Exercise 2.8

Q1. Operation performed on the two-member set G = {0,1) is shown in the adjoining table. Answer the questions: -

i. Name the identity element if it exists?

ii. What is the inverse of 1?

iii. Is the set G, under the given operation a group? Abelian or nob-Abelian?

$\oplus$	0	1
0	0	1
1	1	0

Solution:

i. Identity element = 0

i.e $0\oplus0 = 0$

$1\oplus0 = 0\oplus1=1$

ii. Inverse of 1 = 1

i.e $1\oplus1 = 1 \oplus 1 = 0$

iii. The set G = {0,1}

1. Closure property holds

$0 + 1 = 0$

2. Associative law holds

(0+0) +1 = 0+ (0+1)

0+1 = 0+ (1)

1 = 1

3. Additive inverse exists

4. Inverse law holds.

1+1 =0 = 1+1,

So, set G={0,1} is a group.

5. Commutative law

0+1 = 1+0

1 = 1

commutative law holds

Hence, set G is abelian

Q2. The operation ⊕ as performed on the set {0,1,2,3} is shown in the adjoining table, show that the set is an Abelian group?

⊕	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1

3	3	0	1	2

Solution:

1. Associative property

$1 \oplus (2 \oplus 3) = (1 \oplus 2) \oplus 3$

$1 \oplus 1 = 3 \oplus 3.$

$2 = 2$

Associative law hold

2. Closure law.

$2 \oplus 3 = 1$ (1 belongs to given table)

3. Inverse law

$2 \oplus 2 = 2 \oplus 2 = 0$

$3 \oplus 1 = 1 \oplus 3 = 0$

4. Identity exists

$0 \oplus 1 = 1 \oplus 0 = 1$

$0 \oplus 2 = 2 \oplus 2 = 2$

5. Commutative law

Or

$1 + 2 = 2 + 1$

$3 = 3$

So, commutative law holds

Hence, the set is an Abelian group

Q3. For each of the following sets, determine whether or not the set forms a group with respect to the indicated operation.

<u>**Set**</u>	<u>**Operation**</u>
i. The set of rational numbers	$\times$
ii. The set of rational numbers	$+$
iii. The set of positive rational numbers	$\times$
iv. The set of integers	$+$
v. The set of integers.	$\times$

Solution:

i. The set of rational number under multiplication.

Set of rational number = Q.

Let $a \in Q$.

$a \times 0 = 0 \times a \neq 1$

Therefore, the set of rational numbers is not a group w.r.t the binary operation multiplication.

ii. The set of rational numbers under addition.

a. Closure property

$a + b = c$ [$\because a, b, c \in Q$]

b. Associative property.

$(a+b) + c = a + (b+c)$ [$\because$ a, b, c $\in$ Q]

c. Identity element

$a+0 = 0+a = a$ [$\because$ a, b, c $\in$ Q]

d. Inverse element

$a+(-a) = (-a) + a = 0$ [$\because$ -a, $\in$ Q]

e. Commutative property

$a+b = b+a$ [$\because$ a,b, $\in$ Q]

Therefore, the set of rational numbers under.

iii. The set of positive rational numbers under multiplication.

Let 'Q' denotes the set of positive rational numbers, the "$0 \notin Q$".

i. Closure law

$x.y \in Q$ [$\forall$ x,y $\in$ Q]

product of two rational numbers is also positive rational number.

ii. Associative law

$(x.y) . Z = x. (y.Z)$ [$\forall$ x, y, Z $\in$ Q]

Product of three rational numbers is also positive rational number so, Associative law hold.

iii. Identity element

$1 \in Q \quad [\forall x \in Q]$

as $1 . x = 1 = x . 1$

Hence, '1' is the identity element of Q'.

iv. Inverse element.

$x.\frac{1}{x} = 1 = \frac{1}{x} x, \quad [\because x, \frac{1}{x} \in Q']$

Hence, Q' (positive rational numbers) is group under multiplication.

v. Commutative law

$x.y = y.x \quad [x, y \in Q']$

It is that the group (Q') Abelian group.

iv. The set of integers under addition.

Let 'z' be set of integers $Z = \{0, \pm 1, \pm 2, \pm 3, ...\}$

Then

i. Closure law

$a+b \in Z \quad [\forall a, b \in Z]$

ii. Associative law

$a + (b+c) = (a+b) + c \quad [\forall a, b \in Z]$

iii. Identity element.

$a+(-a) = o = (-a) + a$ [$\forall$ a, b,c $\in$ Z]

iv. Inverse element

$a+(-a) = 0 = (-a) + a$ [$\forall$ a,-a, $\in$ Z]

Hence integers under addition is a croon.

v. Commutative property

$a+b = b+a$ [$\forall$ a, b, $\in$ Z]

Hence, this group is Abelian group.

v. The set of integers under multiplication.

Let 'Z' be set of integers then

$Z = \{0, \pm1, \pm2, \pm3, ...\}$

i. Closure law

$a.b \in Z$ [$\forall$ a,b $\in$ Z]

ii. Associative law

$(a.b). c = a.(b.c)$ [$\forall$ a, b $\in$ Z]

iii. Identity law

$a.1 = 1 = 1.a$ [$\forall$ a, $\in$ Z and 1 $\in$ Z]

iv. Inverse element

$a.\frac{1}{a} = 1 = \frac{1}{a}.a$ [$\forall$ a, $\in$ and $\frac{1}{a} \in Z$]

because a , $\frac{1}{a} \in Z$

by this it show, the set of integers under multiplication.

Q4. Show that the adjoining table represents the sums of the elements of the set {E,0).

What is the identity element of this set? Show that this set is an abelian group.

⊕	E	0
E	E	0
0	0	E

Solution:

a. Sum of the elements of the set {E, 0}

Even ⊕ Even = Even

Even ⊕ odd = odd

Odd ⊕ even = odd

Odd ⊕ odd = even

b. Identity element

E+E = E = E+E [.'. 0 ∈ E)

Or E+0 = E = 0+E

c.

i. closure property

E+ 0= 0 [.'. E, 0 ∈ set]

ii. Associative property

E+(E+0) = (E+E)+0

E+0 = 0+0

E = E

iii. Inverse law

E + E = E

0 + 0 = E

iv. Identity element

E

v. commutative law

E+0 = 0+E = 0

So, set {E, 0} is a abelian element.

Q5. Show that the set {1, ω,ω^2} , when ω^3 = 1 is an Abelian group w.r.t. or multiplication.

$\oplus$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

i. Closure property

$1 \oplus \omega = \omega$

or $\omega^2 \oplus 1 = \omega^2$

Closer property holds

ii. Associative property

$\omega \oplus (\omega^2 \oplus 1) = 1 \oplus 1$

$\omega^3 = 1$ $[\because \omega^3 = 1]$

$1 = 1$

Associative property holds

iii. Inverse law

$\omega \oplus \omega^2 = 1$

Or $\omega^2 \oplus \omega = 1$

Inverse hold

iv. Identity element

$\omega^2 \oplus 1 = \omega^2$

$\omega \oplus 1 = \omega$

Inverse element

v. Commutative property

$\omega \oplus \omega^2 = \omega^2$

Commutative property holds

Hence, set $(1, \omega, \omega^2)$ is an Abelian group w.r.t. multiplication.

Q6. If G is group under the operation $*$ and a, b $\in$ G, find the solutions of the equations.

$a*x = b$, $x*a = b$

Solution:

i. $a * x = b$

multiply a^{-1} by the left side of equation.

$a^{-1} * (a * x) = a^{-1} * b$

$(a^{-1} * a) * x = a^{-1} * b$

$[\because a^{-1} * (a * x) = (a^{-1} * a) * x]$

$e * x = a^{-1} * b$

$[a^{-1} * a = e]$

$\Rightarrow x = (a^{-1} * b)$

$[e * x = x]$

Hence, solution of equation is $x = a^{-1} * b$

ii. $x * a = b$

Multiply a^{-1} by the right side of equation.

$(x * a) * a^{-1} = b * a^{-1}$

$x * (a * a^{-1}) = b * a^{-1}$

$[\because x * a)]=a^{-1} = x * (a * a^{-1})$

$x * e = b * a^{-1}$ $[\because a * a^{-1}=e]$

$x = b *$ $(\because x * e=x]$

$x = b * a^{-1}$

Hence, solution of equation is $x = b * a^{-1}$

Q7. Show that set consisting of elements of the form 2+ $\sqrt{3b}$ (a, b being rational), is an abelian group w.r.t. addition.

Solution:

Let

G= (2+ $\sqrt{3b}$; | , a, b ∈ Q), let x, y, z ∈ G any three elements of G, such that

x= a+ $\sqrt{3b}$; y =c+ $\sqrt{3d}$

z= f+ $\sqrt{3}$f

{a, b, c, d, e, f ∈ Q}

i. closure properties

x+ y = u [∴ U ∈ Q]

∴ closure property hold

ii. Associative property

(x+y)+z = z+(y+z)

∴ Associative property hold

iii. Since, '0' is a rational number, so, additive identity present in set G.

iv. Since, any x = a+ $\sqrt{3b}$ ∈ G, then

-x = -a- $\sqrt{3b}$ ∈ G

So, inverse of each element of `G' is also is G.

v. x + y = y + x; show that commutative property w.r.t. addition holds is G.

Hence G is an abelian group w.r.t. addition.

Q8. Determine whether, (P(S) ∗), where ∗ stands for intersection is a semi-group, a monoid or neither. If it is a monoid, specify its identity.

Solution:

Let '(P(s))' be the power set of any set 's' then 'ø' and 's' will belong to that set.

∴Let ∗ represents intersection then

i. Closure Law

Let A, B ∈ P(s)

Then A ∗ B will contain elements of 's' because ∗ is multiplication which will be some subset of 's' belong to P(s)

ii. Associative Law

Let A, B. C ∈ P(s)

A ∗ (B ∗ C) = (A ∗ B) ∗ C

As A ∩ (B ∩ C) = (A ∩ B) ∩ C

iii. As s ∈ P(s)

Then y A ∈ P(s)

S∗A = A∗S= A [∵ s ∩ A = A = A ∩ A]

→ 's' is the identity element

Hence, P(s) is a monoid, Therefore, a semi group as well.

Its identity is 's'

Q9. Complete the following table to obtain a semi-group under ∗.

∗	a	b	c

A	**c**	**a**	**b**
b	**a**	**b**	**c**
c	**--**	**--**	**a**

Solution:

Let x , y be required element

→ In semi-group 'Associative law' hold

(a ∗ a) ∗ a = a∗ (a∗a) [∵ a∗a = c]

→ c∗a =a∗c

→ c∗ = b

so, x = b.

→ In semi-group 'Associative law' hold

(a∗a) ∗ b = a∗(a∗a)

c∗b =a∗a [∵ a ∗ a = c]

[a∗b = a]

→ y = a [a ∗ a = a]

So, y =

Hence x = b and y = a

Q10. Prove that all 2 x 2 non-singular matrices over the real field from a non-abelian group under multiplication.

Solution:

Let 'S' denotes the set of all non-singular 2 x matrices over the real field.

Then:

i. Closure law

$A_{2\times2} \times B_{2\times2} = C_{2\times2}$ [∵ $A, B, C \in S$]

ii. Associative law

$A.(B.C) = (A.B).C$ [∵ $A, B, C \in S$]

iii. Identity Element

$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, is a non singular matrix.

iv. Inverse law

$A_{2\times2} . A_{2\times2}^{-1} = A_{(2\times2)}^{-1} . A = I_{2\times2}$ [∵ $A \in S$]

v. Commutative law

$A_{(2\times2)} . B_{(2\times2)} \neq B_{(2\times2)} . A_{(2\times2)}$ [∵ $A.B \in S$]

So, by this 'S' the set of 2 x 2 non-singular matrixes over the real field is a non-abelian group under multiplication.