Exercise 12.8

1. Show that

i. $r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ **ii.** $s=4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

i. $r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

Solution

$$\text{R.H.S} = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{(S-a)^2(S-b)^2(S-c)^2}{a^2b^2c^2}}$$

$$=\frac{abc}{\Delta}\frac{(S-a)(S-b)(S-c)}{abc}$$

$$=\frac{abc}{\Delta}\frac{\Delta^2}{S}\frac{1}{abc}$$

$$=\frac{\Delta}{S}$$

$$= r$$

Hence proved

$$r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

ii. $s= 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

Solution

R.H.S = $4R \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

$= 4\frac{abc}{4\Delta}\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{S(S-c)}{ab}}$

$= 4\frac{abc}{4\Delta}\sqrt{\frac{S^3(s-a)(S-b)(S-c)}{a^2b^2c^2}}$

$=\frac{abc}{\Delta}\frac{S\sqrt{S(S-a)(S-b)(S-c)}}{abc}$

$= \frac{\Delta}{S}\Delta$

$= S$

= L.H.S

Hence proved

$s=4R \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

2. Show that $r= a \sin\frac{\beta}{2} \sin\frac{\beta}{2} \sec\frac{\alpha}{2} = b \sin\frac{\gamma}{2} \sin\frac{\alpha}{2} \sec\frac{\beta}{2}$

$= c \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}$

Solution

a. $=a \sin\frac{\beta}{2} \sin\frac{\beta}{2} \sec\frac{\alpha}{2}$

$= a\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}\sqrt{\frac{bc}{S(S-a)}}$

$= a\sqrt{\frac{(S-a)^2(S-b)(S-c)bc}{Sa^2bc(S-a)}}$

$= a\sqrt{\frac{(S-a)^2(S-b)(S-c)bc}{Sa^2}}$

$= \frac{\Delta}{\sqrt{S}\sqrt{S}}$

$= \frac{\Delta}{S}$

$= r$

Hence proved

$r = a \sin\frac{\beta}{2}\ sin\ \frac{\beta}{2}\ sec\ \frac{\alpha}{2}$

b. $\mathbf{b \sin\frac{\gamma}{2}\ sin\ \frac{\alpha}{2}\ sec\ \frac{\beta}{2}}$

$= b\sqrt{\frac{(S-a)(S-b)}{ab}}\ \sqrt{\frac{(S-b)(S-c)}{bc}}\ \sqrt{\frac{ac}{S(S-b)}}$

$= b\sqrt{\frac{(S-b)^2(S-a)(S-c)ac}{S[ab^2c(S-b)]}}$

$= b\sqrt{\frac{S(S-a)(S-b)(S-c)}{Sb^2}}$

$= \frac{\Delta}{S}$

$= r$

Hence proved

$r = b \sin\frac{\gamma}{2}\ sin\ \frac{\alpha}{2}\ sec\ \frac{\beta}{2}$

c. $\mathbf{c \sin\frac{\alpha}{2}\ sin\ \frac{\beta}{2}\ sin\ \frac{\gamma}{2}}$

$= c\sqrt{\frac{(S-b)(S-c)}{bc}}\ \sqrt{\frac{(S-a)(S-c)}{ac}}\ \sqrt{\frac{ab}{S(S-c)}}$

$= b\sqrt{\frac{(S-c)^2(S-a)(S-b)ab}{S[abc^2(S-c)]}}$

$= b\sqrt{\frac{S(S-a)(S-b)(S-c)}{Sc^2}}$

$= \frac{\Delta}{S}$

=r

Hence proved

r= $c \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sec\frac{\gamma}{2}$

3. Show that:

i. $r_1 = 4R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$ **ii.** $r_2 = 4R \cos\frac{\alpha}{2} \sin\frac{\beta}{2} \cos\frac{\gamma}{2}$

iii. $r_3 = 4R \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \sin\frac{\gamma}{2}$

Solution

i. $r_1 = 4R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

R.H.S $= 4R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

$$= 4\,\frac{abc}{4\Delta}\sqrt{\frac{(S-a)(S-c)}{bc}}\sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{S(S-c)}{bc}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2(S-b)^2(S-c)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}\frac{S(S-b)(S-c)}{abc}$$

$$= \frac{1}{\Delta}\frac{\Delta^2}{S-a}$$

$$= \frac{\Delta}{S-a}$$

$$= r_1$$

Hence proved

$r_1 = 4R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

ii. $r_2 = 4R\cos\frac{\alpha}{2}\ \sin\frac{\beta}{2}\ \cos\frac{\gamma}{2}$

Solution

$$\text{R.H.S} = 4R\cos\frac{\alpha}{2}\ \sin\frac{\beta}{2}\ \cos\frac{\gamma}{2}$$

$$= 4\,\frac{abc}{4\Delta}\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{(S-a)(S-c)}{ac}}\ \sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2(S-a)^2(S-c)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}\frac{S(S-a)(S-c)}{abc}$$

$$= \frac{1}{\Delta}\frac{\Delta^2}{S-b}$$

$$= \frac{\Delta}{S-a}$$

$$= r_2$$

Hence proved

$$r_2 = 4R\cos\frac{\alpha}{2}\ \sin\frac{\beta}{2}\ \cos\frac{\gamma}{2}$$

iii. $r_3 = 4R\cos\frac{\alpha}{2}\ \cos\frac{\beta}{2}\ \sin\frac{\gamma}{2}$

Solution

$$\text{R.H.S} = 4R\cos\frac{\alpha}{2}\ \cos\frac{\beta}{2}\ \sin\frac{\gamma}{2}$$

$$= 4\,\frac{abc}{4\Delta}\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{(S-a)(S-b)}{ab}}\sqrt{\frac{S(S-b)}{ac}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2(S-a)^2(S-b)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}\frac{S(S-a)(S-b)}{abc}$$

$= \frac{1}{\Delta}\frac{\Delta^2}{S-c}$

$= \frac{\Delta}{S-c}$

$= r_3$

Hence proved

$r_3 = 4R \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \sin\frac{\gamma}{2}$

4. Show that

i. $r_1 = s\tan\frac{\alpha}{2}$ ii. $r_2 = s\tan\frac{\beta}{2}$ iii. $r_3 = s\tan\frac{\gamma}{2}$

Solution

i. $r_1 = s\tan\frac{\alpha}{2}$

$R.H.S = s\tan\frac{\alpha}{2}$

$= S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$

$= \sqrt{\frac{S^2(S-a)(S-b)(S-c)}{S(S-a)^2}}$

$= \sqrt{\frac{S(S-a)(S-b)(S-c)}{S(S-a)}}$

$= \frac{\Delta}{S-a}$

$= r_1$

= L.H.S

Hence proved

$r_1 = s\tan\frac{\alpha}{2}$

ii. $r_2 = s\tan\frac{\beta}{2}$

Solution

$$R.H.S = s\tan\frac{\beta}{2}$$

$$= S\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}$$

$$= \sqrt{\frac{S^2(S-a)(S-b)(S-c)}{S(S-b)^2}}$$

$$= \sqrt{\frac{S(S-a)(S-b)(S-c)}{S(S-b)}}$$

$$= \frac{\Delta}{S-b}$$

$$= r_2$$

= L.H.S

Hence proved

$$r_2 = s\tan\frac{\beta}{2}$$

iii. $\boldsymbol{r_3 = s\tan\frac{\gamma}{2}}$

Solution

$$R.H.S = s\tan\frac{\gamma}{2}$$

$$= S\sqrt{\frac{(S-a)(S-c)}{S(S-c)}}$$

$$= \sqrt{\frac{S^2(S-a)(S-b)(S-c)}{S(S-c)^2}}$$

$$= \sqrt{\frac{S(S-a)(S-b)(S-c)}{S(S-c)}}$$

$$= \frac{\Delta}{S-c}$$

$$= r_3$$

= L.H.S

Hence proved

$$r_3 = s\tan\frac{\gamma}{2}$$

5. Prove that

i. $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ ii. $rr_1\ r_2\ r_3 = \Delta^2$

iii. $r_1 + r_2 + r_3 - r = 4R$ iv. $r_1r_2r_3 = rs^2$

Solution

i. $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

$R.H.S = r_1r_2 + r_2r_3 + r_3r_1 = s^2$

$= \frac{\Delta}{S-a}\frac{\Delta}{S-b} + \frac{\Delta}{S-b}\frac{\Delta}{S-c} + \frac{\Delta}{S-c}\frac{\Delta}{S-b}$

$= \frac{\Delta^2}{(S-a)(S-b)} + \frac{\Delta^2}{(S-a)(S-b)} + \frac{\Delta^2}{(S-a)(S-c)}$

$= \Delta^2[\frac{S-c+S-a+S-b}{(S-a)(S-b)(S-c)}]$

$=\Delta^2[\frac{3S-(a+b+c)}{(S-a)(S-b)(S-c)}]$

$=\Delta^2[\frac{3S-2S}{(S-a)(S-b)(S-c)}]$

$= [\frac{\Delta^2 S}{(S-a)(S-b)(S-c)}]$

$= [\frac{\Delta^2 S}{S(S-a)(S-b)(S-c)}]$

$= \frac{\Delta^2 S^2}{\Delta^2} = S^2$

$= S^2$

$=$ L.H.S

Hence proved $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

ii. $rr_1\ r_2\ r_3 = \Delta^2$

Solution

L.H.S = $rr_1\ r_2\ r_3$

$= \frac{\Delta}{S}.\frac{\Delta}{S-a}.\frac{\Delta}{S-b}.\frac{\Delta}{S-c}$

$$= \frac{\Delta^4}{S(S-a)(S-b)(S-c)}$$

$$= \frac{\Delta^4}{\Delta^2}$$

$$= \Delta^2$$

= R.H.S

Hence proved $rr_1 \ r_2 r_3 = \Delta^2$

iii. $\boldsymbol{r_1 + r_2 + r_3 - r = 4R}$

Solution

$$\text{L.H.S} = r_1 + r_2 + r_3 - r$$

$$= \frac{\Delta}{S-a} \cdot \frac{\Delta}{S-b} \cdot \frac{\Delta}{S-c} \cdot \frac{\Delta}{S}$$

$$= \Delta\left[\frac{1}{S-a} \cdot \frac{1}{S-b} \cdot \frac{1}{S-c} \cdot \frac{1}{S} \right]$$

$$= \Delta\left[\frac{S-b+S-a}{(S-a)(S-b)} + \frac{S-S+c}{S(S-c)} \right]$$

$$= \Delta\left[\frac{2S-[a+b]}{(S-a)(S-b)} + \frac{c}{S(S-c)} \right]$$

$$= \Delta\left[\frac{c}{(S-a)(S-b)} + \frac{c}{S(S-c)} \right]$$

$$= \Delta c\left[\frac{S(S-c)+(S-a)(S-b)}{S(S-a)(S-b)(S-c)} \right]$$

$$= \Delta c\left[\frac{S^2-Sc+S^2-Sb-aS+ab}{\Delta^2} \right]$$

$$= \frac{c}{\Delta}[2S^2 - S(a+b+c) + ab]$$

$$= \frac{c}{\Delta}[2S^2 - S(2S) + ab]$$

$$= \frac{c}{\Delta}[2S^2 - 2S^2 + ab]$$

$= \frac{c}{\Delta}[ab]$

$= \frac{abc}{\Delta}$

= R

= R.H.S

Hence proved $r_1 + r_2 + r_3 - r$ **= R**

vi. $\boldsymbol{r_1 r_2 r_3 = rs^2}$

Solution

L.H.S = $r_1\, r_2\, r_3$

$= \frac{\Delta}{S-a} \cdot \frac{\Delta}{S-b} \cdot \frac{\Delta}{S-c}$

$= \frac{\Delta^3}{(S-a)(S-b)(S-c)}$

$= \frac{S\Delta^3}{S(S-a)(S-b)(S-c)}$

$= \frac{S\Delta^3}{\Delta^2}$

= SΔ

= S.S .r

$= S^2\, r$

= R.H.S

Hence proved

$r_1 r_2 r_3 = rs^2$

6. Find R, r , r_1, r_2 *and* r_3, if measure of the sides of triangle ABC are

i. a= b , b =14, c =15 **ii. a=34, b=20 , c= 42**

i a=b , b = 14, c= 15

Solution

$$S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

$a.b.c = 13 \times 14 \times 15 = 2730$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{21(21-13)(21-14)(21-15)}$$

$$\Delta = \sqrt{21(8)(7)(6)} = \sqrt{7056} = 84$$

$$R = \frac{abc}{4\Delta} = \frac{(13)(14)(15)}{4\times 84} = \frac{2730}{336} = 8.125$$

$$r = \frac{\Delta}{S} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{S-a} = \frac{84}{8} = 10.5$$

$$r_2 = \frac{\Delta}{S-b} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{84}{7} = 12$$

Hence

R= 8.125 ; r=4; $r_1 = 10.5$; $r_2 = 12$; $r_3 = 14$

ii. a=34 : b =20 : c = 42

$$S = \frac{a+b+c}{2} = \frac{33+20+42}{2} = \frac{96}{2} = 48$$

a.b.c = 34 × 20 ×42 = 28560

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{48(48-34)(48-20)(48-42)}$$

$$\Delta = \sqrt{48(14)(28)(6)} = 336$$

$$R = \frac{abc}{4\Delta} = \frac{28560}{4\times336} = 21.25$$

$$r = \frac{\Delta}{S} = \frac{336}{48} = 7$$

$$r_1 = \frac{\Delta}{S-a} = \frac{336}{14} = 14$$

$$r_2 = \frac{\Delta}{S-b} = \frac{336}{24} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{336}{6} = 56$$

Hence

R= 21.25 ;r =7; $r_1 = 14$; $r_2 = 12$; $r_3 = 56$

7. Prove that in an equilateral triangle

i. r : R : $r_1 = 1:2:3$

ii. r:R:$r_1 : r_2 : r_3 = 1:2:3:3:3$

i. r : R : $r_1 = 1:2:3$

Solution

L.H.S = r : R : r_1

$$= \frac{\Delta}{S} : \frac{abc}{4\Delta} : \frac{\Delta}{S-a}$$

We know that

a=b=c

$$S = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3}{2}a$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{\frac{3}{2}a(\frac{3}{2}a - a)(\frac{3}{2}a - a)(\frac{3}{2}a - a)}$$

$$= \sqrt{\frac{3}{2}a(\frac{a}{2})(\frac{a}{2})(\frac{a}{2})}$$

$$= \sqrt{3}\ \frac{a^2}{4}$$

$$\Delta = \frac{\sqrt{3}}{4}a^2$$

$$S\text{-}a = \frac{3}{2}a - a = \frac{3a-2a}{2} = \frac{a}{2}$$

$$r : R : r_1 = \frac{\Delta}{S} : \frac{abc}{4\Delta} : \frac{\Delta}{S-a}$$

$$= \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3}{2}a} : \frac{a.a.a}{4\frac{\sqrt{3}}{4}a^2} : \frac{\frac{\sqrt{3}}{4}a^2}{\frac{a}{2}}$$

$$= \frac{2a^2}{\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}}{2}a$$

$$= 1 : 2 : 3$$

= R.H.S

Hence proved

$$\mathbf{r : R : r_1 = 1 : 2 : 3}$$

ii. $\mathbf{r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3}$

L.H.S

$$r : R : r_1 : r_2 : r_3 = \frac{\Delta}{S} : \frac{abc}{4\Delta} : \frac{\Delta}{S-a} : \frac{\Delta}{S-b} : \frac{\Delta}{S-c}$$

$$= \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3}{2}a} : \frac{a.a.a}{4\frac{\sqrt{3}}{4}a^2} : \frac{\frac{\sqrt{3}}{4}a}{\frac{a}{2}} : \frac{\frac{\sqrt{3}}{4}a}{\frac{a}{2}} : \frac{\frac{\sqrt{3}}{4}a}{\frac{a}{2}}$$

$$= \frac{2a^2}{\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}}{2}a : \frac{\sqrt{3}}{2}a : \frac{\sqrt{3}}{2}a$$

$$= 1:2:3:3:3$$

= R.H.S

Hence proved

r:R:r_1: r_2: r_3 = 1: 2: 3: 3: 3

8. prove that

i. $\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$ **ii.** $r = s\tan\frac{\alpha}{2} \tan\frac{\beta}{2} \cot\frac{\gamma}{2}$

iii. $\Delta = 4R\cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

i. $\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$

Solution

R.H.S = $r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$

$= r^2 \sqrt{\frac{S(S-a)}{(S-b)(S-c)}} \sqrt{\frac{S(S-b)}{(S-a)(S-c)}} \sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$

$= r^2 \sqrt{\frac{S^3(S-a)(S-b)(S-c)}{(S-a)^2(S-b)^2(S-c)^2}}$

$= r^2 S \sqrt{\frac{S}{(S-a)(S-b)(S-c)}}$

$= r^2 S \frac{\sqrt{S^2}}{\sqrt{S(S-a)(S-b)(S-c)}}$

$= \frac{r^2 S.S}{\Delta}$

$= \frac{r^2 S^2}{\Delta} = \frac{\Delta^2}{\Delta}$

$= \Delta$

= L.H.S

Hence proved

$$\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$$

ii. $$\boldsymbol{r = s \tan\frac{\alpha}{2} \tan\frac{\beta}{2} \cot\frac{\gamma}{2}}$$

R.H.S = $s \tan\frac{\alpha}{2} \tan\frac{\beta}{2} \cot\frac{\gamma}{2}$

$$= S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}\sqrt{\frac{(S-b)(S-a)}{S(S-c)}}$$

$$= S\sqrt{\frac{(S-a)^2(S-b)^2(S-c)^2}{S^3(S-a)(S-b)(S-c)}}$$

$$= S\sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$$

$$= \frac{S}{S}\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S}$$

$$= \frac{\Delta}{S}$$

= r

= L.H.S

Hence proved

$$r = s \tan\frac{\alpha}{2} \tan\frac{\beta}{2} \cot\frac{\gamma}{2}$$

iii. $$\boldsymbol{\Delta = 4R\, r \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}}$$

R.H.S = $4R\, r \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

$$= 4.\frac{abc}{4\Delta}.\frac{\Delta}{S}\sqrt{\frac{S(S-a)}{bc}}.\sqrt{\frac{S(S-b)}{ac}}.\sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{abc}{S}\frac{S\sqrt{S(S-a)(S-b)(S-c)}}{\sqrt{a^2b^2c^2}}$$

$= \frac{abc}{abc}\,\Delta = \Delta$

$=$ L.H.S

Hence proved

$$\Delta = 4R\,r\,cos\,\frac{\alpha}{2}\,cos\,\frac{\beta}{2}\,cos\,\frac{\gamma}{2}$$

9. Show that

i. $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$ **ii.** $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

i. $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

Solution

L.H.S $= \frac{1}{2rR}$

$= \frac{1}{2.\frac{\Delta}{S}.\frac{abc}{4\Delta}}$

$= \frac{2s}{abc}$

$= \frac{a+b+c}{abc}$

$= \frac{a}{abc} + \frac{b}{abc} + \frac{c}{abc}$

$= \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}$

$= \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

$=$ R.H.S

Hence proved

$$\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$$

ii. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

Solution

$$\text{R.H.S} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$= \frac{S-a}{\Delta} + \frac{S-b}{\Delta} + \frac{S-c}{\Delta}$$

$$= \frac{S-a+S-b+S-c}{\Delta}$$

$$= \frac{3S-(a+b+c)}{\Delta}$$

$$= \frac{3S-2S}{\Delta}$$

$$= \frac{S}{\Delta} = \frac{1}{r}$$

= L.H.S

Hence proved

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

10. Prove that

$$r = \frac{a \sin\frac{\beta}{2} \sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{b \sin\frac{\alpha}{2} \sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}} = \frac{c \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

Solution

(a) $= \frac{a \sin\frac{\beta}{2} \sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

$$= \frac{a\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-a)}{bc}}}$$

$$= \frac{a}{a}\sqrt{\frac{(S-a)(S-c)(S-a)(S-b)}{S(S-a)}}$$

$$= \sqrt{\frac{S(S-a)(S-c)(S-a)(S-b)}{S^2}}$$

$$= \frac{\Delta}{S}$$

= L.H.S Hence proved

$$r = \frac{a \sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$$

(b)

$$= \frac{b \sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$$

$$= \frac{b\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-b)}{ac}}}$$

$$= \frac{b}{b}\sqrt{\frac{(S-b)(S-c)(S-a)(S-b)}{S(S-b)}}$$

$$= \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}}$$

$$= \frac{\Delta}{S}$$

= L.H.S Hence proved

$$r = \frac{b \sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$$

(c)

$$= \frac{c \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

$$= \frac{c\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-c)}{ac}}}{\sqrt{\frac{S(S-c)}{ab}}}$$

$$= \frac{c}{c}\sqrt{\frac{(S-b)(S-c)(S-a)(S-c)}{S(S-c)}}$$

$$= \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}}$$

$$= \frac{\Delta}{S}$$

= L.H.S Hence proved

$$r = \frac{c \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

11. Prove that : abc($\sin\alpha + \sin\beta + \sin\gamma$) = $4\Delta s$

Solution

L.H.S $= abc(\sin\alpha + \sin\beta + \sin\gamma)$

$$= abc\left(\frac{2\Delta}{ab} + \frac{2\Delta}{bc} + \frac{2\Delta}{ac}\right)$$

As $\Delta = \frac{1}{2}\, ac \sin\alpha$ and $\sin\alpha = \frac{2\Delta}{bc}$ similarly $\sin\beta = \frac{2\Delta}{ac}$ and $\sin\gamma = \frac{2\Delta}{ab}$

$$= abc\,(2\Delta)\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}\right)$$

$$= abc\,(2\Delta)\left(\frac{a+b+c}{abc}\right)$$

$$= (2\Delta)(2S)$$

$$=4\Delta S$$

= R.H.S

Hence proved

abc($\sin\alpha + \sin\beta + \sin\gamma$) = 4Δ

12. Prove that

i. $(r_1 + r_2)\tan\frac{\gamma}{2} = c$ **ii.** $(r_3 - r)\cot\frac{\alpha}{2} = c$

i. $(r_1 + r_2)\tan\frac{\gamma}{2} = c$

Solution

L.H.S = $(r_1 + r_2)\tan\frac{\gamma}{2}$

$= \left(\frac{\Delta}{S-a} + \frac{\Delta}{S-b}\right)\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$

$= \Delta\left[\frac{S-b+S-c}{(S-a)(S-b)}\right]\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$

$= \Delta\left[\frac{2S-(a+b)}{\sqrt{S(S-a)(S-b)(S-c)}}\right]$

$= \Delta\left[\frac{2S-[2S-c]}{\Delta}\right]$

= 2S-2S +c

= c

= R.H.S

Hence proved

$(r_1 + r_2)\tan\frac{\gamma}{2} = c$

ii. $(r_3 - r)\cot\frac{\alpha}{2} = c$

L.H.S = $(r_3 - r)\cot\frac{\alpha}{2}$

$= \left(\frac{\Delta}{S-c} + \frac{\Delta}{S}\right)\sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$

$= \Delta\left[\frac{S-(S-c)}{S(S-c)}\right]\sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$

$=\Delta\left[\frac{c}{S(S-c)}\right]\frac{\sqrt{S^2(S-c)^2}}{\sqrt{S(S-a)(S-b)(S-c)}}$

$=\Delta\left[\frac{c}{S(S-c)}\right]\frac{S(S-c)}{\Delta}$

= c

= R.H.S

Hence proved

$$(r_3 - r)cot\,\frac{\alpha}{2} = c$$