Exercise 8.2

Q1. Using binomial theorem, expand the following:

i) $(a+2b)^5$ **ii)** $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$ **iii)** $\left(3a-\frac{x}{3a}\right)^4$

iv) $\left(2a-\frac{x^2}{a}\right)^7$ **v)** $\left(\frac{x}{2y}-\frac{2y}{x}\right)^6$ **vi)** $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

i) $(a+2b)^5$

Solution:

$$(a+2b)^5 = \begin{bmatrix}5\\0\end{bmatrix}(a)^5(2b)^0 + \begin{bmatrix}5\\1\end{bmatrix}(a)^4(2b)^1 + \begin{bmatrix}5\\2\end{bmatrix}(a)^3(2b)^2 + \begin{bmatrix}5\\3\end{bmatrix}(a)^2(2b)^3 + \begin{bmatrix}5\\4\end{bmatrix}(a)^1(2b)^4 + \begin{bmatrix}5\\5\end{bmatrix}(a)^0(2b)^5$$

$$= (1)(a^5)(1) + (5)(a^4)(2b) + (10)(a^3)(4b^2) + (10)(a^2)(8b^3) + (5)(a)(16b^4) + (1)(1)(32b^5)$$

$$= a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5$$

Hence

$$(a+2b)^5 = a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5$$

ii) $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$

Solution:

$$\left(\frac{x}{2}-\frac{2}{x^2}\right)^6=\left[\left(\frac{x}{2}\right)+\left(\frac{-2}{x^2}\right)\right]^6$$

$$=\begin{bmatrix}6\\0\end{bmatrix}\left(\frac{x}{2}\right)^6\left(\frac{-2}{x^2}\right)^0+\begin{bmatrix}6\\1\end{bmatrix}\left(\frac{x}{2}\right)^5\left(\frac{-2}{x^2}\right)^1+\begin{bmatrix}6\\2\end{bmatrix}\left(\frac{x}{5}\right)^4\left(\frac{-2}{x^2}\right)^2$$

$$+\begin{bmatrix}6\\3\end{bmatrix}\left(\frac{x}{2}\right)^3\left(\frac{-2}{x^2}\right)^3+\begin{bmatrix}6\\4\end{bmatrix}\left(\frac{x}{2}\right)^2\left(\frac{-2}{x^2}\right)^4+\begin{bmatrix}6\\5\end{bmatrix}\left(\frac{x}{5}\right)^1\left(\frac{-2}{x^2}\right)^5+\begin{bmatrix}6\\6\end{bmatrix}\left(\frac{x}{2}\right)^0\left(\frac{-2}{x^2}\right)^6$$

$$=(1)\left(\frac{x^6}{64}\right)(1)+(6)\left(\frac{x^5}{32}\right)\left(\frac{-2}{x^2}\right)+15\left(\frac{x^4}{16}\right)\left(\frac{4}{x^4}\right)+20\left(\frac{x^3}{8}\right)\left(\frac{-8}{x^6}\right)$$

$$+(15)\left(\frac{x^2}{4}\right)\left(\frac{16}{x^8}\right)+(6)\left(\frac{x}{2}\right)\left(\frac{-32}{x^2}\right)+(1)(1)\left(\frac{64}{x^2}\right)$$

$$=\frac{x^6}{64}-\frac{3}{8}x^3+\frac{15}{4}-\frac{20}{x^3}+\frac{60}{x^6}-\frac{96}{x^9}+\frac{64}{x^{12}}$$

Hence;

$$\left(\frac{x}{2}-\frac{2}{x^2}\right)^6=\frac{x^6}{64}-\frac{3}{8}x^3+\frac{15}{4}-\frac{20}{x^3}+\frac{60}{x^6}-\frac{96}{x^9}+\frac{64}{x^{12}}$$

iii) $\left(3a-\frac{x}{3a}\right)^4$

Solution:

$$\left[(3a)+\left(\frac{-x}{3a}\right)\right]^4=\begin{bmatrix}4\\0\end{bmatrix}(3a)^4\left(\frac{-x}{3a}\right)^0+\begin{bmatrix}4\\1\end{bmatrix}(3a)^3\left(\frac{-x}{3a}\right)^1+\begin{bmatrix}4\\2\end{bmatrix}(3a)^2\left(\frac{-x}{3a}\right)^2$$

$$+\begin{bmatrix}4\\3\end{bmatrix}(3a)^1\left(\frac{-x}{2a}\right)^3+\begin{bmatrix}4\\4\end{bmatrix}(3a)^0\left(\frac{-x}{2a}\right)^4$$

$$=(1)(81a^4)(1)+(4)(27a^3)\left(\frac{-x}{3a}\right)+(6)(9a^2)\left(\frac{x^4}{9a^2}\right)$$

$$+(4)(3a)\left(\frac{-x^3}{8a^3}\right)+(1)(1)\left(\frac{x^4}{16a^4}\right)$$

$$=81a^4-36a^2x+6x^2-\frac{4x^3}{9a^2}+\frac{x^4}{16a^4}$$

Hence;

$$\left[3a-\left(\frac{-x}{3a}\right)\right]^4=81a^4-36a^2x+6x^2-\frac{4x^3}{9a^2}+\frac{x^4}{16a^4}$$

iv) $\left(2a-\frac{x^2}{a}\right)^7$

Solution:

$$\left[(2a)+\left(\frac{-x^2}{a}\right)\right]^7=\begin{bmatrix}7\\0\end{bmatrix}(2a)^7\left(\frac{-x^2}{a}\right)^0+\begin{bmatrix}7\\1\end{bmatrix}(2a)^6\left(\frac{-x^2}{a}\right)^1$$

$$+\begin{bmatrix}7\\2\end{bmatrix}(2a)^5\left(\frac{-x^2}{a}\right)^2+\begin{bmatrix}7\\3\end{bmatrix}(2a)^4\left(\frac{-x^2}{a}\right)^3+\begin{bmatrix}7\\4\end{bmatrix}(2a)^3\left(\frac{-x^2}{a}\right)^4$$

$$+\begin{bmatrix}7\\5\end{bmatrix}(2a)^2\left(\frac{-x^2}{a}\right)^5+\begin{bmatrix}7\\6\end{bmatrix}(2a)^1\left(\frac{-x^2}{a}\right)^6+\begin{bmatrix}7\\7\end{bmatrix}(2a)^0\left(\frac{-x^2}{a}\right)^7$$

$$=(1)(128a^7)+(7)(64a^6)\left(\frac{-x^2}{a^2}\right)+(21)(32a^5)\left(\frac{x^2}{a^2}\right)$$

$$+35(16a^4)\left(\frac{-x^3}{a^3}\right)+35(8a^3)\left(\frac{x^4}{a^4}\right)+(21)(4a^2)\left(\frac{-x^2}{a^5}\right)$$

$$=128a^7-448a^5x+672a^3x^2-560ax^3+\frac{280x^4}{a}-\frac{84x^5}{a^3}+\frac{14x^6}{a^5}-\frac{x^7}{a^7}$$

Hence $\left(2a-\frac{x^2}{a}\right)^7=128a^7-448a^5x+672a^3x^2-560ax^3+\frac{280x^4}{a}-\frac{84x^5}{a^3}+\frac{14x^6}{a^5}-\frac{x^7}{a^7}$

v) $\left(\frac{x}{2y}-\frac{2y}{x}\right)^6$

Solution:

$$\left[\frac{x}{2y}-\frac{2y}{x}\right]^8 = \begin{bmatrix}8\\0\end{bmatrix}\left(\frac{x}{2y}\right)^8\left(\frac{-2y}{x}\right)^0+\begin{bmatrix}8\\1\end{bmatrix}\left(\frac{x}{2y}\right)^7\left(\frac{-2y}{x}\right)^1+\begin{bmatrix}8\\2\end{bmatrix}\left(\frac{x}{2y}\right)^6\left(\frac{-2y}{x}\right)^2$$

$$+\begin{bmatrix}8\\3\end{bmatrix}\left(\frac{x}{2y}\right)^5\left(\frac{-2y}{x}\right)^3+\begin{bmatrix}8\\4\end{bmatrix}\left(\frac{x}{2y}\right)^4\left(\frac{-2y}{x}\right)^4+\begin{bmatrix}8\\5\end{bmatrix}\left(\frac{x}{2y}\right)^3\left(\frac{-2y}{x}\right)^5$$

$$\begin{bmatrix}8\\6\end{bmatrix}\left(\frac{x}{2y}\right)^2\left(\frac{-2y}{x}\right)^6+\begin{bmatrix}8\\7\end{bmatrix}\left(\frac{x}{2y}\right)\left(\frac{-2y}{x}\right)^7+\begin{bmatrix}8\\8\end{bmatrix}\left(\frac{x}{2y}\right)^0\left(\frac{-2y}{x}\right)^8$$

$$=(1)\left(\frac{x^8}{64}\right)(1)+(8)\left(\frac{x^7}{128y^8}\right)\left(\frac{-2}{x}\right)+(28)\left(\frac{x^6}{64x^6}\right)\left(+\frac{4y^2}{x^2}\right)$$

$$+(56)\left(\frac{x^5}{32y^5}\right)\left(\frac{-8y^3}{x^3}\right)+(70)\left(\frac{x^4}{16y^4}\right)\left(\frac{16y^4}{x^4}\right)\left(\frac{16y^4}{x^4}\right)$$

$$+8\left(\frac{x}{2y}\right)\left(\frac{-128y^7}{x^7}\right)+(1)(1)\left(\frac{256y^8}{x^8}\right)$$

$$=\frac{x^8}{256y^8}-\frac{x^6}{8y^6}+\frac{7x^4}{4y^2}-\frac{14x^2}{y^2}+70-\frac{224y^2}{x^2}+\frac{448y^4}{x^4}-\frac{512y^6}{x^6}+\frac{256y^8}{x^8}$$

Hence $\left(\frac{x}{2y}-\frac{2y}{x}\right)^8=\frac{x^8}{256y^8}-\frac{x^6}{8y^6}+\frac{7x^4}{4y^2}-\frac{14x^2}{y^2}+70-\frac{224y^2}{x^2}+\frac{448y^4}{x^4}-\frac{512y^6}{x^6}+\frac{256y^8}{x^8}$

vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

Solution:

$$\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6=\begin{bmatrix}6\\0\end{bmatrix}\left[\frac{\sqrt{a}}{x}\right]^6\left[-\sqrt{\frac{x}{a}}\right]^0+\begin{bmatrix}6\\1\end{bmatrix}\left[\sqrt{\frac{a}{x}}\right]^5\left[-\sqrt{\frac{x}{a}}\right]^1+\begin{bmatrix}6\\2\end{bmatrix}\left[\sqrt{\frac{a}{x}}\right]^4\left[-\sqrt{\frac{x}{a}}\right]^2+$$

$$\begin{bmatrix}6\\3\end{bmatrix}\left[\frac{\sqrt{a}}{x}\right]^3\left[-\sqrt{\frac{x}{a}}\right]^3+\begin{bmatrix}6\\4\end{bmatrix}\left[\sqrt{\frac{a}{x}}\right]^2\left[-\sqrt{\frac{x}{a}}\right]^4+\begin{bmatrix}6\\4\end{bmatrix}\left[\sqrt{\frac{a}{x}}\right]\left[-\sqrt{\frac{x}{a}}\right]^5+\begin{bmatrix}6\\4\end{bmatrix}\left[\sqrt{\frac{a}{x}}\right]^0\left[-\sqrt{\frac{x}{a}}\right]^6$$

$$= (1)\left(\frac{a}{x}\right)^3 + (6)\left(\frac{-a}{x}\right)^2 + (15)\left(\frac{-a}{x}\right) + (20)\left(\frac{-a}{x}\right)^0 +$$

$$(15)\left(\frac{-a}{x}\right)^{-1} + (6)\left(\frac{-a}{x}\right)^{-2} + (1)\left(\frac{a}{x}\right)^{-3}$$

$$= \left(\frac{a^3}{x^3}\right) - \frac{6a^2}{x^2} + \frac{15a}{x} + 20 + \frac{15x}{a} - \frac{6x^2}{a^2} + \frac{x^3}{a^3}$$

Hence

$$\left(\sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}}\right)^6 = \frac{a^3}{x^3} - \frac{6a^2}{x^2} + \frac{15a}{x} + 20 + \frac{15x}{a} - \frac{6x^2}{a^2} + \frac{x^3}{a^3}$$

2. Calculate following by means of binomial theorem:

i. $(0.97)^3$ ii. $(2.02)^4$ iii. $(9.98)^4$ iv. (2.1)

i. $(0.97)^3$

Solution:

$(0.97)^3 = (1 - 0.03)^3$

$$= \binom{3}{0}(1)^3(-0.03)^0 + \binom{3}{1}(-0.03)^1 + \binom{3}{2}(1)^1(-0.03)^2 + \binom{3}{3}(1)^0(-0.03)^3$$

$$= 1 + 3\times(-0.03)^2 + 3\times(0.0009) + 1\times(0)$$

$$= 1 - 0.09 + 0.0027 + 0 = 0.9127$$

Hence

$$(0.97)^3 = 0.9127$$

ii. $(2.02)^4$

Solution:

$$(2.02)^4 = (2+0.02)^4$$

$$= \binom{4}{0}(2)^4(0.02)^0 + \binom{4}{1}(2)^3(0.02)^1 + \binom{4}{2}(2)^2(0.02)^2 + \binom{4}{3}(2)^1(0.02)^3 + \binom{4}{4}(2)^0(0.02)^4$$

$$= 1(16) + 4(8)(0.02) + 6(4)(0.0004)$$
$$= 4(2)(0.000008) + 1(0.00000016)$$
$$= 16 + 0.64 + 0.0096 + 0.000064 + 0.0000016 = 16.649664$$

Hence

$$(0.02)^4 = 16.649664$$

iii. $(9.98)^4$

Solution:

$$(9.98)^4 = (10 - 0.02)^4$$

$$= \binom{4}{0}(10)^4(-0.02)^0 + \binom{4}{1}(10)^3(-0.02)^1 + \binom{4}{2}(10)^2(-0.02)^2 + \binom{4}{3}(10)^1(-0.02)^3 + \binom{4}{4}(10)^0(-0.02)^4$$

$$= 1\times(10000) + 4(1000)(-0.02) + 6(100)(0.0004) + 4(10)(0) + 1\times(0)$$
$$= 10000 - 80 + 0.24 = 9920.24$$

Hence

$$(9.98)^4 = 9920.24$$

iv. (2.1)

Solution:

$$(2.1)^5 = (2 + 0.1)^5$$

$$= \binom{5}{0}(2)^5(0.1)^0 + \binom{5}{1}(2)^4(0.1)^1 + \binom{3}{2}(2)^3(0.1)^2 + \binom{5}{3}(2)^2(0.1)^3$$
$$+ \binom{5}{4}(2)^1(0.1)^4 + \binom{5}{5}(2)^0(0.1)$$

$$= (1)(32)(1) + (5)(16)(0.1) + (10)(8)(0.01) + (10)(4)(0.001) +$$
$$(5)(2)(0.0001) + (1)(1)(0.00001)$$
$$= 32 + 8 + 0.8 + 0.04 + 0.001 + 0.00001$$
$$= 40.84101$$

Hence

$(2.1)^5 = 40.84101$

3. Expand and simplify the following

i. $\left(a+\sqrt{2x}\right)^4 + \left(a-\sqrt{2x}\right)^4$ **ii.** $\left(2+\sqrt{3}\right)^5 + \left(2-\sqrt{3}\right)^5$

iii. $\left(2+1\right)^5 - \left(2-1\right)^5$ **iv.** $\left(x+\sqrt{x^2-1}\right)^3 + \left(x-\sqrt{x^2-1}\right)^3$

i. $\left(a+\sqrt{2x}\right)^4 + \left(a-\sqrt{2x}\right)^4$

Solution:

Take

$$\left(a+\sqrt{2x}\right)^4 = \binom{4}{0}(a)^4(+\sqrt{2x})^0 + \binom{4}{1}(a)^3(+\sqrt{2x})^1 + \binom{4}{2}(a)^2(+\sqrt{2x})^2 + \binom{4}{3}(a)^1$$
$$(+\sqrt{2x})^3 + \binom{4}{4}(a)^0(+\sqrt{2x})^4$$

$$= 1\times a^4 + 4a^3\sqrt{2}x + 6a^2(2x^2) + 4a\left(2^{\frac{3}{2}}x^3\right) + 1\times 2^{\frac{4}{2}}x^4$$

$$\Rightarrow\left(a+\sqrt{2x}\right)^4=a^4+4\sqrt{2}a^3x+12a^2x^2+8\sqrt{2}ax^3+4x^4 \quad \_\_\_\_\_\_(i)$$

$$and\left(a-\sqrt{2x}\right)^4=a^4-4\sqrt{2}a^3x+12a^2x^2-8\sqrt{2}ax^3+4x^4 \quad \_\_\_\_\_\_(ii)$$

Adding equations (i) and (ii), we get

$$\left(a+\sqrt{2x}\right)^4+\left(a-\sqrt{2x}\right)^4=2a^4+24a^2x^2+8x^4=2\left\{a^4+12a^2x^2+4x^4\right\}$$

Hence

$$\left(a+\sqrt{2x}\right)^4+\left(a-\sqrt{2x}\right)^4=2\left(a^4+12a^2x^2+4x^4\right)$$

ii. $\left(2+\sqrt{3}\right)^5+\left(2-\sqrt{3}\right)^5$

Solution:

Take

$$\left(2+\sqrt{3x}\right)^5=\binom{5}{0}(2)^5(\sqrt{3})^0+\binom{5}{1}(2)^4(\sqrt{3})^1+\binom{5}{2}(2)^3(\sqrt{3})^2+\binom{5}{3}(2)^2(\sqrt{3})^3+\binom{5}{4}(2)^1(\sqrt{3})^4+\binom{5}{5}(2)^0(\sqrt{3})^5$$

$$=1\times 32+5\times 16\sqrt{3}+10\times 8(3)+10\times 4(3\sqrt{3})+5\times 2(3^2)+1\times(3^2\sqrt{3})$$

$$\Rightarrow\left(2+\sqrt{3x}\right)^5=32+80\sqrt{3}+240+120\sqrt{3}+90+9\sqrt{3} \quad \_\_\_\_\_\_(i)$$

And $\left(2-\sqrt{3x}\right)^5=32-80\sqrt{3}+240+120\sqrt{3}+90+9\sqrt{3} \quad \_\_\_\_\_\_(ii)$

Adding equations (i) and (ii), we get

$$\left(a+\sqrt{3x}\right)^5+\left(a-\sqrt{3x}\right)^5=2\{32+240+90\}=2\{362\}=724$$

Hence

$$\left(2+\sqrt{3x}\right)^5+\left(2-\sqrt{3x}\right)^5=724$$

iii. $(2+i)^5-(2-i)^5$

Solution:

Take $(2+i)=\binom{5}{0}(2)^5(i)^0+\binom{5}{1}(2)^4(i)^1+\binom{5}{2}(2)^3(i)^2+\binom{5}{3}(2)^2(i)^3+\binom{5}{4}(2)^1(i)^4+\binom{5}{5}(2)^0(i)^5$

= 1 x 32 + 5 x 16i + 10 x 8i^2 + 10 x 4i^3 +5 x 2i^4 + 1 x i^5

$$\Rightarrow (2+i)^5=32+80i+80i^2+40i^3+10i^4+5^i \quad \_\_\_\_\_\_\_(i)$$

$$and(2-i)^5=32-80i+80i^2-40i^3+10i^4-5^i \quad \_\_\_\_\_\_\_(ii)$$

Subtracting equation (ii) from equation (i), we get

$$\begin{aligned}(2+i)^5-(2-i)^5 &= 2\{80i+40i^3+i^5\}\\ &= 2\{80i+40(-i)+\ \}\\ &= 2\{80i-40i+I\}=2\{41i\}=82i\end{aligned}$$

Hence $(2+2)^5$ + $(2-2)^5$ = 82 i

iv. $\left(x+\sqrt{x^2-1}\right)^3+\left(x+\sqrt{x^2-1}\right)^3$

Solution:

Take

$$\left(x+\sqrt{x^2-1}\right)^3=\binom{3}{0}(x)^3\left(\sqrt{x^2-1}\right)^0+\binom{3}{1}(x)^2\left(\sqrt{x^2-1}\right)^1+\binom{3}{2}(x)^1\left(\sqrt{x^2-1}\right)^2+\binom{3}{3}(x)^0\left(\sqrt{x^2-1}\right)^3$$

$$= 1 \times x^3 + 3 \times x^2 \sqrt{x^2-1} + 3 \times x(x^2-1) + 1 \times \left(\sqrt{x^2-1}\right)^3$$

$$\Rightarrow \left(x+\sqrt{x^2-1}\right)^3 = x^3 + 3x^2\sqrt{x^2-1} + 3x(x^2-1) + \left(\sqrt{x^2-1}\right) \quad \_\_\_\_\_\_\_\_(i)$$

$$and \left(x-\sqrt{x^2-1}\right)^3 = x^3 - 3x^2\sqrt{x^2-1} + 3x(x^2-1) - \left(\sqrt{x^2-1}\right) \quad \_\_\_\_\_\_\_\_(ii)$$

Adding equation (i) and (ii), we get

$$\left(x+\sqrt{x^2-1}\right)^3 + \left(x-\sqrt{x^2-1}\right)^3 = 2x^3 + 6x(x^2-1)$$

$$= 2x^3 + 6x^3 - 6x$$

$$= 2x(x^2 + 3x^2 - 3)$$

$$= 2x(4x^2 - 3)$$

Hence

$$\left(x+\sqrt{x^2-1}\right)^3 + \left(x-\sqrt{x^2-1}\right)^3 = 2x(4x^2-3)$$

4. Expand the following in ascending power of x:

i. $(2 + x - x^2)^4$ ii. $(1 - x + x^2)^4$ iii. $(1 - x - x^2)^4$

i. $(2 + x - x^2)^4$

Solution:

$$\left(2 + x - x^2\right)^4 = \binom{4}{0}(2+x)^4(-x^2)^0 + \binom{4}{1}(2+x)^3(-x^2)^1 + \binom{4}{2}(2+x)^2(-x^2)^2 + \binom{4}{3}(2+x)^1(-x^2)^3 + \binom{4}{4}(2+x)^0(-x^2)^4$$

$=1 \times (2 + x)^4 + 4(8 + 12x + 6x^2 + x^3)(-x^2) + 6(4 + 4x + x^2)(x^4) + 4(2 + x)$

$(-x^6) + 1 \times (x^8)$

$= 16 + 32x + 24x^2 + 8x^3 + x^4 - 32x^2 - 48x^3 - 24x^4 - 4x^5 + 24x^4 +$

$24x^5 + 6x^6$

$- 8x^6 - 4x^7 + x^8$

$= 16 + 32x + 24x^2 - 32x^2 + 8x^3 - 24x^3 + x^4 - 24x^4 - 4x^5 + 24x^5 + 6x^6$

$- 8x^6 -$

$4x^7 + x^8$

$= 16 + 32x - 8x^2 - 40x^3 + x4 + 20x^5 - 2x^6 - 4x^7 + x^8$

Hence

$(2 + x - x^2)^4 \quad = 16 + 32x - 8x^2 - 40x^3 + x4 + 20x^5 - 2x^6 - 4x^7 + x^8$

ii. $(1-x+x^2)^4$

Solution:

$$\left(\mathbf{1}-\mathbf{x}+\mathbf{x}^2\right)^4 = \binom{4}{0}(1-x)^4(x^2)^0 + \binom{4}{1}(1-x)^3(x^2)^1 + \binom{4}{2}(1-x^2)(x^2)^2 + \binom{4}{3}$$
$$(1-x)^1(x^2)^3 + \binom{4}{4}(1-x)^0(x^2)^4$$

$$= 1(1-4x+6x^2-4x^3+x^4) + 4(1-3x+3x^2-x^3)(x^2) + 6(1-2x+x^2)(x^4)$$
$$+ 4(1-x)(x^6) + 1\times(x^8)$$

$$= 1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4-4x^5+6x^4-12x^5+6x^6+4x^6-$$
$$4x^7+x^8$$

$$= 1-4x+6x^2+4x^2-4x^3-12x^3+x^4+12x^4+6x^4-4x^5-12x^5+6x^6+4x^6-$$
$$4x^7+x^8$$

$$= 1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$$

Hence

$$(1-x+x^2)^4 = 1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$$

iii. $(1-x-x^2)^4$

Solution:

$$\left(\mathbf{1}-\mathbf{x}-\mathbf{x}^2\right)^4 = \binom{4}{0}(1-x)^4(-x^2)^0 + \binom{4}{1}(1-x)^3(-x^2)^1 + \binom{4}{2}(1-x^2)(-x^2)^2 + \binom{4}{3}(1-x)^1(-x^2)^3 + \binom{4}{4}(1-x)^0(-x^2)^4$$

$= 1 \times (1 - 4x + 6x^2 - 4x^3 + x^3 + x^4) + 4\,(1 - 3x + 3x^2 - x^3)\,(-x^2) + 6(1-$

$2x+$

$x^2)\,(x^4) + 4\,(1 - x)\,(-x^6) + 1 \times (x^8)$

$= 1 - 4x + 6x^2 - 4x^3 + x^4 - 4x^2 + 12x^3 - 12x^4 + 4x^5 + 6x^4 - 12x^5 + 6x^6$

$-4x^6 +$

$4x^7 + x^8$

$= 1 - 4x + 6x^2 - 4x^2 - 4x^3 + 12x^3 + x^4 - 12x^4 + 6x^4 + 4x^5 - 12x^5 + 6x^6 -$

$4x^6 +$

$4x^7 \times x^8$

$= 1 - 4x + 2x^2 + 8x^3 - 5x^4 - 8x^5 + 2x^6 + 4x^7 + x^8$

Hence

$(1-x-x^2)^4 = 1 - 4x + 2x^2 + 8x^3 - 5x^4 - 8x^5 + 2x^6 + 4x^7 + x^8$

5. Expand the following in descending powers of x:

i. $(x^2 + x - 1)^2$ **ii.** $\left(x-1-\frac{1}{x}\right)^3$

i. $(x^2 + x - 1)^2$

Solution:

$$\left(\mathbf{x}^2 + \left(\mathbf{x} - \mathbf{1}\right)\right)^3 = \binom{3}{0}(x^2)^3(x-1)^0 + \binom{3}{1}(x^2)^2(x-1)^1 + \binom{3}{2}(x^2)^1(x-1)^2 + \binom{3}{3}(x^2)^0 (x-1)^3$$

$$= 1\times x^6 + 3x^4(x-1) + 3x^2(x^2-2x+1) + 1\times(x^3-3x^2+3x-1)$$

$$= x^6 + 3x^5 - 3x^4 + 3x^4 - 6x^3 + 3x^2 + x^3 - 3x^2 + 3x - 1$$

$$= x^6 + 3x^5 - 6x^3 + x^3 + 3x - 1$$

$$= x^6 + 3x^5 - 5x^3 + 3x - 1$$

Hence

$$\left(x^2 + (x-1)\right)^3 = x^6 + 3x^5 - 5x^3 + 3x - 1$$

ii. $\left(x - 1 - \frac{1}{x}\right)^3$

Solution:

$$\left[(x-1) - \frac{1}{x}\right]^3 = \binom{3}{0}(x-1)^3\left(-\frac{1}{x}\right)^0 + \binom{3}{1}(x-1)^2\left(-\frac{1}{x}\right)^1 + \binom{3}{2}(x-1)^1\left(-\frac{1}{x}\right)^2 + \binom{3}{3}(x-1)^0\left(-\frac{1}{x}\right)^3$$

$$= 1 \times (x^3 - 3x^2 + 3x - 1) + (x^2 - 2x + 1)\left(-\frac{1}{x}\right) + 3 - 3x^2 + 3x - 1 - 3x - 1 -$$

$$3x + 6 - \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}$$

$$= x^3 - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}$$

Hence

$$\left(x-1-\frac{1}{x}\right)^3 = x^3-3x^2+5-\frac{3}{x^2}-\frac{1}{x^3}$$

6. Find the term involving

i. x^4 in the expansion of $(3-2x)^7$ **ii. x^{-2} in the expansion of $\left(x-\frac{2}{x^2}\right)^{13}$**

iii. a^4 in the expansion of $\left(\frac{2}{x^2}-a\right)^9$ **iv. y^3 in the expansion of $\left(x-\sqrt{y}\right)^{11}$**

i. x^4 in the expansion of $(3-2x)^7$

Solution:

Let T_{r+1} be the required term.

So, $T_{r+1} = \binom{n}{7} a^{n-r} b^r$

$$= \binom{n}{7} (3)^{7-r}(-2x)^r \qquad \_\_\_\_\_\_\_\_ (i)$$

For the term involving x^4, put exponent of x equal to 4

i.e. r = 4

put r = 4 in equation (i) , we get

$$T_{4+1} = \binom{n}{4}(3)^{7-4}(-2)^4 x^4$$

$$T_5 = \frac{7.6.5}{3.2.1} 3^3 (16) x^4$$

$$T_5 = (35)(27)(16)x^4$$

$$T_5 = 15120x^4$$

Hence

$$T_5 = 15120x^4$$

ii. x^{-2} in the expansion of $\left(x - \frac{2}{x^2}\right)^{13}$

Solution:

Let T_{r+1} be the required term.

So, $T_{r+1} = \binom{n}{r} a^{n-r} b^r = \binom{13}{r} x^{13-r} \left(\frac{2}{x^2}\right)^r$

$$= \binom{13}{r} (-2)^r x^{13-r-2r}$$

$$= \binom{13}{r} (-2)^r x^{13-3r} \qquad \_\_\_\_\_\_(i)$$

For the term involving x^{-2} , put exponent of x equal to -2

i.e. 13 - 3r = -2

$\Rightarrow$ 3r = 13 + 2

3r = 15 $\Rightarrow$ r = 5

Put r = 5 in equation (i), we get.

$$T_{5+1} = \binom{13}{5} (-2)^5 x^{13-3(5)}$$

$$T_6 = \frac{13.12.11.10.9}{5.4.3.2.1} (-32) x^{-2}$$

$$T_6 = -41184x^{-2}$$

Hence

$$T_6 = -41184x^{-2}$$

iii. **a^4 in the expansion of $\left(\frac{2}{x^2} - a\right)^9$**

Solution:

Let T_{r+1} be the required term.

So, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$= \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^4$$

$$= \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-1)^r a^r \quad \_\_\_\_\_\_\_\_(i)$$

For the term involving a^4, put exponent of x equal to 4

i.e. r = 4

put r = 4 in equation (i), we get

$$T_{4+1} = \binom{9}{4}\left(\frac{2}{x}\right)^{9-4}(-4)^4 a^4$$

$$T_5 = \frac{9.8.7.6}{4.3.2.1}.\frac{2^5}{x^5}(1)^4 a^4 = 4032\frac{a^4}{x^5}$$

Hence

$$T_5 = 4032\frac{a^4}{x^5}$$

iv. **y^3 in the expansion of $\left(x - \sqrt{y}\right)^{11}$**

Solution:

Let T_{r+1} be the required term.

So, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$= \binom{11}{r} (x)^{11-r} (-\sqrt{y})^r$$

$$= \binom{11}{r} (x)^{11-r} (-1)^r y^{\frac{r}{2}} \quad \_\_\_\_\_\_(i)$$

For the term involving y^3, put exponent of x equal to 3

i.e. $\frac{r}{2} = 3 \Rightarrow r = 6$

put r = 6 in equation (i), we get

$$T_{6+1} = \binom{11}{6} x^{11-6} (-1)^6 y^{\frac{6}{2}}$$

$$T_7 = \frac{11.10.9.8.7}{5.4.3.2.1} (x)^5 = 462x^5y^3$$

Hence

$$T_7 = 462x^5y^3$$

Q7. Find the coefficient of;

i) x^5 *in the expansion of* $\left(x^2 - \frac{3}{2x}\right)^{10}$ **ii)** x^n *in the expansion of* $\left(x^2 - \frac{1}{x}\right)^{2n}$

Solution:

i) x^5 *in the expansion of* $\left(x^2 - \frac{3}{2x}\right)^{10}$

$$\left(x^2 - \frac{3}{2x}\right)^{10}$$

We know that

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$= \binom{10}{r} (x^2)^{10-r} \left(-\frac{3}{2x}\right)^r$$

$$= \binom{10}{r} x^{20-2r} \left(-\frac{3}{2}\right)^r \left(\frac{1}{x}\right)^r$$

$$= \binom{10}{r} \left(-\frac{3}{2}\right)^r .x^{20-2r-r}$$

$$= \binom{10}{r} \left(-\frac{3}{2}\right)^r .x^{20-3r} \quad ....(i)$$

For the term involving x^5 , put exponent of x equal to 5,

i.e. $20 - 3r = 5$

$3r = 20 - 5 \quad => \quad 3r = 15$

$r = 5$

Put r = 5 in equation (i), we get

$$T_{5+1} = \binom{10}{5} \left(-\frac{3}{2}\right)^5 x^{20-3(5)}$$

$$T_6 = \frac{10.9.8.7.6}{5.4.3.2.1} \left(-\frac{3^5}{2^5}\right) x^5$$

$$T_6 = 252\left[-\frac{243}{32}\right] x^5$$

$$T_6 = -\frac{15309}{8} x^5$$

Hence

Co-efficient of $x^5 = -\frac{15309}{8}$

Solution:

ii) x^n *in the expansion of* $\left(x^2 - \frac{1}{x}\right)^{2n}$

$\left(x^2 - \frac{1}{x}\right)^{2n}$

We know that

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r = \binom{2n}{r} (x^2)^{2n-r} \left(-\frac{1}{x}\right)^r$$

$$= \binom{2n}{r} x^{2(2n-r)} (-1)^r \left(\frac{1}{x^r}\right)$$

$$= \binom{2n}{r} (-1)^r x^{4n-2r-r}$$

$$= \binom{2n}{r} (-1)^r x^{4n-3r} \quad ....(i)$$

For the term involving x^n, put exponent of x equal to n,

i.e. $4n - 3r = n$

$\Rightarrow \quad 3r = 4n - n$

$\Rightarrow \quad 3r = 3n$

$\Rightarrow \quad r = n$

Put r = n in equation (i), we get

$$T_{n+1} = \binom{2n}{n}(-1)^n x^{4n-3n}$$

$$T_{n+1} = \binom{2n}{n}(-1)^n x^n$$

Hence

Co-efficient of x^n = $(-1)^n \binom{2n}{n}$ or $(-1)^n \frac{(2n)!}{(n!)^2}$

Q8. Find 6^{th} term in the equation of $\left(x^2 - \frac{3}{2x}\right)^{10}$

Solution:

We know that rth term of general equation is

$$T_r = \binom{n}{r-1} a^{n-(r-1)} b^{r-1}$$

For 6^{th} terms, put r=6, we get

$$T_6 = \binom{10}{5}(x^2)^{10-5}\left(-\frac{3}{2x}\right)^5$$

$$= \frac{10.9.8.7.6}{5.4.3.2.1} x^{2(5)} \left(-\frac{3}{2}\right)^5 \frac{1}{x^5}$$

$$= 252 x^{10-5}\left(-\frac{243}{32}\right)$$

$$T_6 = \frac{15309}{8} x^5$$

Hence, 6^{th} term

$$T_6 = -\frac{15309}{8} x^5$$

Q9. Find the term independent of x in the following expansions.

i) $\left(x-\frac{2}{x}\right)^{10}$ **ii)** $\left(\sqrt{x}+\frac{1}{2x^2}\right)^{10}$ **iii)** $\left(1+x^2\right)^3\left[1+\frac{1}{x^2}\right]^4$

i) $\left(x-\frac{2}{x}\right)^{10}$

Solution:

The term independent of x means x^0

So we are find a term free of x.

Let T_{r+1} be the required term. Then

$$T_{r+1}=\binom{n}{r}a^{n-r}b^r=\binom{10}{r}(x)^{10-r}\left(-\frac{2}{x}\right)^r$$

$$=\binom{10}{r}(x)^{10-r}(-2)^r\left(\frac{1}{x^r}\right)$$

$$=\binom{10}{r}(-2)^r(x)^{10-r-r}$$

$$=\binom{10}{r}(-2)^r(x)^{10-2r}$$

For the term involving x^0, put exponent of x equal to 0

i.e. $10 - 2r = 0$

$\Rightarrow$ $2r = 10$ $\Rightarrow$ $r = 5$

Put r= 5 in equation (i), we get

$$T_{5+1}=\binom{10}{5}(-2)^5(x)^{10-2(5)}$$

$$T_6=\frac{10.9.8.7.6}{5.4.3.2.1}(-32)x^0=-8064$$

Hence

The terms independent of x = -8064

ii) $\left(\sqrt{x}+\frac{1}{2x^2}\right)^{10}$

Solution:

The term independent of x means x^0

So we are find a term free of x.

Let T_{r+1} be the required term. Then

$$T_{r+1}=\binom{n}{r}a^{n-r}b^r=\binom{10}{r}\left(\sqrt{x}\right)^{10-r}\left(-\frac{1}{2x^2}\right)^r$$

$$=\binom{10}{r}x^{\frac{1}{2}(10-r)}\frac{1}{2r}\frac{1}{x^{2r}}$$

$$=\binom{10}{r}\frac{1}{2^r}x^{\left(5-\frac{r}{2}-2r\right)}$$

$$=\binom{10}{r}\frac{1}{2^r}x^{\left(5-\frac{r+4r}{2}\right)}$$

$$=\binom{10}{r}\frac{1}{2^r}x^{\left(5-\frac{5r}{2}\right)} \quad ....(i)$$

For the term involving x^0 , put exponent of x equal to 0

i.e.

$$5-\frac{5r}{2}=0$$
$$\Rightarrow\frac{5r}{2}=5\Rightarrow 5r=10\Rightarrow r=2$$

Put r = 2 in equation (i), we get

$$T_{2+1} = \binom{10}{r} \frac{1}{2^2} x^{5-\frac{5(2)}{2}}$$

$$T_3 = \frac{10.9}{2.1} \frac{1}{4} . x^0 = \frac{90}{8} = \frac{45}{4}$$

Hence

The terms independent of x = $\frac{45}{4}$

iii) $(1+x^2)^3 \left[1+\frac{1}{x^2}\right]^4$

Solution:

$$(1+x^2)^3 \left[1+\frac{1}{x^2}\right]^4$$

$$= (1+x^2)^3 \left[1+\frac{1}{x^2}\right]^4$$

$$= (1+x^2)^3 \frac{(1+x^2)^4}{(x^2)^4} = \frac{1}{x^8}(1+x^2)^7$$

$$= \frac{1}{x}\left(\frac{1+x^2}{x}\right)^7 = \frac{1}{\left(x^{\frac{1}{7}}\right)^7}\left(\frac{1+x^2}{x}\right)^7$$

$$= \left(\frac{1+x^2}{x.x^{\frac{1}{7}}}\right)^7 = \left(\frac{1+x^2}{x^{1+\frac{1}{7}}}\right)^7$$

$$= \left(\frac{1+x^2}{x^{\frac{8}{7}}}\right)^7 = \left(\frac{1}{x^{\frac{8}{7}}} + \frac{x^2}{x^{\frac{8}{7}}}\right)^7$$

$$= \left(\frac{1}{x^{\frac{8}{7}}} + \frac{1}{x^{\frac{8}{7}-2}} \right)^7 = \left(\frac{1}{x^{\frac{8}{7}}} + \frac{x^2}{x^{-\frac{6}{7}}} \right)^7$$

$$= \left(x^{\frac{6}{7}} + \frac{1}{x^{\frac{8}{7}}} \right)^7$$

The term independent of x means x^0. So we are to find a term free of x.

Let T_{r+1} *be the required term.*

Then $$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$= \binom{7}{r} \left(x^{\frac{6}{7}} \right)^{7-r} \left(\frac{1}{x^{\frac{8}{7}}} \right)^r$$

$$= \binom{7}{r} x^{\frac{6}{7}(7-r)} \frac{1}{x^{\frac{8}{7}}}$$

$$= \binom{7}{r} x^{6-\frac{6r}{7}-\frac{8r}{7}} = \binom{7}{r} x^{6-\frac{6r+8r}{7}}$$

$$= \binom{7}{r} x^{6-\frac{14r}{7}} = \binom{7}{r} x^{6-2r} \quad ....(i)$$

For the term involving x^0 , put exponent of x equal to 0.

i.e. $6 - 2r = 0$

$\Rightarrow \quad 2r = 6 \quad \Rightarrow \quad r = 3 \quad \Rightarrow \quad r = 2$

Put r = 2 in equation (i) , we get

$$T_{r+1} = \binom{7}{3} x^{6-2(3)} = \frac{7!}{(7-3)!3!} x^0$$

$$T_4 = \frac{7.6.5.4.3.2.1}{4.3.2.1.3.2.1} = 35$$

Hence

The terms independent of x = 35

Q10. Determine the middle term in the following expansions:

i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$ **ii)** $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$ **iii)** $\left(2x-\frac{1}{2x}\right)^{2m+1}$

i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$

Solution:

The seventh term is the middle term

Thus $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

$$T_{6+1}=\binom{12}{6}\left(\frac{1}{x}\right)^{12-6}\left(-\frac{x^2}{2}\right)^6$$

$$T_7=\frac{12.11.10.9.8.7}{6.5.4.3.2.1}\left(\frac{1}{x}\right)^6\left(-\frac{1}{2}\right)^6.x^{12}$$

$$T_7=\frac{231}{16}x^6$$

Hence

Middle term $(T_7)=\frac{231}{16}x^6$

ii) $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$

Solution:

There are total 11 + 1 = 12 terms in $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$

So 6th and 7th are two middle terms. Now general terms is

$$T_{r+1}=\binom{n}{r}a^{n-r}b^{r}$$

For 6th term, put r = 5

$$T_{5+1}=\binom{11}{5}\left(\frac{3x}{2}\right)^{11-5}\left(-\frac{1}{3x}\right)^{5}$$

$$T_{6}=\frac{11.10.9.8.7}{5.4.3.2.1}\left(\frac{3x}{2}\right)^{6}\left(-\frac{1}{3^{5}x^{5}}\right)$$

$$T_{6}=-462.\frac{3^{6}}{2^{6}}.\frac{1}{3^{5}}x^{6-5}=-\frac{693}{32}x$$

For 7th term, put r = 6

$$T_{6+1}=\binom{11}{6}\left(\frac{3x}{2}\right)^{11-6}\left(-\frac{1}{3x}\right)^{6}$$

$$T_{7}=\frac{11.10.9.8.7}{5.4.3.2.1}\left(\frac{3x}{2}\right)^{5}\left(-\frac{1}{3^{6}x^{6}}\right)$$

$$T_{7}=462\frac{3^{5}x^{5}}{2^{5}}\frac{1}{x^{6}3^{6}}=-\frac{462(3^{5})}{2^{5}.(3^{6})}.\frac{1}{x^{6-5}}=\frac{77}{16}\frac{1}{x}$$

Hence

Middle term = $-\frac{693}{32}x$ and $\frac{77}{16x}$

iii) $\left(2x-\frac{1}{2x}\right)^{2m+1}$

Solution:

There are 2m + 2 terms in $\left(2x-\frac{1}{2x}\right)^{2m+1}$

So $(m+1)^{th}$ and $(m+2)^{th}$ are two middle terms. Now general terms is

$$T_{r+1}=\binom{n}{r}a^{n-r}b^{r}$$

For $(m+1)^{th}$ term, put r = m

$$T_{m+1}=\binom{2m+1}{m}(2x)^{2m+1-m}\left(-\frac{1}{2x}\right)^{m}$$

$$T_{m+1}=\binom{2m+1}{m}2^{m+1}(x)^{m+1}\left(-\frac{1}{2x}\right)^{m}\frac{1}{x^{m}}$$

$$T_{m+1}=\binom{2m+1}{m}2^{m+1}(-1)^{m}\frac{1}{2^{m}}x^{m+1-m}$$

$$T_{m+1}=\binom{2m+1}{m}(-1)^{m}2^{m+1-m}x$$

$$T_{m+1}=\binom{2m+1}{m}(-1)^{m}2.x$$

$$T_{m+1}=2(-1)^{m}\frac{(2m+1)!}{(2m+1-m)!m!}$$

$$T_{m+1}=2(-1)^{m}\frac{(2m+1)!}{m!(m+1)!}$$

And for $(m+2)^{th}$ term, put r = m + 1

$$T_{m+1+1}=\binom{2m+1}{m+1}(2x)^{2m+1-(m+1)}\left(-\frac{1}{2x}\right)^{m+1}$$

$$T_{m+2} = \binom{2m+1}{m}(2x)^{2m+1-m-1}(-1)^{m+1}\frac{1}{2^{m+1}}\frac{1}{x^{m+1}}$$

$$T_{m+2} = \binom{2m+1}{m}2^m x^m(-1)^{m+1}\frac{1}{2^{m+1}}\frac{1}{x^{m+1}}$$

$$T_{m+2} = \binom{2m+1}{m}(-1)^{m+1}\frac{1}{2^{m+1-m}}\frac{1}{x^{m+1-m}}$$

$$T_{m+2} = \binom{2m+1}{m}(-1)^{m+1}\frac{1}{2x}$$

$$T_{m+2} = 2(-1)^{m+1}\frac{(2m+1)!}{m!(m+1)!}\frac{1}{x}$$

Hence

Middle term $\frac{2(-1)^m(2m+1)!}{m!(m+1)!}$ and $\frac{2(-1)^{m+1}(2m+1)!}{m!(m+1)!x}$

Q.11 Find (2n+1)th term from the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$.

Solution:

(2n+1)th term from the end in $\left(x-\frac{1}{2x}\right)^{3n}$ is the (2n+1)th terms

From the beginning in $\left(-\frac{1}{2x}+x\right)^{3n}$

$$T_{r+1} = \binom{n}{r}a^{n-r}b^r$$

For (2n+1)th term, put r = 2n, a = $-\frac{1}{2x}$, b = x

$$T_{2n+1} = \binom{3n}{2n}\left(-\frac{1}{2x}\right)^{3n-2n} x^{2n}$$

$$T_{2n+1} = \binom{3n}{2n}\left(-\frac{1}{2x}\right)^{n} x^{2n}$$

$$T_{2n+1} = \frac{(3n)!}{(3n-n)!n!}(-1)^n \frac{1}{2^n}\frac{1}{x^n} x^{2n}$$

$$T_{2n+1} = \frac{(3n)!}{(2n)!n!}(-1)^n \frac{1}{2^n} x^n = \frac{(-1)^n}{2^n}\frac{(3n)!}{(2n)!n!} x^n$$

Hence

$(2n+1)^{th}$ term $= \dfrac{(-1)^n (3n)!}{2^n (2n)!n!} x^n$

Q.12 Show that the middle term of $(1+x)^{2n}$ is $\dfrac{1.3.5...(2n-1)}{n!} 2^n x^n$.

Solution:

Let the expansion = $(1 + x)^{2n}$

The middle term = $\left[\dfrac{2n}{2}+1\right] = n+1$

$$T_{n+1} = \binom{n}{r} a^{n-r} b^r$$

n = 2n , a = 1 , b = x,

$$T_{n+1} = \binom{2n}{n}(1)^{2n-n}(x)^n$$

$$\Rightarrow T_{n+1} = \binom{2n}{n}(1)^n (x)^n$$

$$T_{n+1} = \binom{2n}{n} x^n$$

$$T_{n+1} = \frac{2n!}{n!(2n-n)!} . x^n$$

$$T_{n+1} = \frac{[2n(2n-1)(2n-2).....3.2.1]}{n!n!} . x^n$$

$$T_{n+1} = \frac{[(2n)(2n-1)(2n-2)...4.2][2(2n-3)...3.1]}{n!n!} . x^n$$

$$T_{n+1} = \frac{[2^n(n)(n-1)(n-2)...2.1][(2n-3)...3.1]}{n!n!} . x^n$$

$$T_{n+1} = \frac{2^n(n!)[(2n-1)(2n-3)...5.3.1]}{n!n!} . x^n$$

$$T_{n+1} = \frac{[1.3.5.....(2n-3)(2n-1)]2^n x^n}{n!}$$

Hence proved

$$Middle\ term = \frac{1.3.5.....(2n-1)}{n!} 2^n x^n$$

Q.13 Show that: $\begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix} + \begin{bmatrix} n \\ 5 \end{bmatrix} + ... + \begin{bmatrix} n \\ n-1 \end{bmatrix} = 2^{n-1}$

Solution:

$$\text{L.H.S} = \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix} + \begin{bmatrix} n \\ 5 \end{bmatrix} + ... + \begin{bmatrix} n \\ n-1 \end{bmatrix}$$

We know that

$$(1+x)^n = \begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix} x + \begin{bmatrix} n \\ 2 \end{bmatrix} x^2 + \begin{bmatrix} n \\ 3 \end{bmatrix} x^3 + ... + \begin{bmatrix} n \\ n \end{bmatrix} x^n \qquad ...(2)$$

Put x = 1 in eq (2)

$$2^n = \begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix} + \ldots\ldots + \begin{bmatrix} n \\ n \end{bmatrix} \qquad \ldots(3)$$

And

Put x = -1 in eq (2)

$$(1-1)^n = \begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix}(-1) + \begin{bmatrix} n \\ 2 \end{bmatrix}(-1)^2 + \begin{bmatrix} n \\ 3 \end{bmatrix}(-1)^3 + \ldots\ldots + \begin{bmatrix} n \\ n \end{bmatrix}(-1)^n$$

$$0 = \begin{bmatrix} n \\ 0 \end{bmatrix} - \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} - \begin{bmatrix} n \\ 3 \end{bmatrix} + \ldots\ldots + (-1)^n \begin{bmatrix} n \\ n \end{bmatrix} \qquad \ldots(4)$$

Subtract eq (4) from (3)

$$2^n = 2\begin{bmatrix} n \\ 1 \end{bmatrix} + 2\begin{bmatrix} n \\ 3 \end{bmatrix} + 2\begin{bmatrix} n \\ 5 \end{bmatrix} + \ldots\ldots + 2\begin{bmatrix} n \\ n-1 \end{bmatrix}$$

$$2^n = 2\begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix} + \begin{bmatrix} n \\ 5 \end{bmatrix} + \ldots\ldots + \begin{bmatrix} n \\ n-1 \end{bmatrix}$$

$$2^{n-1} = \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix} + \begin{bmatrix} n \\ 5 \end{bmatrix} + \ldots\ldots + \begin{bmatrix} n \\ n-1 \end{bmatrix} \qquad \ldots(5)$$

From eq(5)

L.H.S becomes

$$\begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix} + \begin{bmatrix} n \\ 5 \end{bmatrix} + \ldots\ldots + \begin{bmatrix} n \\ n-1 \end{bmatrix} = 2^{n-1}$$

R.H.S

Hence proved

Q.14 Show that: $\begin{bmatrix} n \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} n \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} n \\ 2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} n \\ 3 \end{bmatrix} + \ldots\ldots + \frac{1}{n+1}\begin{bmatrix} n \\ n \end{bmatrix} = \frac{2^{n-1}-1}{n+1}$

Solution:

$$\text{L.H.S} = \begin{bmatrix} n \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} n \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} n \\ 2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} n \\ 3 \end{bmatrix} + ... + \frac{1}{n+1}\begin{bmatrix} n \\ n \end{bmatrix}$$

$$= 1 + \frac{1}{2}n + \frac{1}{3}\frac{n(n-1)}{2!} + \frac{1}{4}\frac{n(n-1)(n-2)}{3!} + ... + \frac{1}{n+1} \quad ...(1)$$

$$= 1 + \frac{1}{1.2} + \frac{n(n-1)}{3.2!} + \frac{n(n-1)(n-2)}{4.3!} + ... + \frac{1}{n+1}$$

$$= 1 + \frac{n}{2!} + \frac{n(n-1)}{3!} + \frac{n(n-1)(n-2)}{4!} + ... + \frac{1}{n+1}$$

$$= \frac{1}{n+1}\left[(n+1) + \frac{n(n+1)}{2!} + \frac{(n+1)n(n-1)}{3!} + \frac{(n+1)n(n-1)(n-2)}{4!} + ... + 1 \right]$$

$$= \frac{1}{n+1}\left[\begin{bmatrix} n+1 \\ 1 \end{bmatrix} + \begin{bmatrix} n+1 \\ 2 \end{bmatrix} + \begin{bmatrix} n+1 \\ 3 \end{bmatrix} + \begin{bmatrix} n+1 \\ 4 \end{bmatrix} + ... + \begin{bmatrix} n+1 \\ n+1 \end{bmatrix} \right] \quad ...(1)$$

Let

$$(1+x)^{n+1} = \begin{bmatrix} n+1 \\ 0 \end{bmatrix} + \begin{bmatrix} n+1 \\ 1 \end{bmatrix}x + \begin{bmatrix} n+1 \\ 2 \end{bmatrix}x^2 + ... + \begin{bmatrix} n+1 \\ n+1 \end{bmatrix}x^2$$

Put x = 1

$$(2)^{n+1} = \begin{bmatrix} n+1 \\ 0 \end{bmatrix} + \begin{bmatrix} n+1 \\ 1 \end{bmatrix} + \begin{bmatrix} n+1 \\ 2 \end{bmatrix} + ... + \begin{bmatrix} n+1 \\ n+1 \end{bmatrix}$$

$$2^{n+1} - 1 = \begin{bmatrix} n+1 \\ 1 \end{bmatrix} + \begin{bmatrix} n+1 \\ 2 \end{bmatrix} + \begin{bmatrix} n+2 \\ 3 \end{bmatrix} + ... + \begin{bmatrix} n+1 \\ n+1 \end{bmatrix}$$

Therefore equation 1 becomes

$$= \frac{1}{n+1}\left[2^{n+1} - 1 \right]$$

$$= \frac{2^{n+1} - 1}{n+1}$$

= R.H.S

Hence proved