Exercise 3.5

Q1. Solve the following systems of linear equations by Cramer's rule

1) $\begin{cases} 2x + 2y + x = 3 \\ 3x - 2y - 2z = 1 \\ 5x + y + z = 2 \end{cases}$ **2)** $\begin{cases} 2x_1 - x_2 - x_3 = 5 \\ 4x_1 + 2x_2 + 3x_3 = 8 \\ 3x_1 - 4x_2 - x_3 = 3 \end{cases}$ **3)** $\begin{cases} 2x_1 - x_2 + x_3 = 5 \\ x_1 + 2x_2 + 2x_3 = 8 \\ x_1 - 2x_2 - x_3 = 1 \end{cases}$

i)
$$\begin{aligned} 2x + 2y + x &= 3 \\ 3x - 2y - 2z &= 1 \\ 5x + y + z &= 2 \end{aligned}$$

Solution

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$AX = B$$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix}$$

$=2(6+2)-2(-9+10)+1(3+10)$

$=2(8)-2(1)+1(13)$

$=16-2+13 = 27 \neq 0$

$$|A_1| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}$$

$=3(6+2)-2(-3+4)+1(1+4)$

$=3(8)-2(1)+1(5)$

$=16-2+13 = 27$

$$|A_2| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$

=2(-3+4)-3(-9-10)+1(6-5)

=2(1)-3(1)+1(1)

=2-3 +1 = 27

$$|A_3| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix}$$

=2(6+2)-2(-9+10)+1(3+10)

=2(8)-2(1)+1(13)

=16-2+13 = 27

Thus $x = \frac{|A_1|}{|A|} = \frac{27}{27} = 1$

$y = \frac{|A_2|}{|A|} = \frac{0}{27} = 0$

$z = \frac{|A_3|}{|A|} = \frac{27}{27} = 1$

ii)
$$\begin{aligned} 2x_1 - x_2 - x_3 &= 5 \\ 4x_1 + 2x_2 + 3x_3 &= 8 \\ 3x_1 - 4x_2 - x_3 &= 3 \end{aligned}$$

Solution

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

AX = B

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$$

=2(-2+12)+1(-4-9)+1(-16-6)

$=2(10)+1(-13)+1(-22)$

$=20-13-22 = -15 \neq 0$

$$|A_1| = \begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$$

$=5(-2+12)+1(-8-9)+1(-32-6)$

$=5(10)+1(-17)+1(-38)$

$=50-17-38 = -5$

$$|A_2| = \begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}$$

$=2(-8-9)+5(-4-9)+1(12-24)$

$=2(-17)+5(-13)+1(-12)$

$= 34+65-12 = 19$

$$|A_3| = \begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & -3 \end{vmatrix}$$

$=2(6-32)+1(12-24)+5(-16-6)$

$=2(38)+1(-12)+5(-22)$

$= 76 -12 -110 = -46$

Thus $x_1 = \frac{|A_1|}{|A|} = \frac{-5}{-15} = \frac{1}{3}$

$x_2 = \frac{|A_2|}{|A|} = \frac{19}{-15} = \frac{-19}{15}$

$x_3 = \frac{|A_3|}{|A|} = \frac{-46}{-15} = \frac{46}{15}$

iii.
$$\begin{aligned} 2x_1 - x_2 + x_3 &= 5 \\ x_1 + 2x_2 + 2x_3 &= 8 \\ x_1 - 2x_2 - x_3 &= 1 \end{aligned}$$

Solution

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$=2(-2+4)+1(-1-2)+1(-2-2)$

$=2(2)+1(-3)+1(-4)$

$=4-3-4 = -3 \neq 0$

$$|A_1| = \begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$=8(-2+4)+1(-6-2)+1(-12-2)$

$=8(2)+1(-8)+1(-14)$

$= 16-8-14 = -6$

$$|A_2| = \begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$

$=2(-6-2)-8(-1-2)+1(1-6)$

$=2(-8)-8(-3)+1(-5)$

$= -16+24-5 = 3$

$$|A_3| = \begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}$$

$=2(2+12)+1(1-6)+8(-2-2)$

$=2(14)+1(-5)+8(-4)$

$= 28 -5 -32 = -9$

Thus $x_1 = \frac{|A_1|}{|A|} = \frac{-6}{-3} = 2$

$x_2 = \frac{|A_2|}{|A|} = \frac{3}{-3} = -1$

$x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$

Q2. Use matrices to solve the following systems

i) $\begin{cases} \mathbf{x - 2y + x = -1} \\ \mathbf{3x + y - 2z = 4} \\ \mathbf{y - z = 1} \end{cases}$ ii) $\begin{cases} \mathbf{2x_1 + x_2 + 3x_3 = 3} \\ \mathbf{x_1 + x_2 - 2x_3 = 0} \\ \mathbf{-3x_1 - x_2 + 2x_3 = -4} \end{cases}$ iii) $\begin{cases} \mathbf{x + y = 2} \\ \mathbf{2x - z = 1} \\ \mathbf{2y - 3z = -1} \end{cases}$

i) $\mathbf{x - 2y + x = -1}$
$\mathbf{3x + y - 2z = 4}$
$\mathbf{y - z = 1}$

Solution

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 1(-1+2)+2(-3-2)+1(3-2)$

$=1-6+3$

$= -2$

$A_{11} = (-1)^{1+1}\begin{vmatrix}1 & -2\\1 & -1\end{vmatrix}$

$= (-1)^2(-1+2) = 1$

$A_{12} = (-1)^{1+2}\begin{vmatrix}3 & -2\\0 & -1\end{vmatrix} = (-1)(-3+0) = 3$

$A_{13} = (-1)^{1+3}\begin{vmatrix}3 & 1\\0 & 1\end{vmatrix} = 1(3-0) = 3$

$A_{21} = (-1)^{2+1}\begin{vmatrix}-2 & 1\\1 & -1\end{vmatrix} = (-1)\ (2-1) = -1$

$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 1\\0 & -1\end{vmatrix} = 1(-1) = -1$

$A_{23} = (-1)^{2+3}\begin{vmatrix}1 & -2\\0 & 1\end{vmatrix} = -(1-0) = -1$

$A_{31} = (-1)^{3+1}\begin{vmatrix}-2 & 1\\1 & -2\end{vmatrix} = 1(4-1) = 3$

$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 1\\3 & -2\end{vmatrix} = (-1)(-2-3) = -5$

$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & -3\\3 & 1\end{vmatrix} = 1+6=7$

$$A = \begin{bmatrix}1 & 3 & 3\\-1 & -1 & -1\\3 & 5 & 7\end{bmatrix}$$

Thus, $\text{Adj } A = \begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}$

$$A^{-1} = \frac{Adj\ A}{|A|} = -\ \frac{1}{2}\begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}$$

$X = A^{-1}B$

$$= -\frac{1}{2}\begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}\begin{bmatrix}-1\\4\\1\end{bmatrix}$$

$$= -\ \frac{1}{2}\begin{bmatrix}-1-4+3\\-3-4+5\\-3-4+7\end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix}-2\\-2\\0\end{bmatrix}$$

$$= \begin{bmatrix}{-2}/{-2}\\{-2}/{-2}\\{0}/{-2}\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

ii)

$$\begin{aligned}2x_1 + x_2 + 3x_3 &= 3\\ x_1 + x_2 - 2x_3 &= 0\\ -3x_1 - x_2 + 2x_3 &= -4\end{aligned}$$

Solution

$$A = \begin{bmatrix}2 & 1 & -3\\1 & 1 & -2\\-3 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\0\\-4\end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$A = \begin{bmatrix}2 & 1 & -3\\1 & 1 & -2\\-3 & -1 & 2\end{bmatrix}$$

$$|A| = \begin{vmatrix}2 & 1 & -3\\1 & 1 & -2\\-3 & -1 & 2\end{vmatrix}$$

$$= 2(2-2)-1(2-6)-3(-1+3)$$

$$= 4-6 = -2$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix}1 & -2\\1 & 2\end{vmatrix}$$

$$= (-1)^2(2-2) = 0$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}1 & -2\\-3 & 2\end{vmatrix} = (-1)(2-6) = 3$$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix}$ =1(-1+3) = 2

$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix}$ =(-1) (2-3) = 1

$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}$ = 1(4-9) = -5

$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix}$ = (-1)(-2+3) = 1

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix}$ =1(-2+3) = 1

$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix}$ = (-1)(-4+3) = 1

$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$ = 1(2-1) = 1

$$A = \begin{bmatrix} 0 & 4 & 2 \\ 1 & -5 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus, Adj A $= \begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{Adj\ A}{|A|} = -\ \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= -\frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$= -\ \frac{1}{2}\begin{bmatrix} 0+0-4 \\ 12+0-4 \\ 6+0-4 \end{bmatrix}$$

$$= -\ \frac{1}{2}\begin{bmatrix} -4 \\ 8 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$$

iii)
$$\begin{aligned} x + y &= 2 \\ 2x - z &= 1 \\ 2y - 3z &= -1 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix}$$

$$= 1(0+2)-1(-6)+0$$

$$= 2+6 = -8$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= (-1)^2(2) = 2$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} = (-1)(-6) = 6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 1(4) = 4$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = (-1)\ (-3) = 3$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = 1(-3) = -3$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}1 & 1\\0 & 2\end{vmatrix} = (-1)(2) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}1 & 1\\0 & -1\end{vmatrix} = 1(-1) = -1$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 0\\2 & -1\end{vmatrix} = (-1)(-1) = 1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & 1\\2 & 0\end{vmatrix} = 1(-2) = -2$$

$$A = \begin{bmatrix}2 & 6 & 4\\3 & -3 & -2\\-1 & 1 & -2\end{bmatrix}$$

Thus, $\text{Adj } A = \begin{bmatrix}2 & 3 & -1\\6 & -3 & 1\\4 & -2 & -2\end{bmatrix}$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{8}\begin{bmatrix}2 & 3 & -1\\6 & -3 & 1\\4 & -2 & -2\end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{8}\begin{bmatrix}2 & 3 & -1\\6 & -3 & 1\\4 & -2 & -2\end{bmatrix}\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix}4+3+1\\12-3-1\\8-2+2\end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix}8\\8\\8\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Q3. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms.

i. $\begin{aligned} x_1 - 2x_2 - 2x_3 &= 1 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_1 - 4x_2 - 3x_3 &= 1 \end{aligned}$ ii. $\begin{aligned} x + 2y + z &= 2 \\ 2x + y + 2z &= -2 \\ 2x + 3y - z &= 9 \end{aligned}$ iii. $\begin{aligned} x_1 + 4x_2 + 2x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 9 \\ 2x_1 - 2x_2 - 2x_3 &= 12 \end{aligned}$

i. $\begin{aligned} x_1 - 2x_2 - 2x_3 &= 1 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_1 - 4x_2 - 3x_3 &= 1 \end{aligned}$

Solution

In augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 2 & 3 & 1 & 1 \\ 5 & -4 & -3 & 1 \end{array}\right]$$

Now reduced in echelon form

$$R_{el} = \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 7 & 5 & 3 \\ 0 & 6 & 7 & 6 \end{array}\right] \begin{matrix} R_2 + 2R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 6 & 7 & 6 \\ 0 & 7 & 5 & 3 \end{array}\right] R_{23}$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 1 & 7/6 & 1 \\ 0 & 7 & 5 & 1 \end{array}\right] \frac{1}{6}R_3$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 1 & 7/6 & 1 \\ 0 & 7 & 19/6 & -4 \end{array}\right] R_3 - 7R_2$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 1 & 7/6 & 1 \\ 0 & 0 & 1 & 24/19 \end{array}\right] \frac{-6}{19}R_3$$

(Alternative ways)

The equivalent system in row echelon form is

$$x_1 - 2x_2 - 2x_3 = -1$$

$$x_2 + \frac{7}{6}x_3 = 1$$

$$x_3 = \frac{24}{19}$$

Put the value of 'x_3'

$$x_2 + \frac{7}{6}(\frac{24}{19}) = 1$$

$$x_2 + \frac{28}{19} = 1$$

$$x_2 = 1 + \frac{28}{19} = \frac{19-28}{19} = -\frac{9}{19}$$

Put the value of x_2, x_3

$$x_1 - 2(-\frac{9}{19}) - 2(\frac{24}{19}) = -1$$

$$x_1 + \frac{18}{19} - \frac{48}{19} = -1$$

$$x_1 + \frac{18-48}{19} = -1$$

$$x_1 - \frac{30}{19} = -1$$

$$x_1 = -1 + \frac{30}{19} = \frac{11}{19}$$

Thus $x_1 = \frac{11}{19}$, $x_2 = \frac{9}{19}$ and $x_3 = \frac{24}{19}$

On alternative way

Now reduced the matrix to reduced echelon from

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 1 & 7/6 & 1 \\ 0 & 0 & 1 & 24/19 \end{array}\right]$$

$$R_{el} = \begin{bmatrix} 1 & -2 & -2 & \big| & -29/19 \\ 0 & 1 & 7/6 & \big| & 1 \\ 0 & 0 & 1 & \big| & 24/19 \end{bmatrix} R_1 + 2R_1$$

$$R_{el} = \begin{bmatrix} 1 & 0 & 0 & \big| & 11/19 \\ 0 & 1 & 7/6 & \big| & 1 \\ 0 & 0 & 1 & \big| & 24/19 \end{bmatrix} R_1 + 2R_2$$

$$R_{el} = \begin{bmatrix} 1 & 0 & 0 & \big| & 11/19 \\ 0 & 1 & 0 & \big| & -9/19 \\ 0 & 0 & 1 & \big| & 24/19 \end{bmatrix} R_2 - \frac{7}{6}R_3$$

The equivalent system in row echelon form is

Hence $x_1 = \frac{11}{19}$, $x_2 = \frac{9}{19}$ and $x_3 = \frac{24}{19}$

ii.
$$\begin{aligned} x + 2y + z &= 2 \\ 2x + y + 2z &= -2 \\ 2x + 3y - z &= 9 \end{aligned}$$

Solution

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 1 & \big| & 2 \\ 2 & 1 & 2 & \big| & -1 \\ 2 & 3 & -1 & \big| & 9 \end{bmatrix}$$

Now reduced in echelon form

$$R = \begin{bmatrix} 1 & 2 & 1 & \big| & 2 \\ 0 & -3 & 0 & \big| & -5 \\ 0 & -1 & -3 & \big| & 5 \end{bmatrix} \begin{matrix} R_2 + 2R_1 \\ R_3 - R_1 \end{matrix}$$

$$R_{el} = \begin{bmatrix} 1 & 2 & 1 & \big| & 2 \\ 0 & -1 & -3 & \big| & 5 \\ 0 & -3 & 0 & \big| & -5 \end{bmatrix} R_{23}$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 3 & 0 & 5 \end{array}\right] \begin{array}{l} (-1)R_2 \\ (-2)R_3 \end{array}$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 0 & -5 & 12 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & -9 & 20 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \\ R_3 - 3R_2 \end{array}$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 0 & -5 & 12 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -20/9 \end{array}\right] -\frac{1}{9}R_3$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8/9 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & -20/9 \end{array}\right] \begin{array}{l} R_1 + 5R_3 \\ R_2 - 3R_3 \end{array}$$

The equivalent system in reduced (row) echelon form is

Hence $x = \frac{8}{9}$, $y = \frac{5}{3}$ and $z = \frac{-20}{9}$

iii.

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 9 \\ 2x_1 - 2x_2 - 2x_3 &= 12 \end{aligned}$$

Solution

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 2 & 1 & -2 & 9 \\ 3 & 2 & -2 & 12 \end{array}\right]$$

Now reduced in echelon form

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & -7 & -6 & 5 \\ 0 & -10 & -8 & 5 \end{array}\right] \begin{array}{l} R_3 + 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 7 & 6 & -5 \\ 0 & 5 & 4 & -3 \end{array}\right] \begin{array}{l} (-1)R_2 \\ (-2)R_3 \end{array}$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 5 & 4 & -3 \end{array}\right] \frac{1}{7}R_2$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 0 & -10/7 & 34/7 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & -2/7 & 4/7 \end{array}\right] \begin{array}{l} R_1 + 4R_2 \\ R_3 - 5R_2 \end{array}$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 0 & -10/7 & 34/7 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 1 & -2 \end{array}\right] -\frac{7}{2}R_3$$

$$R_{el} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} R_1 + \frac{10}{7}R_3 \\ R_2 - \frac{7}{6}R_3 \end{array}$$

The equivalent system in reduced (row) echelon form is

Hence $x_1 = 2$, $x_2 = 1$ and $x_3 = -2$

Q4. Solve the following system of homogenous linear equations

i) $\begin{aligned} x + 2y - 2z &= 0 \\ 2x + y + 5z &= 0 \\ 5x + 4y + 8z &= 0 \end{aligned}$ **ii)** $\begin{aligned} x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ 3x_1 + 2x_2 - 4x_3 &= 0 \end{aligned}$ **iii)** $\begin{aligned} x_1 - 2x_2 - x_3 &= 0 \\ x_1 + x_2 + 5x_3 &= 0 \\ x_1 - x_2 + 4x_3 &= 0 \end{aligned}$

i) $\begin{aligned} x + 2y - 2z &= 0 \\ 2x + y + 5z &= 0 \\ 5x + 4y + 8z &= 0 \end{aligned}$

Solution

In matrix form

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{vmatrix}$$

$=1(8-20)-2(16-25)-2(8-5)$

$=1(-12)-2(-9)-2(3)$

$= 0$

So, the system has infinite no. of solution

$x + 2y - 2z = 0$

$\underline{\pm 4x \ \pm 2y \ \pm 10z} = 0$

$-3x \qquad -12z = 0$

$-3x = 12z$

$x = -4z$

$2x + 4y - 4z = 0$

$\underline{\pm 2x \ \pm y \ \pm 5z = 0}$

$3y \qquad 9z = 0$

$3y = 9z$

Thus x -4z and y = 3z

Put the value of x and y in any equation

$5(-4z)+4(3z)+8z = 0$

$-20z+12z+8z = 0$

$-8z+8z = 0$

$0=0$ (satisfied)

Put z = t

So, x -4t, y =3t and z =t

Q4 Part-II Goes Here

Part-3

$$\mathbf{x_1 - 2x_2 - x_3 = 0}$$
$$\mathbf{x_1 + x_2 + 5x_3 = 0}$$
$$\mathbf{x_1 - x_2 + 4x_3 = 0}$$

Solution

In matrix foam

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{vmatrix}$$

=1(4+5)+2(4-10)-1(-1-2)

=1(9)+2(-6)-1(-3)

= 12-12 = 0

So, the system has infinite no. of solution

$$x_1 - 2x_2 - x_3 = 0$$

$$\underline{\pm x_1 \pm x_2 \pm 5x_3} = 0$$

$$-3x_2 \ -6x_3=0$$

$$x_2 = -x_3$$

$$x_1 + x_2 + 5x_3 = 0$$

$$\underline{\pm 2x_1 \pm x_2 \pm 4x_3} = 0$$

$$3x_1 + 9x_3 = 0$$

$$x_1 = -3x_3$$

Thus $x_1 = -3x_3$ and $x_2 = -2x_3$

Put the value of x_1 *and* x_2 in any equation

$$-3x_3 - 2(-2x_3) - x_3 = 0$$

$$-3x_3 + 4x_3 - x_3 = 0$$

$$0 = 0 \quad \text{(satisfied)}$$

Put $x_3 = t$

So, $x_1 = -3t,\ x_2 = -2t,\ x_3 = t$

Q5. Find the value of λ for which the following system does not possess a unique solution. Also solve the system for the value of λ

i)
$$\begin{aligned} x + y + z &= 0 \\ 2x + \lambda y + z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

ii)
$$\begin{aligned} x_1 + 4x_2 + \lambda x_3 &= 0 \\ 2x_1 + x_2 - 2x_3 &= 0 \\ 3x_1 + \lambda x_2 - 4x_3 &= 0 \end{aligned}$$

i)
$$\begin{aligned} x + y + z &= 0 \\ 2x + y + \lambda z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

Solution

The matrix of coefficients of the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix}$$

Now reduced in echelon form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2-2 & 1-2 & -\lambda-2 \\ 1-1 & 2-1 & -2-1 \end{bmatrix} \begin{matrix} R_2-2R_1 \\ R_3-R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -\lambda-2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \lambda+2 \\ 0 & 1 & -3 \end{bmatrix} \text{-}1R_2$$

$$\begin{bmatrix} 1 & 1 & 1+3 \\ 0 & 1 & \lambda+2 \\ 0 & 1 & -3 \end{bmatrix} R_1 - R_3$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & \lambda+2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & \lambda+2 \\ 0 & 1-1 & -3-\lambda-2 \end{bmatrix} R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & \lambda+2 \\ 0 & 0 & -\lambda-5 \end{bmatrix}$$

The system has a non-trivial solution, if the rank of matrix is less 3.

The rank is less than 3 if

$$\cdot \quad \boldsymbol{\lambda - 5 = 0}$$
$$\boldsymbol{\lambda = -5}$$

then the equivalent system in reduced echelon form is

x+0y+4z = 0 …..(i)

0x+y-3z = 0 ……(ii)

From (i) and (ii) the system has non-trivial solution of the form

x = -4t

y= 3t

z = t

ii)
$$\begin{aligned} x_1 + 4x_2 + \lambda x_3 &= 0 \\ 2x_1 + x_2 - 2x_3 &= 0 \\ 3x_1 + \lambda x_2 - 4x_3 &= 0 \end{aligned}$$

Solution

The matrix of coefficients of the system

$$\begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{bmatrix}$$

Now reduced in echelon form

$$\begin{bmatrix} 1 & 4 & \lambda \\ 0 & -7 & -3-2\lambda \\ 0 & \lambda-12 & -4-3\lambda \end{bmatrix} \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 4 & \lambda \\ 0 & 1 & \frac{1}{7}(2\lambda+3) \\ 0 & \lambda-12 & -4-3\lambda \end{bmatrix} \frac{1}{7}R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7}(-12-\lambda) \\ 0 & 1 & \frac{1}{7}(2\lambda+3) \\ 0 & 0 & \frac{-2}{7}(\lambda^2-4) \end{bmatrix} \begin{matrix} R_1-4R_2 \\ R_1-(12+\lambda)R_2 \end{matrix}$$

The system(ii) has a non-trivial solution, if the rank of matrix is less 3 where rank is the number of non-zero rows in reduced echelon form

$$\frac{-2}{7}(\lambda^2-4) = 0$$

$$(\lambda^2-4) = 0$$

$$\boldsymbol{\lambda = \sqrt{4} = \pm 2}$$

if $\lambda = 2$ then the matrix in reduced echelon form is

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then the equivalent system is

$$x_1 + 0x_2 - 2x_3 = 0 \;.....(i)$$

$$x_2 + x_3 = 0 \;......(ii)$$

From (i) and (ii) the system has non-trivial solution of the form

$$x_1 = 2x_3$$

$$x_2 = x_3$$

Let $x_3 = t, x_2 = -t , x_3 = t$

if $\lambda = 2$ then the matrix in reduced echelon form is

$$\begin{bmatrix} 1 & 0 & \frac{-10}{7} \\ 0 & 1 & \frac{-1}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

The equivalent system is

$$x_1 - \frac{10}{7}x_3 = 0$$

$$x_1 = \frac{10}{7}x_3$$

$$x_2 - \frac{1}{7}x_3 = 0$$

$$x_2 = \frac{1}{7}x_3$$

Q6. Find the value of λ for which the following system does not possess a unique solution. Also solve the system for the value of λ

$$\begin{aligned} \mathbf{x_1 + 4x_2 + \lambda x_3} &= \mathbf{2} \\ \mathbf{2x_1 + x_2 - 2x_3} &= \mathbf{11} \\ \mathbf{3x_1 + 2x_2 - 2x_3} &= \mathbf{16} \end{aligned}$$

Solution

Let the coefficients of the system in matrix form is

$$A = \begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & 2 \\ 3 & 2 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & 2 \\ 3 & 2 & -2 \end{vmatrix}$$

$= 1(-2+4) -4(-4+6)+\lambda(4-3)$

$= 1(2) -4(2) + \lambda(1)$

$= 2-8 + \lambda = -6 + \lambda$

The system will not have a unique solution if $|A| = 0$

$-6+\lambda = 0 \Rightarrow \lambda = 6$

Put the value of $\lambda = 6$ in the system

$$\begin{aligned} x_1 + 4x_2 + \lambda x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 11 \\ 3x_1 + 2x_2 - 2x_3 &= 16 \end{aligned}$$

In matrix form it is

$$\begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & 2 \\ 3 & 2 & -2 \end{bmatrix}$$

We convert it into reduced echelon form

$$\begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & -7 & -14 & 7 \\ 0 & -10 & -20 & 10 \end{bmatrix} \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{matrix} -\frac{1}{7}R_2 \\ \frac{1}{10}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} R_1-4R_2 \\ R_3-R_2 \end{matrix}$$

So the equivalent system is

$$x_1 - 2x_3 = 6$$

$$x_2 + 2x_3 = -1$$

Let $x_3 = t$ then

$$x_1 - 2t = 6$$

$$x_1 = 6+2t$$

$$x_2 = -1 - 2t$$

Hence

$$x_1 = 6+2t$$

$$x_2 = -1 - 2t$$

$$x_3 = t$$