Exercise 6.10

1. Find the 9th term of the harmonic sequence

i. $\frac{1}{3},\frac{1}{5},\frac{1}{7},$ .... **Is a H.P** **ii.** $\frac{-1}{5},\frac{-1}{3},-1,$.......

Solution:

Let $\frac{1}{3},\frac{1}{5},\frac{1}{7},$ .... **Is a H.P**

Then 3, 5, 7, ... Is an A.P.

Here $a = 3$; $d = 5 - 3 = 2$; $n = 9$

We know that

$$a_n = a_1 + (n-1)d$$

$$a_9 = 3 + (9-1)(2)$$

$$a_9 = 3 + (8)(2)$$

$$= 3 + 16 = 19$$

$$a_9 = 19$$

So, $H.P_9 = \frac{1}{19}$

Hence the required 9th term of H.P.is $= 19$.

ii. $\frac{-1}{5},\frac{-1}{3},-1,$.......

Let $\frac{-1}{5},\frac{-1}{3},-1,$....... **Is an H.P.**

Than -5, -3, -1, ... is a A.P.

$$a_n = 5;\ n = 9\ ;\ d = -3 \quad 5 = -3 + 5 = 2$$

We know that

$a_n = a_1 + (n-1)d$

$a_9 = -5 + (9-1)(2)$

$a_9 = -5 + (8)(2)$

$= -5 + 16 = 11$

$a_9 = 11$

So, $H.P_9 = \frac{1}{11}$

Hence the required 9[th] term of H.P.is $= \frac{1}{11}$

2. Find the 12[th] term of the harmonic sequence

i. $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots\ldots\ldots$ **ii.** $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots\ldots\ldots$ **iii.** $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots\ldots$

Solution:

Let $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots\ldots\ldots$ **Is an H.P.**

Than 2, 5, 8, ….. is a A.P.

$a_n = 2$; $d = 5 - 2 = 3$; $n = 12$

We know that

$a_n = a_1 + (n-1)d$

$a_{12} = 2 + (12-1)(3)$

$a_{12} = 2 + (11)(3)$

$= 2 + 33 = 35$

$a_{12} = 35$

So, $H.P_{12} = \frac{1}{35}$

Hence the required 9[th] term of H.P.is $= \frac{1}{35}$

$ii.\ \frac{1}{4}\ and\ \frac{1}{24}$

Let, $\frac{1}{4}$, H.P$_1$, H.P$_2$, H.P$_3$, H.P$_4$, H.P$_5$, $\frac{1}{24}$ is an H.P. then 4, a_1, a_2, a_3, a_4, a_5, 24, is an A.P.

Here $a = 5$ $a_7 = 24$

$$a_n = a + (n-1)d$$

$$24 = 4 + (7-1)d$$

$$24 - 4 = 6d$$

$$20 = 6d$$

$$d = \frac{20}{6} = \frac{10}{3}$$

$$d = \frac{10}{3}$$

So, $a_1 = a + d = 4 + \frac{10}{3} = \frac{12+10}{3} = \frac{22}{3}$

$$a_2 = a_1 + d = \frac{22}{3} + \frac{10}{3} = \frac{22+10}{3} = \frac{32}{3}$$

$$a_3 = a_2 + d = \frac{32}{3} + \frac{10}{3} = \frac{32+10}{3} = \frac{42}{3} = 14$$

$$a_4 = a_3 + d = \frac{42}{3} + \frac{10}{3} = \frac{42+10}{3} = \frac{52}{3}$$

$$a_5 = a_4 + d = \frac{52}{3} + \frac{10}{3} = \frac{52+10}{3} = \frac{62}{3}$$

Thus 4, $\frac{22}{3}$, $\frac{32}{3}$, 14, $\frac{52}{3}$, $\frac{62}{3}$, 24 is a A.P.

$\frac{1}{4}, \frac{3}{22}, \frac{3}{32}, \frac{1}{14}, \frac{3}{52}, \frac{3}{62}, \frac{1}{24}, is\ a\ H.P.$

Hence the required H.P. $\frac{1}{4}, \frac{3}{22}, \frac{3}{32}, \frac{1}{14}, \frac{3}{52}, \frac{3}{62}, \frac{1}{24}$

$iii.\quad \frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots\ldots$

Let $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}$,....... **Is a H.P.**

Then 3, $\frac{2}{9}$, 6, ... Is an A.P.

$$a = 3;\ \ d = \frac{9}{2} - 3 = \frac{9-6}{2} = \frac{3}{2}\ ;\ \ n = 12$$

We know that

$$a_n = a_1 + (n-1)d$$

$$a_{12} = 3 + (12-1)\left(\frac{3}{2}\right)$$

$$= 3 \times 11 \times \frac{3}{2}$$

$$= 3 + \frac{33}{2}$$

$$= \frac{6+33}{2} = \frac{39}{2}$$

$$a_{12} = \frac{39}{2}$$

Than $H.P_{12} = \frac{2}{39}$

Hence, the required 12th term of H.P. = $\frac{2}{39}$

3. Insert five harmonic means between the following given numbers,

i. $\frac{-2}{5}$ ***and*** $\frac{2}{13}$ **ii.** $\frac{1}{4}$ ***and*** $\frac{1}{24}$

Solution:

i. $\frac{-2}{5}$ ***and*** $\frac{2}{13}$

Let, $\frac{-2}{5}$, $H.P_1$, $H.P_2$, $H.P_3$, $H.P_4$, $H.P_5$, $\frac{2}{3}$ is an H.P.

then $\frac{-5}{2}$, a_1, a_2, a_3, a_4, a_5, $\frac{13}{2}$, is an A.P.

Here $a = \frac{-5}{2}$ $a_7 = \frac{13}{2}$

$a_n = a + (n-1)d$

$\frac{13}{2} = \frac{5}{2} + (7-1)d$

$(7-1)d = \frac{13}{2} + \frac{5}{2} = \frac{13+5}{2} = \frac{18}{2} = 9$

$6d = 9$

$d = \frac{9}{6} = \frac{3}{2}$

So, $a_1 = a + d = \frac{-5}{2} + \frac{3}{2} = \frac{-5+3}{2} = \frac{-2}{2} = -1$

$a_2 = a_1 + d = -1 + \frac{3}{2} = \frac{-2+3}{2} = \frac{1}{2}$

$a_3 = a_2 + d = \frac{1}{2} + \frac{3}{2} = \frac{1+3}{2} = \frac{4}{2} = 2$

$a_4 = a_3 + d = 2 + \frac{3}{2} = \frac{4+3}{2} = \frac{7}{2}$

$a_5 = a_4 + d = \frac{7}{2} + \frac{3}{2} = \frac{7+3}{2} = \frac{10}{2} = 5$

Thus $\frac{-5}{2}, -1, \frac{1}{2}, 2, \frac{7}{2}, 5, \frac{13}{2}$ is a A.P.

$\frac{-2}{5}, -1, 2, \frac{1}{2}, \frac{2}{7}, \frac{1}{5}, \frac{2}{13}, is\ a\ H.P.$

Hence the required H.P. $\frac{-2}{5}, -1, 2, \frac{1}{2}, \frac{2}{7}, \frac{1}{5}, \frac{2}{13}$

4. Insert five harmonic means between the following given numbers,

i. $\frac{1}{3}\ and\ \frac{1}{23}$ **ii.** $\frac{7}{3}\ and\ \frac{7}{11}$ **iii.** $4\ and\ 20$

Solution:

$i.\ \frac{1}{3}\ and\ \frac{1}{23}$

Let, $\frac{1}{3}$, $H.P_1$, $H.P_2$, $H.P_3$, $H.P_4$, $H.P_5$, $\frac{1}{23}$ is an H.P.

then 3, a_1, a_2, a_3, a_4, a_5, 23, is an A.P.

$a = 3$; $a_4 = 23$; $n = 6$

$a_n = a + (n-1)d$

$a_4 = 3 + (6-1)d$

$23 = 3 + 5d$

$5d = 20$

$d = \frac{20}{5} = 4$

Thus, $a = 3$

$a_1 = a + d = 3 + 4 = 7$

$a_2 = a_1 + d = 7 + 4 = 11$

$a_3 = a_2 + d = 11 + 4 = 15$

$a_4 = a_3 + d = 15 + 4 = 19$

Thus 3, 7, 11, 15, 19 is a A.P.

$\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19}, \frac{1}{23}, is\ a\ H.P.$

Hence the required H.P. $\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19}, \frac{1}{23}$

ii. $\frac{7}{3}$ ***and*** $\frac{7}{11}$

Let, $\frac{7}{11}$, H.P$_1$, H.P$_2$, H.P$_3$, H.P$_4$, $\frac{7}{11}$ is an H.P.

Then $\frac{3}{7}$, a_1, a_2, a_3, a_4, $\frac{11}{7}$, is an A.P.

$a = \frac{3}{7}$; $a_4 = \frac{11}{7}$; $n = 6$

We know that

$a_n = a + (n-1)d$

$$\frac{11}{7} = \frac{3}{7} + (6-1)d$$

$$5d = \frac{11}{7} - \frac{3}{7} = \frac{11-1}{3} = \frac{8}{7}$$

$$d = \frac{8}{7\times 5} = \frac{8}{35}$$

Then, $a = \frac{3}{7}$

$$a_1 = \frac{3}{7} + \frac{8}{35} = \frac{15+8}{35} = \frac{23}{35}$$

$$a_2 = \frac{23}{35} + \frac{8}{35} = \frac{23+8}{35} = \frac{31}{35}$$

$$a_3 = \frac{31}{35} + \frac{8}{35} = \frac{31+8}{35} = \frac{39}{35}$$

$$a_4 = \frac{39}{35} + \frac{8}{35} = \frac{39+8}{35} = \frac{47}{35}$$

Thus, $\frac{3}{7}, \frac{23}{35}, \frac{31}{35}, \frac{39}{35}, \frac{47}{35}, \frac{7}{11}$ is A.P.

$$\frac{7}{3}, \frac{35}{23}, \frac{35}{31}, \frac{35}{39}, \frac{35}{47}, \frac{11}{7} \text{ is } H.P.$$

Hence, required H.P., $\frac{35}{23}, \frac{35}{31}, \frac{35}{39}, \frac{35}{47}$

iii. 4 *and* 20

Let, 4, $H.P_1$, $H.P_2$, $H.P_3$, $H.P_4$, 20 is an H.P.

Then $\frac{1}{4}$, a_1, a_2, a_3, a_4, $\frac{1}{20}$, is an A.P.

$$a = \frac{1}{4};\ a_4 = \frac{1}{20};\ n = 6$$

We know that

$$a_n = a_1 + (n-1)d$$

$$a_4 = \frac{1}{4} + (6-1)d$$

$$\frac{1}{20} = \frac{1}{4} + 5d$$

$$5d = \frac{1}{20} - \frac{1}{4} = \frac{1-5}{20} = \frac{-4}{20} = -\frac{1}{5}$$

$$d = \frac{1}{5\times5} = \frac{1}{25}$$

Then, $a = \frac{1}{4}$

$$a_1 = \frac{1}{4} - \frac{1}{25} = \frac{25-4}{100} = \frac{21}{100}$$

$$a_2 = \frac{21}{100} - \frac{1}{25} = \frac{21-4}{100} = \frac{17}{100}$$

$$a_3 = \frac{17}{100} - \frac{1}{25} = \frac{17-4}{100} = \frac{13}{100}$$

$$a_4 = \frac{13}{100} - \frac{1}{25} = \frac{25-4}{100} = \frac{21}{100}$$

Thus, $\frac{1}{4}, \frac{21}{100}, \frac{17}{100}, \frac{13}{100}, \frac{9}{100}, 20$ is A.P.

$$4, \frac{100}{21}, \frac{100}{17}, \frac{100}{13}, \frac{100}{9}, \frac{1}{20} \text{ is } H.P.$$

Hence, required H.P. $\frac{100}{21}, \frac{100}{17}, \frac{100}{13}, \frac{100}{9}$

5. If the 7th and 10th term of an H.P. are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14th term.

Solution:

$H.P_7 = \frac{1}{3}$ and $H.P_{10} = \frac{5}{21}$

$a_7 = 3$ and $a_{10} = \frac{21}{5}$

We know that

$$a_n = a_1 + (n-1)d$$

$$a_7 = a_1 + (7-1)d$$

$$3 = a_1 + 6d$$

And

$$a_7 = a_1 + (n-1)d$$

$$\frac{21}{5} = a_1 + (10-1)d$$

$$\frac{21}{5} = a_1 + 9d$$

$$a_1 + 6d = 3$$

$$\underline{\pm a_1 + \pm 9d = \pm\frac{21}{5}}$$
$$-3d = 3 - \frac{21}{5}$$ (by subtracting)

$$-3d = \frac{15-21}{5}$$

$$-3d = \frac{-6}{5}$$

$$d = \frac{2}{5}$$

Put the value of 'd'

$$3 = a_1 + 6\left(\frac{2}{5}\right)$$

$$3 = a_1 + \frac{12}{5}$$

$$a_1 = 3 - \frac{12}{5} = \frac{15-12}{5} = \frac{3}{5}$$

Thus $a_1 = \frac{3}{5}$ and $d = \frac{2}{5}$

And $a_{14} = a_1 + (14-1)d$

$$a_{14} = \frac{3}{5} + 13\left(\frac{2}{5}\right)$$

$$= \frac{3}{5} + \frac{26}{5}$$

$$= \frac{3+26}{5} = \frac{29}{5}$$

$$a_{14} = \frac{29}{5}$$

And $H.P_{\mathbf{14}} = \frac{\mathbf{5}}{\mathbf{29}}$

Hence required $H.P_{\mathbf{14}} = \frac{\mathbf{5}}{\mathbf{29}}$

6. The first term of an H.P. is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.

Solution:

$$H.P_1 = -\frac{1}{3} \quad \text{and} \quad H.P_5 = \frac{1}{5}$$

$$a_1 = 5 \quad \text{and} \quad a_5 = 5$$

We know that

$$a_n = a_1 + (n-1)d$$

$$a_5 = a_1 + 4d$$

$$5 = (-3) + 4d$$

$$4d = 5 + 3 = 8$$

$$d = \frac{8}{4} = 2$$

And

$$a_n = a_1 + (n-1)d$$

$$= -3 + (9-1)(2)$$

$$= -3 + (8)(2)$$

$$= -3 + 16 = 13$$

$$a_n = 13$$

So, $H.P_n = \frac{1}{13}$

Hence the required H.P.is $= \frac{1}{13}$

7. If 5 is the harmonic mean between 2 and b, find b.

Solution:

$$a = 2; \qquad b = b$$

We know that

$$H.M. = \frac{2ab}{a+b}$$

$$5 = \frac{2(2)b}{2+b}$$

$(2+b)5 = 4b$

$10 + 5b = 4b$

$10 = 4b - 5b$

$-b = 10$

$b = -10$

Hence the value of b $= -10$

8. If the number $\frac{1}{k}, \frac{1}{2k+1}$ *and* $\frac{1}{4k-1}$ are in harmonic sequence, find k.

Solution:

Let

$\frac{1}{k}, \frac{1}{2k+1}$ *and* $\frac{1}{4k-1}$ *is an H.P.*

Than $k, 2k+1, 4k-1$ is an A.P.

$$d = 2k + 1 - k = k + 1$$

We know that

$$a_n = a_1 + (n-1)d$$

$$a_5 = a_1 + 2d$$

$4k - 1 = k + 2(k+1)$

$4k - 1 = k + 2k + 2$

$4k - 1 = 3k + 2$

$4k - 3k = 2 + 1$

$$k = 3$$

Hence the value of $k = 3$.

9. Find n so that $\frac{a^{a+1}+b^{a+1}}{a^n+b^n}$ may be H.M. between a and b.

Solution:

We know that H.M of two terms 'a' and 'b' is

$$H.M = \frac{2ab}{a+b}$$

$$H.M = \frac{a^{a+1}+b^{a+1}}{a^n+b^n}$$

$$\frac{a^{a+1}+b^{a+1}}{a^n+b^n} = \frac{2ab}{a+b}$$

$$(a^{a+1} + b^{a+1})(a + b) = 2ab(a^n + b^n)$$

$$a^{n+2} + b^{a+1} + ab^{n-1} + b^{n+2} = 2a^{n+1}b + 2ab^{n+1}$$

$$a^{n+2} + b^{n+2} = 2a^{n+1}b - a^{n+1}b + 2ab^{n+1} - ab^{n+1}$$

$$a^{n+2} + b^{n+2} = a^{n+1}b + ab^{n+1}$$

$$a^{n+2} - a^{n+1}b = a^{n+1}b + ab^{n+1} - b^{n+2}$$

$$a^{n+2}(a - b) = b^{n+1}(a - b)$$

$$a^{n+1} = b^{n+1}$$

$$\frac{a^{n+1}}{b^{n+1}} = 1$$

$$\left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^0$$

$$n + 1 = 0$$

$$n = -1$$

Hence the value of $n = -1$

10. If a^2, b^2 *and* c^2 are in A.P. show that $a+b, c+a$ *and* $b+c$ are in H.P.

Solution:

Let $a+b, c+a$ *and* $b+c$ are in H.P

Than $d = d$

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{c+a} - \frac{1}{a+b}$$

$$\frac{b+c-c-a}{(a+c)(b+c)} = \frac{a+b-a-c}{(a+b)(c+a)}$$

$$\frac{b-a}{b+a} = \frac{b-c}{a+b}$$

$$(b-a)(b+a) = (b-c)(b+a)$$

$$b^2 - a^2 = b^2 - c^2$$

this show that

a^2, b^2, c^2 are in A.P. which Is given

hence proved

11. The sum of the first and fifth terms of the harmonic sequence is $\frac{4}{7}$, if the first term is $\frac{1}{2}$, find the sequence.

Solution:

Let $H.P._1 + H.P._2 = \frac{4}{7}$

And $H.P._1 = \frac{1}{2}$

By putting the value of 'H.P'

We get

$$\frac{1}{2} + H.P._5 = \frac{4}{7}$$

$$= \frac{4}{7} - \frac{1}{2} = \frac{8-7}{14} = \frac{1}{14}$$

$H.P._5 = \frac{1}{14}$

$a_1 = 2; \qquad a_5 = 14$

We know that

$a_n = a_1 + (n-1)d$

$a_5 = 2 + (5-1)d$

$14 = 2 + 4d$

$4d = 14 - 2 = 12$

$d = \frac{12}{4} = 3$

$a_2 = a_1 + d = 2 + 3 = 5$

$a_3 = a_2 + d = 5 + 3 = 8$

$a_4 = a_3 + d = 8 + 3 = 11$

$2, 5, 8, 11$ is an A.P.

So, $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}$ is H.S

Hence required harmonic sequence = $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots\ldots$

12. If A, G and H are the arithmetic, geometric and harmonic means between a and b respectively, show that $G^2 = AH$.

Solution:

Let $A.M = A$

$G.M = G$

and $H.M = H$

then $G^2 = A.H$

$L.H.S = G^2$

$$= \left[\sqrt{ab}\right]^2 = ab$$

and

$$R.H.S = A.H$$

$$\left[\frac{a+b}{2}\right]\left[\frac{2ab}{a+b}\right] = ab = L.H.S$$

Hence proved $G^2 = A.H$

13. Find A, G, H and show that $G^2 = A.H$. if

i. $\boldsymbol{a = -2, b = -6}$ ii. $\boldsymbol{a = 3i, b = 4i}$ iii. $\boldsymbol{a = 9, b = 4}$

Solution:

i. $\boldsymbol{a = -2, b = -6}$

So, $A = A.M = \frac{a+b}{2} = \frac{-2+(-6)}{2} = \frac{-8}{2} = -4$

and $G = G.M = \sqrt{ab} = \sqrt{(-2)(-6)} = \sqrt{12} = 2\sqrt{3}$

and $H = H.M = \frac{2ab}{a+b} = \frac{2(-2)(-6)}{(-2)+(-6)} = \frac{24}{-8} = -3$

and $G^2 = A.H$

$$\left(2\sqrt{3}\right)^2 = (-3)(-4)$$

$$12 = 12$$

Hence proved $G^2 = A.H$

And $A = -4\,; G = 2\sqrt{3}\ and\ H = -3.$

ii. $\boldsymbol{a = 3i, b = 4i}$

$$A = A.M = \frac{a+b}{2} = \frac{2i+4i}{2} = \frac{6i}{2} = 3i$$

$$G = G.M = \sqrt{ab} = \sqrt{(2i)(4i)} = \sqrt{8i^2} = 2i\sqrt{2}$$

$H = H.M = \frac{2ab}{a+b} = \frac{2(2i)(4i)}{2i+4i} = \frac{16i^2}{6i} = \frac{-8}{3i} = \frac{-8}{3i^2} = \frac{8i}{3}$

$G^2 = A.H$

$\left(2i\sqrt{2}\right)^2 = (3i)\left(\frac{8}{3}i\right)$

$\mathbf{8i^2 = 8i^2}$

Hence proved $G^2 = A.H$

And $A = 3i\,; G = 2i\sqrt{2}$ *and* $H = \frac{8i}{3}$.

iii. $\boldsymbol{a = 9, b = 4}$

So, $A = A.M = \frac{9+4}{2} = \frac{13}{2}$

$G = G.M = \sqrt{4 \times 9} = \sqrt{36} = 6$

and $H = H.M = \frac{2\times4\times9}{4+9} = \frac{72}{13}$

and $G^2 = A.H$

$(6)^2 = \left(\frac{13}{2}\right)\left(\frac{72}{13}\right)$

$36 = 36$

Hence proved $G^2 = A.H$

And $A = \frac{13}{2}\,; G = -6$ *and* $H = \frac{12}{13}$.

14. Find A, G, H and verify that $\boldsymbol{A > G > H(G > 0)}$, if

i. $\boldsymbol{a = 2, b = 8}$ *ii.* $\boldsymbol{a = \frac{2}{5}, b = \frac{8}{5}}$

Solution:

i. $\boldsymbol{a = 2, b = 8}$

$$A = A.M = \frac{2+8}{2} = \frac{10}{2} = 5$$

$$G = G.M = \sqrt{2 \times 8} = \sqrt{16} = \pm 4 = 4 \qquad [\therefore G > 0]$$

$$H = H.M = \frac{2 \times 2 \times 8}{2+8} = \frac{32}{10} = \frac{16}{5} = 3.2$$

It shows

$H < G < A$

Or $A > G > H$

Hence proved

ii. $\boldsymbol{a = \frac{2}{5}, b = \frac{8}{5}}$

$$A = A.M = \frac{\frac{2}{5}+\frac{8}{5}}{2} = \frac{\frac{10}{5}}{2} = \frac{2}{2} = 1$$

$$G = G.M = \sqrt{\frac{2}{5} \times \frac{8}{5}} = \sqrt{\frac{16}{25}} = \frac{4}{5} = 0.75$$

$$H = H.M = \frac{2\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)}{\frac{2}{5}+\frac{8}{5}} = \frac{\frac{32}{25}}{\frac{10}{5}} = \frac{32}{50} = \frac{16}{25}$$

It shows

$H < G < A$

Or $A > G > H$

Hence proved

15. Find A, G, H and verify that $A < G < H (G < 0)$, if

i. $\boldsymbol{a = -2, b = -8}$ ***ii.*** $\boldsymbol{a = \frac{-2}{5}, b = \frac{-8}{5}}$

Solution:

i. $\boldsymbol{a = -2, b = -8}$

$$A = A.M = \frac{-2+(-8)}{2} = \frac{-10}{2} = -5$$

$$G = G.M = \sqrt{(-2)(-8)} = \sqrt{16} = \pm 4 = 4 [\therefore G > 0]$$

$$H = H.M = \frac{2(-2)(-8)}{(-2)+(-8)} = \frac{32}{-10} = \frac{-16}{5}$$

It shows

$$A < G < H$$

Hence proved

ii. $\boldsymbol{a = \frac{-2}{5}, b = \frac{-8}{5}}$

$$A = A.M = \frac{\frac{-2}{5} - \frac{8}{5}}{2} = \frac{\frac{10}{5}}{2} = \frac{-2}{2} = -1$$

$$G = G.M = \sqrt{\left(\frac{-2}{5}\right)\left(\frac{-8}{5}\right)} = \sqrt{\frac{16}{25}} = \pm\frac{4}{5} = -\frac{4}{5} [\therefore G > 0]$$

$$H = H.M = \frac{2\left(-\frac{2}{5}\right)\left(-\frac{8}{5}\right)}{\left(-\frac{2}{5}\right)+\left(-\frac{8}{5}\right)} = \frac{-\frac{32}{5}}{-\frac{10}{5}} = \frac{-32}{-50} = \frac{16}{5}$$

It shows

$$A < G < H$$

Hence proved

16. If the H.M between two numbers are 4 $and\,\frac{9}{2}$ respectively, find the numbers.

Solution:

$$H.M = 4$$

& $A.M = \frac{9}{2}$

$\frac{2ab}{a+b} = 4$

$2\left[\frac{ab}{a+b}\right] = 4$

$2\left[\frac{\frac{ab}{a+b}}{2}\right] = 4$

$\frac{\frac{ab}{a+b}}{2} = 4$

$\frac{ab}{A.M} = 4$

$\frac{ab}{\frac{9}{2}} = 4$

$ab = 4 \times \frac{9}{2} = 2 \times 9 = 18$

$ab = 18$

And $\frac{a+b}{2} = \frac{9}{2}$

$ab = 9$

$a = \frac{18}{b}$

$\frac{18}{b} + b = 9$

$18 + b^2 = 9b$

$b^2 - 9b + 18 = 0$

$b(b-6) - 3(b-6) = 0$

$(b-6)(b-3) = 0$

$b - 6 = 0$ or $b - 3 = 0$

Put the value of 'b' in equation

$ab = 18$

$9(6) = 18$

$a = \frac{18}{6} = 3$

and $a(3) = 18$

$3a = 18$

$a = \frac{18}{3} = 6$

Hence $a = 6 \ or \ 3$

And $b = 3 \ or \ 6$

17. If the (positive) G.M and H.M between two numbers are 4 and $\frac{16}{5}$, find the numbers.

Solution:

$G.M = 4$

$\sqrt{ab} = 4$

$ab = 16$

$b = \frac{16}{a}$

And $H.M = \frac{16}{5}$

$\frac{2ab}{a+b} = \frac{16}{5}$

$\frac{2(16)}{(a+b)} = \frac{16}{5}$

$\frac{32}{a+b} = \frac{16}{5}$

$\frac{2}{a+b} = \frac{1}{5}$

$\frac{1}{a+b} = \frac{1}{5\times2}$

$$a + b = 10$$

and $$a + \frac{16}{a} = 10$$

$$a^2 + 16 = 10a$$

$$a^2 - 10a + 16 = 0$$

$$a^2 - 8a - 2a + 16 = 0$$

$$a(a-8) - 2(a-8) = 0$$

$$(a-8)(a-2) = 0$$

$a - 8 = 0$ or $a - 2 = 0$

$a = 8$ $a = 2$

Put the value of 'a' in equation

$$b = \frac{16}{a}$$

$b = \frac{16}{8}$ or $b = \frac{16}{2}$

$b = 2$ or $b = 8$

Hence the value if $a = 2 \ or \ 8$

And $b = 8 \ or \ 9$

18. If the numbers $\frac{1}{2}, \frac{4}{21}, and \frac{1}{36}$ are subtracted from the three consecutive terms of a G.P, the resulting numbers in H.P. find the numbers if their product is $\frac{1}{27}$.

Solution:

Let the three consecutive terms of G.P. be

$$ar^{-1}, \ a, ar$$

Then, their product

$$ar^{-1} \times a \times ar = \frac{1}{27}$$

$$a^3 = \frac{1}{27}$$

$$a^3 = \left(\frac{1}{3}\right)^3$$

$$a = \frac{1}{3}$$

Now, subtracting the numbers $\frac{1}{2}, \frac{4}{21}, \boldsymbol{and}\, \frac{1}{36}$; from three consecutive terms of G.P.

$$ar^{-1} - \frac{1}{2},\ a = -\frac{4}{21}, ar - \frac{1}{36}$$

So, H.P

$$ar^{-1} - \frac{1}{2},\ a = -\frac{4}{21}, ar - \frac{1}{36}$$

Or $\frac{1}{3}r^{-1} - \frac{1}{2}, \frac{1}{3} - \frac{4}{21}, \frac{1}{3}r - \frac{1}{36}$

$$\frac{1}{3r} - \frac{1}{2}, \frac{7-4}{21}, \frac{1}{3}r - \frac{1}{36}$$

$\frac{1}{3r} - \frac{1}{2}, \frac{3}{21}, \frac{1}{3}r - \frac{1}{36}$ is a H.P.

$\frac{1}{3r} - \frac{1}{2}, \frac{1}{7}, \frac{1}{3}r - \frac{1}{36}$ is a H.P.

$\frac{1-3r}{2}, \frac{1}{7}, \frac{12r-1}{36}$ is H.P.

And $\frac{2}{2-3r}, 7, \frac{36}{12r-1}$ is A.P.

Therefore

$$7 - \frac{2}{2-3r} = \frac{36}{12r-1} - 7$$

$$= \frac{2}{2-3r} - \frac{36}{12r-1} = -7 - 7$$

$$\frac{2}{2-3r} + \frac{36}{12r-1} = 14$$

$$\frac{2(12r-1)+36(2-3r)}{(2-3r)(12r-1)} = 14$$

$$24r - 2 + 72 - 108r = 14(2 - 3r)(12r - 1)$$

$$70 - 83r = 14[\,24r - 2 - 36r + 3r^2]$$

$$14(5 - 6r) = 14\,[3r^2 - 12r - 2]$$

$$5 - 6r = 3r^2 - 12r - 2$$

$$3r^2 - 12r + 6r - 2 - 7 = 0$$

$$3r^2 - 6r - 9 = 0$$

$$3r^2 - 9r + 3r - 9 = 0$$

$$3r(r - 3) + 3(r - 3) = 0$$

$$(3r + 3)(r - 3) = 0$$

$3r + 3 = 0$ or $r - 3 = 0$

$r = \frac{-3}{3}$ $r = 3$

$r = -1$ $r = 3$

Therefore $a = \frac{1}{3}$ $r = \frac{1}{3}$

Terms $ar^{-1} = \frac{1}{3}\left(\frac{1}{3}\right)^{-1} = \frac{1}{3} \times 3 = 1$

$a = \frac{1}{3} = \frac{1}{3}$

Hence the numbers are 1, $\frac{1}{3}, \frac{1}{9}$