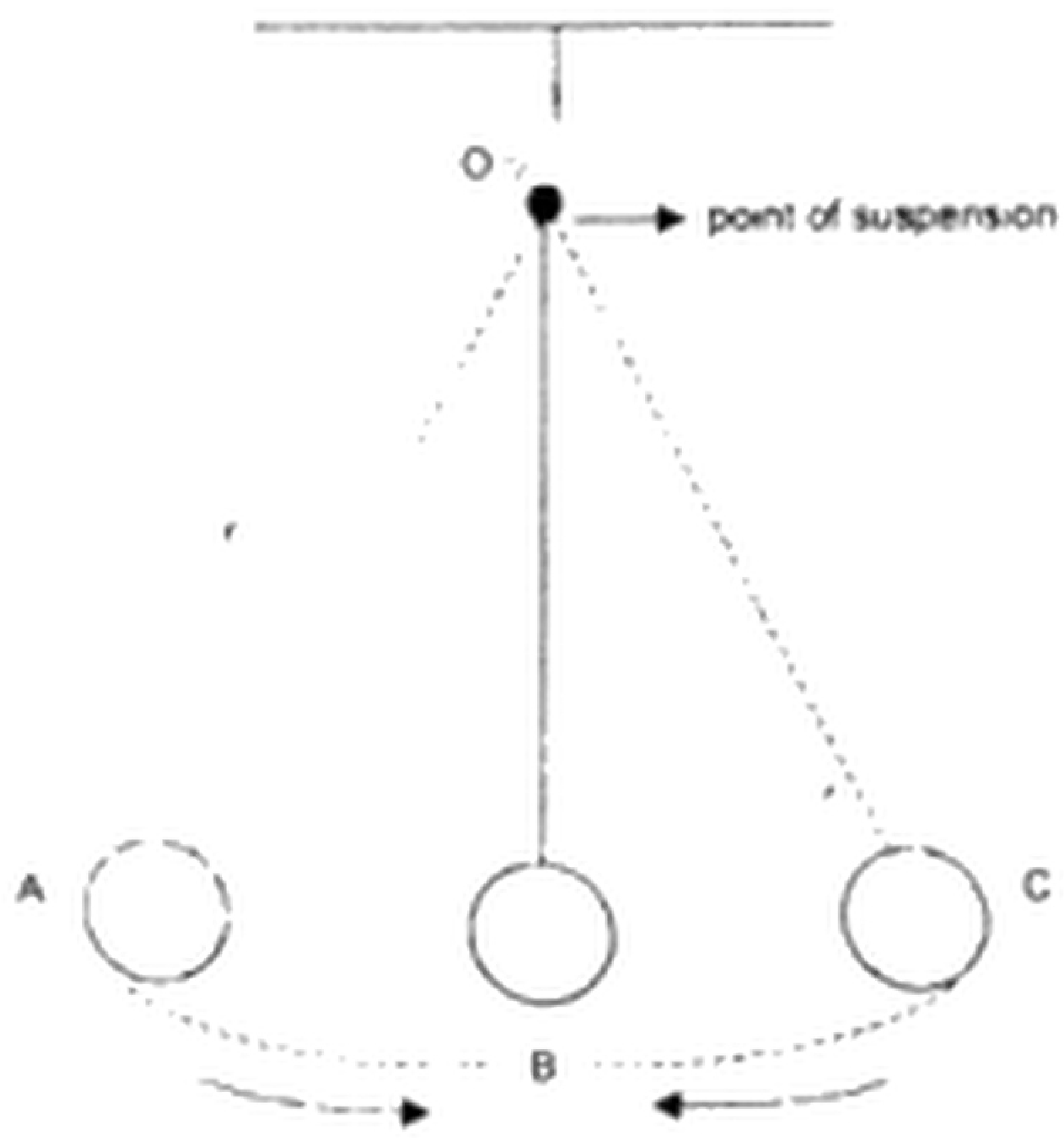
This equation gives the displacement of SHM, but in this case the point N is starting its motion form extreme position instead of the mean position, as shown in figure.

Q.7 What is simple pendulum? Show that the motion of pendulum is SHM. Also find relations for its time period and frequency.

Answer

Simple Pendulum

An ideal simple pendulum consists of a small heavy mass suspended by a weightless and inextensible string fixed with a frictionless support and medium. Practically, the above-mentioned conditions are incompatible and we use the light weight and les extensible string.



Motion of Simple Pendulum is SHM

Consider an object of mass m attached with the end of a light weight string.

Length of the Pendulum

The length of the pendulum *l* is the distance between the point of suspension and the center of the bob.

Working

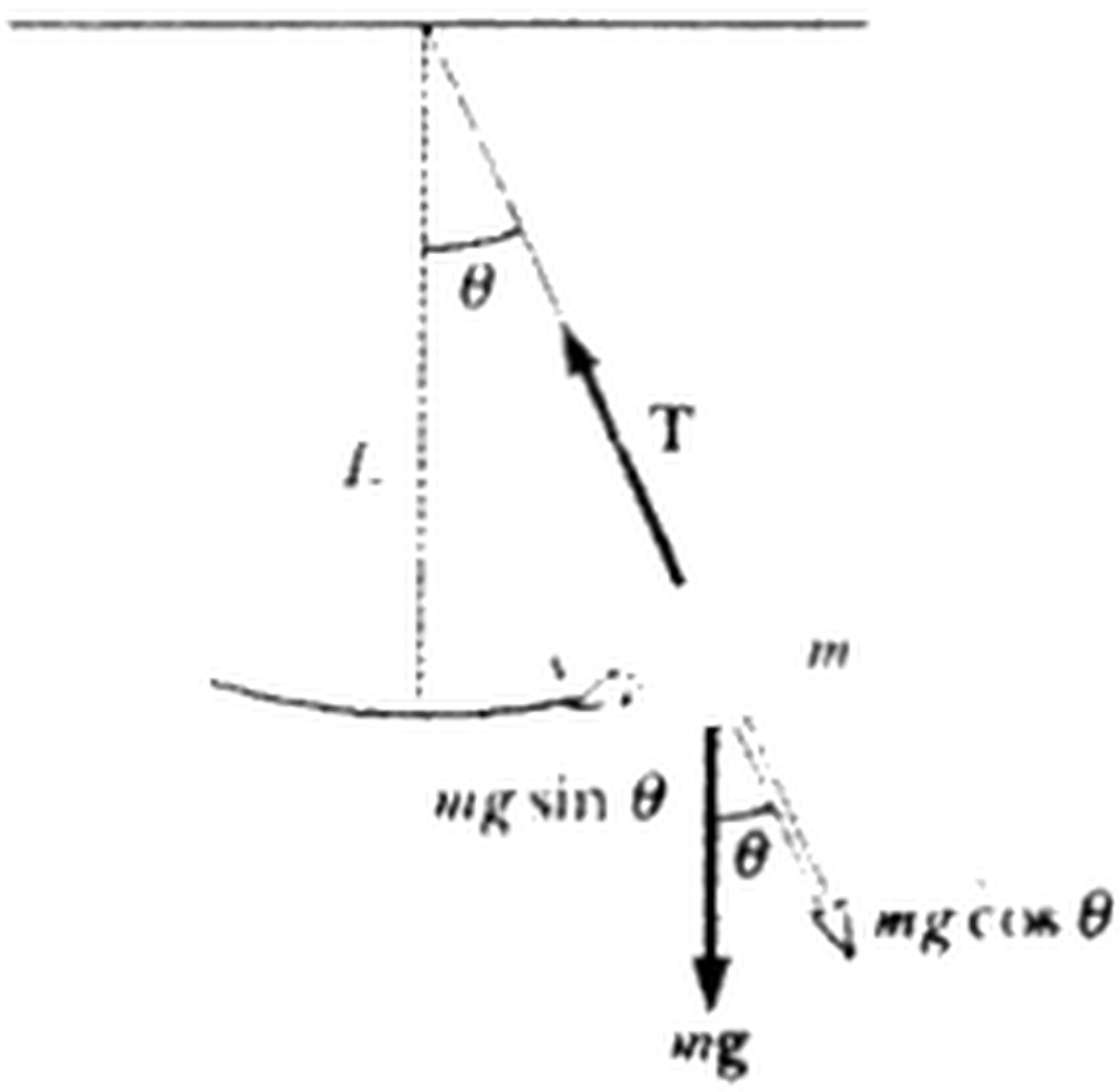
When the pendulum is displaced from its mean position through a small angle θ and released then it starts to oscillate to and fro about mean position.

Components of weight

Resolve the weight mg into two components mg cosθ and mg sinθ.

The other force in this case is the tension T in the string.

mg cosθ and T are equal and opposite to each other. So, they cancel the effect of each other i.e. mg cosθ = T.



Restoring force

The only force responsible for motion of the pendulum is mg sinθ which brings the bob back towards its mean position. So, the restoring force for the bob is

$$F = -mg \sin\theta \qquad \text{........ (1)}$$

Negative sign shows that force is directed towards mean position.

Also we know that

$$F = ma \qquad \text{.........} \ (2)$$

Comparing above two equations, we get

$$ma = -mg \sin\theta$$

Or $\quad a = -g \sin\theta$

For small value of angel θ, sinθ≈0

So, $\quad a = -g\theta \qquad \text{..........} \ (3)$

From figure $\quad \theta = \frac{AB}{l} \qquad \left[\because s = r\theta \Rightarrow \theta = \frac{s}{R}\right]$

$$\theta = \frac{x}{l} \qquad \left[\because \theta \text{ is small so are } AB \approx x\right]$$

So equation (3) becomes

$$a = -g\left(\frac{x}{l}\right)$$

$$a = -\left(\frac{g}{l}\right)x \qquad \text{........} (4)$$

$$a = -\text{constant} \times x \qquad \left[\because \frac{g}{l} = \text{constant}\right]$$

Or $\quad \vec{a} \propto -\vec{x}$

This proves that the motion of pendulum is SHM.

Time Period

We know that for a body having SHM.

$$a = -\omega^2 x \qquad \text{.........} (5)$$

Comparing equation (4) and (5), we have

$$-\omega^2 x = -\left(\frac{g}{l}\right)$$

Or $$\omega^2 = \frac{g}{l}$$

Or $$\omega = \sqrt{\frac{g}{l}} \qquad \ldots\ldots.(6)$$

As the time period for SHM can be expressed as

$$T = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\sqrt{\frac{g}{l}}}$$

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \ldots\ldots(7)$$

Dependence of Time Period

This equation shows that time period of the pendulum depends on

- Length of pendulum
- Acceleration due to gravity

Note

Time period of the pendulum is independent of mass and amplitude.

Frequency

As the reciprocal of the time period is called frequency. So,

$$F = \frac{1}{T}$$

Putting value of T, we get

$$F = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \qquad \ldots\ldots(8)$$

Note:

Second Pendulum

The pendulum whose time period is 2 seconds is called second pendulum.

So, $T = 2\ sec$

Frequency of second Pendulum

$$f = \frac{1}{T}$$

$$f = \frac{1}{2}$$

$$f = 0.5\ Hz$$

Length of Second Pendulum

As $T = 2\pi\sqrt{\frac{l}{g}}$

Or $T^2 - 4\pi^2 \frac{l}{g}$

$$l = \frac{gT^2}{4\pi^2}$$

As $T = 2\ sec$

$$l = \frac{9.8 \times (2)^2}{4 \times (3.14)^2} = 0.992m$$

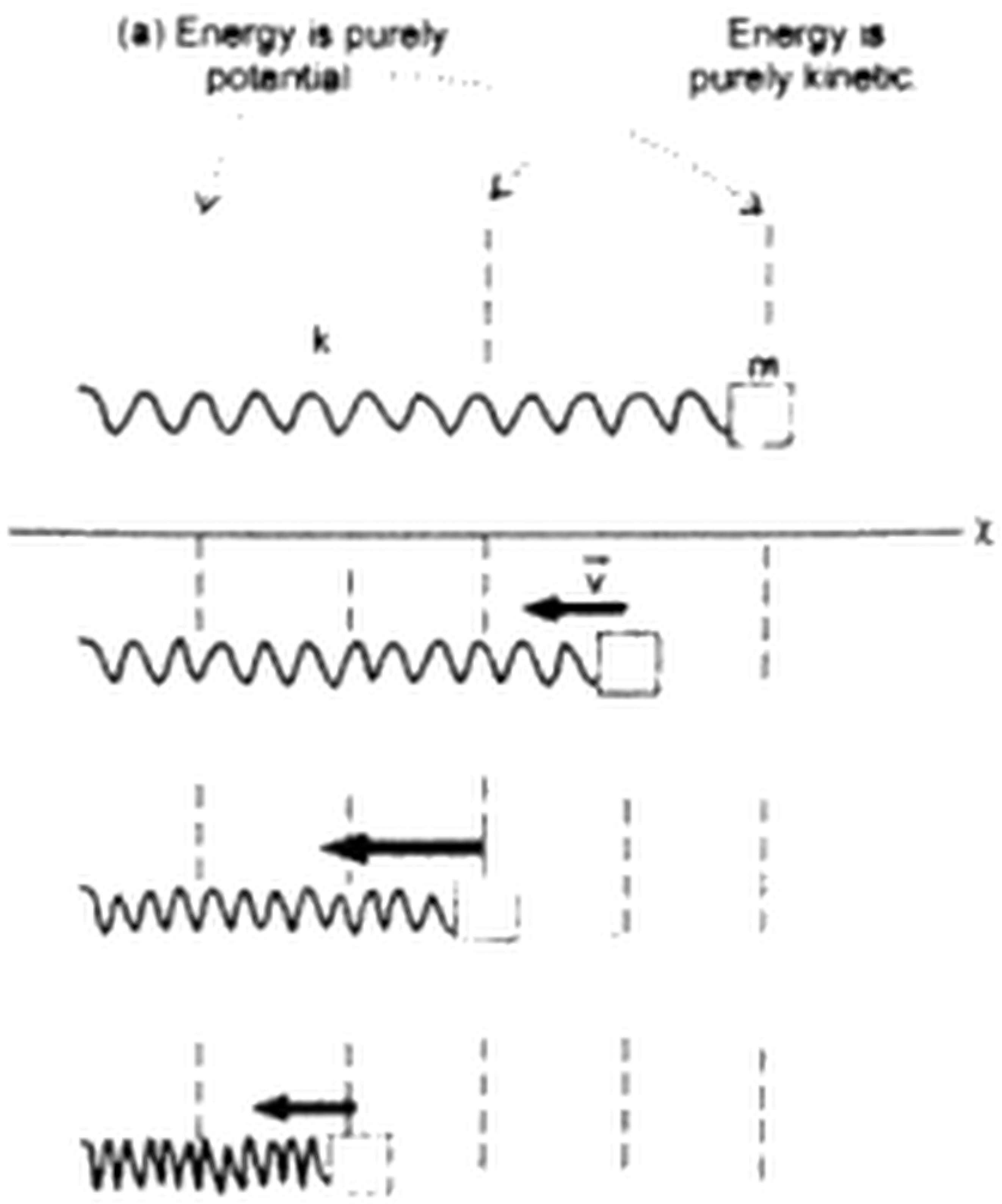
Or $l = 99.2\ cm$

Q.8 Prove that the mechanical energy is conserved in SHM?

Answer

Energy Conservation in SHM

Consider the vibrating mass-spring. When the mass m is pulled slowly, the spring is stretched by an amount x_0 along a horizontal frictionless table. As it is assumed that stretching is done slowly, so that acceleration is zero, because change in velocity will be very small.



Instantaneous P.E.

Let for any instant t, the mass m is at a distance x from mean position,

So according to Hook's law

$$F = kx$$

When displacement $= 0$

Then $F = 0$

When displacement $= x$

Then $F = Kx$

So average force is

$$F = \frac{0 + kx}{2}$$

$$F = \frac{1}{2}(kx)$$

Hence the work done in displacing the mass through displacement x is

$$W = Fd$$

$$= \frac{1}{2}(kx)(x)$$

$$= \frac{1}{2}kx^2$$

Work done appears as elastic P.E. so,

$$\boxed{(P.E)_{ins} = \frac{1}{2}kx^2}$$

Maximum P.E.

P.E. is zero if the displacement x=0, i.e. the mass is at mean position, thus

$$(P.E)_{min} = \frac{1}{2}k(0)^2 = 0$$

Instantaneous K.E.

We know that

$$(K.E)_{ins} = \frac{1}{2}mv^2$$

Since $$v = x_0\sqrt{\frac{k}{m}\left(1-\frac{x^2}{x_0^2}\right)}$$

Or $$(K.E)_{ins} = \frac{1}{2}m\left(x_0\sqrt{\frac{k}{m}\left(1-\frac{x^2}{x_0^2}\right)}\right)^2$$

$$(K.E)_{ins} = \frac{1}{2}mx_0^2\left(\frac{k}{m}\left(1-\frac{x^2}{x_0^2}\right)\right)$$

$$\boxed{(K.E)_{ins} = \frac{1}{2}kx_0^2\left(1-\frac{x^2}{x_0^2}\right)}$$

$$(K.E)_{ins} = \frac{1}{2}kx_0^2 - \frac{1}{2}kx^2$$

Maximum K.E.

The K.E. is maximum at mean position where x=0

$$(K.E)_{max} = \frac{1}{2}kx_0^2\left(1-\frac{0^2}{x_0^2}\right)$$

$$(K.E)_{max} = \frac{1}{2}kx_0^2$$

Minimum K.E.

The K.E is minimum at extreme position where $x = x_0$

$$(K.E)_{min} = \frac{1}{2}kx_0^2\left(1-\frac{x_0^2}{x_0^2}\right)$$

$$(K.E)_{min} = \frac{1}{2}kx_0^2(1-1)$$

$$(K.E)_{min} = 0$$

Total Energy

At any position, the total energy is sum of partly P.E. and partly K.E.

So,

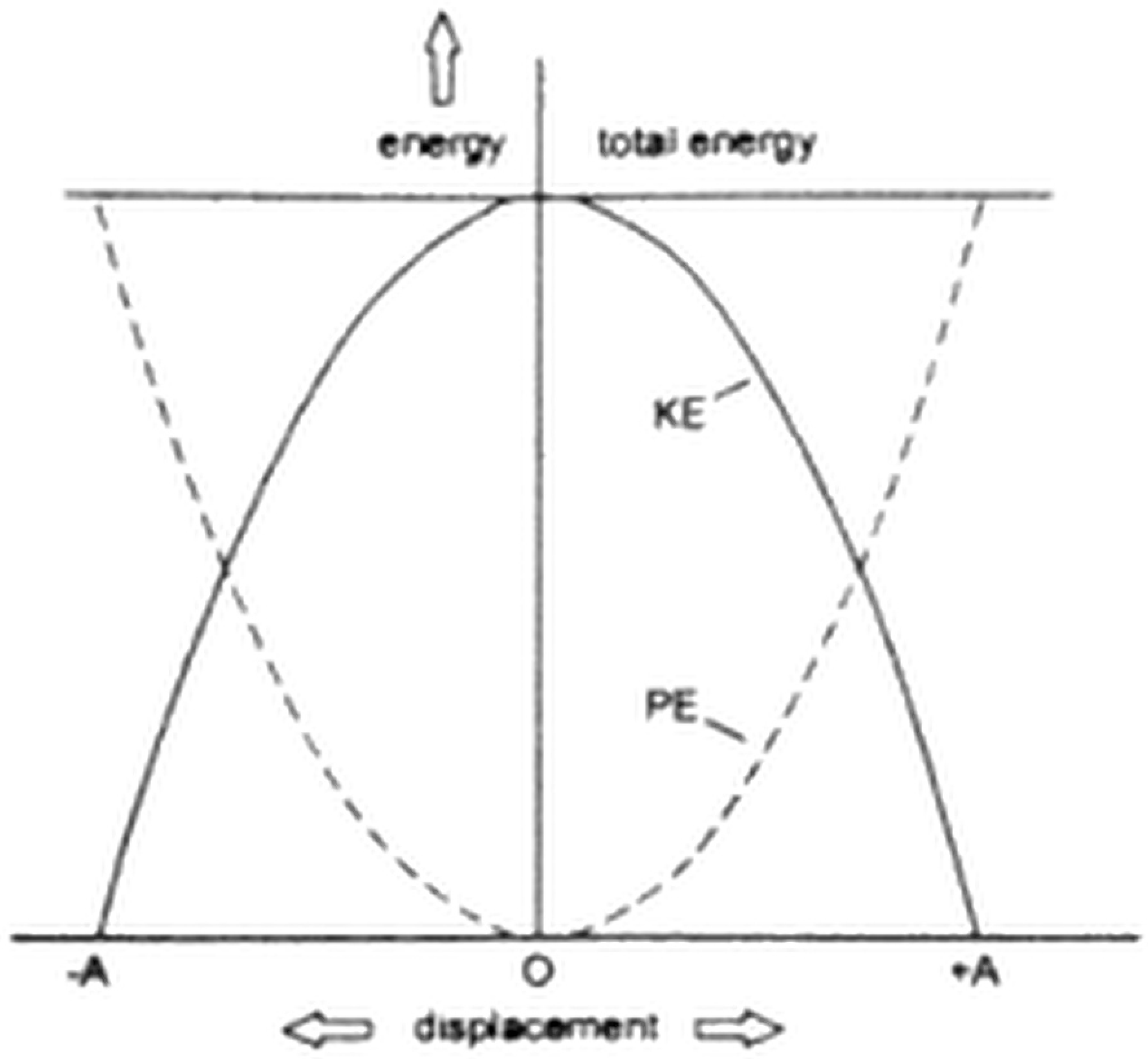
$$E = P.E + K.E$$

$$E = \frac{1}{2}kx^2 + \frac{1}{2}kx_0^2\left(1-\frac{x^2}{x_0^2}\right)$$

$$E = \frac{1}{2}kx^2 + \frac{1}{2}kx_0^2 - \frac{1}{2}kx^2$$

$$\boxed{E = \frac{1}{2}kx_0^2}$$

Thus, the total energy of vibrating mass spring system always remains constant.



Note

During the oscillatory motion where the K.E is maximum, and the P.E. is zero and when the P.E is maximum, K.E is zero. The change of P.E and K.E with displacement is required for maintaining the oscillation.

Thus, periodic exchange of energy is the property of all oscillatory systems.

Q.9 What are free and forced oscillations? Also define driven harmonic oscillator.

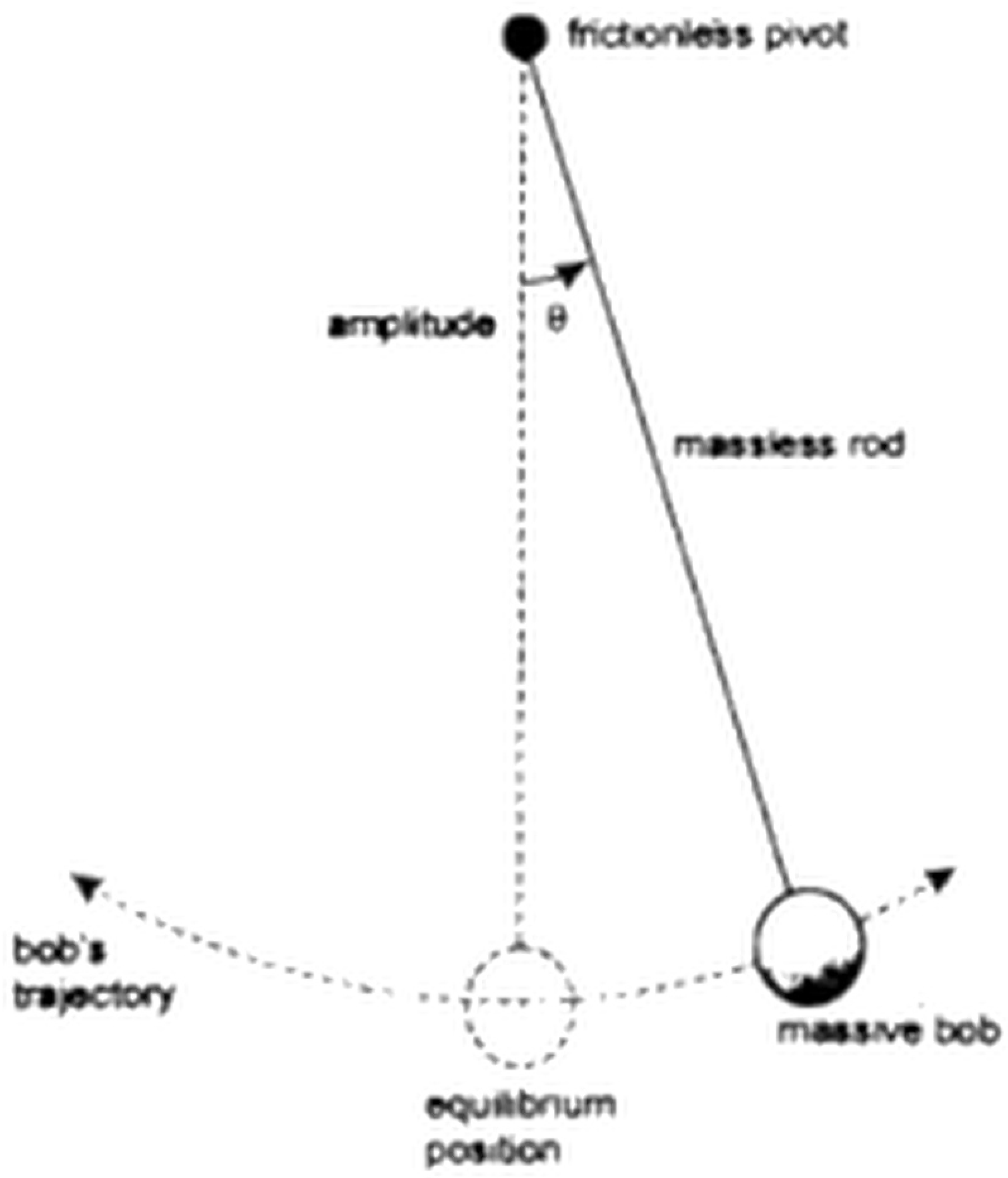
Answer

Free Oscillations

A body is said to be executing free vibrations if it oscillated with its natural frequency without the interference of an external force.

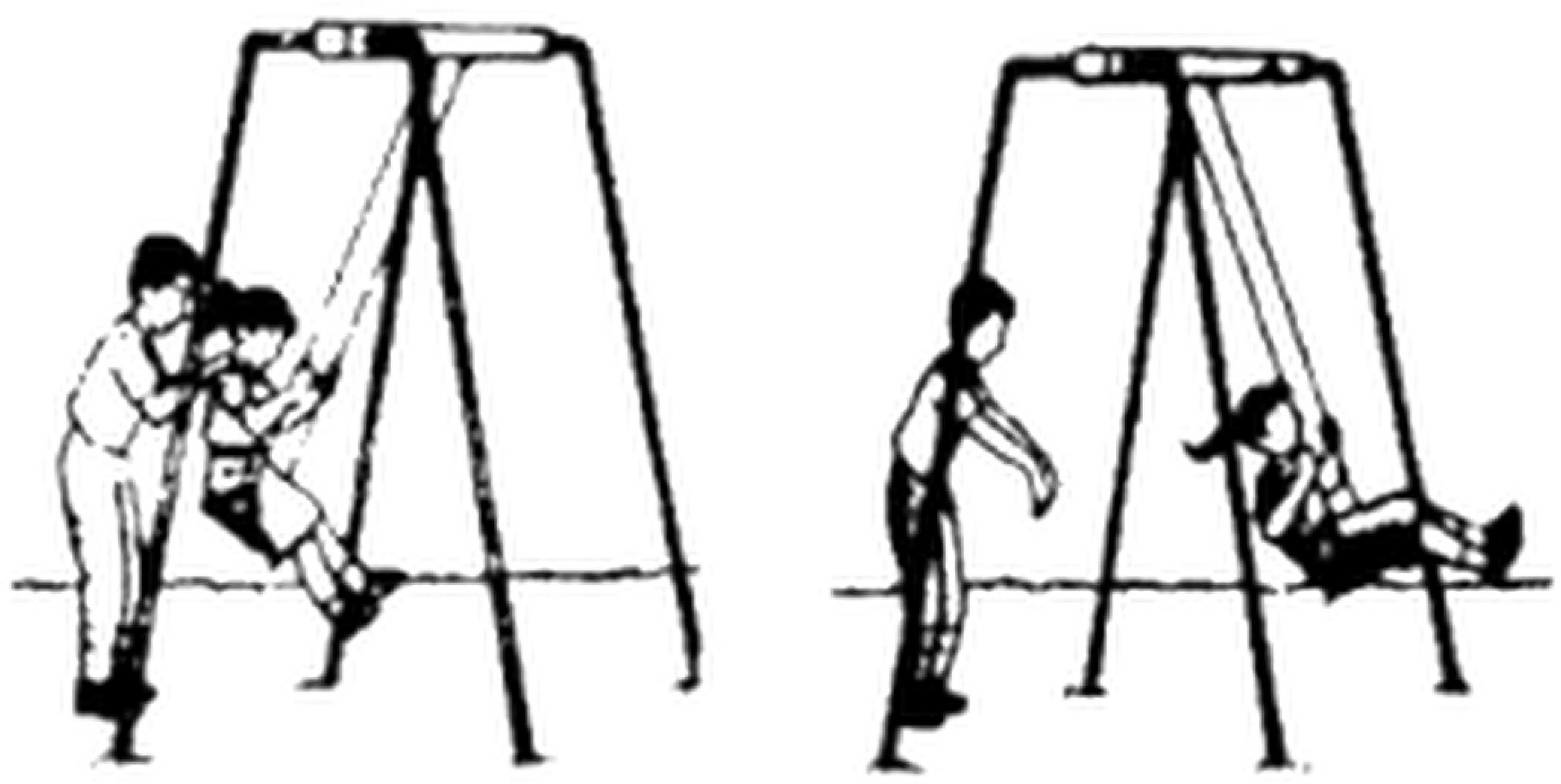
For example

A simple pendulum vibrates freely with its natural frequency that depends only upon the length of the pendulum.



Forced oscillations

A body is said to be executing forced vibrations if it oscillates with the interference of an external force.



(a) (b)

For example

- It the mass of vibrating pendulum is struck repeatedly, then forced vibrations are produced.
- The vibration of factory floor caused by the running of heavy machinery is another example.

Driven harmonic oscillator

The physical system undergoing forced vibrations is known as driven harmonic oscillator.

Q.10 What is resonance phenomenon? Explain it with examples?

Answer

Resonance

Resonance is the specific response of a vibrating system to a periodic force acting with natural vibrating period of system.

OR

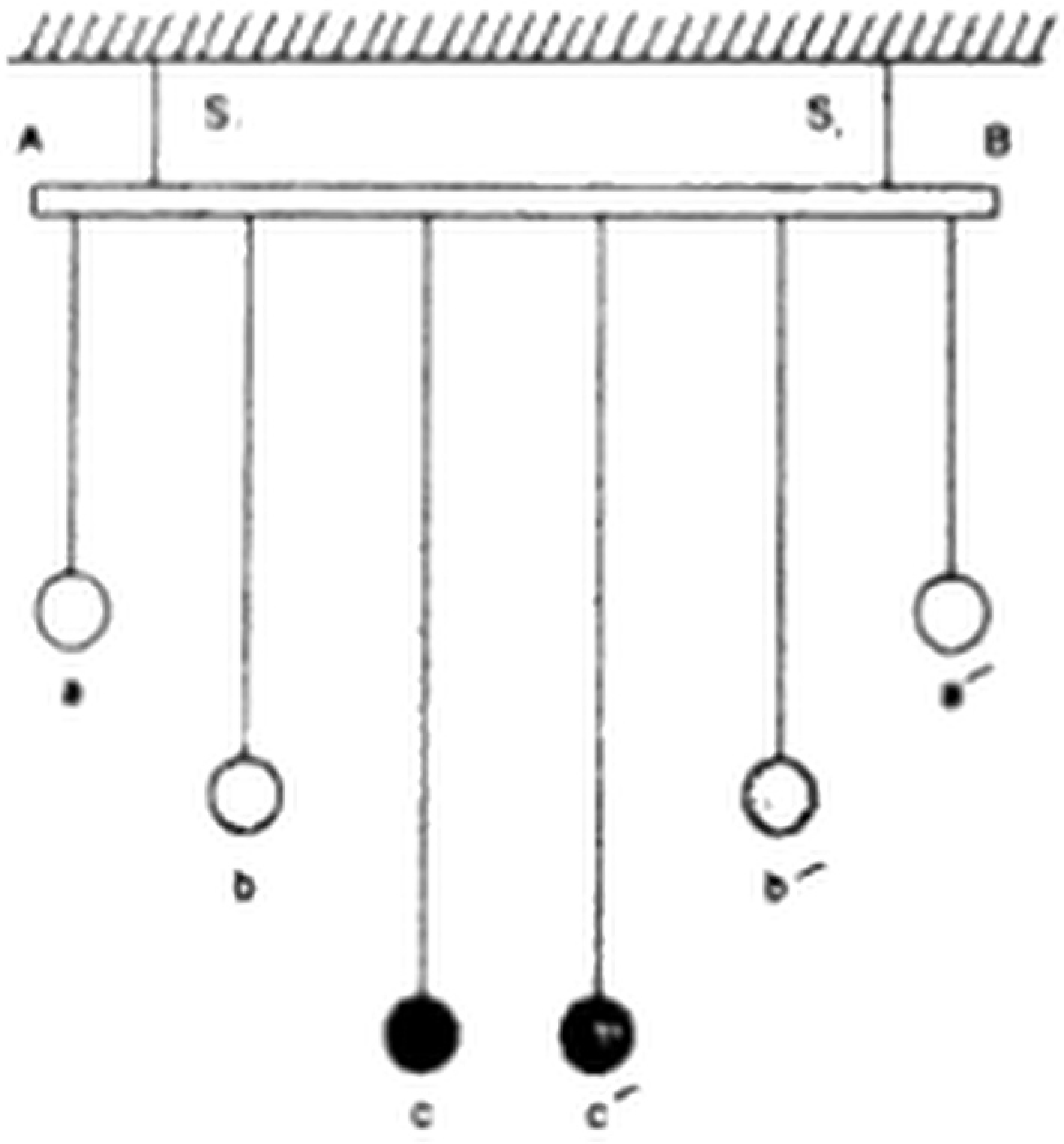
The phenomena in which the amplitude of a vibrating body increases when the frequency of applied force is equal to the natural frequency of the harmonic oscillator.

Experiment to demonstrate resonance

Consider a harmonic rod AB is supported by strings S_1 and S_2. Three pairs of pendulum aa', bb' and cc' are suspended to rod 'AB'.

- The length of each pair is same but different for different pairs.
- Displace pendulum c in a direction perpendicular to the plane of the paper.
- A small force acts on all the pendulum through the rod AB.
- All the pendulum will oscillate with the pendulum c but with slight periodic motion.
- The pendulum c', whose length and hence period is exactly the same as that of c oscillates with larger amplitude equal to c.

- The amplitude of other pendulums remains small because their natural periods are not same as that of the disturbing force due to rod AB.



Examples

Motion of swing (mechanical resonance)

- A swing is a good example of mechanical resonance.
- We apply a periodic force on swing.
- When the frequency of periodic force becomes equal to the natural frequency of the swing, resonance is produced.
- So, energy absorption is maximum hence the amplitude of vibrations increased.

Magnetic Resonance Imaging

Magnetic Resonance Imaging, or MRI, is a method of imaging the interior of structures noninvasively. An MRI device consists of a magnet, magnetic gradient coils, an RF (radio frequency) transmitter and receiver, and a computer that controls the acquisition of signals and computes the MR images. The full name, Nuclear Magnetic Resonance Imaging, usually shortened to MRI, describes the technique. If an atomic Nucleus is exposed to a static Magnetic field, it resonates when a varying electromagnetic field is applied at the proper frequency. An Image is computed from the resonance signals of which the frequency and phase (timing) contain space information. MRI is important because it is noninvasive, safe, and yields information that cannot be obtained with any other techniques. Its most common use by far is in diagnostic medicine but MRI has other applications, particularly in the oil and food industries.

Collapse of suspended bridge

On a big span bridge the soldiers crossing the bridge are ordered to break their steps, if the frequency of steps coincides with natural frequency of the bridge. Then there is chance to collapse the bridge due to resonance.

Turning a radio (Electrical resonance)

- Turning of radio is a good example of electrical resonance. To tune a radio,
- We turn the knob of a radio.
- It changes the natural frequency of electrical circuit of receiver until it becomes equal to the frequency of transmitter.
- Now the resonance is produced and energy absorption is maximum.
- Hence a station is tuned.

Cooking by microwave oven

- Resonance plays an important role in heating and cooking food by microwave oven.
- The microwaves produce by oven are absorbed due to resonance by water and fats molecules in the food.
- This increases the internal energy of the molecules.
- They get heat up and so food is cooked.

Note

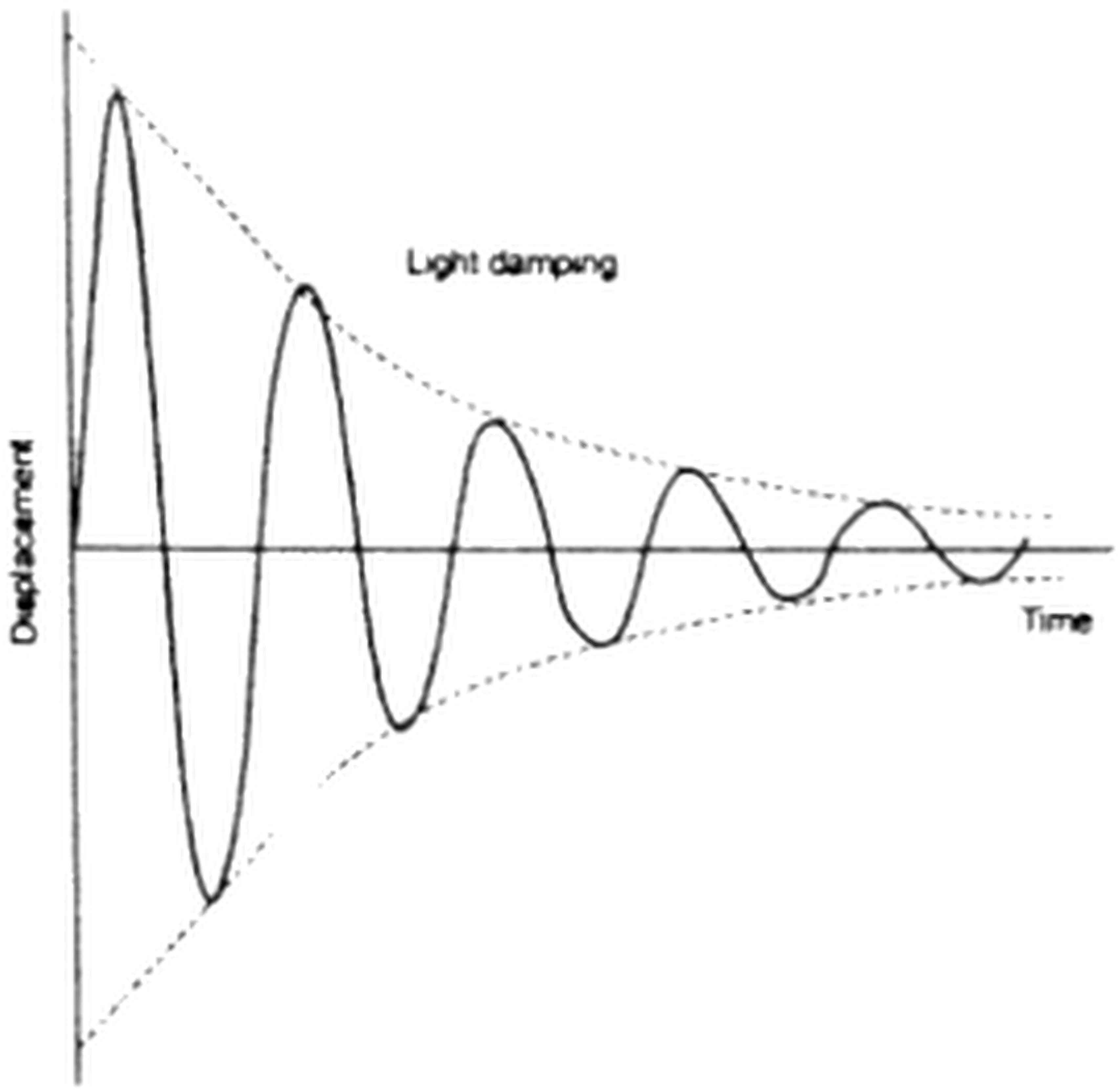
The wave length of the microwaves produced in this type of oven is 12 cm and frequency is 2450 MHz.

Q.11 What are damped and undamped oscillations? What is damping?

Answer

Damped Oscillations

Oscillations in which amplitude decreases with time due to energy dissipation are called damped oscillations.



Explanation

The amplitude of the oscillatory body gradually becomes smaller and smaller because of friction and air resistance. As the energy of the oscillator is used up in doing work against the resistive forces, that is why the amplitude decreases with time till it becomes zero.

Applications

An application of damped oscillation is the shock absorber of a car which provides a damping force to stop the excessive oscillations.

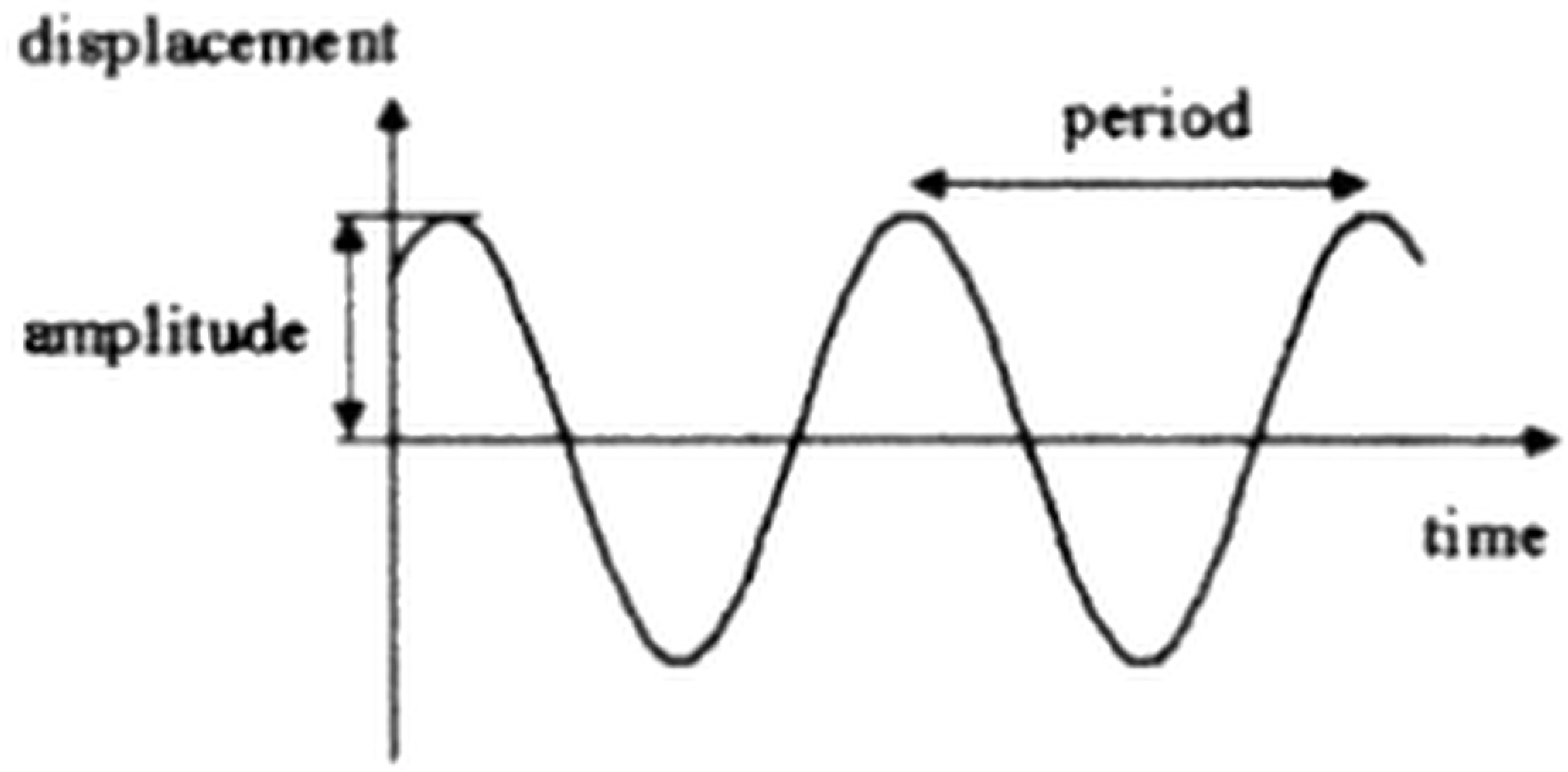
Damping

Damping is the process by which energy is lost by the oscillatory system.

Undamped oscillations

Oscillations in which the amplitude remains same with time is called undamped oscillations.

In undamped oscillations energy is not dissipated from the oscillatory system.

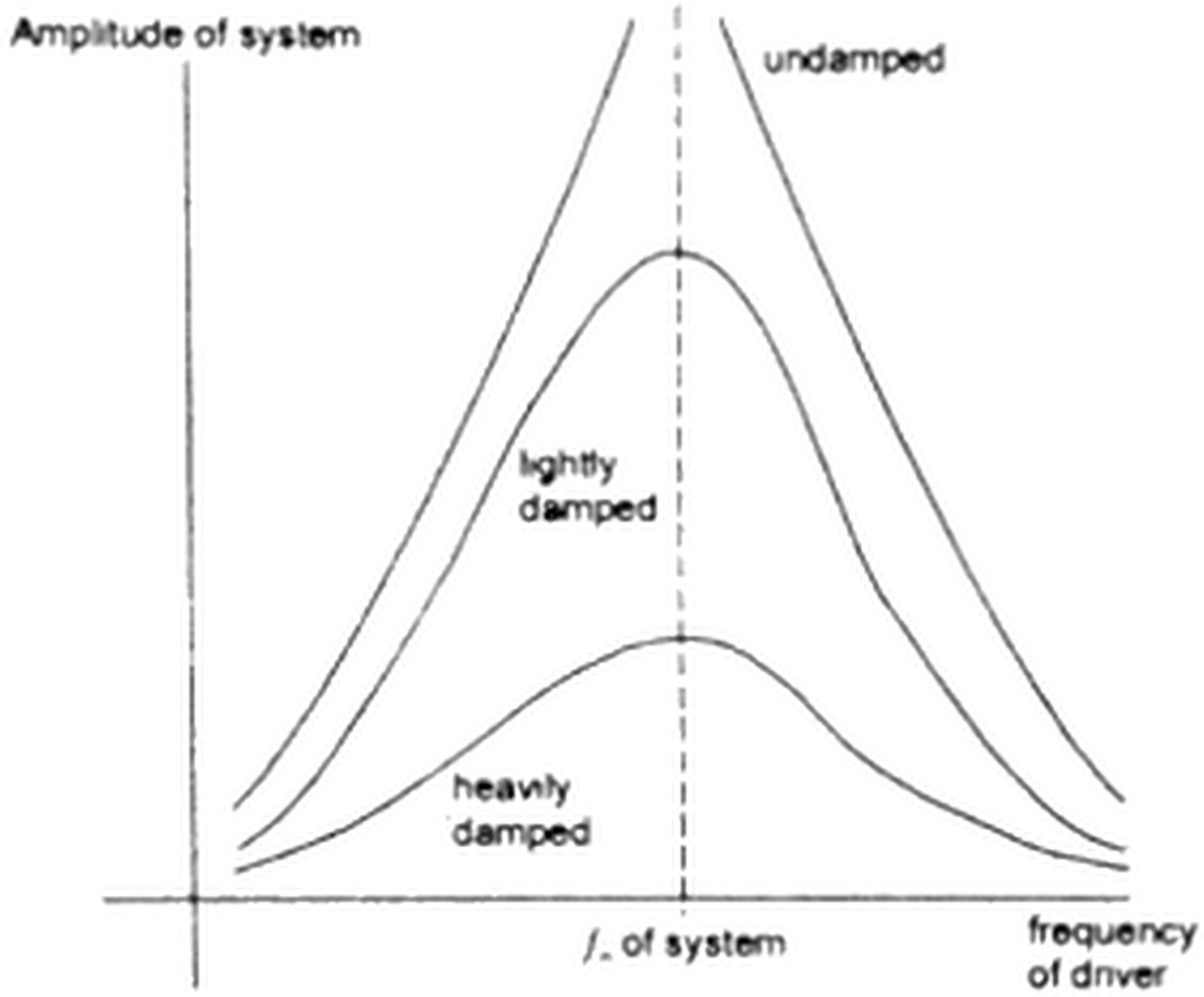


Example

Oscillations of an ideal simple pendulum is an example of undamped oscillations.

Sharpness of Resonance

- The amplitude of vibration of a body increase when the damping is small.
- Thus, the presence of damping prevents the amplitude from becoming sufficiently large.
- The amplitude decreases rapidly at a frequency slightly different from resonance frequency.
- The amplitude as well as sharpness depends upon damping.
- A heavily damped system has fairly flat resonance curve.



Example to see the effect of damping

- Attach a pendulum having very light mass such a pith ball and another of same length with a heavy mass of equal size such as lead ball.
- Set them into vibration by third pendulum of equal length and attached to the same rod.
- It is observed that the amplitude of the heavy ball is much greater than the light ball.
- So, the sharpness of the resonance curve of resonating system depends on energy loss due to friction.