Chapter 5

Partial Fraction

Exercise 5.1

Resolve the following into Partial Fractions.

1. $\frac{1}{x^2-1}$

2. $\frac{(x^2-1)}{(x+1)(x-1)}$

3. $\frac{2x+1}{(x-1)(2x-1)(x+3)}$

4. $\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$

5. $\frac{1}{(x-1)(2x-1)(3x-1)}$

6. $\frac{x}{(x-a)(x-b)(x-c)}$

7. $\frac{6x^2+5x^2-7}{2x^2-x-1}$

8. $\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$

9. $\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$

10. $\frac{1}{(1-ax)(1-bx)(1-cx)}$

11. $\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2-d^2)}$

Hint: Put x^2=y to make factors of the denomination linear.

1. $\frac{1}{x^2-1}$

Solution

Let $\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$ ________________(i)

By multiplying (x+1) (x-1) both sides, we get

$1=A(x-1)+B(x+1)$ ____________________(ii)

$x=A(x-1)+B(x+1)$

$1=A(1-1)+B(1+1)$

$1=A(0)+B(2)$

$B=\frac{1}{2}$

And

$x=-1$ ________$(in_eq(ii))$

$1=A(-1-1)+B(-1+1)$

$1=A(-2)+B(0)$

$-2A=1$

$A=-\frac{1}{2}$

So

$A=-\frac{1}{2}$

$B=+\frac{1}{2}$

Therefore, by putting the value of A&B in eq (i) we get

$$\frac{1}{x^2-1}=\frac{-\frac{1}{2}}{x-1}+\frac{\frac{1}{2}}{x+1}$$

Hence; $$\frac{1}{x^2-1}=-\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$$

2. $\frac{(x^2-1)}{(x+1)(x-1)}$

Solution:

$$\frac{(x^2-1)}{(x+1)(x-1)}=\frac{x^2+1}{x^2-1}$$

Therefore

$$\frac{x^2+1}{x^2-1}=1+\frac{2}{x^2-1}$$,

$$x^2-1\overline{)x^2+1}\ \ 1$$
$$\underline{\pm x^2 ml}$$
$$2$$

<u>And</u> $\frac{2}{x^2-1}=\frac{2}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$ ________________ (i)

By multiplying $(x-1)(x+1)$ **both sides, we get**

$2=A(x+1)+B(x-1)$ ________________ (ii)

$x=-1\_in\_eq(ii)$

$x=A(-1+1)+B(-1-1)$

$x=A(0)+B(-2)$

$2=-2B$

$B=-1$

and

$x=1\_in\_eq(ii)$

$2=A(1+1)+B(1-1)$

$2=A(2)+B(0)$

$A2=2$

$A=1$

So A=1 ; B= -1

Therefore, by putting the value of A &B in eq.(i) becomes

$$\frac{2}{(x+1)(x-1)} = \frac{1}{x-1} + \frac{-1}{x+1}$$

$$\frac{x^2+1}{x^2-1} = 1 + \frac{1}{x-1} - \frac{1}{x+1}$$

3. $\frac{2x+1}{(x-1)(2x-1)(x+3)}$

Solution

Let

$$\frac{2x+1}{(x-1)(2x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3} \_\_\_\_\_\_\_\_\_\_\_\_ (i)$$

By multiplying $(x-1)(2x-1)(x+3)$ **both sides, we get**

$$2x+1 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2) \_\_\_\_\_\_\_\_ (ii)$$

$$put(x=1)in\_eq(ii)$$

$$2(1)+1 = A(1+2)(1+3) + B(1-1)(1+3) + C(1-1)(1+2)$$

$$3 = A(3)(4) + B(0)(4) + C(0)(3)$$

$$3 = A12$$

$$A = \frac{1}{4}$$

Put $(x=-2)$ **in eq, (ii)**

$$2(-2)+1 = A(-2+2)(-2+3) + B(-2-1)(-2+3) + C(-2-1)(-2+2)$$

$$-4+1 = A(0)(1) + B(-3)(1) + C(-3)(0)$$

$$-3 = B(-3)$$

$$B = 1$$

Put $(x=-3)$**in eq, (ii)**

$2(-3)+1=A(-3+2)(-3+3)+B(-3-1)(-3+3)+C(-3-1)(-3+2)$

$-6+1=A(1)(0)+B(-4)(0)+C(-4)(-1)$

$-5=C(4)$

$c=-\frac{5}{4}$

So $A=\frac{1}{4};B=1\_and\_C=\frac{-5}{4}$

Therefore, by putting the values of A, B and C in eq, (i)

$$\frac{2x+1}{(x-1)(2x-1)(x+3)}=\frac{\frac{1}{4}}{x-1}+\frac{1}{x+2}+\frac{\frac{-5}{4}}{x+3}$$

Hence; $$\frac{2x+1}{(x-1)(2x-1)(x+3)}=\frac{1}{4(x-1)}+\frac{1}{(x+2)}-\frac{5}{4(x+3)}$$

4. $\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}=\frac{3x^2-4x-5}{(x-2)(x^2+2x+5x+10)}$$

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}=\frac{3x^2-4x-5}{(x-2)(x(x+2)+5(x+2))}$$

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}=\frac{3x^2-4x-5}{(x-2)(x+2)(x+5)}$$

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x+5}\_\_\_\_\_\_\_\_\_\_(i)$$

by multiplying $(x-2)(x+2)(x+5)$ **both sides, we get**

$$3x^2-4x-5=A(x+2)(x+5)+B(x-2)(x+5)+C(x-2)(x+2)\_\_\_\_\_\_\_\_\_\_(ii)$$

put x=2 in eq, (ii)

$$3(2)^2-4(2)-5=A(2+2)(2+5)+B(2-2)(2+5)+C(2-2)(2+2)$$
$$12-8-5=A(4)(7)+B(0)(7)+C(0)(4)$$
$$-1=A21$$
$$A=-\tfrac{1}{21}$$

Put x=-2 in eq(ii)

$$3(-2)^2-4(-2)-5=A(-2+2)(-2+5)+B(-2-2)(-2+5)+C(-2-2)(-2+2)$$
$$12+8-5=A(0)(3)+B(-4)(3)+C(-4)(0)$$
$$15=B(-12)$$
$$B=-\tfrac{5}{4}$$

Put x=-5 in eq. (ii)

$$3(-5)^2-4(-5)-5=A(-5+2)(-5+5)+B(-5-2)(-5+5)+C(-5-2)(-5+2)$$
$$75+20-5=A(-3)(0)+B(-7)(0)+C(-7)(-3)$$
$$90=C21$$
$$C=\tfrac{30}{7}$$

So $A=\tfrac{-1}{28}, B=\tfrac{-5}{4}\ and\ C=\tfrac{30}{7}$

Therefore, by putting the values of these in eq.(i)

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}=\frac{-1/28}{x-2}+\frac{-5/4}{x+2}+\frac{30/7}{x+5}$$
$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}=-\frac{1}{28(x-2)}-\frac{5}{4(x+2)}+\frac{30}{7(x+5)}$$

5. $\dfrac{1}{(x-1)(2x-1)(3x-1)}$

Let $\dfrac{1}{(x-1)(2x-1)(3x-1)}=\dfrac{A}{x-1}+\dfrac{B}{2x-1}+\dfrac{C}{3x-1}$

By multiplying $(x-1)(2x-1)(3x-1)$ both sides we get

$$1 = A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1) \text{-------------(ii)}$$

$Put\_x = \frac{1}{2}\_in\_eq.(ii)$

$$1 = A\left(2\left(\frac{1}{2}\right)-1\right)\left(3\left(\frac{1}{2}\right)-1\right)+B\left(\left(\frac{1}{2}\right)-1\right)\left(3\left(\frac{1}{2}\right)-1\right)+C\left(\left(\frac{1}{2}\right)-1\right)\left(2\left(\frac{1}{2}\right)-1\right)$$

$$1 = 0 + B\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)+0$$

$$1 = -\frac{1}{4}B$$

$$B = -4$$

$$Put\_x = \tfrac{1}{3}\_in\_eq.(ii)$$

$$1 = A\left(2\left(\tfrac{1}{3}\right)-1\right)\left(3\left(\tfrac{1}{3}\right)-1\right)+B\left(\left(\tfrac{1}{3}\right)-1\right)\left(3\left(\tfrac{1}{3}\right)-1\right)+C\left(\left(\tfrac{1}{3}\right)-1\right)\left(2\left(\tfrac{1}{3}\right)-1\right)$$

$$1 = A(0)+B(0)+C\left(\frac{-2}{3}\right)\left(\frac{-1}{3}\right)$$

$$1 = \frac{2}{9}C$$

$$C = \frac{9}{2}$$

$$Put\ x = 1\_in\_eq.(ii)$$

$$1 = A(2(1)-1)(3(1)-1)+B((1)-1)(3(1)-1)+C((1)-1)(2(1)-1)$$

$$1 = A(1)(2)+B(0)+C(0)$$

$$1 = 2A$$

$$A = \frac{1}{2}$$

So,

$$A = \tfrac{1}{2}, B = 4, C = \frac{9}{2}$$

Therefore by putting these value in eq (i)

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{\tfrac{1}{2}}{x-1}+\frac{4}{2x-1}+\frac{\tfrac{9}{2}}{3x-1}$$

Hence; $$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{1}{2(x-1)}+\frac{4}{(2x-1)}+\frac{9}{2(3x-1)}$$

6. $\dfrac{x}{(x-a)(x-b)(x-c)}$

Solution

Let $\dfrac{x}{(x-a)(x-b)(x-c)}=\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$ ____________ (i)

By multiplying $(x-a)(x-b)(x-c)$ **both sides we get**

$x=A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b)$ ______ (ii)

Put $x=a\_in\_eq.(ii)$

$$a=A(a-b)(a-c)+B(a-a)(a-c)+C(a-a)(a-b)$$
$$a=A(a-b)(a-c)$$
$$A=\frac{a}{(a-b)(a-c)}$$

Put $x=b\_in\_eq.(ii)$

$$b=A(b-b)(b-c)+B(b-a)(b-c)+C(b-a)(b-b)$$
$$b=A(0)(b-c)+B(b-a)(b-c)+C(b-a)(0)$$
$$b=B(b-a)(b-c)$$
$$B=\frac{b}{(b-a)(b-c)}$$

Put $x=c\_in\_eq.(ii)$

$$c=A(c-b)(c-c)+B(c-a)(c-c)+C(c-a)(c-b)$$
$$c=A(c-b)(0)+B(c-a)(0)+C(c-a)(c-b)$$
$$c=C(c-a)(c-b)$$
$$C=\frac{c}{(c-a)(c-b)}$$

So , $A=\dfrac{a}{(a-b)(a-c)}$; $B=\dfrac{b}{(b-a)(b-c)}$ **and** $C=\dfrac{c}{(c-a)(c-b)}$

Therefore, by putting these values in eq.(i)

$$\frac{x}{(x-a)(x-b)(x-c)}=\frac{\frac{a}{(a-b)(a-c)}}{x-a}+\frac{\frac{b}{(b-a)(b-c)}}{x-b}+\frac{\frac{c}{(c-a)(c-b)}}{x-c}$$

Hence; $$\frac{x}{(x-a)(x-b)(x-c)}=\frac{a}{(x-a)(a-b)(a-c)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(x-c)(c-a)(c-b)}$$

7. $\frac{6x^2+5x^2-7}{2x^2-x-1}$

Solution

The highest power of x in the numerator is 3 and in the denominator is 2. So, we will have to divide the numerator by its denominator.

$$\begin{array}{r|l} & 3x+4 \\ 2x^2-x-1 & 6x^3+5x^2-7 \\ & \underline{\pm 6x^3 m^3x^2 \_\_\_ m3x} \\ & 8x^2+3x-7 \\ & \underline{\pm 8x^2 m^{4x} m^4} \\ & 7x-3 \end{array}$$

$$\frac{6x^3+5x^2-7}{2x^2-x-1}=3x+4+\frac{7x+3}{2x^2-x-1}$$

Let

$$\frac{7x+3}{2x^2-x-1}=\frac{7x+3}{2x^2-2x+x-1}=\frac{7x+3}{2x(x-1)+1(x-1)}=\frac{7x+3}{(2x+1)(x-1)}$$

Therefore

$$\frac{7x+3}{2x^2-x-1}=\frac{A}{2x+1}+\frac{B}{x-1}\_\_\_\_\_\_\_\_\_(i)$$

By multiplying $(2x+1)(x-1)$ **both sides we get**

$$7x-3=A(x-1)+B(2x+1)\_\_\_\_\_\_\_\_\_(ii)$$

Put x=1 in eq.(ii)

$$7(1)-3=A(1-1)+B(2(1)+1)$$
$$7-3=A(0)+B(3)$$
$$4=B3$$
$$B=\frac{4}{3}$$

And put $x=-\frac{1}{2}$ in eq (ii)

$$7\left(\frac{-1}{2}\right)-3=A\left(-\frac{1}{2}-1\right)+B\left(2\left(-\frac{1}{2}\right)+1\right)$$
$$-\frac{7}{2}-3=A\left(\frac{-3}{2}\right)+B(0)$$
$$\frac{-7-6}{2}=A\left(\frac{-3}{2}\right)$$

$$-13=-3A$$

$$A=\frac{-13}{-3}=A=\frac{13}{3}$$
So
$$A=\frac{13}{3}\ \_\_\_\ and\ \_\_\_\_\ B=\frac{4}{3}$$

Therefore, by putting these values in eq.(i)

$$\frac{7x+3}{2x^2-x-1}=\frac{13/3}{2x+1}+\frac{4/3}{x-1}$$

Hence; $$\frac{7x+3}{2x^2-x-1}=\frac{13}{3(2x+1)}+\frac{4}{3(x-1)}$$

8. $\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$

Solution:

The power of numerator and denominator is same 1.e ,3.

Therefore, we get proper friction by dividing them.

$$\begin{array}{r} 3x+4 \\ 2x^3-x^2-3x\overline{)2x^3+x^2-5x} \\ \underline{\pm 2x^3 m^{x^2} m^{3x}} \\ -2x+3 \end{array}$$

Therefore

$$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}=3x+4+\frac{(-2x+3)}{2x^3+x^2-3x}$$

Take factorization of $(2x^3+x^2-3x)$

$$2x^3+x^2-3x=x(2x^2+x-3)$$
$$2x^3+x^2-3x=x(2x^2+3x-2x-3)$$
$$2x^3+x^2-3x=[x(2x+3)-1(2x-3)]$$
$$2x^3+x^2-3x=(x-1)(2x+3)$$

So,

$$=\frac{-2x+3}{2x^3+x^2-3x}$$

$$\frac{-2x+3}{x(x-1)(2x+3)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{2x+3}\_\_\_\_\_\_\_\_\_\_(i)$$

By multiplying $x(x-1)(2x+3)$ **both sides we get**

$$-2x+3=A(x-1)(2x+3)+Bx(2x+3)+Cx(x-1)\_\_\_\_\_\_\_\_\_\_(ii)$$

Put x=1 in eq (ii)

$$-2(1)+3=A(1-1)(2(1)+3)+B(1)(2(1)+3)+C(1)((1)-1)$$
$$-2+3=A(0)+B(5)+C(0)$$
$$1=B5$$
$$b=\frac{1}{5}$$

Put x=0 in eq.(ii)

$$-2(0)+3=A(0-1)(2(0)+3)+B(0)(2(0)+3)+C(0)((0)-1)$$
$$3=A(1)(5)+B(0)+C(0)$$
$$3=5A$$
$$A=3/5$$

Put $x=-3/2$ **in eq. (ii)**

$$-2\left(-3/2\right)+3=A\left(-3/2-1\right)\left(2\left(-3/2\right)+3\right)+B\left(-3/2\right)\left(2\left(-3/2\right)+3\right)+C\left(-3/2\right)\left(-3/2-1\right)$$
$$3+3=A(0)+B(0)+C\left(-5/2\right)\left(-3/2\right)$$
$$6=C\,15/4$$
$$C=\frac{(6)(4)}{15}$$
$$C=\frac{8}{5}$$

So $A=-1, B=1/5, C=8/5$

Therefore, by putting these values in eq.(i)

$$\frac{-2x+3}{x(x-1)(2x+3)}=\frac{-1}{x}+\frac{1/5}{x-1}+\frac{8/5}{2x+3}$$

Hence; $$\frac{-2x+3}{x(x-1)(2x+3)}=-\frac{1}{x}+\frac{1}{5(x-1)}+\frac{8}{5(2x+3)}$$

9. $\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$

Solution

The power of numerator and denominator is same i.e. 3. We divide them to get the proper fraction.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad 1 \\ x^3-12x^2+44x-48 & x^3-9x^2+23x-15 \triangle ABC \\ & \underline{\pm x^3 m^{12x^2} \pm 44^x m^{4x}} \\ & 3x^2-21x+33 \end{array}$$

$$So, \quad \frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48} = 1 + \frac{3x^3-21x^2+33x}{x^3-12x^2+44x-48}$$

Therefore;

$$\frac{3x^3-21x^2+33x}{x^3-12x^2+44x-48} = \frac{3x^3-21x^2+33x}{(x-2)(x-4)(x-6)}$$

$$\frac{3x^3-21x^2+33x}{(x-2)(x-4)(x-6)} = \frac{A}{(x-1)} + \frac{B}{(x-4)} + \frac{C}{(x-6)} \_\_\_\_\_\_\_ (i)$$

By multiplying $(x-2)(x-4)(x-6)$**both sides we get.**

$$3x^2-21x+33 = A(x-4)(x-6) + B(x-2)(x-6) + C(x-2)(x-4) \_\_\_\_\_\_\_ (ii)$$

Put x=2 in eq. (ii)

$$3(2)^2-21(2)+33 = A((2)-4)(2-6) + B(2-2)(2-6) + C(2-2)(2-4)$$
$$12-42+33 = A(2)(4) + B(0) + C(0)$$
$$3 = A8$$
$$A = \frac{3}{8}$$

Put x=4 in eq. (ii)

$$3(4)^2-21(4)+33 = A(4-4)(4-6) + B(4-2)(4-6) + C(4-2)(4-4)$$
$$48-84+33 = A(0) + B(2)(-2) + C(0)$$
$$-3 = B(-4)$$
$$B = \frac{3}{4}$$

Put x=6 in eq.(ii)

$$3(6)^2-21(6)+33=A(6-4)(6-6)+B(6-2)(6-6)+C(6-2)(6-4)$$
$$108-126+33=A(0)+B(0)+C(4)(2)$$
$$15=C8$$
$$C=\frac{15}{8}$$

So $A=\frac{3}{8}$; $B=\frac{4}{3}$; $C=\frac{15}{8}$

Therefore, by putting these values in eq. (i)

$$\frac{3x^3-21x^2+33x}{(x-2)(x-4)(x-6)}=1+\frac{3}{4(x-4)}+\frac{15}{8(x-6)}$$

Hence; $$\frac{3x^3-21x^2+33x}{(x-2)(x-4)(x-6)}=1+\frac{3}{8(x-2)}+\frac{3}{4(x-4)}+\frac{15}{8(x-6)}$$

10. $\frac{1}{(1-ax)(1-bx)(1-cx)}$

Solution

Let $\frac{1}{(1-ax)(1-bx)(1-cx)}=\frac{A}{(1-ax)}+\frac{B}{(1-bx)}+\frac{C}{(1-cx)}$ ____ (i)

by multiplying $(1-ax)(1-bx)(1-cx)$ **both sides we get.**

$$1=A(1-bx)(1-cx)+B(1-ax)(1-cx)+C(1-ax)(1-bx)\_\_\_\_\_(ii)$$

Put $x=\frac{1}{a}$ **in eq, (ii)**

$$1=A\left(1-b\frac{1}{a}\right)\left(1-c\frac{1}{a}\right)+B\left(1-a\frac{1}{a}\right)\left(1-c\frac{1}{a}\right)+C\left(1-a\frac{1}{a}\right)\left(1-b\frac{1}{a}\right)$$
$$1=A(0)+B\left(1-\frac{a}{a}\right)\left(1-\frac{c}{a}\right)+c(0)$$

$$1 = B\left(\frac{b-a}{b}\right)\left(\frac{b-c}{b}\right)$$

$$B = \frac{b^2}{(b-a)(b-c)}$$

Put $x = \frac{1}{c}$ **in eq (ii)**

$$1 = A\left(1-b\frac{1}{c}\right)\left(1-c\frac{1}{c}\right) + B\left(1-a\frac{1}{c}\right)\left(1-c\frac{1}{c}\right) + C\left(1-a\frac{1}{c}\right)\left(1-b\frac{1}{c}\right)$$

$$1 = A(0) + B(0) + C\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)$$

$$1 = C\left(\frac{c-a}{c}\right)\left(\frac{c-b}{c}\right)$$

$$C = \frac{c^2}{(c-b)(c-a)}$$

So,

$$A = \frac{a^2}{(a-b)(a-c)} \quad ; \quad B = \frac{b^2}{(b-a)(b-c)} \quad ; \quad C = \frac{c^2}{(c-a)(c-b)}$$

Therefore, by putting these values in eq. (i)

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{\frac{a^2}{(a-b)(a-c)}}{(1-ax)} + \frac{\frac{b^2}{(b-a)(b-c)}}{(1-bx)} + \frac{\frac{c^2}{(c-a)(c-b)}}{(1-cx)}$$

Hence;

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{a^2}{(a-b)(a-c)(1-ax)} + \frac{b^2}{(b-a)(b-c)(1-bx)} + \frac{c^2}{(c-a)(c-b)(1-cx)}$$

11. $\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$

Solution

Put $x^2=y$ in equation above we get $\frac{x^2+a^2}{\left(x^2+b^2\right)\left(x^2+c^2\right)\left(x^2+d^2\right)}$

Let $\frac{y+a^2}{\left(y+b^2\right)\left(y+c^2\right)\left(y+d^2\right)}=\frac{A}{y+b^2}+\frac{B}{y=c^2}+\frac{C}{y+d^2}$ ________ (i)

By multiplying $\left(y+b^2\right)\left(y+c^2\right)\left(y-d^2\right)$ **both sides we get**

$$y+a^2=A\left(y+c^2\right)\left(y+d^2\right)+B\left(y+b^2\right)\left(y+d^2\right)+C\left(y+b^2\right)\left(y+c^2\right) \_\_\_\_\_\_\_ (ii)$$

Put $y=-b^2$ in eq. (ii)

$$-b^2+a^2=A\left(-b^2+c^2\right)\left(-b^2+d^2\right)+B\left(-b^2+b^2\right)\left(-b^2+d^2\right)+C\left(-b^2+b^2\right)\left(-b^2+c^2\right)$$

$$-b^2+a^2=A\left(-b^2+c^2\right)\left(-b^2+d^2\right)+B(0)+C(0)$$

$$A=\frac{\left(-b^2+a^2\right)}{\left(-b^2+c^2\right)\left(-b^2+d^2\right)}$$

$$A=\frac{\left(a^2-b^2\right)}{\left(c^2-b^2\right)\left(d^2-b^2\right)}$$

Put $y = -c^2$ in eq. (ii)

$$-c^2+a^2=A\left(-c^2+c^2\right)\left(-c^2+d^2\right)+B\left(-c^2+b^2\right)\left(-c^2+d^2\right)+C\left(-c^2+b^2\right)\left(-c^2+c^2\right)$$

$$-c^2+a^2=A(0)+B\left(-c^2+b^2\right)\left(-c^2+d^2\right)+C(0)$$

$$B=\frac{a^2-c^2}{\left(b^2-c^2\right)\left(d^2-c^2\right)}$$

Put $y= -d^2$ in eq. (ii)

$$-d^2+a^2=A\left(-d^2+c^2\right)\left(-d^2+d^2\right)+B\left(-d^2+b^2\right)\left(-d^2+d^2\right)+C\left(-d^2+b^2\right)\left(-d^2+c^2\right)$$

$$-d^2+a^2=A(0)+B(0)+C\left(-d^2+b^2\right)\left(-d^2+c^2\right)$$

$$C=\frac{a^2-d^2}{\left(b^2-d^2\right)\left(c^2-d^2\right)}$$

So,

$$A=\frac{\left(a^2-b^2\right)}{\left(c^2-b^2\right)\left(d^2-b^2\right)} \quad ; \quad ,B=\frac{a^2-c^2}{\left(b^2-c^2\right)\left(d^2-c^2\right)} \quad ; \quad C=\frac{a^2-d^2}{\left(b^2-d^2\right)\left(c^2-d^2\right)}$$

Therefore, by putting these values in equation (i)

$$\frac{y+a^2}{\left(y+b^2\right)\left(y+c^2\right)\left(y+d^2\right)}=\frac{\frac{\left(a^2-b^2\right)}{\left(c^2-b^2\right)\left(d^2-b^2\right)}}{y+b^2}+\frac{\frac{a^2-c^2}{\left(b^2-c^2\right)\left(d^2-c^2\right)}}{y=c^2}+\frac{\frac{a^2-d^2}{\left(b^2-d^2\right)\left(c^2-d^2\right)}}{y+d^2}$$

Hence;

$$\frac{y+a^2}{\left(y+b^2\right)\left(y+c^2\right)\left(y+d^2\right)}=\frac{\left(a^2-b^2\right)}{\left(c^2-b^2\right)\left(d^2-b^2\right)\left(y+b^2\right)}+\frac{a^2-c^2}{\left(b^2-c^2\right)\left(d^2-c^2\right)\left(y=c^2\right)}+\frac{a^2-d^2}{\left(b^2-d^2\right)\left(c^2-d^2\right)\left(y+d^2\right)}$$