Chapter 8

Mathematical Induction and Binomial Theorem

Exercise 8.1

- **Use mathematical induction to prove the following formula for every positive integer n.**

1. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

2. $1 + 3 + 5 + \cdots + (3n - 2) = n^2$

3. $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$

4. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$

5. $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2\left(1 - \frac{1}{2^n}\right)$

6. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$

7. $2 + 6 + 18 + \cdots + 2 \times 3^{n-1} = 3^n - 1)$

8. $1 \times 3 + 2 \times 5 + 3 \times 7 + \cdots + n \times (2n + 1) = \frac{n(n+1)(4n+5)}{6}$

9. $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n + 1) = \frac{n(n+1)(n+2)}{3}$

10. $1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + (2n - 1) \times 2n = \frac{n(n+1)(4n+2)}{3}$

11. $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$

12. $\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

13. $\frac{1}{2\times5} + \frac{1}{5\times8} + \frac{1}{8\times11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$

14. $r + r^2 + r^3 + \cdots + r^n = \frac{r(1-r^n)}{1-r}$

15. $a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$

16. $1|1 + 2|2 + 3|3 + \cdots + n|n = |n + 1 - 1$

17. $a_n = a_1 + (n - 1)d$ when, $a_1, a_1, a_1 + 2d, \ldots$ form an A.P.

18. $a_n = a_1 r^{n-1}$ when, $a_1, a_1 r, a_1 r^2, \dots$ form an G. P.

19. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$

20. $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \cdots + \binom{n+2}{3} = \binom{n+2}{4}$

1. $1 + 5 + 9 + \cdots + (4n-3) = n(2n-1)$

Solution

C-1

Put $n = 1$

$$[4(1) - 3] = 1[2(1) - 1]$$

$$[4 - 3] = [2 - 1]$$

$$1 = 1$$

Its true for $n = 1$

C-2

Put $n = k$

$1 + 5 + 9 + \cdots + (4k-3) = k(2k-1)$ ___________ (1)

C-3

Put $n = k + 1$

$$4(k+1) - 3 = 4k + 4 - 3$$

$$= 4k + 1$$

Add $(4k+1)$ both sides in eq. (1), we get

$$1 + 5 + 9 + \cdots + (4k-3) + [4k+1] = k(2k-1)(4k+1)$$

$$= 2k^2 - k + 4k + 1$$

$$= 2k^2 + 3k + 1$$

$$= 2k^2 + 2k + k + 1$$

$$= 2k(k + 1) + 1(k + 1)$$

$$= (2k + 1)(k + 1)$$

$$= [2(k + 1) - 1](k + 1)$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

2. $\mathbf{1 + +3 + 5 + \cdots + (3n - 2) = n^2}$

Solution

C-1

Put $n = 1$

$$[3(1) - 2] = 1^2$$

$$[3 - 2] = 1$$

$$1 = 1$$

Its true for $n = 1$

C-2

Put $n = k$

$1 + 5 + 9 + \cdots + (2k - 1) = k^2$ ___________ (1)

C-3

Put $n = k + 1$

$$(2n - 1) = [2(k + 1) - 1]$$

$$= 2k + 2 - 1$$

$$= 2k + 1$$

Add (2k + 1) both sides in eq. (1), we get

$$1 + 5 + 9 + \cdots + (2k - 1) + [2k + 1] = k^2 + 2k + 1 = (k + 1)^2$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

3. $\mathbf{1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}}$

Solution

C-1

Put $n = 1$

$$[3(1) - 2] = \frac{1[3(1)-1]}{2}$$

$$[3 - 2] = \frac{[3-1]}{2}$$

$$1 = \frac{2}{2}$$

$$1 = 1$$

Its true for $n = 1$

C-2

Put $n = k$

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k-1)}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_ (1)}$$

C-3

Put $n = k + 1$ in (3n-2)

$$3n - 2 = 3(k + 1) - 2$$

$$= 4k + 3 - 2 = 3k + 1$$

Add $(4k + 1)$ both sides in eq. (1), we get

$$1 + 5 + 9 + \cdots + (3k - 1) + [3k + 1] = \frac{k(3k-1)}{2} + (3k + 1)$$

$$= \frac{3k^2-k}{2} + 3k + 1$$

$$= \frac{3k^2-k+6k+2}{2}$$

$$= \frac{3k^2+5k+2}{2}$$

$$= \frac{3k^2+3k+2k+2}{2}$$

$$= \frac{3k(k+1)+2(k+1)}{2}$$

$$= \frac{(3k+2)(k+1)}{2}$$

$$= \frac{[3(k+1)-1](k+1)}{2}$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

4. $\mathbf{1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1}$

Solution

C-1

Put $n = 1$

$2^{1-1} = 2^1 - 1$

$2^0 = 2 - 1$

$1 = 1$

Its true for $n = 1$

C-2

Put $n = k$

$1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1$ __________ (1)

C-3

Put $\quad n = k + 1$

$$3^{n-1} = 3^{k-1+1} = 2^k$$

Add (2^k) both sides in eq. (1), we get

$$1 + 2 + 4 + \cdots + \left(2^{k-1}\right) + \left[2^k\right] = 2^k - 1 + 2^k$$

$$= 2.2^k - 1$$

$$= 2^{k+1} - 1$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

5. $\mathbf{1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2\left(1 - \frac{1}{2^n}\right)}$

Solution

C-1

Put $\quad n = 1$

$$\frac{1}{2^{1-1}} = 2\left[1 - \frac{1}{2}\right]$$

$$\frac{1}{2^0} = 2\left[1 - \frac{1}{2^1}\right]$$

$$\frac{1}{1} = [1]$$

$$1 = 1$$

Its true for $\quad n = 1$

C-2

Put $\quad n = k$

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}} = 2\left[1 - \frac{1}{2^k}\right] \_\_\_\_\_\_\_\_\_\_ (1)$$

C-3

Put $n = k + 1$ in (3n-2)

$$\frac{1}{2^{n-1}} = \frac{1}{2^{k+1-1}} = \frac{1}{2^k}$$

Add (2^k) both sides in eq. (1), we get

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}} + \frac{1}{2^k} = 2\left[1 - \frac{1}{2^k}\right] + \frac{1}{2^k}$$

$$= 2 - \frac{2}{2^k} + \frac{1}{2^k}$$

$$= 2 - \left[\frac{2-1}{2^k}\right]$$

$$= 2 - \frac{1}{2^k} = 2 - \frac{2}{2^{k+1}}$$

$$= 2\left[1 - \frac{1}{2^k}\right]$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

6. $\mathbf{2 + 4 + 6 + \cdots + 2n = n(n + 1)}$

Solution

C-1

Put $n = 1$

$$2(1) = 1[1 + 1]$$

$$2 = 1(2)$$

$$2 = 2$$

Its true for $n = 1$

C-2

Put $n = k$

$$2 + 4 + 6 + \cdots + 2k = k(k + 1) \quad \text{\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

C-3

Put $n = k + 1$ in 2n

$2n = 2(k+1)$

Add 2(k + 1) both sides in eq. (1), we get

$2 + 4 + 6 + \cdots + 2k + 2[k+1] = k(k+1) + 2(k+1)$

$= k^2 + 2k + k + 2$

$= k(k+2) + 1(k+2)$

$= (k+1)(k+2)$

$= [(k+1)+1](k+1)$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

7. $\mathbf{2 + 6 + 18 + \cdots + 2 \times 3^{n-1} = 3^n - 1}$

Solution

C-1

Put $n = 1$

$2 \times 3^{1-1} = 3 - 1$

$2 \times 3^0 = 3 - 1$

$2 \times 1 = 2$

$2 = 2$

Its true for $n = 1$

C-2

Put $n = k$

$2 + 6 + 18 + \cdots + 2 \times 3^{k-1} = 3^k - 1$ __________ (1)

C-3

Put $n = k + 1$

$$2 \times 3^{n-1} = 2 \times 3^{k}$$

Add 2×3^{k} both sides in eq. (1), we get

$$2 + 6 + 18 + \cdots + 2.3^{k-1} + 2 \times 3^{k} = 3^{k} - 1 + 2 \times 3^{k}$$

$$= 3^{k}(1 + 2) - 1$$

$$= 3^{k}.3 - 1$$

$$= 3^{k+1} - 1$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

8. $\mathbf{1 \times 3 + 2 \times 5 + 3 \times 7 + \cdots + n \times (2n + 1) = \frac{n(n+1)(4n+5)}{6}}$

Solution

C-1

Put $n = 1$

$$1 \times [2(1) + 1] = \frac{1(1+1)[4(1)+5]}{6}$$

$$1 \times [3] = \frac{1(2)(9)}{6}$$

$$3 = 3$$

Its true for $n = 1$

C-2

Put $n = k$

$$1 \times 3 + 2 \times 5 + 3 \times 7 + \cdots + k \times (2k + 1) = \frac{k(k+1)(4k+5)}{6} \quad \_\_\_\_\_\_\_\_ (1)$$

C-3

Put $\quad n = k + 1 \quad$ in $n\times(2n+1)$

$$n \times [2n + 1] = (k + 1)[2(k + 1) + 1]$$

$$= (2k + 3)(k + 1)$$

Add $(k + 1)(2k + 3)$ both sides in eq. (1), we get

$$1 \times 3 + 2 \times 5 + 3 \times 7 + \cdots + k \times (2k + 1) + [k + 1](2k + 3)$$

$$= \frac{k(k + 1)(4k + 5)}{6} + (k + 1)(2k + 3)$$

$$= \frac{(k+1)}{6}[k(4k + 5) + 6(2k + 3)]$$

$$= \frac{(k+1)}{6}[4k^2 + 5k + 12k + 18]$$

$$= \frac{(k+1)}{6}[4k^2 + 17k + 18]$$

$$= \frac{k+1}{6}[4k^2 + 8k + 9k + 18]$$

$$= \frac{k+1}{6}[4k(k + 2) + 9(k + 2)]$$

$$= \frac{k+1}{6}[(4k + 9)(k + 2)]$$

$$= \frac{k+1}{6}[[(k + 1) + 1][4(k + 1) + 5]]$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

9. $\mathbf{1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n + 1) = \frac{n(n+1)(n+2)}{3}}$

Solution

C-1

Put $\quad n = 1$

$$1\times[1+1]=\frac{1(1+1)[(1)+2]}{3}$$

$$1\times\ [2]=\frac{(2)(3)}{3}$$

$$2=2$$

Its true for $n=1$

C-2

Put $n=k$

$$1\times 2+2\times 3+3\times 4+\cdots+k\times(k+1)=\frac{k(k+1)(k+2)}{3}\ \_\_\_\_\_\_\_\_\_\_\ (1)$$

C-3

Put $n=k+1$ in n× (n+1)

$$n\times[n+1]=(k+1)[(k+1)+1]$$

$$=(k+2)(k+1)$$

Add $(k+1)(k+2)$ both sides in eq. (1), we get

$$1\times 2+2\times 3+3\times 4+\cdots+k\times(k+1)+[k+1](k+2)=\frac{k(k+1)(k+2)}{3}+$$
$$(k+1)\times(k+2)$$

$$=(k+1)(k+2)\left[\frac{k}{3}+3\right]$$

$$=(k+1)(k+2)\left[\frac{k+3}{3}\right]$$

$$=\frac{(k+1)(k+2)[(k+1)+2]}{3}$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

10. $\mathbf{1\times 2+3\times 4+5\times 6+\cdots+2n\times(2n-1)=\frac{n(n+1)(4n+2)}{3}}$

Solution

C-1

Put $n = 1$

$$2(1) \times [2(1) - 1] = \frac{1(1+1)[4-1]}{3}$$

$$1 \times [2] = \frac{(2)(3)}{3}$$

$$2 = 2$$

Its true for $n = 1$

C-2

Put $n = k$

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + 2k \times (2k - 1) = \frac{k(k+1)(4k+2)}{3} \quad \text{\_\_\_\_\_\_\_\_\_\_} (1)$$

C-3

Put $n = k + 1$ in 2n× (2n-1)

$$2n \times [2n - 1] = 2(k + 1)[2(k + 1) - 1]$$

$$= [2k + 2 - 1]2(k + 1)$$

$$= (2k + 1)2(k + 1)$$

Add $2(k + 1)(2k + 1)$ both sides in eq. (1), we get

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + 2k \times (2k - 1) + 2[k + 1](2k + 1)$$

$$= \frac{k(k + 1)(4k - 1)}{3} + 2(k + 1)(2k + 1)$$

$$= \frac{(k+1)}{3}[k(4k - 1) + 6(2k + 1)]$$

$$= \frac{(k+1)}{3}[4k^2 - k + 12k + 6]$$

$$= \frac{(k+1)}{3}[4k^2 + 11k + 6]$$

$$= \frac{k+1}{3}[4k^2 + 8k + 3k + 6]$$

$$= \frac{k+1}{3}[4k(k+2) + 3(k+2)]$$

$$= \frac{k+1}{3}[(4k+3)(k+2)]$$

$$= \frac{k+1}{3}\big[[(k+1)+1][4(k+1)-1]\big]$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

11. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n\times(n+1)} = 1 - \frac{1}{n+1}$

Solution

C-1

Put $n = 1$

$$\frac{1}{1\times(1+1)} = 1 - \frac{1}{1+1}$$

$$\frac{1}{2} = 1 - \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2}$$

Its true for $n = 1$

C-2

Put $n = k$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{k\times(k+1)} = 1 - \frac{1}{k+1} \quad \text{\_\_\_\_\_\_\_\_\_\_ (1)}$$

C-3

Put $n = k+1$ in $\frac{1}{n(n+1)}$

$$\frac{1}{n\times(n+1)} = \frac{1}{(k+1)[(k+1)+1]}$$

$$= \frac{1}{(k+2)(k+1)}$$

Add $\frac{1}{(k+1)(k+2)}$ both sides in eq. (1), we get

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{k\times(k+1)}+\frac{1}{[k+1](k+2)} = 1-\frac{1}{k+1}+\frac{1}{(k+1)\times(k+2)}$$

$$= 1-\frac{1}{(k+1)}\left[1-\frac{1}{k+2}\right]$$

$$= 1-\frac{1}{(k+1)}\left[\frac{k+2-1}{k+2}\right]$$

$$= 1-\frac{1}{k+1}\left[\frac{k+1}{k+2}\right]$$

$$= 1-\frac{1}{k+2}$$

$$= 1-\frac{1}{[(k+1)+1]}$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

12. $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2n-1)\times(2n+1)} = 1-\frac{n}{2n+1}$

Solution

C-1

Put $n = 1$

$$\frac{1}{[2(1)-1][2(1)+1]} = \frac{1}{2(1)+1}$$

$$\frac{1}{(1)(3)} = \frac{1}{3}$$

$$\frac{1}{3} = \frac{1}{3}$$

Its true for $n = 1$

C-2

Put $n = k$

$$\frac{1}{(2k+7)(2k+1)} = \frac{k}{2k+1} \text{ ______ } (1)$$

C-3

Put $n = k + 1$ in $\frac{1}{(2n-1)(2n+1)}$

$$\frac{1}{(2n-1)[2n+1]} = \frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$= \frac{1}{[2k+2-1][2k+2+1]}$$

$$= \frac{1}{(2k+3)(2k+1)}$$

Add $\frac{1}{(2k+1)(2k+3)}$ both sides in eq. (1), we get

$$\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \cdots + \frac{1}{(2k+1)(2k+1)} + \frac{1}{[2k+1](2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)\times(2k+3)}$$

$$= \frac{k}{(2k+1)}\left[k + \frac{1}{2k+3}\right]$$

$$= \frac{1}{(2k+1)}\left[\frac{2k^2+3k+1}{2k+3}\right]$$

$$= \frac{1}{2k+1}\left[\frac{2k^2+2k+k+1}{2k+3}\right]$$

$$= \frac{1}{2k+1}\left[\frac{2k(k+1)+1(k+1)}{2k+3}\right]$$

$$= \frac{k+1}{2k+3}$$

$$= \left[\frac{k+1}{2(k+1)+1}\right]$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

13. $\frac{1}{2\times5} + \frac{1}{5\times8} + \frac{1}{8\times11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$

Solution

C-1

Put $n = 1$

$$\frac{1}{(3-1)(3+2)} = \frac{1}{2(3+2)}$$

$$\frac{1}{10} = \frac{1}{10}$$

Its true for $n = 1$

C-2

Put $n = k$

$$\frac{1}{2\times5} + \frac{1}{5\times8} + \frac{1}{8\times11} + \cdots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{2(3k+2)} \text{ ——— } (1)$$

C-3

Put $n = k + 1$ in $\frac{1}{(3n-1)(3n+2)}$

$$\frac{1}{(3n-1)[3n+2]} = \frac{1}{[3(k+1)-1][3(k+1)+2]}$$

$$= \frac{1}{[3k+3-1][3k+3+2]}$$

$$= \frac{1}{(3k+2)(3k+5)}$$

Add $\frac{1}{(3k+5)(3k+2)}$ both sides in eq. (1), we get

$$\frac{1}{2\times5} + \frac{1}{5\times8} + \frac{1}{8\times11} + \cdots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{[3k+5](3k+2)} = \frac{k}{2(3k+2)} + \frac{1}{(3k+5)(3k+2)}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{1}{(3k+2)}\left[\frac{k}{2} + \frac{1}{(3k+5)}\right]$$

$$= \frac{1}{(3k+2)}\left[\frac{k(3k+5)+2}{2(3k+5)}\right]$$

$$= \frac{1}{(3k+2)}\left[\frac{3k^2+5k+2}{2(3k+5)}\right]$$

$$= \frac{3k^2+3k+2k+2}{2(3k+2)(3k+5)}$$

$$= \frac{3k(k+1)+2(k+1)}{2(3k+2)(3k+5)}$$

$$= \frac{(3k+2)(k+1)}{2(3k+2)(3k+5)}$$

$$= \frac{k+1}{[3(k+1)-1]}$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

14. $\mathbf{r + r^2 + r^3 + \cdots + r^n = \frac{r(1-r^n)}{1-r}}$

Solution

C-1

Put $n = 1$

$$r^1 = \frac{r(1-r^1)}{1-r}$$

$$r = \frac{r(1-r)}{1-r}$$

$$r = r$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$$n = K + \in \mathbb{N}$$

i.e. $r + r^2 + r^3 + \cdots + r^k = \frac{r(1-r^k)}{1-r}$ ________ (1)

C-3

Put $n = k + 1$

$$r + r^2 + r^3 + \cdots + r^{k+1} = \frac{r(1-r^{k+1})}{1-r}$$

to prove that it is true or n=k+1

Add r^{k+1} both sides in eq. (1), we get

$$r + r^2 + r^3 + \cdots + r^k + r^{k+1} = \frac{r(1-r^k)}{1-r} + r^{k+1}$$

$$= \frac{r(1-r^k)+r^{k+1}(1-r)}{1-r}$$

$$= \frac{r-r^{k+1}+r^{k+1}-r^{k+2}}{1-r}$$

$$= \frac{r-r^{k+2}}{1-r}$$

$$= \frac{r-(1-r^{k+1})}{1-r}$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

15. $\mathbf{a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]}$

Solution

C-1

Put $n = 1$

$$a + (1 - 1)d = \frac{1}{2}[2a + (1 - 1)d]$$

$$a = \frac{1}{2}(2a)$$

$$a = a$$

Its true for $n = 1$

C-2

Suppose the statement is true for

Put $n = k$

$$a + (a + d) + (a + 2d) + \cdots + [a + (k - 1)d] = \frac{k}{2}[2a + (k - 1)d] \text{ \_\_\_\_\_\_\_\_\_\_ } (1)$$

C-3

To prove that it is true for n=k+1

Put $n = k + 1$

$$a + (a + d) + \cdots + a + kd = \frac{k-1}{2}[2a + \{(k + 1) - 1\}d]$$

Add $a + kd$ both sides in eq. (1), we get

$$a + (a + d) + \cdots + [a + (k - 1)d] + a + kd = \frac{k}{2}[2a + (k - 1)d] + a + kd$$

$$= \frac{k}{2}[2a + kd - d] + a + kd$$

$$= ak + \frac{k^2 d}{2} - \frac{kd}{2} + a + kd$$

$$= a + ak + \frac{kd}{2} + \frac{k^2 d}{2}$$

$$= a(k + 1) + \frac{kd}{2}(1 + k)$$

$$= a(k + 1) + \frac{kd}{2}(k + 1)$$

$$= \frac{k+1}{2}[2a + kd]$$

$$= \frac{(k+1)}{2}[2a + [(k + 1) - 1]d]$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

16. $\mathbf{1.1! + 2.2! + 3.3! + \cdots + n.n! = (n + 1)! - 1}$

Solution

C-1

Put $n = 1$

$$1.(1)! = [1+1]! - 1$$

$$1 = (2)! - 1$$

$$1 = 2 - 1$$

$$1 = 1$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$$n = k \in \mathbb{N}$$

i.e. $1.1! + 2.2! + 3.3! + \cdots + k.k! = (k+1)! - 1$ ___________ (1)

C-3

To prove that it is true for n=k+1

Add $(k+1)(k+1)!$To both sides in eq. (1), we get

$$1.1! + 2.2! + 3.3! + \cdots + k.k! + [k+1](k+1)! = (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)! + (k+1)(k+1)! - 1$$

$$= (k+1)!\,(1+k+1) - 1$$

$$= (k+1)!\,(k+2) - 2$$

$$= (k+2)! - 1$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

17. $\mathbf{a_n = a_1 + (n-1)d}$

Solution

when, $a_1, a_1, a_1 + 2d, \ldots$ form an A. P.

C-1

Put $n = 1$

$$a_1 = a_1 + (1-1)d$$

$$a_1 = a_1$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$$n = k$$

i.e. $a_k = a_1(k-1)d$ ____________ (1)

C-3

Put n=k+1

$$a_{k+1} = a_1 + (k+1-1)d$$

$$= a_1 + kd$$

To prove that it is true for n=k+1

Add d To both sides in eq. (1), we get

$$a_k + d = a_1 + (k-1)d + d$$

$$= a_1 + kd - d + d$$

$$= a_1 + kd$$

$$= a_1 + [(k+1) - 1]d$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

18. $\mathbf{a_n = a_1 r^{n-1}}$

Solution

when, $a_1, a_1 r, a_1 r^2, \ldots$ form an G. P.

C-1

Put $n = 1$

$$a_1 = a_1 + r^0$$

$$a_1 = a_1$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$$n = k$$

i.e. $a_k = a_1 r^{k-1}$ ____________ (1)

C-3

Put n=k+1

$$a_n = a_{k+1} = a_1 r^{(n+1)-1}$$

$$a_n = a_1 r^{n-1}$$

Multiplying both sides of eq. (1) by n, we get

$$a_k r = a_1 r^{k-1}.r$$

$$a_{k+1} = a_1 r^{k+1-1}$$

$$= a_1 r^{(k+1)-1}$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

19. $\mathbf{1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}}$

Solution

C-1

Put $n = 1$

$$[2(1) - 1]^2 = \frac{1[4(1)^2 - 1]}{3}$$

$$[2 - 1]^2 = \frac{1(4-1)}{3}$$

$$(1)^2 = \frac{3}{3}$$

$$1 = 1$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$n = k \in \mathbb{N}$

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k(4k^2-1)}{3} \quad \text{\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

C-3

Put $n = k + 1$

To prove that it is true for n=k+1

Add $(2k + 1)^2$ both sides in eq. (1), we get

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2 = \frac{k(4k^2-1)}{3} + (2k+1)^2$$

$$= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2$$

$$= \frac{(2k+1)}{3}[k(2k+1) + 3(2k+1)]$$

$$= \frac{(2k+1)}{3}[2k^2 - k + 6k + 3]$$

$$= \frac{2k+1}{3}[2k^2 + 5k + 3]$$

$$= \frac{2k+1}{3}[2k^2 + 2k + 3k + 3]$$

$$= \frac{2k+1}{3}[2k(k+1) + 3(k+1)]$$

$$= \frac{2k+1}{3}[(2k+3)(k+1)]$$

$$= \frac{(k+1)(2k+3)(2k+3)}{3}$$

$$= \frac{k+1}{3}[4k^2 + 2k + 6k + 3]$$

$$= \frac{k+1}{3}[4k^2 + 8k + 3]$$

$$= \frac{k+1}{3}[4k^2 + 8k + 4 - 1]$$

$$= \frac{k+1}{3}[4(k^2 + 2k + 1) - 1]$$

$$= \frac{k+1}{3}[4(k+1)^2 - 1]$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

20. $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \cdots + \binom{n+2}{3} = \binom{n+2}{4}$

Solution

C-1

Put $n = 1$

$$\begin{bmatrix} 1+2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Its true for $n = 1$

C-2

Suppose the statement is true for

Put for $n = k$

$$\begin{bmatrix}3\\3\end{bmatrix} + \begin{bmatrix}4\\3\end{bmatrix} + \begin{bmatrix}5\\3\end{bmatrix} + \cdots + \begin{bmatrix}k+2\\3\end{bmatrix} = \begin{bmatrix}k+3\\4\end{bmatrix} \quad \text{\_\_\_\_\_\_\_\_ (1)}$$

C-3

Put $n = k + 1$

$$\begin{bmatrix}3\\3\end{bmatrix} + \begin{bmatrix}4\\3\end{bmatrix} + \begin{bmatrix}5\\3\end{bmatrix} + \cdots + \begin{bmatrix}k+2\\3\end{bmatrix} = \begin{bmatrix}k+3\\4\end{bmatrix}$$

To prove that it is true for n=k+1

Add $\begin{bmatrix}k+3\\3\end{bmatrix}$ both sides in eq. (1), we get

$$\begin{bmatrix}3\\3\end{bmatrix} + \begin{bmatrix}4\\3\end{bmatrix} + \begin{bmatrix}5\\3\end{bmatrix} + \cdots + \begin{bmatrix}k+2\\3\end{bmatrix} + \begin{bmatrix}k+3\\3\end{bmatrix} = \begin{bmatrix}k+3\\4\end{bmatrix} + \begin{bmatrix}k+3\\3\end{bmatrix}$$

$$= \begin{bmatrix}k+4\\4\end{bmatrix} \qquad [\because \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

$$= \begin{bmatrix}(k+1)+3\\4\end{bmatrix}$$

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

21. Prove by mathematical induction that for all positive integer values of n

(i) $n^2 + n$ **is divisible by 2.** **(ii)** $5^n - 2^n$ **is divisible by 3.**

(iii) $5^n - 1$ **is divisible by 4.** **(iv)** $8 \times 10^2 - 2$ **is divisible by 6.**

(v) $n^3 - n$ **is divisible by 6.**

22. $\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$

23. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 + \cdots + (-1)^{n-1} = n^2 = \frac{(-1)^{n\ \ 1} n(n+1)}{2}$

24. $1^3 - 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2[2n^2 - 1]$

25. $\mathbf{x + 1}$ **is a factor of** $\mathbf{x^{2n} - 1; (x \neq -1)}$

26. $\mathbf{x - y}$ **is a factor of** $\mathbf{x^n - y^n; (x \neq y)}$

27. $\mathbf{x + y}$ **is a factor of** $\mathbf{x^{2n-1} + y^{2n-1}; (x \neq -y)}$

Solution

(i) $n^2 + n$ **is divisible by 2.**

C-1 Put $n = 1$

$(1)^2 + 1 = 1 + 1 = 2$

Clearly 2 is divisible by '2'.

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$

i.e. $k^2 + k$ is divisible by 2

$\Rightarrow$ $k^2 + k = 2(q)$, where `q' is some quotient

C-3 now we want to prove for $n = k + 1$

i.e. $(k + 1)^2 + (k + 1)$ is divisible by 2.

Or $(k + 1)^2 + (k + 1) = 2(\text{some quotient})$

Consider $(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1$

$= (k^2 + k) + 2k + 2 = k(k + 1) + 2(k + 1)$

$= 2(q) + 2(k + 1)(\because k(1 + 1)$ is an even integer)

$= 2(q + k + 1);$ where $q + k + 1$ is some quotient

$= 2$ (some quotient)

$\Rightarrow$ $(k + 1)^2 + (k + 1)$ is divisible by 2.

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

(ii) $5^n - 2^n$ is divisible by 3.

C-1 Put $n = 1$

$$5^1 - 2^1 = 3$$

Clearly 3 is divisible by '3'.

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$

i.e. $5^k - 2^k$ is divisible by 3.

$\Rightarrow$ $5^k - 2^k = 3(q)$, where `q' is some quotient

C-3 now we want to prove for $n = k + 1$

i.e. $5^{k+1} - 2^{k+1}$ is divisible by 3.

Or $5^{(k+1)} - 2^{k+1} = 3(\text{some quotient})$

Consider $5^{(k+1)} - 2^{k+1} = 5.5^k - 2.2^k$

$$= (3 + 2)5^k - 2.2^k$$

$$= 3.5^k + 2.5^k - 2.2^k$$

$$= 3.5^k + 2(5^k - 2^k)$$

$$= 3.5^k + 2.3(q) = 3(5^k + 2q)$$

$$= 3\ (\text{some quotient})$$

$\Rightarrow$ $5^{k+1} - 2^{k+1}$ is divisible by 3.

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

(iii) $5^2 - 1$ is divisible by 4.

C-1 Put $n = 1$

$5^1 - 1 = 4$

Clearly 4 is divisible by '4'.

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$

i.e. $5^k - 1$ is divisible by 4.

$\Rightarrow$ $5^k - 1 = 4(q)$, where `q' is some quotient

C-3 now we want to prove for $n = k + 1$

i.e. $5^{k+1} - 1$ is divisible by 4.

Or $5^{k+1} - 1 = 4(\text{some quotient})$

Consider $5^{k+1} - 1 = 5.5^k - 1 = (4 + 1)5^k - 1$

$= 4.5^k + (5^k - 1)$

$= 4.5^k + 4(q) = 4(5^k + q)$

$= 4\ (\text{some quotient})$

$\Rightarrow$ $5^{k+1} - 1$ is divisible by 2.

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

(iv) $8 \times 10^n - 2$ is divisible by 2.

C-1 Put $n = 1$

$8 \times 10^1 - 2 = 80 - 2 = 78 = 6(13)$

Clearly 78 is divisible by '6'.

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$

i.e. $8 \times 10^k - 2$is divisible by 6

$\Rightarrow$ $8 \times 10^k - 2 = 6(q)$, where `q′is some quotient

C-3 now we want to prove for $n = k + 1$

i.e. $8 \times 10^{k+1} - 2$is divisible by 6.

Or $8 \times 10^{k+1} - 2 = 6(\text{some quotient})$

Consider $8 \times 10^{k+1} - 2$

$$= 8 \times 10^k . 10 - 2$$

$$= 10 \times 8^k \times 10^k - 2 + 20 - 20$$

$$= 10 \times 8^k \times 10^k - 2 + 18 = 10\{8 \times 10^k - 2\} + 18$$

$$= 10\{6(q)\} + 6 \times 3; \text{ where q is some quotient}$$

$$= 6\{10q + 3\} = 6 \text{ (some quotient)}$$

$\Rightarrow$ $8 \times 10^{k+1} - 2$is divisible by 6.

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

(v) $n^3 + n$ is divisible by 6.

C-1 Put $n = 1$

$$n^3 - n = (1)^3 - 1 = 0$$

Clearly 0 is divisible by '6'.

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$

i.e. $k^3 - k$ is divisible by 6

$\Rightarrow$ $k^3 - k = 6(q)$, where `q' is some quotient

C-3 now we want to prove for $n = k + 1$

i.e. $(k + 1)^3 - (k + 1)$ is divisible by 6.

Consider $(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1$

$= (k^3 - k) + (3k^2 + 3k)$

$= (k^3 - k) + 3k(k + 1)$

$= 6(q) + 3(2m) \text{Q} \; k(k + 1) = 2m = \text{an even integer}$

$= 6q + 6m$

$= 6(q + m)$ where $(q + m)$ is an integer.

$\Rightarrow$ $(k + 1)^3 - (k + 1)$ is divisible by 6.

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

22. $\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left[1 - \frac{1}{3^n}\right]$

Solution

C-1

Put $n = 1$

$\frac{1}{3} = \frac{1}{2}\left[1 - \frac{1}{3^1}\right]$

$\frac{1}{3} = \frac{1}{2}\left[\frac{2}{3}\right]$

$$\frac{1}{3}=\frac{1}{3}$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$n = k \in \mathbb{N}$

i.e. $\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^k}=\frac{1}{2}\left[1-\frac{1}{3^k}\right]$ __________ (1)

C-3

Put $n = k + 1$

To prove that it is true for n=k+1

i.e. $\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{k+1}}=\frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$

Add $\frac{1}{3^{k+1}}$ both sides in eq. (1), we get

$$\left\{\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^k}\right\}+\frac{1}{3^{k+1}}=\frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{3^k}\right]+\frac{1}{3^{k+1}}$$

$$=\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{3^k}+\frac{1}{3}\cdot\frac{1}{3^k}=\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{3}\right)\frac{1}{3^k}=\frac{1}{2}-\left[\frac{3-2}{6}\right]\frac{1}{3^{k+1}}=\frac{1}{2}-\left[\frac{1}{6}\right]\frac{1}{3^k}$$

$$=\frac{1}{2}-\left(\frac{1}{2}\cdot\frac{1}{3}\right)\frac{1}{3^k}=\frac{1}{2}-\frac{1}{2}\frac{1}{3^{k+1}}$$

$$=\frac{1}{2}\left(1-\frac{1}{3^{k+1}}\right)$$

Truth for n=k implies the truth for n=k+1

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

23. $\mathbf{1^2-2^2+3^2-4^2+\cdots+(-1)^{n-1}n^2=\frac{n(-1)^{n-1}(n+1)}{2}}$

Solution

C-1

Put $n = 1$

$$[1]^2 = \frac{1[-1]^{1\;\;1}(1+1)}{2}$$

$$1 = \frac{1(-1)^0(2)}{2}$$

$$1 = 1$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$$n = k \in \mathbb{N}$$

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1}(k)^2$$

$$= \frac{k(-1)^{k\;\;1}(k+1)}{2} \text{ \_\_\_\_\_\_\_\_\_\_ (1)}$$

C-3

Put $n = k + 1$

i.e. $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^k(k+1)^2 = \frac{(k+2)(-1)^k(k+1)}{2}$

Add $(-1)^k(k+1)^2$ both sides in eq. (1), we get

$$\left\{1^2 + 3^2 + 5^2 + \cdots + (-1)^k(k+1)^2\right\} + (-1)^k(k+1)^2 = \frac{k(-1)^{k-1}(k+1)}{2} + (-1)^k(k+1)^2$$

$$= \frac{k(k+1)(-1)^k(k)^{\;\;1}}{2} + (-1)^k(k+1)^2$$

$$= \frac{(-1)^k(-1)k(k+1)}{2} + (-1)^k(k+1)^2$$

$$= (-1)^k(k+1)\left\{-\frac{k}{2} + (k+1)\right\}$$

$$= (-1)^k(k+1)\left\{\frac{-k+2k+2}{2}\right\}$$

$$= (-1)^k(k+1)\left\{\frac{k+2}{2}\right\} = \frac{\left((-1)^k(k+1)(k+2)\right)}{2}$$

Truth for n=k implies the truth for n=k+1

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

24. $\mathbf{1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)}$

Solution

C-1

Put $n = 1$

$[1]^3 = 1^2[2(1)^2 - 1]$

$1 = 1[2 - 1]$

$$1 = 1$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$n = k \in \mathbb{N}$

$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2 - 1)$__________ (1)

C-3

Put $n = k + 1$

To prove that it is true for n=k+1

$$1^3 + 3^3 + 5^3 + \cdots + (2k)^3 = (k+1)^2(2(k+1)^2 - 1)$$

Add $(2k+1)^3$ both sides in eq. (1), we get

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^3 + (2k+1)^3 = k^2(2k^2-1) + (2k+1)^3$$

$$= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1$$

$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

$$= 2k^4 + 2k^3 + 6k^3 + 6k^2 + 5k^2 + 5k + k + 1$$

$$= 2k^3(k+1) + 6k^2(k+1) + 5k(k+1) + (k+1)$$

$$= (k+1)[2k^3 + 6k^2 + 5k + 1]$$

$$= (k+1)[2k^3 + 2k^2 + 4k^2 + 4k + (k+1)]$$

$$= (k+1)[(2k^2(k+1) + 4k(k+1) + (k+1)]$$

$$= (k+1)^2[2k^2 + 4k + 1]$$

Truth for n=k implies the truth for n=k+1

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

25. **$x + 1$ is a factor of $x^{2n} - 1$; $(x \neq -1)$**

C-1 Put $n = 1$

$$x^{2n} - 1 = (x)^2 - 1 = (x+1)(x-1)$$

Clearly $(x+1)$ is a factor of $(x^2 - 1)$

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$

i.e. $(x+1)$ is a factor of $(x^{2k} - 1)$

$\Rightarrow$ $k^3 - k = 6(q)$, where `q' is some quotient

C-3 now we want to prove that statement is true for $n = k + 1$

i.e. $(x+1)$ is a factor of $(x^{2k+2} - 1)$

Consider $x^{2k+2} - 1 = x^2 . x^{2k} - 1$

$= \{(x^2 - 1) + 1\}. x^{2k} - 1$

$= (x^2 - 1). x^{2k} + x^{2k} - 1$

$= \{(x + 1)(x - 1). x^{2k}\} + x^{2k} - 1$

Now $(x + 1)$ is a factor of $\{(x + 1)(x - 1). x^{2k}\}$

And $(x + 1)$ is a factor of $x^{2k} - 1$ $(\because x^{2k} - 1 = (x + 1)$

$\Rightarrow$ $(x + 1)$ is a factor of $\{(x + 1)(x - 1). x^{2k}\} + \{x^{2k} - 1\}$

$\Rightarrow$ $(x + 1)$ is a factor of $\{x^{2k+2} - 1\}$

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

26. $x - y$ is a factor of $x^n - y^n$; $(x \neq y)$

C-1 Put $n = 1$

$x^n - y^n = x^1 - y^1 = (x - y)$

Clearly $(x - y)$ is a factor of $(x - y)$

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$

i.e. $(x - y)$ is a factor of $(x^k - y^k)$

$\Rightarrow$ $x^k - y^k = (x - y)(q)$, where `q' is some quotient

C-3 now we want to prove that statement is true for $n = k + 1$

i.e. $(x - y)$ is a factor of $(x^{k+1} - y^{k+1})$

Consider $x^{k+1} - y^{k+1} = x. x^k - y. y^k$

$= x.x^k - x^ky + x^ky - y.y^k$

$= (x - y).x^k + (x^k - y^k)y$

$= (x - y).x^k + (x - y)qy$

$= (x - y)(x^k - qy) \quad where\ [x^k - qy]\ is\ some\ quotient$

$\Rightarrow \quad (x - y)\ \ is\ a\ factor\ of\ \{x^{k+1} - y^{k+1}\}$

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

27. $\mathbf{x + y}$ is a factor of $\mathbf{x^{2n-1} + y^{2n-1}; (x \neq -y)}$

C-1 Put $n = 1$

$x^{2n-1} + y^{2n-1} = x^{2-1} + y^{2-1} = (x + y)$

Clearly (x + y) is a factor of (x + y)

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$

i.e. (x + y) is a factor of $(x^{2k-1} + y^{2k-1})$

$\Rightarrow \quad x^{2k-1} + y^{2k-1} = (x + y)(q)$, where `q' is some quotient

C-3 now we want to prove that statement is true for $n = k + 1$

i.e. $(x + y)$ is a factor of $(x^{2k+1} + y^{2k+1})$

Consider $x^{2k+1} + y^{2k+1}$

$= x^{2k+2-1} + y^{2k+2-1}$

$= x^2.x^{2k-1} + y^2.y^{2k-1}$

$= x^2.x^{2k-1} + x^2y^{2k-1} - x^2y^{2k-1} + y^2.y^{2k-1}$

$$= \left(x^{2k-1} + y^{2k-1}\right).x^2 - (x^2 - y^2)y^{2k-1}$$

$$= (x + y).x^2q - (x + y)(x - y)y^{2k-1}$$

$$= (x + y)\left(x^2q - (x - y)y^{2k-1}\right)$$

$\Rightarrow$ $(x + y)$ is a factor of $\left\{x^{2k+1} - y^{2k+1}\right\}$

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

28. Use mathematical induction to show that

$$\mathbf{1 + 2 + 2^2 + \cdots + (2)^n = (2^{n+1} - 1) \text{ for all non} - \text{negtive integers n.}}$$

29. If A and B are square matrices and AB=BA, then show by mathematical induction that $AB^n = B^nA$ for any positive integer n.

30. Prove by the mathematical induction that $n^2 - 1$ is divisible by 8 when n is an odd positive integer.

31. Use the principle of mathematical induction to prove that $\ln x^n = n \ln x$ for any integer n≥ 0 if x is a positive integer.

28. Use mathematical induction to show that

$$\mathbf{1 + 2 + 2^2 + \cdots + (2)^n = (2^{n+1} - 1) \text{ for all non} - \text{negtive integers n.}}$$

Solution

C-1

Put $n = 1$

$$1 = [2^1 - 1]$$

$$1 = 1$$

Its true for $n = 1$

C-2

Suppose the statement is true for

$$n = k \in \mathbb{N}$$

$1 + 2 + 2^2 + \cdots + (2)^k = (2^{k+1} - 1)$__________ (1)

C-3

Now we want to prove that it is true for n=k+1

$$1 + 2 + 2^2 + \cdots + (2)^{k+1} = 2^{k+2} - 1$$

L.H.S. $= 1 + 2 + 2^2 + \cdots + (2)^{k+1}$

$= \{1 + 2 + 2^2 + \cdots + 2^k\}(2)^{k+1}$

$= 2^{k+1} - 1 + 2^{k+1}$

$= 2.2^{k+1} - 1 = 2^{k+2} - 1$=**R.H.S.**

Truth for n=k implies the truth for n=k+1

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

29. If A and B are square matrices and AB=BA, then show by mathematical induction that $AB^n = B^nA$ for any positive integer n.

Solution

C-1

Put $n = 1$

$AB^1 = B^1A$ i.e. $AB = BA$

Its true for $n = 1$

C-2 Suppose the statement is true for $n = k \in \mathbb{N}$

$AB^k = B^kA$

C-3 Now we want to prove that it is true for n=k+1

$$AB^{k+1} = B^{k+1}A$$

L.H.S. $= AB^{k+1}$

$= AB^kB = (AB^k)B = (B^kA)B$

$= B^k(AB) = B^k(BA) = (B^kB)A$

$= B^{k+1}A$=**R.H.S.**

Truth for n=k implies the truth for n=k+1

Hence; by the principal of mathematical induction the formula is true for all positive integers n.

30. Prove by the mathematical induction that $n^2 - 1$ is divisible by 8 when n is an odd positive integer.

Solution

C-1 Put $n = 1$

$n^2 - 1 = (1)^2 - 1 = 0$

Clearly 0 is divisible by '8'.

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k \in \mathbb{N}$ where k is odd

i.e. $k^2 - 1$ is divisible by 8

⇒ $k^2 - 1 = 8(q)$, where `q' is some quotient

C-3 now we want to prove for $n = k + 2$

i.e. $(k + 2)^2 - 1$ is divisible by 8.

Consider $(k + 2)^2 - 1$

$$= k^2 + 4k + 4 - 1$$

$= (k^2 - 1) + 4k + 4$

$= 8(q) + 4(k + 1)(\because (k + 1)$ is an even integer)

$= 8q + 4(2m);$ where m is an even integer.

$= 8q + 8m = 8(q + m)$

$\Rightarrow \quad (k + 2)^2 - 1$ is divisible by 8.

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

31. Use the principle of mathematical induction to prove that $\ln x^n = n \ln x$ for any integer n≥ 0 if x is a positive integer.

Solution

C-1 Put $n = 1$

L.H.S$=\ln x^n = \ln x^0$

$= \ln(1) = 0$

R.H.S $= n\ln x = 0\ln x$

Clearly 0 = 0.

Thus statement is true for $n = 1$

C-2 suppose statement is true for $n = k$

i.e. $\ln x^k = k\ln x$ .... (i)

$\Rightarrow$ $k^2 - 1 = 8(q)$, where `q'is some quotient

C-3 now we want to prove for $n = k + 2$

i.e. $\ln x^{k+1} = (k + 1)\ln x$

Adding 'nx' on both sides of (i), we get

$\ln x^k + nx = k\ln x + nx$

$\text{Ln}(x^k.x) = (k+1)\ln x$

$\ln x^{k+1} = (k+1)\ln x$

Truth for n=k implies the truth for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n.

Use the principle of expended mathematical induction to prove that:

32. $n! > 2^n - 1$ **for integral value of** $n \geq 4$

Solution

C-1 Put $n = 4$

L.H.S $n! = 4!$

$= 4.3.2.1 = 24$

R.H.S $2^n - 1 = 2^4 - 1$

$16 - 1 = 15$

Thus statement is true for $n = 4$

C-2 suppose statement is true for $n = k$

$k! > 2^k - 1$

C-3 now we want to prove for $n = k + 1$

L.H.S $= (k+1)!$

$= (k+1)k!$

$= (kd! + k!)$

We know that $= k!\, 2^k - 1$

$= k(2^k - 1) + k!$

$= 2^{k+1} - 1 + (k! - 1)2^{k+1} - 1$

Hence by the principle of mathematical induction the statement is true for all positive integral values of n≥ 4.

33. $n^2 > n + 3$ for integral value of $n \geq 3$

Solution

C-1 Put $n = 3$

L.H.S $n^2 = 3^2 = 9$

R.H.S $n + 3 = 3 + 3 = 6$

Thus statement is true for $n \geq 3$

C-2 suppose statement is true for $n = k$

$k^2 > k + 3$ ………… (i)

C-3 now we want to prove for $n = k + 1$

$(k + 1)^2 > k + 1 + 3$

$(k + 1)^2 > k + 4$

Adding (2k+1) both sides of (i)

We get

$k^2 + (2k + 1) > (k + 3) + (2k + 1)$

$k^2 + 2k + 1 > k + 3 + 2k + 1$

$(k + 1)^2 > 3k + 4$

$(k + 1)^2 > 3(k + 1) + 1$

Hence by the principle of mathematical induction the statement is true for all positive integral values of n≥ 3.

34. $3^n > 3 + 2^{n-1}$ **for integral value of** $n \geq 2$

Solution

C-1 Put $n = 2$

L.H.S $4n = 4.2 = 16$

R.H.S $= 3^2 + 2^{2-1}$

$= 9 + 2^1 = 9 + 2 = 11$

Thus statement is true for $n = 2$

C-2 suppose statement is true for $n = k$

$4^k > 3^k + 2^{k-1}$ .... (i)

Multiply eq. (i) by 4

$4.4^k > 4.3^k + 4.2^{k-1}$

$4^{k+1} > (3 + 1)3^k + 2^2.2^{k+1}$

$4^{k+1} > 3^{k+1} + 2^{k+3} + 3^k$

$4^{k+1} > 3^{k+1} + 2^{k+3}.\{3^k > 0\ as\ k > 0\}$

$4^{k+1} > 3^{k+1} + 2^{k+3}$

$4^{k+1} > 3^{k+1} + 4.2^{k+1}$

$4^{k+1} > 3^{k+1} + (3 + 1)2^{k+1}$

$4^{k+1} > 3^{k+1} + 3.2^{k+1} + 2^{k+1}$

$4^{k+1} > 3^{k+1} + 2^{k+1} + [2^{k+1} > 2^k \text{for } k \geq 2]$

$4^{k+1} > 3^{k+1} + 2^{(k+1)-1}$

Hence by the principle of mathematical induction the statement is true for all positive integral values of n≥ 2.

35. $3^n < n!$ **for integral value of** $n > 6$

Solution

C-1 Put $n = 7$

L.H.S $= 3^n$

$= 3^7 = 2187$

R.H.S $= n!$

$= 7!$

$= 7.6.5.4.3.2.1 = 5040$

Thus statement is true for $n = 7$

C-2 suppose statement is true for $n = k$

$3^n < n!$

$3^k < k!$ .... (i)

Multiply eq. (i) by 3

$3^{k+1} < 3k!$

$3^{k+1} < 3k!$ $[3 < k + 1\ as\ k < 6]$

$3^{k+1} < (k + 1)!$

$n3 < n!$ $[it\ is\ true\ for\ n = k + 1$

Hence by the principle of mathematical induction the statement is true for all positive integral values of n> 6

36. $n! > n^2$ **for integral value of** $n \geq 4$

Solution

Put $n = 4$

L.H.S $n! = 4!$

$= 4.3.2.1 = 24$

R.H.S $n^2 = (4)^2 = 16$

Thus statement is true for $n \geq 4$

suppose statement is true for $n = k$

$k! > k^2$ .... (i)

multiply (i) by $(k+1)$, we get

$(k+1)k! > k^2(k+1) > (k+1)(k+1)$ [as $k^2 > k+1$

$(k+1)! > (k+1)^2$

It is true for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n≥ 4.

37. $\mathbf{3 + 5 + 7 + \cdots + (2n + 5) = (n + 2)(n + 4)}$ **for integral values of** $n \geq 1$

Solution

Put $n = -1$

$2(-1) + 5 = (-1 + 2)[-1 + 4]$

$-2 + 5 = 1(3)$

$3 = 3$

Its true for $n = -1$

Let us suppose that $n = k$

$3 + 5 + 7 + \cdots + (2k + 5) = (k + 2)(k + 4)$ ___________ (1)

Add $(2k + 7)$ both sides in eq. (1), we get

$$3 + 5 + 7 + \cdots + (2k + 5) + [2k + 7] = (k + 2)(k + 4) + (2k + 7)$$

$$= k^2 + 6k + 8 + 2k + 7$$

$$= k^2 + 8k + 15$$

$$= k^2 + 3k + 5k + 15$$

$$= k(k + 3) + 5(k + 3)$$

$$= (k + 3)(k + 5)$$

$$= [(k + 1) + 2][(k + 1) + 4]$$

It is true for n=k+1

Hence; by the principal of mathematical induction the formula is true for all positive integers $n \geq 1$.

38. for any positive number 'x' $(1 + x)^n \geq 1 + nx$ for any positive integral $n \geq 2$ And x>-1

Solution

Put $n = 2$

L.H.S $(1 + x)^n = (1 + x)^2 = 1 + 2x + x^2$

And

R.H.S $= 1 + nx = 1 + 2x$

Thus statement is true for $n = 2$

suppose statement is true for $n = k$

$$(1 + x)k \geq 1 + kx \quad \text{............. (i)}$$

Multiply eq. (i) by $(1 + x)$

$$(1 + x)^k(1 + x) \geq (1 + kx)(1 + x)$$

$(1 + x)^{k+1} > 1 + x + kx + kx^2$

$(1 + x)^{k+1} > 1 + (k + 1)x + kx^2$

$(1 + x)^{k+1} > (k + 1)x + 1 + kx^2$

$(1 + x)^{k+1} > (k + 1)x + 1 \quad [\text{as } kx^2 > 0 \text{ } for \text{ } k > 2]$

It is true for n=k+1

Hence by the principle of mathematical induction the statement is true for all positive integral values of n≥ 2 and n>-1.