EXERCISE 5.3

1. Resolve the following into Partial Fractions:

i. $\dfrac{9x-7}{(x^2+1)(x+3)}$

ii. $\dfrac{1}{(x^2+1)(x+1)}$

iii. $\dfrac{3x+7}{(x^2+4)(x+3)}$

iv. $\dfrac{x^2+15}{(x^2+2x+5)}$

v. $\dfrac{x^2}{(x^2+4)(x+2)}$

vi. $\dfrac{x^2+1}{x^3+1}$

vii. $\dfrac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$

viii. $\dfrac{x^2-2x+3}{x^4+x^2+1}$

ix. $\dfrac{x^4}{1-x^4}$

x. $\dfrac{1}{(x-1)^2(x^2+2)}$

1. $\dfrac{9x-7}{(x^2+1)(x+3)}$

Solution

Let $\dfrac{9x-7}{(x^2+1)(x+3)}$ **=** $\dfrac{A}{x+3}+\dfrac{Bx+C}{x^2+1}$ **............(1)**

By multiplying $(x^2+1)(x+3)$ both sides we get

$$9x-7=A(x^2+1)+(Bx+C)..........(2)$$
$$put(x)=-3$$
$$9(-3)^{-7}=A\left((-3)^2+1\right)+(B(-3)+C)(-3+3)$$
$$-27^{-7}=A(10)+(-3B+C)(0)$$
$$-34=10A$$
$$\Rightarrow A=\frac{-34}{10}$$
$$A=\frac{-17}{5}$$
$$9x-7=Ax^2+A+Bx^2+3Bx+Cx+3C$$
$$9x-7=(A+B)x^2+(3B+C)x+(A+3C)$$

By equating the coefficient of x^2,x and constant.

$$A+B=0AAA$$
$$\Rightarrow B=-A$$
$$B=-\left(\frac{-17}{5}\right)=\frac{17}{5}$$

and

$$A+3C=-7$$
$$\frac{-17}{5}+3C=-7$$
$$3C=-7+\frac{17}{5}$$
$$3C=\frac{-35+17}{5}$$
$$3C=\frac{-18}{5}$$
$$C=\frac{-6}{5}$$

So, $A = -\frac{-17}{5}; B = \frac{17}{5}; C = \frac{-6}{5}$

Therefore, putting these values in equation (1)

$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{\frac{-17}{5}}{x+3} + \frac{17/5x - 6/5}{x^2+1}$$

Hence

$$\frac{9x-7}{(x^2+1)(x+1)} = \frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

2. $\frac{1}{(x^2+1)(x+1)}$

Solution

Let $\frac{1}{(x^2+1)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$............(1)

By multiplying (x+1)(x^2+1) both sides , we get

$$1 = A(x^2+1) + (Bx+C)(x+1)...........(2)$$

$$1 = A\left[(-1)^2+1\right] + \left[B(-1)+C\right](-1+1)$$

$$1 = A[1+1][-B+C](0)$$

$$1 = 2A$$

$$A = \frac{1}{2}$$

$$1 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

By equating the coefficient of x^2,x and constant

$$A + B = 0$$

$$B = -A$$

$$B = \frac{-1}{2}$$

and

$$A + C = 1$$

$C = 1 - A$

$C = 1 - \frac{1}{2} = \frac{2-1}{2}$

$C = \frac{1}{2}$

so,

$A = \frac{1}{2}, B = \frac{-1}{2}, C = \frac{1}{2}$

Therefore by putting these values in eq (1)

$$\frac{1}{(x^2+1)(x+1)} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1}$$

Hence

$$\frac{1}{(x^2+1)(x+1)} = \frac{1}{2(x+1)} - \frac{(x-1)}{2(x^2+1)}$$

3. $\frac{3x+7}{(x^2+4)(x+3)}$

Solution

Let $\frac{3x+7}{(x^2+4)(x+3)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$

$3x + 7 = A(x^2+4) + (Bx+C)(x+3)..........(1)$

$3(-3) + 7 = A\left[(-3)^2 + 4\right] + \left[B(-3) + C\right](-3+3)$

$-9 + 7 = A\left[9+7\right] + \left[-3B + C\right](0)$

$-2 = BA$

$A = \frac{-2}{13}$

$3x + 7 = Ax^2 + 4A + Bx^2 + 3Bx + Cx + 3C$

$3x + 7 = (A+B)x^2 + (3B+C)x + (4A+3C)$

By equating the coefficient of x, x and constant

$A + B = 0$

$B = -A$

$B = -\left(\frac{-2}{13}\right)$

$B = \frac{2}{13}$

$3B + C = 3$

$3\left(\frac{2}{13}\right) + C = 3$

$C = 3 - \frac{6}{13}$

$C = \frac{39 - 6}{13}$

$C = \frac{33}{13}$

So,

$$A = \frac{\frac{-2}{13}}{x+3} + \frac{\frac{2}{13x} + \frac{33}{13}}{x^2 + 4}$$

Hence

$$\frac{3x+7}{(x+3)(x^2+4)} = \frac{-2}{13(x+3)} + \frac{2x+33}{13(x^2+4)}$$

4. $\frac{x^2+15}{(x^2+2x+5)(x-1)}$

Solution

Let $\frac{x^2+15}{(x^2+2x+5)(x-1)}$ **=** $\frac{A}{x-1} + \frac{Bx+C}{x^2+2x+5}$............(1)

By multiplying $(x-1)(x^2+2x+5)$ **both sides we get**

$x^2 + 15 = A(x^2 + 2x + 5) + (Bx + C)(x - 1)$..........(2)

$(1)^2 + 15 = A\left((1)^2 + 2(1) + 5\right)(B(1) + C)(1-1)$

$1 + 15 = A\left(1 + 2 + 5\right) + (B + C)(0)$

$16 = A(8)$

$A = \frac{16}{8} = 2$

$x^2 + 15 = Ax^2 + 2Ax + 5A + Bx^2 - Bx + Cx - C$

$x^2 + 15 = (A + B)x^2 (2A - B + C)x + (5A - C)$

By equating the coefficient of x^2, x and constant

$A + B = 1$

$2 + B = 1$

$B = 1 - 2 = -1$

$B = -1$

and

$5A - C = 15$

$5(2) - C = 15$

$10 - C = 15$

$-C = 15 - 10$

$C = -5$

so,

$A = 2, B = -1, C = -5$

Therefore, by putting these values in eq (1)

$$\frac{x^2 + 15}{(x^2 + 2x + 5)(x - 1)} = \frac{2}{x - 1} + \frac{(-1)x(-5)}{x^2 + 2x + 5}$$

Hence

$$\frac{x^2 + 15}{(x^2 + 2x + 5)} = \frac{2}{x - 1} + \frac{x + 5}{x^2 + 2x + 5}$$

5. $\frac{x^2}{(x^2 + 4)(x + 2)}$

Solution

Let $\frac{x^2}{(x^2 + 4)(x + 2)}$ **=** $\frac{A}{x + 2} + \frac{Bx + C}{x^2 + 4}$........(1)

By multiplying $(x + 2)(x^2 + 4)$ **both sides we get**

$$x^2 = A(x^2+4)+(Bx+C)(x+2)..............(2)$$
$$(-2)^2 = A\left[(-2)^2+4\right]+\left[B(-2)+C\right]\left[-2+2\right]$$
$$4 = A(4+4)+(-2B+C)(0)$$
$$4 = 8A$$
$$A = \frac{4}{8} = \frac{1}{2}$$
$$x^2 = Ax^2+4A+Bx^2+2Bx+Cx+2x$$
$$x^2 = (A+B)x^2+(2B+C)x+(4A+2C)$$

By equating the coefficient of x^2, x and constant

$$A+B=1$$
$$\frac{1}{2}+B=1$$
$$B = 1-\frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$$
$$B = \frac{1}{2}$$
$$2B+C=0$$
$$C=-2B$$
$$C=-2\left(\frac{1}{2}\right)$$
$$C=-1$$
so,
$$A=\frac{1}{2}, B=\frac{1}{2}, C=-1$$

Therefore, by putting these values in equation (1)

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{\frac{1}{2}}{x+1}+\frac{\frac{1}{2}x-1}{x^2+4}$$

Hence
$$\frac{x^2}{(x^2+4)(x+2)} = \frac{1}{2(x+2)}+\frac{x-2}{2(x^2+4)}$$

6. $\frac{x^2+1}{x^3+1}$

Solution

Let $\frac{x^2+1}{x^3+1}$ **=** $\frac{x^2+1}{(x+1)(2x-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$ ....(1)

By multiplying (x+1) (x^2-x+1) both sides we get

$x^2+1=A(x^2-x+1)+(Bx+c)(x+1)$............(2)

$(-1)^2+1=A\left[(-2)^2-(-1)+1\right]+\left[B(-1)+C\right]\left[-1+1\right]$

$1+1=A\left[1+1+1\right]+\left[-B+C\right](0)$

$2=3A+0$

$A=\frac{2}{3}$

$x^2+1=Ax^2-Ax+A+Bx^2+Bx+Cx+C$

$x^2+1=(A+B)x^2+(-A+B+C)x+(A+C)$

By equating the coefficient of x^2,x and constant

$A+B=1$

$B=1-A$

$B=1-\frac{2}{3}=\frac{3-2}{3}$

$B=\frac{1}{3}$

$A+C=1$

$\frac{2}{3}+C=1$

$C=1-\frac{1}{2}$

$=\frac{2-1}{3}$

$C=\frac{1}{3}$

$so,$

$A=\frac{2}{3},B=\frac{1}{3},C=\frac{1}{3}$

Therefore, by putting these values in equation (1)

$$\frac{x^2+1}{x^3+1}=\frac{\frac{2}{3}}{x+1}+\frac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}$$

Hence

$$\frac{x^2+1}{x^3+1}=\frac{2}{3(x+1)}+\frac{x+1}{3(x^2-x+1)}$$

7. $\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$

Solution

Let $\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$ **=** $\frac{A}{x+1}+\frac{B}{x-1}+\frac{Cx+D}{x^2+3}$..........(1)

By multiplying $(x^2+3)(x+1)(x-1)$ both sides we get

$$x^2+2x+2=A(-1-1)((-1)^2+3)+B(x+1)(x^2+3)+(Cx+1)(x+1)(x-1).........(2)$$

$$(-1)^2+2(-1)+2=A(-1-1)((-1)^2+3)+B(-1+1)((-1)^2+3)\left[((-1)+1))(-1+1)(-1-1)\right]$$

$$1-2+2=A(-2)(4)+B(0)(4)+(-C+D)(0)(-2)$$

$$1=8A$$

$$A=\frac{-1}{8}$$

$$x^2+2x+2=A(x^3-x^2+3x-3)+B(x^3+x^2+3x+3)Cx(x^2-1)+D(2x-1)$$

$$x^2+2x+2=Ax^3-Ax^2+3Ax-3A+Bx^3+Bx^2+3Bx+3B+Cx^3-Cx+Dx^2-D$$

$$x^2+2x+2=(A+B+C)x^3(-A+B+D)x^2+(3A+3B-C)x+(-3A+3B-D)$$

By equating the coefficient of x^3, x^2, x and constant

$$A+B+C=0$$

$$\frac{-1}{8}+\frac{5}{8}+C=0$$

$$\frac{-1+15}{8}+C=C=0$$

$$\frac{4}{8}+C=0$$

$$C=\frac{-4}{8}=\frac{-1}{2}$$

And

$-A+B+C=1$

$-\left(\frac{-1}{8}\right)+\frac{5}{8}+D=1$

$\frac{1}{8}+\frac{5}{8}+D=1$

$\frac{1+5}{8}+D=1$

$\frac{6}{8}+D=1$

$D=1-\frac{3}{4}$

$D=\frac{4-3}{4}=\frac{1}{4}$

$D=\frac{1}{4}$

so,

$A=\frac{-1}{8};B=\frac{5}{8};C=\frac{-1}{2};D=\frac{1}{4}$

Therefore by putting these values in eq (1)

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}=\frac{\frac{-1}{8}}{x+1}+\frac{\frac{5}{8}}{x-1}+\frac{\frac{-1}{2x}+\frac{1}{4}}{x^2+3}$$

Hence

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}=\frac{-1}{8(x-1)}+\frac{5}{8(x-1)}-\frac{2x-1}{4(x^2+3)}$$

8. $\frac{1}{(x-1)^2(x^2+2)}$

Solution

Let $\frac{1}{(x-1)^2(x^2+2)}$ **=** $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}$...........(1)

By multiplying $(x-1)^2(x^2+2)$ **both sides we get**

$$1 = A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2$$
$$1 = A(1-1)(1+2) + B(1+2) + (Cx+D)(x-1)^2$$
$$1 = A(0)(3) + B(3) + (C+D)(1-1)^2$$
$$1 = 0 + 3B + 0$$
$$B = \frac{1}{3}$$
$$1 = A(x^3 + 2x - x^2 - 2) + (Bx^2 - 2B)(Cx+D)(x^2 - 2x + 1)$$
$$1 = (Ax^3 - Ax^2 + 2Ax + 2A) + (Bx^2 + 2B) + (Cx^3 - 2Cx^2 + Cx) + (Dx^2 - 2Dx + 1)$$
$$1 = (A+C)x^3 + (-A+B-2C+D)x^2(2A+C-2D)x + (-2A+2B+D)$$

By equating the coefficient of x^3,x^2,x and constant

$$A + B = 0$$
$$A = -C$$
and
$$-A + B - 2C + D = 0$$
$$-(-C) + \frac{1}{3} - 2C + D = 0$$
$$C + \frac{1}{3} - 2C + D = 0$$
$$D = 0 - \frac{1}{3} + C$$

and
$$2A + C - 2D = 0$$
$$2(-C) + C - 2\left(-\frac{1}{3} + C\right) = 0$$
$$-2C + C + \frac{2}{3} - 2C = 0$$
$$-3C = -\frac{2}{3}$$
$$C = \frac{2}{9}$$
$$A = -C = -\left(\frac{2}{9}\right) = \frac{-2}{9}$$
$$D = \frac{1}{3} + C$$

$$D = -\frac{1}{3} + \frac{2}{9}$$
$$D = \frac{-3+2}{9}$$
so,
$$A = \frac{-2}{9}; B = \frac{1}{3}; C = \frac{2}{9}; D = \frac{-1}{9}$$

Therefore by putting these values in eq (1)

Hence,

$$\frac{1}{(x-1)^2(x^2+2)} = \frac{-2}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{2x-1}{9(x^2+2)}$$

9. $\frac{x^4}{1-x^4}$

Solution

Since the order of numerator and denominator are same i.e. 4 , by dividing them to convert the improper fraction into proper fraction .

$$1-x^4 \overline{\big) x^4} \qquad -1$$
$$\pm x^4 m 1$$
$$1$$

$$\frac{x^4}{1-x^4} = -1 + \frac{1}{1-x^4}$$
$$\frac{1}{1-x^4} = \frac{1}{(1-x^2)(1+x^2)} = \frac{1}{(1-x)(1+x)(1+x^2)}$$
$$\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

By multiplying $(1-x)(1+x)(1+x^2)$ **both sides we get**

$$1 = A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1-2x)........(2)$$

$$1 = A(1+1)(1+1) + B(1-1)(1+1) + (C+D)(1-1)$$

Put $1 = A(2)(2) + B(0)(2) + (C+D)(2)(0)$

$$1 = 4A + 0 + 0$$

$$A = \frac{1}{4}$$

And put x=-1 in eq (2)

$$1 = A(1-1)(1+1) + B(1+1)(1+1) + (-C+D)(1-1)$$

$$1 = A(0)(2) + B(2)(2) + (-C+D)(2)(0)$$

$$1 = 0 + 4B + 0$$

$$B = \frac{1}{4}$$

$$1 = A(1+x^2+x+x^3) + B(1-x+x^2-x^3) + Cx(1-x^2) + D(1-x^2)$$

$$1 = Ax^3 + Ax^2 + Ax + A - Bx^3 + Bx^2 - Bx + B - Cx^3 - Dx^2 + Cx + D$$

$$1 = (A-B)x^3 + (A+B-D)x^2 + (A-B+C)x + (A+B+D)$$

By equating the coefficient of x3,x2,x and constant

$$A + B - D = 0$$

$$D = A + B = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$D = \frac{1}{2}$$

$$A - B + C = 0$$

$$\frac{1}{4} - \frac{1}{4} + C = 0$$

$so,$

$$A = \frac{1}{4}; B = \frac{1}{4}; C = 0 and D = \frac{1}{2}$$

Therefore, by putting these values in equation (1)

$$\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{1/4}{(1-x)} - \frac{1/4}{(1+x)} + \frac{(0)x + 1/2}{(1+x^2)}$$

Hence

$$\frac{x^4}{1-x^4} = -1 + \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}$$

10. $\dfrac{x^2-2x+3}{x^4+x^2+1}$

Solution

$$= \frac{x^2-2x+3}{x^4+x^2+1+x^2}$$

$$= \frac{x^2-2x+3}{x^4+2x^2+1-x^2}$$

$$= \frac{x^2-2x+3}{(x^2-1)^2-x^2}$$

$$= \frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)}$$

$$\Rightarrow \frac{x^2-2x+3}{x^4+x^2+1} = \frac{x^2-2x+3}{(x^4+x+1)(x^2-x+1)}$$

$$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)} = \frac{Ax+B}{(x^2+x+1)} + \frac{Cx+D}{(x^2-x+1)}$$

$$x^2-2x+3 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1)$$

$$x^2-2x+3 = Ax(x^2-x+1) + B(x^2-x+1) + Cx(x^2+x+1) + D(x^2+x+1)$$

$$x^2-2x+3 = Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D$$

$$x^2-2x+3 = (A+C)x^3 + (-A+B+C+D)x^2 + (A-B+C+D)x + (B+D)$$

By equating the co-efficient of x3, x2, x and constant

$A + C = 0 \Rightarrow A = -C$

$B + D = 3$

$\Rightarrow B = 3 - D$

$A - B + C + D = -2$

$$(A+C)-B+D=-2$$
$$(0)-(3-D)+D=-2$$
$$-3+D+D=-2$$
$$2D=-2+3$$
$$2D=1$$
$$D=\frac{1}{2}$$
$$B=3-D$$
$$B=3-\frac{1}{2}=\frac{6-1}{2}=\frac{5}{2}$$

$$-A+B+C+D=1$$
$$-(-C)+\frac{5}{2}+C+\frac{1}{2}=1$$
$$C+C+\frac{1}{2}+\frac{5}{2}=1$$
$$2C=1-\frac{1}{2}-\frac{5}{2}$$
$$2C=\frac{2-1-5}{2}$$

$$2C=\frac{-4}{2}$$
$$\rightarrow \quad C=\frac{-4}{2\times 2}=-1$$
$$C=-1$$

Therefore, by putting these values in equation (1)

$$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)}=\frac{(1)x+5/2}{(x^2+x+1)}+\frac{(-1)x+1/2}{(x^2-x+1)}$$

Hence

$$\frac{x^2-2x+3}{(x^4+x^2+1)}=\frac{2x+5}{2(x^2+x+1)}+\frac{1-2x}{x^2-x+1}$$