Exercise 3.4

Q1. If A = $\begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$ **and B =** $\begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$

Then show that A+B is symmetric.

Solution

$$A+B = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & -2+1 & 5+(-2) \\ -2+1 & 3+0 & -1+(-1) \\ 5+(-2) & -1-1 & 0+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & -2+1 & 5-2 \\ -2+1 & 3+0 & -1-1 \\ 5-2 & -1-1 & 0+2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}$$

$$(A+B)^t = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}$$

By above

$$A+B = (A+B)^t$$

Hence A+B is symmetric

Q2. If A = $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$ **, show that**

i. **A+A^t is symmetric** **ii. A-A^t is skew-symmetric**

Solution

i. $A+A^t$ is symmetric

$$A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A+A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+3 & 0-1 \\ 3+2 & 2+2 & -1+3 \\ -1+0 & 3+(-1) & 2+2 \end{bmatrix}$$

$$A+A^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$(A+A^t)^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

By above $A+A^t = (A+A^t)^t$

Hence proved $A+A^t$ *is symmetric*

ii. $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{3} & \mathbf{2} \end{bmatrix}$

$$A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A\text{-}A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 2-3 & 0+1 \\ 3-2 & 2-2 & -1-3 \\ -1-0 & 3+1 & 2-2 \end{bmatrix}$$

$$A\text{-}A^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

$$(A - A^t\ )^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}^t$$

$$= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$$

Taking common -1

$$= (-1) \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix} = (-1)\ [A\text{-}A^t]$$

Thus $A - A^t = (-1)[A - A^t]$

Hence proved $A - A^t$ *is skew – symmetric*

Q3. If A is any square matrix of order 3, show that

i. A+A^t is symmetric **ii. A-A^t is skew-symmetric**

Solution

i. A+A^t is symmetric

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + a_{11} & a_{12} + a_{21} & a_{13} + a_{31} \\ a_{21} + a_{12} & a_{22} + a_{22} & a_{23} + a_{32} \\ a_{31} + a_{13} & a_{32} + a_{23} & a_{33} + a_{32} \end{bmatrix}$$

$$(A + A^t)^t = \begin{bmatrix} a_{11} + a_{11} & a_{12} + a_{21} & a_{13} + a_{31} \\ a_{21} + a_{12} & a_{22} + a_{22} & a_{23} + a_{32} \\ a_{31} + a_{13} & a_{32} + a_{23} & a_{33} + a_{32} \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11} + a_{11} & a_{12} + a_{21} & a_{13} + a_{31} \\ a_{12} + a_{21} & a_{22} + a_{22} & a_{23} + a_{32} \\ a_{13} + a_{31} & a_{32} + a_{23} & a_{33} + a_{32} \end{bmatrix}$$

By above

$$A + A^t = (A + A^t)^t$$

Hence $A + A^t$ is symmetric

ii. $\quad$ **A** $= \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & \boldsymbol{a_{33}} \end{bmatrix}$

$$A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{21} & a_{13} - a_{31} \\ a_{21} - a_{12} & a_{22} - a_{22} & a_{23} - a_{32} \\ a_{31} - a_{13} & a_{32} - a_{23} & a_{33} - a_{32} \end{bmatrix}$$

$$A-A^t = \begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & 0 & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & 0 \end{bmatrix}$$

$$(A-A^t)^t = (-1)\begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & 0 & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & 0 \end{bmatrix}$$

$$(A-A^t)^t = (-1)\begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & 0 & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & 0 \end{bmatrix}$$

By above

$(-1)\ (A-A^t) = (A-A^t)^t$

Hence $A-A^t$ is skew- symmetric

Q4. If the matrices A and B are symmetric and AB = BA, show that AB is symmetric

Solution

Given $A = A^t$

And $B = B^t$

Then prove that

$(AB)^t = AB$

L.H.S = $(AB)^t$

$= A^t . B^t$

=BA

=A.B

=R.H.S

Hence proved

$(AB)^t = AB$

Q5. Show that AA^t and A^tA are symmetric for any matrix of order 2×3

Solution

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{bmatrix}$

$$A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$A.A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a^2{}_{11} + a^2{}_{12} + a^2{}_{13} & a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} \\ a_{21}.a_{11} + a_{22}.a_{12} + a_{23}.a_{13} & a^2{}_{21} + a^2{}_{22} + a^2{}_{23} \end{bmatrix}$$

$$(A.A^t)^t = \begin{bmatrix} a^2{}_{11} + a^2{}_{12} + a^2{}_{13} & a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} \\ a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} & a^2{}_{21} + a^2{}_{22} + a^2{}_{23} \end{bmatrix}$$

$$(A.A^t)^t = (A.A^t)$$

Hence proved $A.A^t$ is symmetric

ii $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{bmatrix}$

$$A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$A^t.A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a^2{}_{11} + a^2{}_{12} + a^2{}_{13} & a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} \\ a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} & a^2{}_{21} + a^2{}_{22} + a^2{}_{23} \end{bmatrix}$$

$$(A^t.A)^t = \begin{bmatrix} a^2{}_{11} + a^2{}_{12} + a^2{}_{13} & a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} \\ a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} & a^2{}_{21} + a^2{}_{22} + a^2{}_{23} \end{bmatrix}$$

$$(A.A^t)^t = (A^t.A)$$

Hence $A^t.A$ is a symmetric

Q6. If A = $\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$ show that

i. A +$(\bar{A})^t$ ii. A - $(\bar{A})^t$ is shew- symmetric

Solution

$$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & +i \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & +i \end{bmatrix}$$

$$A + (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & +i \end{bmatrix}$$

$$= \begin{bmatrix} -i-i & 1+i+1 \\ 1+1-i & -i+i \end{bmatrix}$$

$$A + (\bar{A})^t = \begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$\overline{[\ A + (\bar{A})]^t} = \begin{bmatrix} 0 & 2-i \\ 2+i & 0 \end{bmatrix}$$

$$[\overline{[\ A + (\bar{A})]^t}]^t = A + (\bar{A})^t$$

Hence $A + (\bar{A})^t$ is Hermitian

ii. A = $\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & +i \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & +i \end{bmatrix}$$

$$A - (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 1-i & +i \end{bmatrix}$$

$$= \begin{bmatrix} i+i & 1+i-1 \\ 1-1+i & -i+i \end{bmatrix}$$

$A-(\bar{A})^t = \begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$

$\overline{[\ A - (\bar{A})]^t} = \begin{bmatrix} -2i & -i \\ i & 2i \end{bmatrix}$

$[\overline{[\ A - (\bar{A})]^t}]^t = \begin{bmatrix} -2i & -i \\ i & 2i \end{bmatrix} = -\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$

$= A-(\bar{A})^t$

Hence $A-(\bar{A})^t$ is skew-Hermitian

Q7. If A is symmetric or skew-symmetric, shoe that A^2 is symmetric

Solution

Given $A = A^t$ (Symmetric)

And $A = -A^t$ (skew- symmetric)

Prove that A^2 is symmetric

$A^2 = (A^2)^t$

R.H.S $= (A^2)^t$

$= (A.A)^t$ or $(-A.-A)^t$

$= A^t.A^t$ or $-A^t.-A^t$

$= A.A$ [given $A=A^t$ and $-A^t = A$]

$= A^2$

$=$L.H.S

Hence proved A^2 is symmetric

Q8. If A = $\begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$

Solution

$$A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}^t = [1 \quad 1-i \quad -i]$$

$$A.(\bar{A})^t = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}.[1 \quad 1-i \quad -i]$$

$$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -i(1+i) \\ i & i(1-i) & -i(i) \end{bmatrix}$$

$$A.(\bar{A})^t = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1+1 & -i-i^2 \\ i & i-i^2 & -i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & -i+1 \\ i & i+1 & 1 \end{bmatrix}$$

Hence $A.(\bar{A})^t = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & -i+1 \\ i & i+1 & 1 \end{bmatrix}$

Q9. Find the inverses of the following matrices. Also find their inverses by using row and column operations.

i. $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ ii. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ iii. $\begin{bmatrix} 1 & -3 & 2 \\ 2 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Solution

i. $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{vmatrix}$

$= 1(-4+0)-2(0-0)-3(0-4)$

$= 1(-4)-2(0)-3(-4)$

$=-4-0+12$

$= 8$

Coefficient of Matrix A

$C_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix}$

$= (-1)^2(-4) = -4$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} 0$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} =(-1)^4(-4) = -4$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} =(-1)\ (4-6) = 2$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} = 2-6 = -4$

$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix} =6$

$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} =0$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = (-2)$

Co-efficient of matrix

$$C = \begin{bmatrix} -4 & 0 & -4 \\ 2 & -4 & 0 \\ -6 & 0 & -2 \end{bmatrix}$$

Thus, $Adj\ A = C = \begin{bmatrix} -4 & 0 & -4 \\ 2 & -4 & 0 \\ -6 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & 0 & -2 \end{bmatrix}$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{8}\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{bmatrix}$

Inverse by using elementary row operation:

Hence

$$A^{-1} = \begin{bmatrix} -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{bmatrix}$$

Inverse by using column operations

Hence

$$A^{-1} = \begin{bmatrix} -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{bmatrix}$$

Inverse by using elementary row operation

$$A = \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_{el=}\left[\begin{array}{ccc|ccc}1 & 2 & -3 & 1 & 0 & 0\\ 0 & -2 & 0 & 0 & 1 & 0\\ 0 & 2 & -4 & 2 & 0 & 1\end{array}\right] R_3+2R_1$$

$$R_{el=}\left[\begin{array}{ccc|ccc}1 & 2 & -3 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0\\ 0 & 2 & -4 & 2 & 0 & 1\end{array}\right] -\frac{1}{2}R_2$$

$$R_{el=}\left[\begin{array}{ccc|ccc}1 & 0 & -3 & 1 & -1 & 0\\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & -4 & 2 & -1 & 1\end{array}\right] \begin{array}{l}R_1-2R_2\\ R_3-2R_2\end{array}$$

$$R_{el=}\left[\begin{array}{ccc|ccc}1 & 0 & -3 & 1 & -1 & 0\\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4}\end{array}\right] -\frac{1}{4}R_3$$

$$R_{el=}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1-\frac{3}{2} & 1-\frac{3}{4} & 0-\frac{3}{4}\\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4}\end{array}\right] R_1-3R_3$$

$$R_{el=}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4}\\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}\end{array}\right]$$

Hence is

$$A^{-1}=\begin{bmatrix}-1/2 & 1/4 & -3/4\\ 0 & -1/2 & 0\\ -1/2 & -1/4 & -1/4\end{bmatrix}$$

Inverse by using elementary. Column operation

$$A=\left[\begin{array}{ccc}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & -2 & 2\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]$$

$$C_{el}=\left[\begin{array}{ccc}1 & 2-2 & -3+3\\ 0 & -2+0 & 0+0\\ -2 & -2+4 & 2-6\\ \hline 1 & 0-2 & 0+3\\ 0 & 1-0 & 0+0\\ 0 & 0-0 & 1+0\end{array}\right]\begin{array}{l}C_2-2C_1\\ C_3+2C_1\end{array}$$

$$C_{el}=\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & -2 & 0\\ -2 & 2 & 4\\ \hline 1 & -2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]$$

$$C_{el}=\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & -2 & 0\\ -2 & 2 & 4\\ \hline 1 & -1 & 3\\ 0 & -1/2 & 0\\ 0 & 0 & 1\end{array}\right]-\frac{1}{2}C_2$$

$$C_{el}=\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 1 & -4\\ \hline 1 & -1 & 3\\ 0 & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{1}{4}\end{array}\right]-\frac{1}{4}C_2$$

$$C_{el} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & -4 \\ \hline 1 & -1 & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$$

$$C_{el} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 1-\frac{3}{2} & 1 & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix} C_1 + 2C_3$$

$$C_{el} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline -\frac{1}{2} & 1-\frac{3}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2}-0 & 0 \\ -\frac{1}{4} & 0-\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$C_{el} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline -\frac{1}{2} & 1-\frac{3}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2}-0 & 0 \\ -\frac{1}{4} & 0-\frac{1}{4} & -\frac{1}{4} \end{bmatrix} C_2 - C_3$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{bmatrix}$$

ii. $\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

$|\mathbf{B}| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{vmatrix}$

$= 1(-2)-2(-3)-1(1)$

$= -2 +6-1$

$|B| = 3$

Coefficient of Matrix B

$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix}$

$= (-1)^2(-2) = -2$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} = (-1)(-3)= 3$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} =(-1)^4(1) = 1$

$$C_{21} = (-1)^{2+1}\begin{vmatrix}2 & -1\\0 & 2\end{vmatrix} = (-1)\ (4) = -4$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & -1\\1 & 2\end{vmatrix} = (1)(2+1) = 3$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}1 & 2\\1 & 0\end{vmatrix} = (-1)(-2) = 2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}2 & -1\\-1 & 3\end{vmatrix} = (1)(6-1) = 5$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}1 & -1\\0 & 3\end{vmatrix} = (-1)(3) = -3$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & 2\\0 & -1\end{vmatrix} = (1)(-1) = -1$$

Co-efficient of matrix

$$C = \begin{bmatrix}-2 & 3 & 1\\-4 & 3 & 2\\5 & -3 & -1\end{bmatrix}$$

Thus, $\text{Adj } B = C = \begin{bmatrix}-2 & -4 & 5\\3 & 3 & -3\\1 & 2 & -1\end{bmatrix}$

$$B^{-1} = \frac{Adj\ B}{|B|} = \frac{1}{3}\begin{bmatrix}-2 & -4 & 5\\3 & 3 & -3\\1 & 2 & -1\end{bmatrix}$$

$$= \begin{bmatrix}-1/3 & -4/3 & 5/3\\1 & 1 & 1\\1/3 & 2/2 & 5/3\end{bmatrix}$$

Hence $B^{-1} = \begin{bmatrix}-1/3 & -4/3 & 5/3\\1 & 1 & 1\\1/3 & 2/2 & 5/3\end{bmatrix}$

Inverse by using elementary row operation

$$B = \left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\0 & -1 & 3 & 0 & 1 & 0\\1 & 0 & 2 & 0 & 0 & 1\end{array}\right]$$

$$R_{el} = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & -2 & 3 & -1 & 0 & 1 \end{array}\right] R_3 - R_1$$

$$R_{el} = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -3 & 0 & -1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{array}\right] (-1)R_1$$

$$R_{el} = \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right] \begin{array}{l} R1 - 2R_2 \\ R_3 + 2R_2 \end{array}$$

$$R_{el} = \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{array}\right] -\frac{1}{3} R_3$$

$$R_{el} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - 5R_3 \\ R_2 - 2R_3 \end{array}$$

Hence $B^{-1} = \begin{bmatrix} -1/3 & -4/3 & 5/3 \\ 1 & 1 & 1 \\ 1/3 & 2/2 & 5/3 \end{bmatrix}$

Inverse by using elementary column operations

$$B = \left[\begin{array}{ccc} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$C_{el} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -2 & 3 \\ \hline 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \begin{array}{l} C_2 - 2C_1 \\ C_3 - C_1 \end{array}$$

$$C_{el} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ \hline 1 & 2 & 1/3 \\ 0 & -1 & 0 \\ 1 & 0 & 1/3 \end{array}\right] \begin{array}{l} (-1)C_1 \\ \frac{1}{3}C_3 \end{array}$$

$$C_{el} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ \hline 1 & 2 & -5/3 \\ 0 & -1 & +1 \\ 1 & 0 & +1/3 \end{array}\right] C_3 - C_1$$

$$C_{el} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ \hline 1 & 2 & 5/3 \\ 0 & -1 & -1 \\ 1 & 0 & -1/3 \end{array}\right] (-1)C_3$$

$$C_{el} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2/3 & -4/3 & 5/3 \\ 1 & 1 & 1 \\ 1/3 & 2/3 & -1/3 \end{array}\right] C_2 - 2C_3$$

Hence $B^{-1} = \begin{bmatrix} -2/3 & -4/3 & 5/3 \\ 1 & 1 & 1 \\ 1/3 & 2/2 & 5/3 \end{bmatrix}$

iii. **C =** $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$|C| = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$

$=1(1-0)-3(2-0)+2(-2-0)$

$= 1(1)+2(2)+2(-2)$

$= 1+6-4$

$|C| = 3$

Coefficient of Matrix C

$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$= (-1)^2(1-0) = 1$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = (-1)(2-0) = -2$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = (-1)^4(-2-0) = -2$

$C_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 2 \\ -1 & 1 \end{vmatrix} = (-1)\ (-3+2) = 1$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (1)(1) = 1$

$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} = (-1)(-1) = 1$

$C_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 2 \\ 1 & 0 \end{vmatrix} = (1)(-2) = -2$

$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = (-1)(-4) = 4$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & -3\\2 & 1\end{vmatrix} = (1)(7) = 7$$

Co-efficient of matrix

$$C = \begin{bmatrix}1 & -2 & -2\\1 & 1 & 1\\-2 & 4 & 7\end{bmatrix}$$

Thus, $C = \left[\begin{bmatrix}1 & -2 & -2\\1 & 1 & 1\\-2 & 4 & 7\end{bmatrix}\right]$

$$C^{-1} = \frac{Adj\ C}{|C|} = \frac{1}{3}\left[\begin{bmatrix}1 & 1 & -2\\-2 & 1 & 4\\-2 & 1 & 7\end{bmatrix}\right]$$

$$= \begin{bmatrix}1/3 & 1/3 & -2/3\\-2/3 & 1/3 & 4/3\\-2/3 & 1/3 & 7/3\end{bmatrix}$$

Hence $C^{-1} = \begin{bmatrix}1/3 & 1/3 & -2/3\\-2/3 & 1/3 & 4/3\\-2/3 & 1/3 & 7/3\end{bmatrix}$

Inverse by using elementary row operation

$$C=\left[\begin{array}{ccc|ccc}1 & -3 & 2 & 1 & 0 & 0\\2 & 1 & 0 & 0 & 1 & 0\\0 & -1 & 1 & 0 & 0 & 1\end{array}\right]$$

$$R_{el} \sim \left[\begin{array}{ccc|ccc}1 & -3 & 2 & 1 & 0 & 0\\0 & 7 & -4 & -2 & 1 & 0\\0 & 1 & 1 & 0 & 0 & 1\end{array}\right] R_2 - 2R_1$$

$$R_{el} \sim \left[\begin{array}{ccc|ccc}1 & -3 & 2 & 1 & 0 & 0\\0 & -1 & -1 & 0 & 0 & 1\\0 & 7 & -4 & -2 & 1 & 1\end{array}\right] R_{23}$$

$$R_{el=}\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 7 & -4 & -2 & 1 & 1 \end{array}\right](-1)R_2$$

$$R_{el=}\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -3 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 3 & -2 & 1 & 7 \end{array}\right]\begin{matrix} R_1+3R_2 \\ R_3-7R_2 \end{matrix}$$

$$R_{el=}\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -3 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{-2}{3} & \frac{1}{3} & \frac{7}{3} \end{array}\right]-1/3R$$

$$R_{el=}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} \\ 0 & 1 & 0 & \frac{-2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & \frac{-2}{3} & \frac{1}{3} & \frac{7}{3} \end{array}\right]\begin{matrix} R_1+R_3 \\ R_2+R_3 \end{matrix}$$

Hence $C^{-1} = \begin{bmatrix} 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 4/3 \\ -2/3 & 1/3 & 7/3 \end{bmatrix}$

Inverse by using elementary column operations

$$C = \left[\begin{array}{ccc} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$C_{cl} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & -4 \\ 0 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} C_3 + 3C_1 \\ C_3 - 2C_1 \end{matrix}$$

$$C_{cl} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 7 \\ 0 & 1 & -1 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} C_{23}$$

$$C_{cl} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 7 \\ 0 & \frac{-1}{4} & -1 \\ 1 & \frac{1}{2} & 3 \\ 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{-1}{4} & \frac{3}{4} \\ 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{4} & \frac{7}{4} \end{bmatrix} \begin{matrix} C_1 + 2C_2 \\ C_3 - 7C_2 \end{matrix}$$

$$C_{el} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{-1}{4} & 1 \\ 1 & \frac{1}{2} & \frac{-2}{3} \\ 0 & 0 & \frac{4}{3} \\ \frac{1}{2} & -\frac{1}{4} & \frac{7}{3} \end{bmatrix} \frac{4}{3}C_3$$

$$C_{el} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{-2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \begin{matrix} C_1 - \frac{1}{2}C_3 \\ C_2 + \frac{1}{4}C_2 \end{matrix}$$

Hence $C^{-1} = \begin{bmatrix} 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 4/3 \\ -2/3 & 1/3 & 7/3 \end{bmatrix}$

Q10. Find the rank of the following matrices

i. $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$ **ii.** $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$ **iii.** $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$

Solution

i. **A=** $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$

Rank of matrix A

$$R_{el} = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2-2 & -6+2 & 5-4 & 1-2 \\ 3-3 & 5+3 & 4-6 & -3-3 \end{bmatrix} \begin{matrix} R_2+2R_1 \\ R_3-3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & -6 \end{bmatrix}$$

$$R_{el} = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 8 & -2 & -6 \end{bmatrix} -\frac{1}{4}R_2$$

$$R_{el} = \begin{bmatrix} 1 & -1+1 & -\frac{1}{4}+2 & 1+\frac{1}{4} \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 8 & -2 & -6 \end{bmatrix} R_1+R_2$$

$$R_{el} = \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 8 & -2 & -6 \end{bmatrix}$$

$$R_{el} = \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 4 & -1 & -3 \end{bmatrix} \frac{1}{2}R_3$$

$$R_{el} = \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 4-4 & -1+1 & -3-1 \end{bmatrix} R_3-4R_2$$

$$= \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$R_{el} = \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} -\frac{1}{4}R_3$$

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, the rank of given matrix A is 3.

ii. **let B=** $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$

$$R_{el} = \begin{bmatrix} 1 & -4 & -7 \\ 2-2 & -5+8 & 1+14 \\ 1-1 & -2+4 & 3+7 \\ 3-3 & -7+12 & 4+21 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -4 & -7 \\ 0 & 3 & 15 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix}$$

$$R_{el} = \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix} \begin{matrix} \frac{1}{3}R_2 \\ \frac{1}{2}R_3 \\ \frac{1}{5}R_4 \end{matrix}$$

$$R_{el} = \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_3 - R_2 \\ R_4 - R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence rank of given matrix B is 2

iii. **let C=** $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$

$$R_{el} = \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & -1 & 3 & 0 & -1 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} R_{21}$$

$$R_{el} = \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3-3 & -1-6 & 3+3 & 0+9 & -1+6 \\ 2-2 & 3-4 & 4+2 & 2+6 & 5+4 \\ 2-2 & 5-4 & -2+2 & -3+6 & 3+4 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 5 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & 1 & 0 & 3 & 7 \end{bmatrix}$$

$$R_{el}=\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & -1 & 6 & 8 & 9\\ 0 & -7 & 6 & 9 & 5\end{bmatrix}R_{24}$$

$$R_{el}=\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & -1+1 & 6+0 & 8+3 & 9+7\\ 0 & -7 & 6 & 9 & 5\end{bmatrix}R_3+R_2$$

$$=\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 6 & 11 & 16\\ 0 & -7 & 6 & 9 & 5\end{bmatrix}$$

$$R_{el}=\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 6 & 11 & 16\\ 0 & -7+7 & 6+0 & 9+21 & 5+49\end{bmatrix}R_4+7R_2$$

$$=\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 6 & 11 & 16\\ 0 & 0 & 6 & 30 & 54\end{bmatrix}$$

$$R_{el}=\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 6 & 11 & 16\\ 0 & 0 & 6-6 & 30-11 & 54-16\end{bmatrix}R_4-3R_3$$

$$=\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 6 & 11 & 16\\ 0 & 0 & 0 & 19 & 38\end{bmatrix}$$

$$R_{el}=\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 6 & 11 & 16\\ 0 & 0 & 0 & 1 & 2\end{bmatrix}\frac{1}{19}R_4$$

$$R_{el} = \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11-11 & 16-22 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} R_3 - 11R_4$$

$$= \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 0 & -6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_{el} = \begin{bmatrix} 1 & 2-2 & -1-0 & -3-6 & -2-14 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 0 & -6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} R_1 - 2R_2$$

$$R_{el} = \begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 0 & -6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \frac{1}{6} R_1$$

$$R_{el} = \begin{bmatrix} 1 & 0 & -1+1 & 0-9 & -16-1 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} R_1 + R_3$$

$$= \begin{bmatrix} 1 & 0 & 0 & -9 & -17 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_{el} = \begin{bmatrix} 1 & 0 & 0 & -9+9 & -17+18 \\ 0 & 1 & 0 & 3-3 & 7-6 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 + 9R_4 \\ R_2 - 3R_4 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Hence the rank of given matrix C is 4.