Exercise 1.2

Q1. Verify the addition properties of complex numbers.

Solution:

Let

z_1, z_2 and z_3 be three complex number.

$z_1 = a_1+b_1i$

$z_2 = a_2+b_2i$

$z_3 = a_3+b_3i$

Property 1: closure property w.r.t addition

$z_1 + z_2 = (a_1+b_1i) + (a_2+b_2i)$

$= (a_1+a_2) + (b_1+b_2)i \in Z$

So, complex numbers has closure property w.r.t addition.

Property 2: Associative properties w.r.t. addition.

$z_1 + (z_2 + z_3) = a_1+b_1i + \{(a_2+b_2i) + (a_3+b_3i)\}$

$= a_1+b_1i + \{(a_2+a_3) + (b_2+b_3)i\}$

$= [(a_1+a_2+a_3) + (b_1+b_2+b_3)i]$ (i)

$(z_1 + z_2) + z_3 =[(a_1+b_1i + (a_2+b_2i)] + (a_3+b_3i)$

$=[(a_1+a_2) + (b_1+b_2)i] + (a_3+b_3i)$

$= [(a_1+a_2+a_3) + (b_1+b_2+b_3)i]$ (ii)

By eq (i) & (ii)

Proved $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

So, complex number has associative properties w.r.t addition

Property 3 Identity law of addition

$z_1 + 0 = 0 + z_1 = z_1$

L.H.S $z_1 + 0 = (a_1+b_1i) + (0+0i)$

$= (a_1+0) + (b_1+0)i$

$= a_1+b_1i = z_1$

R.H.S $0+z_1 = (0+0i)+(a_1+b_1i)$

$= (0+a_1) + (0+b_1)i$

$= a_1+b_1i = z_1$

So, complex numbers has identity law w.r.t addition

$z_1 + (-z_1) = (-z_1) + z_1 = 0$

Property 4: Inverse law of addition

Let $z_1 = a+bi$ & $-z_1 = -a-bi$

$z_1 + (-z_1) = (a+bi) + (-a-bi)$

$= (a-a) + (b-b)i$

$=0+0i = 0$

$(-z_1) + z_1 = (-a-bi) + (a+bi)$

$= (-a+a) + (-b+b)i$

$=0+0i = 0$

So, inverse law of addition holds on complex numbers.

Property 5: Commutative law of addition.

$z_1 + z_2 = z_2 + z_1$

L.H.S $= z_1 + z_2 = (a_1+b_1i) + (a_2+b_2i)$

$= (a_1+a_2) + (b_1+b_2)i$

R.H.S $= z_2 + z_1 = (a_2+b_2i) + (a_1+b_1i)$

$= (a_2+a_1) + (b_2+b_1)i$

Thus $z_1 + z_2 = z_2 + z_1$

So, commutative law of addition holds on complex numbers.

Q2. Verify the multiplication properties of complex numbers.

Solution:

Properties 1 Closure properties w.r.t multiplication

$z_1 z_2 = (a_1+b_1i)(a_2+b_2i) = a_1a_2 + a_1b_2i + b_1a_2i + b_1b_2i^2$

$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i \in C$

So, closure property w.r.t multiplication holds on complex numbers.

Property 2 Associative law w.r.t multiplication.

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$

L.H.S $= (z_1 z_2) z_3$

$= [(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i] (a_3+b_3i)$

$= [(a_1 a_2 - b_1 b_2) a_3 + a_3 (a_1 b_2 + a_2 b_1)i] + [(a_1 a_2 - b_1 b_2) b_3i - (a_1 b_2 + a_2 b_1)b_3]$

$= [a_1a_2a_3 - b_1b_2b_3 - a_1b_2b_3 - a_2b_1b_3 + (a_1a_3b_2 + a_2a_3b_1)i + (a_1a_2b_3 - b_1b_2b_3)i$

$= [a_1a_2a_3 - b_1b_2b_3 - a_1b_2b_3 - a_2b_1b_3 + (a_1a_3b_2 + a_2a_3b_1 + a_1a_2b_3 - b_1b_2b_3)i$

$= [(a_1a_2a_3 - a_1a_2b_3) - (a_3b_1b_2 + a_2b_1b_3)] + [(a_2a_3b_1 - b_1b_2b_3) + (a_1a_3b_2 + a_1a_2b_3)]i$

R.H.S = $(z_1 z_2) z_3 = (a_1+b_1i) [(a_2+b_2i) (a_3+b_3i)]$

$= (a_1+b_1i) [(a_2 a_3 - b_2 b_3) + (a_3 b_2 + a_2 b_3)i]$

$= [a_1 (a_2 a_3 - b_2 b_3) - b_1 (a_3 b_2 + a_2 b_3)] + [b_1 (a_2 a_3 - b_2 b_3) + a_1 (a_3 b_2 + a_2 b_3)]i$

$= [(a_1a_2a_3 - a_1a_2b_3) - (a_3b_1b_2 + a_2b_1b_3)] + [(a_2a_3b_1 - b_1b_2b_3) + (a_1a_3b_2 + a_1a_2b_3)]i$

Thus $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

So, Associative properties w.r.t multiplication holds on complex numbers.

Property 3 identity law of multiplication

$z.1 = 1.z = z$

L.H.S $= z.1 = (a+bi) (1+0i) = (a+bi) = z$

R.H.S $= 1.z = (1+0i) (a+bi) = (a+bi) = z$

So, identities law of multiplication holds on complex numbers.

Property 4 Inverse law of multiplication $z.z^{-1} = 1$

L.H.S $= z.z^{-1} = a+bi \times \frac{1}{a+bi} = \frac{a+bi}{a+bi} = 1$

$$R.H.S = z^{-1}.z = \frac{1}{a+bi} \times a+bi = \frac{a+bi}{a+bi} = 1$$

So, Inverse law of multiplication holds on complex numbers.

Property 5 commutative law of multiplication.

$z_1 z_2 = z_2 z_1$

L.H.S $= z_1 z_2 = (a_1+b_1i)(a_2+b_2i)$

$= (a_2 a_1 - b_1 b_2) + (b_1 b_2 + a_1 b_2)i$

$= (a_1 a_2 - b_1 b_2) + (a_2 b_1 + a_1 b_2)i$

$= z_1 z_2$

Thus $z_1 z_2 = z_2 z_1$

So, commutative law multiplication holds on complex numbers.

Q3. Verify the distributive law of complex numbers.

(a,b)[(c,d) + (e,f)] = (a,b) (c,d) + (a,b) (e,f)

Solution:

(a,b)[(c,d) + (e,f)] = (a,b) (c,d) + (a,b) (e,f)

L.H.S = (a,b)[(c,d) + (e,f)]

= (a,b)(c+d)(e+f)

= [a(c+e) – b(d+f),a(d+f) + (c+e)]

= [ac+ ae – bd – bf.ad + af + bc + be]

= [ac – bd, ad + bc] + [ae – bf, af + be]

= (a,b) (c,d) + (a,b)(e,f)

= R.H.S

Hence proved

$$(a,b)\ [(c,d) + (e,f)] = (a,b)\ (c,d) + (a,b)\ (e,f)$$

Q4. Simply the following.

i. i^9 **ii.** i^{14} **iii)** $(-i)^{19}$ **iv.** $(-1)^{-21/2}$

Solution:

i. $i^9 = i^{8+1}$

$= i^8.i$

$= (i^2)^4\ i$

$= (-1)^4 i \qquad [\because i^2 = -1]$

$= 1.i$

$= i$

Hence $i^9 = i$

ii. $i^{14} = (i^2)^7$

$= (-1)^7 \qquad [\because i^2 = -1]$

$= -1$

Hence $i^{14} = -1$

iii. $(-i)^{19} = -(i)^{19}$

$= -(i)^{18}\ .\ i$

$= -(i^2)^9\ i$

$= -(-1)^9.i$

$= -(-1).i = i$

Hence $(-i)^{19} = i$

iv. $(-1)^{-21/2} = [(-1)^{1/2}]^{-21}$

$= (i)^{-21} \quad [\because (-1)^{1/2} = i]$

$= \frac{1}{i^{21}}$

$= \frac{1}{i^{20}.i}$

$= \frac{1}{(i^2)^{10}.i}$

$= \frac{1}{(-1)^{10}.i}$

$= \frac{1}{i}$

$= \frac{1}{i}.\frac{i}{i}$

$= \frac{i}{i^2}$

$= \frac{i}{-1}$

$= -i$

Hence $(-1)^{-21/2} = -i$

Q5. Write in terms of i.

i) $\sqrt{-1b}$ **ii)** $\sqrt{-5}$ **iii)** $\sqrt{\frac{-16}{25}}$ **iv)** $\sqrt{\frac{1}{-4}}$

Solution:

i) $\sqrt{-1b}$ **= b i** $[\because i = \sqrt{-1}]$

Hence $\sqrt{-1b} = i\,b$

ii) $\sqrt{-5} = \sqrt{-1 \times 5}$

$= \sqrt{-1} \ . \ \sqrt{5}$

$= i \ . \sqrt{5}$ $[\therefore i = \sqrt{-1}]$

Hence $\sqrt{-5} = \sqrt{5}\, i$

iii) $\sqrt{\frac{-16}{25}} = \sqrt{-1.\frac{16}{25}}$

$= \sqrt{-1} \ . \sqrt{\frac{16}{25}}$

$= i \ . \frac{4}{5}$

$= \frac{4}{5} i$

Hence $\sqrt{\frac{-16}{25}} = \frac{4}{5} i$

iv) $\sqrt{\frac{1}{-4}} = \sqrt{\frac{1}{(-1).4}}$

$= \frac{1}{\sqrt{-1} \times \sqrt{4}}$

$= \frac{1}{i.2} = \frac{1}{2.i}$

$= \frac{i}{2i.i}$

$= \frac{i}{2(-1)}$

$= \frac{-i}{2}$

Hence $\sqrt{\frac{1}{-4}} = \frac{-i}{2}$

Q6. Simplify the following

6. (7,9) + (3,-5) 7. (8,-5) - (-7,4) 8. (2,6) (3,7) 9. (5,-4) (-3,-2)

10. (0,3) (0,5) 11. (2,6) ÷ (3,7) 12. (5,-4) ÷ (-3,-8)

Solution:

6. (7,9) + (3,-5)

= (7+3 , (9+(-5)]

= (10,9-5)

= (10,4)

Hence (7,9) + (3,-5) = (10,4)

7. (8,-5) - (-7,4)

= [8-(-7) , -5-4)]

= (8+7, – 5-4)

= (15,-9)

Hence (8,-5) - (-7,4) = (15,-9)

8. (2,6) . (3,7)

= (2.3 – 6.7, 2.7 + 6.3)

= (6-42, 14+18)

= (-36, 32)

Hence (2,6) . (3,7) = (-36,32)

9. (5,-4) (-3,-2)

= [(5)(-3) – (-4)(-2), (5)(-2) + (-4)(-3)]

= [-155-8, -10+12]

= [-23,2]

Hence (5,-4) (-3,-2) = (-23,2)

10. (0,3).(0,5)

= [0.0 – 3.5, 0.5 + 3.0]

= (0-15, 0+0)

= (-15,0)

Hence (0,3) (0,5) = (-15,0)

11. (2,6) ÷ (3,7)

$= (2,6) \div (3,7) = (2,6) \cdot \frac{1}{3,7}$

$= (\frac{2,6}{3,7}) \cdot (\frac{3,-7}{3,-7})$

$= \frac{(2).(3)-(6)(-7),(2)(-7)+(6)(3)}{|3\times3-7.(-7),(3).(-7)+(7).(3)}$

$= \frac{6-(-42),-14+18}{9-(-49),-21+21}$

$= \frac{(6+42,-14+18)}{(9+49,0)}$

$= \frac{(48,4)}{(58,0)}$

$= (\frac{48}{58})(\frac{4}{58})$

$= (\frac{24}{29})(\frac{2}{29})$

Hence $(2,6)\div(3,7) = (\frac{24}{29})(\frac{2}{29})$

12. (5,-4) ÷ (-3,-8)

$= (5,-4) \div (-3,-8) = (5,-4) \cdot \frac{1}{-3,-8}$

$= (\frac{5,-4}{-3,-8}) . (\frac{-3,8}{-3,8})$

$= \frac{(5).(-3)-(-4)(8),(5)(8)+(-4)(-3)}{[-3\times(-3)-(-8).(8),(-3).(8)+(-8).(-3)}$

$= \frac{(-15+32,40+12]}{(9+64,-24+24)}$

$= \frac{(17,52)}{(73,0)}$

$= (\frac{17}{73})(\frac{52}{73})$ $[\therefore (73,0) = 73]$

Hence $(5,4)\div(-3,-8) = (\frac{17}{73})(\frac{52}{73})$

Q13. Prove that the sum as well as the product of any two conjugate complex numbers is real number.

Solution:

Let,

$z = a+bi$ (complex number)

$\overline{z} = a - bi$ (conjugate of complex number)

a) Sum of complex number & conjugate of complex number.

$z + \overline{z} = a + bi + a - bi$

$= 2a \in R$

So, sum of complex number & conjugate of complex number is real number.

b) Product of complex number & conjugate of complex number.

$z \cdot \bar{z} = (a + bi)(a - bi)$

$= a(a-bi) + bi(a-bi)$

$= a^2 - abi + abi - bi^2$

$= a^2 - b^2 \in R$

So, Product of complex and conjugate of complex number is a real number.

Q14. Find the multiplicative inverse of each of the following numbers:

i. (-4,7) ii. $(\sqrt{2}, \sqrt{-5})$ iii. (1,0)

Solution:

i. (-4,7)

The multiplicative inverse of (-4,7) is

$$= \frac{1}{(-4,7)}$$

$$= \frac{1}{(-4,7)} \times \frac{(-4,-7)}{(-4,-7)}$$

$$= \frac{(-4,7)}{[(-4)(-4)-(7)(-7),(-4)(-7)+(-4)(7)]}$$

$$= \frac{(-4,7)}{(16+49,28-28)}$$

$$= \frac{(-4,7)}{(65,0)} = (\frac{-4}{65}, \frac{7}{65})$$

The multiplicative inverse of (-4,7) is $(\frac{-4}{65}, \frac{7}{65})$

ii. $(\sqrt{2}, \sqrt{-5})$

The multiplicative inverse of $(\sqrt{2}, \sqrt{-5})$ is

$$= \frac{1}{(\sqrt{2}, \sqrt{-5})}$$

$$= \frac{1}{(\sqrt{2}, \sqrt{-5})} \times \frac{(\sqrt{2}, -\sqrt{-5})}{(\sqrt{2}, -\sqrt{-5})}$$

$$= \frac{(\sqrt{2}, -\sqrt{-5})}{[\sqrt{2}.\sqrt{5} - (-\sqrt{5})\sqrt{5}, \sqrt{2}.\sqrt{5} + (\sqrt{-5})(\sqrt{2})]}$$

$$= \frac{(\sqrt{2}, -\sqrt{-5})}{(2+5, \sqrt{10} - \sqrt{10})}$$

$$= \frac{(\sqrt{2}, \sqrt{5})}{(7,0)} = (\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7})$$

The multiplicative inverse of $(\sqrt{2}, \sqrt{-5})$ is $(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7})$

iii. (1,0)

The multiplicative inverse of (1,0) is

$$= \frac{1}{(1,0)}$$

$$= \frac{1}{(1,0)} \times \frac{(1,0)}{(1,0)}$$

$$= \frac{(1,0)}{(1+0, 0-0)}$$

$$= \frac{(1,0)}{(1,0)}$$

$$= (1,0)$$

The multiplicative inverse of (1,0) is (1,0)

Q15. Factorize the following.

i. $a^2 + 4b^2$ ii. $9a^2 + 16b^2$ iii. $3x^2 + 3y^2$

Solution:

i. $a^2 + 4b^2 = a^2 - (-1)4b^2$

$= a^2 - 4b^2i^2$

$= (a)^2 - (2bi)^2$

= (a + 2bi)(a - 2bi)

Hence, $a^2 + 4b^2$ = (a+2bi)(a-2bi)

ii. $9a^2 + 16b^2 = 9a^2 - (-1)16b^2$

$= 9a^2 - 16i^2$

$= (3a)^2 - (4bi)^2$

$= (3a + 4bi)(3a - 4bi)$

Hence, $9a^2 + 16b^2 = (3a+4bi)(3a-4bi)$

iii. $3x^2 + 3y^2 = 3(x^2 + y^2)$

$= 3[x^2 - (-1)y^2]$

$= 3[x^2 - y^2i^2]$

$= 3[(x)^2 - (yi)^2]$

$= 3(x + yi)(x - yi)$

Hence, $3x^2 + 3y^2 = 3(x+yi)(x-yi)$

Q16. Separate into real and imaginary parts (write as a simple complex number.)

i. $\frac{2-7i}{4+5i}$ ii. $\frac{(-2+3i)^2}{(1+i)}$ iii. $\frac{i}{1+i}$

Solution:

i. $\frac{2-7i}{4+5i}$

$= \frac{2-7i}{4+5i} \times \frac{4-5i}{4-5i}$

$= \frac{(2-7i)(4-5i)}{(4+5i)(4-5i)}$

$= \frac{2(4-5i)-7i(4-5i)}{4^2-(5i)^2}$

$= \frac{8-10i-28i+25i^2}{16-25i^2}$

$= \frac{8-38i-35}{16+25}$

$= \frac{8-35-38i}{41}$

$= \frac{-27-38i}{41}$

$= \frac{-27}{41} - \frac{38}{41}i$

Thus real part = $\frac{-27}{41}$

Thus real part = $\frac{-38}{41}$

ii. $\frac{(-2+3i)^2}{(1+i)}$

$= \frac{(-2+3i)^2}{(1+i)}$

$= \frac{4+9i^2+2(2)(3i)}{(1+i)}$

$= \frac{4-9+12i}{(1+i)}$

$= \frac{-5+12i}{(1+i)} \times \frac{1-i}{1-i}$

$= \frac{1(-5+12i)-i(-5+12i)}{1+i^2}$

$= \frac{-5+12+12i+5i}{1-(-1)}$

$= \frac{7+7i}{1+1}$

$= \frac{7+17i}{2}$

$= \frac{7}{2} + \frac{17i}{2}$

Thus real part = $\frac{7}{2}$

Imaginary part = $\frac{17}{2}$

iii. $\frac{i}{1+i}$

$= \frac{i}{1+i} \times \frac{1-i}{1-i}$

$$= \frac{i(1-i)}{1^2-(i)^2}$$

$$= \frac{i-i^2}{1-i^2}$$

$$= \frac{i-(-1)}{1-(-1)}$$

$$= \frac{i+1}{1+1}$$

$$= \frac{1+i}{2}$$

$$= \frac{1}{2} + \frac{1}{2}i$$

Thus real part = $\frac{1}{2}$