Exercise 8.3

1. **Expand the following up to 4 terms taking the values of 'x' such that the expansions in each case.**

i) $(1-x)^{\frac{1}{2}}$

ii) $(1+2x)^{-1}$

iii) $(1+x)^{-\frac{1}{3}}$

iv) $(4-3x)^{\frac{1}{3}}$

v) $(8-2x)^{-1}$

vi) $(2-3x)^{-2}$

vii) $\frac{(1-x)^{-1}}{(1+x)^{2}}$

viii) $\frac{\sqrt{1+2x}}{1-x}$

ix) $\frac{(4+2x)^{\frac{1}{2}}}{2-x}$

x) $(1+x-2x^2)^{\frac{1}{2}}$

xi) $(1-2x+3x^2)^{\frac{1}{2}}$

i. $(1-x)^{\frac{1}{2}}$

Solution:

$(1-x)^{\frac{1}{2}}$

We know that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$(1-x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^3 + ...$$

$$(1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - ....$$

$$Hence, (1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^2}{16} - ....$$

ii. $(1+2x)^{-1}$

Solution

$(1+2x)^{-1}$

We know that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$(1+2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-1-1)}{2!}(2x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!} + ...$$

$$(1+2x)^{-1} = 1 - 2x + 4x^2 - 8x^3 + ...$$

Hence,

$$(1+2x)^{-1} = 1 - 2x + 4x^2 - 8x^3 + ...$$

iii. $(1+x)^{-\frac{1}{3}}$

Solution

$(1+x)^{-\frac{1}{3}}$

We know that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$(1+x)^{-\frac{1}{3}} = 1 + \left(-\frac{1}{3}\right)(x) + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!}(x)^2 + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}(x)^3 + ...$$

$$(1+x)^{-\frac{1}{3}} = 1 - \frac{x}{3} + \frac{2x^2}{9} - \frac{14x^3}{81} + ...$$

Hence

$$(1+x)^{-\frac{1}{3}} = 1 - \frac{x}{3} + \frac{2x^2}{9} - \frac{14x^3}{81} + ...$$

iv. $(4-3x)^{\frac{1}{2}}$

Solution

$$(4-3x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{\frac{1}{2}}$$

$$= 2\left(1-\frac{3x}{4}\right)^{\frac{1}{2}}$$

We know that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$2\left(1-\frac{3x}{4}\right)^{\frac{1}{2}} = 2\left\{1 + \left(\frac{1}{2}\right)\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{3!}\left(-\frac{3x}{4}\right)^3 + ...\right\}$$

$$= 2 - \frac{3x}{4} - \frac{9x^2}{64} - \frac{27x^3}{512} - ....$$

Hence

$$(4-3x)^{\frac{1}{2}} = 2 - \frac{3x}{4} - \frac{9x^2}{64} - \frac{27x^3}{512} - ....$$

v. $(8-2x)^{-1}$

Solution

$$(8-2x)^{-1} = 8^{-1}\left(1-\frac{2x}{8}\right)^{-1} = \frac{1}{8}\left(1-\frac{x}{4}\right)^{-1}$$

We know that

$$(1+8)^{n} = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}+.........$$

$$\frac{1}{8}\left(1-\frac{x}{4}\right)^{-1} = \frac{1}{8}\left\{1+(-1)\left(-\frac{x}{4}\right)+\frac{(-1)(-1-1)}{2!}\left(-\frac{x}{4}\right)^2+\frac{(-x)(-1-1)(-1-2)}{3!}\left(-\frac{x}{4}\right)^3+.........\right\}$$

$$\frac{1}{8}\left(1-\frac{x}{4}\right)^{-1} = \frac{1}{8}\left\{1+\frac{x}{4}+\frac{x^2}{16}+\frac{x^3}{64}+.....\right\}$$

$$\frac{1}{8}\left(1-\frac{x}{4}\right)^{-1} = \frac{1}{8}+\frac{x}{32}+\frac{x^2}{128}+\frac{x^3}{512}+......$$

Hence

$$(8-2x)^{-1} = \frac{1}{8}+\frac{x}{32}+\frac{x^2}{128}+\frac{x^3}{512}+......$$

vi. $(2-3x)^{-2}$

Solution

$$(2-3x)^{-2} = 2^{-2}\left(1-\frac{3x}{2}\right)^{-2} = \frac{1}{4}\left(1-\frac{3x}{2}\right)^{-2}$$

We know that

$$(1+x)^{n} = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+.........$$

$$\frac{1}{4}\left(1-\frac{3x}{2}\right)^{-2} = \frac{1}{4}\left\{1+(-2)\left(-\frac{3x}{2}\right)+\frac{(-2)(-2-1)}{2!}\left(-\frac{3x}{2}\right)^2+\frac{(-2)(-2-1)(-2-2)}{3!}\left(-\frac{3x}{2}\right)^3+........\right\}$$

$$\frac{1}{4}\left(1-\frac{3x}{2}\right)^{-2} = \frac{1}{4}\left\{1+3x+\frac{27x^2}{4}+\frac{27x^3}{2}+.......\right\}$$

$$\frac{1}{4}\left(1-\frac{3x}{2}\right)^{-2} = \frac{1}{4}+\frac{3x}{4}+\frac{27x^2}{16}+\frac{27x^3}{8}+......$$

Hence

$$(2-3x)^{-2} = \frac{1}{4}+\frac{3x}{4}+\frac{27x^2}{16}+\frac{27x^3}{8}+......$$

vii. $\dfrac{(1-x)^{-1}}{(1+x)^{2}}$

Solution

$$\frac{(1-x)^{-1}}{(1+x)^{2}} = (1-x)^{-1}(1+x)^{-2}$$

We know that

$$(1+x)^{n} = 1 + nx + \frac{n(n-1)}{2!}x^{2} + \frac{n(n-1)(n-2)}{3!}x^{3} + \dots\dots\dots$$

$$\frac{(1-x)^{-1}}{(1+x)^{2}} = (1-x)^{-1}(1+x)^{-2}$$

$$(1-x)^{-1}(1+x)^{2} = \left\{1 + (-1)(-x)\frac{(-1)(-1-1)}{2!}(-x)^{2} + \frac{(-1)(-1-1)(-1-2)}{3!}(-x)^{3} + \dots\dots\right\}$$

$$\left\{1 + (-2)(-x)\frac{(-2)(-2-1)}{2!}x^{2} + \frac{(-2)(-2-1)(-2-3)}{3!}x^{3} + \dots\right\}$$

$$= \{1 + x + x^{2} + \dots\dots\}\{1 - 2x + 3x^{2} - 4x^{3} + \dots\dots\}$$

$$= 1 + 2x + x + 3x^{2} - x^{2} - 4x^{3} + 3x^{3} - 2x^{3} + x^{3} + \dots\dots\dots$$

$$= 1 - x + 2x^{2} - 2x^{3} + \dots\dots\dots$$

Hence

$$\frac{(1-x)^{-1}}{(1+x)^{2}} = 1 - x + 2x^{2} - 2x^{3} + \dots\dots\dots$$

viii. $\dfrac{\sqrt{1+2x}}{1-x}$

Solution

$$\frac{\sqrt{1+2x}}{1-x} = (1+2x)^{\frac{1}{2}}(1-x)^{-1}$$

We know that

$$(1+x)^{n} = 1 + nx + \frac{n(n-1)}{2!}x^{2} + \frac{n(n-1)(n-2)}{3!}x^{3} + \frac{\sqrt{1+2x}}{1-x}$$

$$= (1+2x)^{\frac{1}{2}}(1-x)^{-1}$$

$$(1+2x)^{\frac{1}{2}}(1-x)^{-1}=\left\{1+\frac{1}{2}(2x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(2x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(2x)^3+......\right\}\times$$

$$\left\{1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+.....\right\}$$

$$=\left\{1+x+\frac{x^2}{2}+\frac{x^3}{2}+......\right\}\left\{1+x+x^2+x^3+.....\right\}$$

$$(1+2x)^{\frac{1}{2}}(1-x)^{-1}=1+x+x+x^2+x^2-\frac{x^2}{2}+\frac{x^2}{2}+x^3+x^3+\frac{x^3}{2}+........$$

$$(1+2x)^{\frac{1}{2}}(1-x)^{-1}=1+2x+\frac{3x^2}{2}+2x^3+......$$

Hence

$$\frac{\sqrt{1+2x}}{1-x}=1+2x+\frac{3x^2}{2}+2x^3+......$$

ix. $\dfrac{(4+2x)^{\frac{1}{2}}}{2-x}$

Solution

$$\frac{(4+2x)^{\frac{1}{2}}}{2-x}=(4+2x)^{\frac{1}{2}}(2-x)^{-1}$$

We know that

$$(1+x)^n=1+2x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+.........$$

$$\frac{(4+2x)^{\frac{1}{2}}}{2-x}=(4+2x)^{\frac{1}{2}}(2-x)^{-1}$$

$$(4+2x)^{\frac{1}{2}}(2-x)^{-1}=\left[4\left(1+\frac{x}{2}\right)\right]^{\frac{1}{2}}\left[2\left(1-\frac{x}{2}\right)\right]^{-2}$$

$$=(2^2)^2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}.2^{-1}\left(1-\frac{x}{2}\right)^{-1}$$

$$= 2.2^{-1}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$(4+2x)^{\frac{1}{2}}(2-x)^{-1} = \left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1} = \left\{1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3+.....\right\}$$

$$= \left\{1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3+....\right\}$$

$$= \left\{1+\frac{x}{2^2}+\frac{x^2}{2^5}+\frac{x^3}{2^7}+....\right\}\times\left\{1+\frac{x}{2}+\frac{x^2}{2^2}+\frac{x^3}{2^3}+....\right\}$$

$$= 1+\frac{x}{2}+\frac{x}{2^2}+\frac{x^2}{2^2}+\frac{x^2}{2^3}-\frac{x^2}{2^5}+\frac{x^3}{2^3}+\frac{x^3}{2^4}-\frac{x^3}{2^6}+\frac{x^3}{2^7}+......$$

$$= 1+\frac{2x+x}{2^2}+\frac{2^3x^2+2^2x^2-x^2}{2^5}+\frac{2^4x^3+2^3x^3-2x^3+x^3}{2^7}+.....$$

$$= 1+\frac{3}{2^2}x+\frac{11}{2^5}x^2+\frac{23}{2^7}x^3+......$$

$$\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1} = 1++\frac{3}{4}x+\frac{11}{32}x^2+\frac{23}{128}x^3+......$$

Hence

$$\frac{(4+2x)^{\frac{1}{2}}}{2-x} = 1++\frac{3}{4}x+\frac{11}{32}x^2+\frac{23}{128}x^3+......$$

x. $(1+x-2x^2)^{\frac{1}{2}}$

Solution

$$(1+x-2x^2)^{\frac{1}{2}} = \left[1+\left(x-2x^2\right)\right]^{\frac{1}{2}}$$

We know that

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+.........$$

$$\left[1+\left(x-2x^2\right)\right]^{\frac{1}{2}} = 1+\frac{1}{2}\left(x-2x^2\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(x-2x^2\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(x-2x^2\right)^3+.....$$

$$= 1 + \frac{1}{2}x - x^2 - \frac{1}{2^3}\left(x^2 - 4x^3 + 4x^4\right) + \frac{1}{2^4}\left(x^3 - 6x^4 + 12x^5 - 8x^6\right) + ...$$

$$= 1 + \frac{1}{2}x - x^2 - \frac{x^2}{8} + \frac{1}{2}x^3 - \frac{x^4}{2} - \frac{x^3}{16} - \frac{3x^4}{8} + ...$$

$$\left[1 + \left(x - 2x^2\right)\right]^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{9}{8}x^2 + \frac{9}{16}x^3 - \frac{7}{8}x^4 + ...$$

Hence

$$\left(1 + x - 2x^2\right)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{9}{8}x^2 + \frac{9}{16}x^3 + ...$$

xi. $\left(1 - 2x + 3x^2\right)^{\frac{1}{2}}$

Solution

$$\left(1 - 2x + 3x^2\right)^{-\frac{1}{3}} = \left[1 - \left(2x - 3x^2\right)\right]^{-\frac{1}{3}}$$

We know that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$\left[1 - \left(2x - 3x^2\right)\right]^{-\frac{1}{3}} = 1 + \left(-\frac{1}{3}\right)\left(-\left(2x - 3x^2\right)\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3} - 1\right)}{2!}$$

$$\left(-\left(2x - 3x^2\right)\right) + \frac{\left(\frac{-1}{3}\right)\left(-\frac{1}{3} - 1\right)\left(-\frac{1}{3} - 2\right)}{3!}\left[-(2x - 3x)\right]^3 + ...$$

$$= 1 + \frac{1}{3}\left(2x - 3x^2\right) + \frac{2}{9}\left(4x^2 - 12x^3 + 9x^4\right) + \frac{14}{81}\left(8x^3 - 36x^4 + 54x^5 - 27x^6\right) + ...$$

$$\left[1 - \left(2x - 3x^2\right)\right]^{-\frac{1}{3}} = 1 + \frac{2}{3}x + \left[-1 + \frac{8}{9}\right]x^2 + \left[-\frac{8}{3} + \frac{112}{81}\right]x^3 + ...$$

$$\left[1 - \left(2x - 3x^2\right)\right] = 1 + \frac{2}{3}x - \frac{1}{9}x^2 - \frac{104}{81}x^3 + ...$$

Hence

$$\left(1 - 2x + 3x^2\right)^{\frac{1}{3}} = 1 + \frac{2}{3}x - \frac{1}{9}x^2 - \frac{104}{81}x^3 + ...$$

2. Using Binomial theorem find the value of the following to the three places of decimals.

1 – $\sqrt{99}$

2 – $(0.98)^{\frac{1}{2}}$

3 – $(1.03)^{\frac{1}{3}}$

4 – $3\sqrt{65}$

5 – $4\sqrt{17}$

6 – $5\sqrt{31}$

7 – $\dfrac{1}{3\sqrt{998}}$

8 – $\dfrac{1}{5\sqrt{525}}$

9 – $\dfrac{\sqrt{7}}{\sqrt{8}}$

10 – $(0.998)^{\frac{1}{3}}$

11 – $\dfrac{1}{6\sqrt{486}}$

12 – $(1280)^{\frac{1}{4}}$

1. $\sqrt{99}$

Solution

$$(99)^{\frac{1}{2}} = (100-1)^{\frac{1}{2}}$$

$$= \left[100\left(1-\frac{1}{100}\right)\right]^{\frac{1}{2}}$$

$$= 10\left(1-\frac{1}{100}\right)^{\frac{1}{2}}$$

$$= 10\left\{1+\frac{1}{2}\left(-\frac{1}{100}\right)+\frac{2\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{100}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{1}{100}\right)^3+\dots\right\}$$

$$= 10\left\{1-\frac{1}{200}-\frac{1}{80000}-\frac{1}{16000000}-\dots\right\}$$

$$= 10\{1-0.005\dots\}$$

$$= 10(0.995)$$

$= 9.95$ approximately

Hence

$$\sqrt{99} = 9.95$$

2. $(0.98)^{\frac{1}{2}}$

Solution

$(0.98)^{\frac{1}{2}} = (1-0.02)^{\frac{1}{2}}$

$= 1+\frac{1}{2}(-0.02)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-0.02)^2+....$

$= 1-0.01-\frac{1}{8}(0.0004)-....$

$= 1-0.01-0.0001-0-.....$

$= 0.99$ *approximately*

Hence

$(0.98)^{\frac{1}{2}} = 0.99$

3. $(1.03)^{\frac{1}{3}}$

Solution

$(1.03)^{\frac{1}{3}} = (1+0.03)^{\frac{1}{3}}$

$= 1+\frac{1}{3}(0.03)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.03)^2+......$

$= 1+0.-1-\frac{1}{9}(0.009)+......$

$= 1+0.01-0.0001+......$

$= 1.00099(\text{approximately})$

Hence

$(1.03)^2 = 1.0009 \cong 1.001$

4. $3\sqrt{65}$

Solution

$3\sqrt{65} = (65)^{\frac{1}{3}} = (64+1)^{\frac{1}{3}}$

$$= \left[64\left(1+\frac{1}{64}\right)\right]^{\frac{1}{3}}$$

$$= 4\left(1+\frac{1}{64}\right)^{\frac{1}{3}}$$

$$= 4\left\{1+\frac{1}{3}\left(\frac{1}{64}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{64}\right)^2+.....\right\}$$

$$= 4\left\{1+0.005-\frac{1}{9}\left(\frac{1}{4096}\right)+......\right\}$$

$$= 4(1+0.005-0+......)$$

$$4(1.005)$$

$$= 4.020(aproximately)$$

Hence

$$3\sqrt{65} = 4.020$$

5. $4\sqrt{17}$

Solution

$$4\sqrt{17} = (17)^{\frac{1}{4}} = (16+1)^{\frac{1}{4}}$$

$$= \left[16\left(1+\frac{1}{16}\right)\right]^{\frac{1}{4}}$$

$$= (2^2)^{\frac{1}{4}}\left(1+\frac{1}{16}\right)^{\frac{1}{4}}$$

$$= 2\left[1+\frac{1}{4}\left(\frac{1}{16}\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(\frac{1}{16}\right)^2+.....\right]$$

$$= 2\left[1+0.016-\frac{3}{32}\left(\frac{1}{256}\right)+....\right]$$

$= 2[1 + 0.016 - 0 + ...]$

$= 2(1.016)$

$= 2.032(\text{approximately})$

Hence

$4\sqrt{17} = 2.032$

6. $5\sqrt{31}$

Solution

$5\sqrt{31} = (31)^{\frac{1}{5}} = (32-1)^{\frac{1}{5}}$

$= \left(32\left(1 - \frac{1}{32}\right)\right)^{\frac{1}{5}}$

$= (2^5)^{\frac{1}{5}}\left(1 - \frac{1}{32}\right)^{\frac{1}{5}}$

$= 2\left(1 + \frac{1}{5}\left(-\frac{1}{32}\right) + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2!}\left(-\frac{1}{32}\right)^2 + ...\right)$

$= 2\left(1 - \frac{1}{160} - \frac{2}{25}\left(\frac{1}{1024}\right) - ....\right)$

$= 2(1 - 0.006 - ......)$

$= 2(1 - 0.006)$

$= 2(0.994) = 1.988(\textit{approximately})$

Hence

$5\sqrt{31} \cong 1.988$

7. $\frac{1}{3\sqrt{998}}$

Solution

$$\frac{1}{3\sqrt{998}} = (998)^{-\frac{1}{3}} = (1000-2)^{\frac{1}{3}}$$

$$= \left(1000\left(1-\frac{2}{1000}\right)\right)^{-\frac{1}{3}}$$

$$= \left(10^3\right)^{-\frac{1}{3}}(1-0.002)^{-\frac{1}{3}}$$

$$= 10^{-1}\left[1+\left(-\frac{1}{3}\right)(-0.002)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(-0.002)^2+\ldots..\right]$$

$$= \frac{1}{10}\left[1+0.001+\frac{2}{9}(0)+\ldots..\right]$$

$$= \frac{1}{10}\left[1+0.001+\frac{2}{9}(0)+\ldots..\right]$$

$$= \frac{1}{10}(1+0.01) = 0.101(\textit{approximtely})$$

Hence

$$\frac{1}{3\sqrt{998}} \cong 0.101$$

8. $\frac{1}{5\sqrt{998}}$

Solution

$$\frac{1}{5\sqrt{998}} = \frac{1}{(252)^{\frac{1}{5}}}$$

$$= (243+9)^{-\frac{1}{5}}$$

$$= \left[243\left(1+\frac{9}{243}\right)\right]^{-\frac{1}{5}}$$

$$= \left(3^5\right)^{-\frac{1}{5}}[1+0.037]^{\frac{1}{5}}$$

$$3^{-1}\left[1+\left(-\frac{1}{5}\right)(0.037)+\frac{\left(\frac{-1}{5}\right)\left(-\frac{1}{5}-1\right)}{2!}(0.037)^2+\ldots\ldots.\right]$$

$$= \frac{1}{3}\left[1 - 0.007 + \frac{3}{25}(0.001) + ...\right]$$

$$= \frac{1}{3}[1 - 0.007 + 0 + ...]$$

$$= \frac{1}{3}[0.993] = 0.331(\textit{approxiately})$$

Hence

$$\frac{1}{5\sqrt{998}} \cong 0.331$$

9. $\dfrac{\sqrt{7}}{\sqrt{8}}$

Solution

$$\frac{\sqrt{7}}{\sqrt{8}} = \frac{\sqrt{7}}{\sqrt{8}} = \sqrt{1 - \frac{1}{8}} = \left(1 - \frac{1}{8}\right)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}\left(-\frac{1}{8}\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(-\frac{1}{8}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3!}\left(-\frac{1}{8}\right)^3 + ...$$

$$= 1 - \frac{1}{16} - \frac{1}{8}\left(\frac{1}{64}\right) - \frac{1}{16}\left(\frac{1}{512}\right) + ...$$

$$= 1 - 0.063 - 0.002 - 0 + ...$$

$$= 0.935(\textit{approximately})$$

Hence

$$\frac{\sqrt{7}}{\sqrt{8}} = 0.935$$

10. $(0.998)^{-\frac{1}{3}}$

Solution

$$(0.998)^{-\frac{1}{3}} = (1-0.002)^{-\frac{1}{3}}$$

$$= 1 + \left(-\frac{1}{3}\right)(-0.002) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(-0.002)^2 + ...$$

$$= 1 + 0.001 + \frac{2}{9}(0) + ...$$

$$= 1 + 0.001 + 0 + ...$$

$$= 1.001(aproximately)$$

11. $\frac{1}{6\sqrt{486}}$

Solution

$$\frac{1}{6\sqrt{486}}$$

$$= \frac{1}{(486)^{\frac{1}{6}}} = (486)^{-\frac{1}{6}}$$

$$= (729-243)^{-\frac{1}{6}}$$

$$= (729)^{-\frac{1}{6}}\left[1-\frac{243}{729}\right]^{-\frac{1}{6}}$$

$$= (3^6)^{-\frac{1}{6}}\left[1-\frac{1}{3}\right]^{-\frac{1}{6}}$$

$$= (3)^{-1}\left[1-\frac{1}{3}\right]^{-\frac{1}{6}} = \frac{1}{3}\left[1-\frac{1}{3}\right]^{-\frac{1}{6}}$$

We know that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$\frac{1}{3}\left[1-\frac{1}{3}\right]^{-\frac{1}{6}}=\frac{1}{3}\left[1+\left(-\frac{1}{6}\right)\left(-\frac{1}{3}\right)+\frac{\frac{1}{6}\left(-\frac{1}{6}-1\right)}{2!}\left(-\frac{2}{3}\right)^2+\frac{-\frac{1}{6}\left(-\frac{1}{6}-1\right)}{3!}\left(-\frac{1}{3}\right)^3+........\right]$$

$$=\frac{1}{3}\left[1+\frac{1}{18}+\frac{7}{648}+\frac{91}{34392}+....\right]$$

$$=\frac{1}{3}(1+0.0555+0.0108+0.00265+....)$$

$$=\frac{1}{3}[1.06895]=0.3563(apriximately)$$

Hence

$$\frac{1}{6\sqrt{486}}\cong 0.3563$$

12. $(1280)^{\frac{1}{4}}$

Solution

$$(1280)^{\frac{1}{4}}$$

$$=(1296-16)^{\frac{1}{4}}$$

$$=(1296)^{\frac{1}{4}}\left(1-\frac{16}{1296}\right)^{\frac{1}{4}}$$

$$=\left(6^4\right)^{\frac{1}{4}}\left\{1-\frac{1}{81}\right\}^{\frac{1}{4}}=6\left\{1-\frac{1}{81}\right\}^{\frac{1}{4}}$$

Using the formula

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+.........$$

Then

$$6\left[1-\frac{1}{81}\right]^{\frac{1}{4}}$$

$$=6\left[1+\frac{1}{4}\left(-\frac{1}{81}\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(-\frac{1}{81}\right)^2+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}\left(-\frac{1}{81}\right)^3+........\right]$$

$$=6\left[1-\frac{1}{324}-\frac{1}{69384}-\frac{7}{68024448}-......\right]$$

$$=6\left[1-0.003084-0-0-.....\right]$$

$$=6\left[0.99316\right]=5.981496(aproximately)$$

Hence

$$(1280)^{\frac{1}{4}}\cong 5.98$$

3. Find the coefficient of x^n the expansion of

i) $\frac{1+x^2}{(1+x)^2}$

ii) $\frac{(1+x)^2}{(1-x)^2}$

iii) $\frac{(1+x)^3}{(1-x)^2}$

iv) $\frac{(1+x)^2}{(1-x)^3}$

v) $\left(1-x+x^2+........\right)^2$

i. $\frac{1+x^2}{(1+x)^2}$

Solution

$\frac{1+x^2}{(1+x)^2}=(1+x^2)(1+x)^{-2}$

$=(1+x^2)\left[1+(-2)x+\frac{(-2)(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-2-2)}{3!}x^3+.......\right]$

$=(1+x^2)\left[1-2x+3x^2-4x^3+........\right]$

$=(1+x^2)\{1-2x+3x^2-4x^3+.....$

$+(-1)^{n-2}(n-1)x^{n-2}+(-1)^{n-1}nx^{n-1}$

$+(-1)^n(n+1)x^n+.......\}$

Now the team involving x^n is

$(-1)^n(n+1)x^n+(-1)^{n-2}(n-1)x^n$

$=(-1)^n\left[n+1+(-1)^{-2}(n-1)\right]x^n$

$=(-1)^n[n+1+n-1]x^n=(-1)^n[2n]x^n$

The coefficient of $x^n=(-1)^n[2n]$

Hence,

ii. $\frac{(1+x)^2}{(1-x)^2}$

Solution

$\frac{(1+x)^2}{(1-x)^2}=(1+x)^2(1-x)^{-2}$

$=(1+2x+x^2)\left[1+(-2)(-x)+\frac{(-2)(-2-1)}{2!}(-x^2)+\frac{(-2)(-2-1)(-2-2)}{3!}(-x^3)+......\right]$

$=(1+2x+x^2)\left[1+2x+3x^2+4x^3+......\right]$

$=(1+2x+x^2)\left[1+2x+3x^2+4x^3+......\right]$

$+(n-1)x^{n-2}+nx^{n-1}+(n+1)x^n+.........]$

Now the term involving x^n is $(n+1)x^n+2nx^n+(n-1)x^n$

$=[(n+1)+2n+n-1]$

$=[4n]x^n$

Hence the turm involving $x^n=4n$

iii. $\dfrac{(1+x)^3}{(1-x)^2}$

Solution

$$\frac{(1+x)^3}{(1-x)^2}=(1+x)^3(1-x)^{-2}$$

$$=\left(1+3x+3x^2+x^3\right)$$

$$\left[1+(-2)(-x)\frac{(-2)(-2-1)}{2!}(-x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3+.......\right]$$

$$=\left(1+3x+3x^2+x^3\right)\left[1+2x+3x^2+4x^3+.....\right]$$

$$=\left(1+3x+3x^2+x^3\right)[1+2x+3x^2+4x^3+.....$$

$$+(n-2)x^{n-3}+(n-1)x^{n-2}+x^{n-1}+(n-2)x^n+........]$$

Now the term involving x^n **is**

$$(n+1)x^n+3nx^n+3(n-1)x^n+(n-2)x^n$$

$$=[n+1+3n+3n-3+n-2]x^n$$

$$=[8n-4]x^n$$

$$=4[2n-1]x^n$$

Hence the coefficient of x^n **is** $4(2n-1)$

iv. $\dfrac{(1+x)^2}{(1-x)^3}$

Solution

$$\frac{(1+x)^2}{(1-x)^3}=(1+x)^2(1-x)^3$$

$$=\left(1-2x+x^2\right)\left[1+(-3)(-x)+\frac{(-3)(-3-1)}{2!}(-x)^2+\frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3+.....\right]$$

$$=\left(1-2x+x^2\right)\left[1+\frac{3}{1}x+\frac{3.4}{2!}x^2+\frac{3.4.5}{3!}x^3+.....\right]$$

$$=\left(1-2x+x^2\right)\left[1+\frac{2.3}{2}x+\frac{2.3.4}{2.2!}x^2+\frac{2.3.4.5}{2.3!}x^3+.....\right]$$

$$=\left(1-2x+x^2\right)\left[1+\frac{3!}{2}x+\frac{4!}{2.2!}x^2+\frac{5!}{2.3!}x^3+.......\right]$$

$$=\left(1-2x+x^2\right)\left[1+\frac{3!}{2}x+\frac{4!}{2.2!}x^2+\frac{5!}{2.3!}x^3+.......\right]$$

$$=+\frac{n!}{2(n-2)}x^{n-2}+2.\frac{(n+1)!}{2(n-1)!}x^{n-1}+\frac{(n+2)!}{2n!}x^n+......$$

Now the term involving x^n is

$$=+\frac{(n+2)!}{2.n!}x^n+2.\frac{(n+1)!}{(n-1)!}x^n+\frac{n!}{2.(n-2)!}x^n$$

$$=\left[\frac{(n+2)(n+1)n!}{2.n!}+\frac{(n+1)n(n-1)!}{(n-1)!}-\frac{n(n-1)(n-2)!}{2.(n-2)!}\right]x^n$$

$$=\left[\frac{(n+2)(n+1)n!}{2}+\frac{(n+1)n(n-1)!}{1}-\frac{n(n-1)(n-2)!}{2}\right]x^n$$

$$=\frac{1}{2}\left[\left(n^2+n+2n+2\right)+2\left(n^2+n\right)+\left(n^2-n\right)\right]x^n$$

$$=\frac{1}{2}\left[\left(n^2+3n+2+2n^2+2n+n^2-n\right)\right]x^n$$

$$=\frac{1}{2}\left[4n^2+4n+2\right]x^n=\left[2n^2+2n+1\right]x^n$$

Hence the coefficient of $x^n=2n^2+2n+1$

v. $\left(1-x+x^2-x^3+........\infty\right)^2$

Solution

$$\left(1-x+x^2-x^3+........\infty\right)=(1+x)^{-1}$$

$$\therefore\left(1-x+x^2-x^3+........\infty\right)^2=(1+x)^{-2}$$

Now x^n occurs in

$$=\left(1+2x+x^2\right)\left[1+\frac{2.3}{2}x+\frac{2.3.4}{2.2!}x^2+\frac{2.3.4.5}{2.3!}x^3+......\right]$$

$$=\left(1+2x+x^2\right)\left[1+\frac{3!}{2}x+\frac{4!}{2.2!}x^2+\frac{5!}{2.3!}x^3+......\right]$$

$$=\left(1+2x+x^2\right)[1+\frac{3!}{2}x+\frac{4!}{2.2!}x^2+\frac{5!}{2.3!}x^3+......$$

$$\ldots\ldots\ldots\frac{n!}{2(n-2)}x^{n-2}+2.\frac{(n+1)!}{2(n-1)!}x^{n-1}+\frac{(n+2)!}{2n!}x^{n}+]$$

Q4. If x is so small that its square and higher powers can be neglected, then show that

Solution

i) $\frac{1-x}{\sqrt{1-x}}\approx 1-\frac{3}{2}x$

ii) $\frac{\sqrt{1+2x}}{\sqrt{1-x}}\approx 1+\frac{3}{2}x$

iii) $\frac{(9+7x)^{\frac{1}{2}}-(4-3x)^{\frac{1}{4}}}{4+5}\approx\frac{1}{4}-\frac{17}{384}$

iv) $\frac{\sqrt{4+x}}{(1-x)^{3}}\approx 2+\frac{25}{4}x$

v) $\frac{(1+x)^{1/2}(9-4x)^{3/2}}{(8+5x)^{1/3}}\approx 4\left[1=\frac{5x}{6}\right]$

vi) $\frac{(1-x)^{1/2}(9-4x)^{3/2}}{(8+3x)^{1/3}}\approx 4\frac{3}{2}-\frac{61}{48}x$

vii) $\frac{\sqrt{4-x}+(8-x)^{1/3}}{(8-x)^{1/3}}\approx 2-\frac{1}{12}x$

i. $\frac{1-x}{\sqrt{1-x}}\approx 1-\frac{3}{2}x$

Solution

L.H.S $\frac{1-x}{\sqrt{1-x}}=(1-x)(1-x)^{-1/2}$

$$=(1-x)\left[1+\left(-\frac{1}{2}\right)(x)+\frac{\left(-\frac{1}{2}\right)\left(1\frac{1}{2}-1\right)}{2!}x^2+.....\right]$$

$=(1-x)\left(1-\frac{x}{2}\right)$; Neglecting x^2 and higher power of x

$=1-\frac{x}{2}-x+\frac{x^2}{2}$

$=1-\frac{x+2x}{2}$; Neglecting x^2

$=1-\frac{3x}{2}$

$=$ R.H.S

Hence proved

$\frac{1-x}{\sqrt{1-x}}\approx 1-\frac{3}{2}x$

ii. $\frac{\sqrt{1+2x}}{\sqrt{1-x}}\approx 1+\frac{3}{2}x$

Solution

L.H.S $\frac{\sqrt{1+2x}}{\sqrt{1-x}}=(1+2x)^{1/2}(1-x)^{-1/2}$ ....(i)

Take

$$(1+2x)^{1/2}=\left[1+2\frac{1}{2}(2x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(2x)^2+...\right]$$

$=[1+x]$; **Neglecting** x^2 **and higher powers of x**

Also $(1-x)^{-1/2} = \left[1+\left(-\frac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(2x)^2+.....\right]$

$=\left[1+\frac{x}{2}\right]$; **Neglecting** x^2 **and higher powers of x**

Putting in equation (i) we get

$$=1+[1+x]\left[1+\frac{x}{2}\right]$$

L.H.S

$$=1+\frac{x}{2}+x+\frac{x^2}{2}$$

$$=1+\frac{x+2x}{2}; \text{Neglecting } x^2$$

$$=1+\frac{3x}{2}=\text{R.H.S}$$

Hence proved

$$\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1+\frac{3}{2}x$$

iii. $\frac{(9+7x)^{\frac{1}{2}}-(4-3x)^{\frac{1}{4}}}{4+5} \approx \frac{1}{4}-\frac{17}{384}$

Solution

L.H.S

$$=\frac{(9+7x)^{\frac{1}{2}}-(4-3x)^{\frac{1}{4}}}{4+5}$$

$$=\left[(9+7x)^{\frac{1}{2}}-(4-3x)^{\frac{1}{4}}\right][4+5]^{-1}\ ....(i)$$

Take $(9+7x)^{1/2}=\left[9\left(1+\frac{7x}{9}\right)\right]^{\frac{1}{2}}$

$$= \left(3^2\right)^{\frac{1}{2}}\left[1+\frac{7x}{9}\right]^{\frac{1}{2}}$$

$$= 3\left[1+2\left(\frac{7x}{9}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{7x}{9}\right)^2+\ldots\ldots\right]$$

$$= 3\left[1+\frac{7x}{18}\right]$$ **Neglecting x^2 and higher powers of x**

$$= 3+\frac{7x}{6}$$

Also $(16+3x)^{\frac{1}{4}} = \left[16\left(1+\frac{3x}{16}\right)\right]^{\frac{1}{4}} = \left(2^4\right)^{\frac{1}{4}}\left(1+\frac{3x}{16}\right)^{\frac{1}{4}}$

$$= 2\left[1+\frac{1}{4}\left(\frac{3x}{16}\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(\frac{3x}{16}\right)+\ldots\ldots\right]$$

$$= 2\left[1+\frac{3x}{64}\right]$$ **Neglecting x^2 and higher powers of x**

$$= 2+\frac{3x}{32}$$

$$(4+5x)^{-1} = \left[4\left(1+\frac{5x}{4}\right)\right]^{-1} = 4^{-1}\left(1+\frac{5x}{4}\right)^{-1}$$

$$= \frac{1}{4}\left[1+(-1)\left(\frac{5x}{4}\right)+\frac{(-1)(-1-1)}{2!}\left(\frac{5x}{4}\right)^2+\ldots\ldots\right]$$

$$= \frac{1}{4}\left[1-\left(\frac{5x}{4}\right)\right]$$ **Neglecting x^2 and higher powers of x**

$$= \frac{1}{4}\left(\frac{1}{4}\right)-\frac{5x}{16}$$

Putting all these in equation (i), we get

L.H.S

$$=\left[\left(3+\frac{7x}{6}\right)-\left(2+\frac{3x}{32}\right)\right]\left(\frac{1}{4}-\frac{5x}{16}\right)$$

$$=\left[3+\frac{7x}{6}-2\frac{3x}{32}\right]\left[\frac{1}{4}-\frac{5x}{16}\right]$$

$$=\left[1+\frac{112x-9x}{96}\right]\left[\frac{1}{4}-\frac{5x}{16}\right]$$

$$=\left[1+\frac{103}{96}x\right]\left[\frac{1}{4}-\frac{5x}{16}\right]$$

$$=\frac{1}{4}\left(\frac{5x}{16}\right)+\frac{103}{384}x-\frac{515}{1536}x^2$$

$$=\frac{1}{4}-\frac{120x-103x}{384};\ \text{Neglecting } x^2$$

$$=\frac{1}{4}-\frac{17}{384}x=\text{R.H.S}$$

iv. $\dfrac{\sqrt{4+x}}{(1-x)^3}\approx 2+\dfrac{25}{4}x$

Solution

L.H.S $=\dfrac{\sqrt{4+x}}{(1-x)^3}=(4+x)^{1/2}(1-x)^{-3}$ ....(i)

Take $(4+x)^{1/2}=\left[4\left(1+\frac{x}{4}\right)\right]^{\frac{1}{2}}=(2^2)^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$

$$=2\left[1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{4}\right)^2+\ldots\ldots\right]$$

$=2\left[1+\left(\frac{x}{8}\right)\right]$; **Neglecting** x^2 **and higher powers of x**

$$=2+\frac{x}{4}$$

$$also \quad (1-x)^{-3} = \left[1+(-3)(-x)+\frac{(-3)(-1)}{2!}(-x)^2+.....\right]$$

$=\left[1+3x\right]$ **Neglecting** x^2 **and higher powers of x**

Putting all these in equation (i), we get

L.H.S $=\left[2+\frac{x}{4}\right]\left[1+3x\right]$

$$=2+6x+\frac{x}{4}+\frac{3}{4}x^2$$

$$=2+\frac{24x+x}{4} \qquad \text{Neglecting } x^2$$

$=2+\frac{25}{4}$ **= R.H.S**

v. $\dfrac{(1+x)^{\frac{1}{3}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}} \approx 4\left[1-\frac{5x}{6}\right]$

Solution:

$$L.H.S = \frac{(1+x)^{\frac{1}{3}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}}$$

$$=(1+x)^{\frac{1}{2}}\left[4\left(1-\frac{3x}{4}\right)\right]^{\frac{3}{2}}(8+5x)^{\frac{-1}{3}}$$

$$=(1+x)^{\frac{1}{2}}\left[4\left(1-\frac{3x}{4}\right)\right]^{\frac{3}{2}}\left[8\left(1+\frac{5x}{8}\right)\right]\frac{-1}{3}$$

$$=(1+x)^{\frac{1}{2}}(8)\left(1-\frac{3x}{4}\right)^{\frac{3}{2}}(8)^{\frac{-1}{3}}\left[1+\frac{5x}{8}\right]^{\frac{-1}{3}}$$

$$=8(1+x)^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{\frac{3}{2}}(8)^{\frac{-1}{3}}\left[1+\frac{5x}{8}\right]^{\frac{-1}{3}}$$

$$= 8(1+x)^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)(2)^{-1}\left[1+\frac{5x}{8}\right]^{\frac{-1}{3}}$$

$$= 4(1+x)^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{\frac{3}{2}}\left[1+\frac{5x}{8}\right]^{\frac{-1}{3}} \text{............(1)}$$

$$Take\ (1+x)^{\frac{1}{2}} = 1+\frac{1}{2}(x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x)^3 \text{............(2)}$$

$$= 1+\frac{x}{2}-\frac{x^2}{8}-\text{..........}$$

$$= 1+\frac{x}{2} \qquad \text{Neglecting } x^2 \text{ and higher powers of x}$$

$$\text{Also}\left[1-\frac{3x}{4}\right]^{\frac{3}{2}} = 1+\frac{3}{2}\left(-\frac{3x}{4}\right)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(-\frac{3x}{4}\right)^2+\text{......}$$

$$= 1-\frac{9}{8}x-\frac{27}{128}x^2+\text{............}$$

$$= 1-\frac{9}{8}x \text{ Neglecting } x^2 \text{ and higher powers of x}$$

Putting all these in eq. (1), we get

$$L.H.S = 4\left(1+\frac{1}{2}x\right)\left(1-\frac{9}{8}x\right)\left(1-\frac{5}{24}x\right)$$

$$= 4\left(1+\frac{1}{2}x\right)\left(1-\frac{5x}{24}-\frac{9x}{8}+\frac{45x^2}{192}\right)$$

$$= 4\left(1+\frac{1}{2}x\right)\left(1-\frac{5x+27x}{24}\right) Neglecting x^2$$

$$= 4\left(1+\frac{1}{2}x\right)\left(1-\frac{4x}{3}\right)$$

$$= \left(1-\frac{4x}{3}+\frac{x}{2}\right) = 4\left(1-\frac{8x-3x}{6}\right)$$

Neglecting x^2

$$= 4\left\{1 - \frac{5x}{6}\right\}$$

$$= R.H.S$$

Hence proved

$$\frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}} \cong 4\left[1 - \frac{5x}{6}\right]$$

vi. $$\frac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{\frac{1}{3}}} \cong \frac{3}{2} - \frac{61}{48}x$$

Solution:

$$L.H.S = \frac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{\frac{1}{3}}}$$

$$= \left[(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}\right](8+3x)^{\frac{-1}{3}} ....(1)$$

Take

$$(1-x)^{\frac{1}{2}} = \left[1 + \frac{1}{2}(-x)\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + ...\right]$$

$$= 1 - \frac{x}{2} \text{ Neglecting } x^2 \text{ and higher power of x}$$

Also

$$(9-4x)^{\frac{1}{2}} = \left[9\left(1-\frac{4x}{9}\right)\right]^{\frac{1}{2}} = (3^2)^{\frac{1}{2}}\left(1-\frac{4x}{9}\right)^{\frac{1}{2}}$$

$$= \left[1+\frac{1}{2}\left(-\frac{4x}{9}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^2+..........\right]$$

$$= 3\left[1-\frac{2x}{9}\right] \text{ Neglecting } x^2 \text{ and higher powers of x}$$

$$= 3-\frac{2}{3}x$$

Also

$$(8-3x)^{-\frac{1}{3}} = \left[8\left(1-\frac{3x}{8}\right)\right]^{-\frac{1}{3}} = (2^2)^{-\frac{1}{3}}\left(1-\frac{3x}{8}\right)^{-\frac{1}{3}}$$

$$= \left[1+\frac{1}{2}\left(-\frac{4x}{9}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^2+..........\right]$$

$$= \frac{1}{2}\left[1-\frac{x}{8}\right] \text{ Neglecting } x^2 \text{ and higher powers of x}$$

$$= \frac{1}{2}-\frac{x}{16}$$

Putting all these in eq. (1), we get

$$L.H.S = \left(1-\frac{x}{2}\right)\left(3-\frac{2x}{3}\right)\left(\frac{1}{2}-\frac{x}{16}\right)$$

$$= \left(1-\frac{x}{2}\right)\left(\frac{3}{2}-\frac{3x}{16}-\frac{x}{3}+\frac{x^2}{24}\right)$$

$$= \left(1-\frac{x}{2}\right)\left(\frac{3}{2}-\frac{25}{48}x\right) \text{ Neglecting } x^2$$

$$= \frac{3}{2}-\frac{25x+36x}{48}$$

$$= \frac{3}{2}-\frac{61}{48}x$$

$$= R.H.S.$$

Hence proved

$$\frac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{3}}}{(8-3x)^{\frac{1}{3}}} \cong \frac{3}{2}-\frac{61}{48}x$$

vii. $\frac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}} \cong 2-\frac{1}{12}x$

Solution:

$$L.H.S = \frac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}}$$

$$= \left[\sqrt{4-x}+(8-x)^{\frac{1}{3}}\right](8-x)^{\frac{-1}{3}}$$

$$= \left[\left[4\left(1-\frac{x}{4}\right)\right]^{\frac{1}{2}}+\left[8\left(1-\frac{x}{8}\right)\right]^{\frac{1}{3}}\right]\left[8\left(1-\frac{x}{8}\right)\right]^{\frac{-1}{3}}$$

$$= \left[\left[2^2\left(1-\frac{x}{4}\right)\right]^{\frac{1}{2}}+2\left(1-\frac{x}{8}\right)^{\frac{1}{3}}\right]2^{-1}\left(1-\frac{x}{8}\right)^{\frac{-1}{3}}$$

$$= 2\left[\left(1-\frac{x}{4}\right)^{\frac{1}{2}}+\left(1-\frac{x}{8}\right)^{\frac{1}{3}}\right]\frac{1}{2}\left(1-\frac{x}{8}\right)^{\frac{-1}{3}}$$

$$= \left[\left(1-\frac{x}{4}\right)^{\frac{1}{2}}+\left(1-\frac{x}{8}\right)^{\frac{1}{3}}\right]\left(1-\frac{x}{8}\right)^{\frac{1}{3}} \text{.......(1)}$$

$$Take\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$$

$$= \left[1-\frac{x}{8}+1-\frac{x}{24}\right]\left(1-\frac{x}{24}\right)$$

$$= \left[2-\frac{3x+x}{24}\right]\left(1+\frac{x}{24}\right)$$

$$=\left[2-\frac{x}{8}\right]\left(1+\frac{x}{24}\right)$$

$$=2+\frac{x}{12}-\frac{x}{8}\quad \text{neglected } x^2$$

$$=2+\frac{x-2x}{12}$$

$$=2-\frac{x}{12}$$

$$=R.H.S$$

Hence proved

$$\frac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}}\cong 2-\frac{1}{12}x$$

Q.5 If x is so small that its cube and higher power can be neglected then show that

i) $\sqrt{1-x-2x^2}\cong 1-\frac{1}{2}x-\frac{9}{8}x^2$

ii) $\sqrt{\frac{1+x}{1-x}}\cong 1+x+\frac{1}{2}x^2$

i) $\sqrt{1-x-2x^2}\cong 1-\frac{1}{2}x-\frac{9}{8}x^2$

Solution:

$$L.H.S=\sqrt{1-x-2x^2}=\left[1-\left(x+2x^2\right)\right]^{\frac{1}{2}}$$

$$=1+\frac{1}{2}\left\{-\left(x+2x^2\right)\right\}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left\{-\left(x+2x^2\right)\right\}^2$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left\{-(x+2x^2)\right\}^3+\ldots\ldots\ldots\ldots$$

$=1-\frac{1}{2}x-x^2-\frac{1}{8}(x+2x^2)^2+$ *terms* containing x^3 and high powers of x

$=1-\frac{1}{2}x-x^2-\frac{1}{8}\left(x^2+4x^3+4x^4\right)+$ *terms* containing x^3 and high powers of x

$=1-\frac{1}{2}x-x^2-\frac{1}{8}x^2$; *Neglecting* x^3 and higher powers of x

$$=1-\frac{1}{2}x-\frac{8x^2+x^2}{8}$$

$$=1-\frac{1}{2}x\frac{9x^2}{8}$$

Hence proved

$$\sqrt{1-x-2x^2}\cong 1-\frac{1}{2}x-\frac{9}{8}x^2$$

ii) $\sqrt{\frac{1+x}{1-x}}\cong 1+x+\frac{1}{2}x^2$

Solution:

$$L.H.S=\sqrt{\frac{1+x}{1-x}}=\frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$$

$$=(1+x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}\ldots\ldots\ldots.(1)$$

take

$$(1+x)^{\frac{1}{2}}=\left\{1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+\ldots\ldots\ldots\right\}$$

$=\left\{1+\frac{1}{2}x-\frac{1}{2}x^2\right\}$; **Neglecting x^3 and high powers of x**

$$=1+\frac{1}{2}\left(-\frac{x}{4}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{x}{4}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{x}{4}\right)^3\ldots\ldots\ldots$$

$=1-\frac{x}{8}$ **Neglecting x^2 and higher powers of x**

Also

$$\left(1-\frac{x}{8}\right)^{\frac{1}{3}}=1+\frac{1}{3}\left(-\frac{x}{8}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(-\frac{x}{8}\right)^{2}+\ldots\ldots$$

$$=1-\frac{1}{24}x \text{Neglecting } x^2 \text{and higher powers of x}$$

Also

$$\left(1-\frac{x}{8}\right)^{-\frac{1}{3}}=1+\left(-\frac{1}{3}\right)\left(-\frac{x}{8}\right)+\frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!}\left(-\frac{x}{8}\right)^{2}+\ldots\ldots$$

$$=1+\frac{x}{24}+\frac{x^2}{228}\ldots\ldots$$

$$=1+\frac{x}{24}$$ Neglecting x^2 and higher powers of x

Putting all these in equation (i) , we get

$$L.H.S=\left\{\left(1-\frac{x}{8}\right)+\left(1-\frac{x}{24}\right)\right\}\left\{1+\frac{x}{24}\right\}$$

$$=\left\{2-\frac{3x+x}{24}\right\}\left\{1+\frac{x}{24}\right\}$$

$$=\left\{2-\frac{x}{6}\right\}\left\{1+\frac{x}{24}\right\}=2+\frac{x}{12}+\frac{x}{6}-\frac{x^2}{144}$$

$$=2+\frac{x}{12}$$ Neglecting x^2

$$=2+\frac{x}{12}$$

=R.H.S

Hence proved

$$\sqrt{\frac{1+x}{1-x}}\cong 1+x+\frac{1}{2}x^2$$

Q.6 If x is very nearly equal 1, then prove that

$$px^p - qx^q \approx (p-q)x^{p+q}$$

Solution:

Let x=1+h, where h is so small that its square and higher power can be neglected.

$$\begin{aligned}
&L.H.S = px^p - qx^q \\
&= p(1+h)^p - q(1+h)^q \\
&= p[1 + ph + ...] - q[1 + qh + ...] \\
&= p + p^2h - q - q^2h \\
&= (p-q) + p^2h - q^2h \\
&= (p-q) + h(p^2 - q^2) \\
&= (p-q) + h(p-q)(p+q) \\
&= (p-q) + [1 + h(p+q)] \\
&= (p-q) + [1+h]^{p+q} \\
&= (p-q) + (x)^{p+q} \\
&= R.H.S
\end{aligned}$$

Hence proved.

Q.7 If p-q is small when compared with p or q, show that

$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q} \gg \left[\frac{p+q}{2q}\right]^{\frac{1}{x}}$$

Solution:

L.H.S= $\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q}$

Let p-q=h where 'h' is so small its higher power can be neglected.

$$= \frac{2np + p + 2nq - q}{2np - p + 2nq + q}$$

$$= \frac{2n(p+q) + (p-q)}{2n(p+q) - (p-q)}$$

$$= \frac{2n(p+p+h) + (h)}{2n(p+p+h) - (h)}$$

$$= \frac{2n(2p+h) + h}{2n(2p+h) - h}$$

$$= \frac{4np + 2nh + h}{4np + 2nh - h}$$

$$= \frac{4np + h(2n+1)}{4np + h(2n-1)}$$

$$= \frac{4np\left[1 + \frac{h}{2np}(2n+1)\right]}{4np\left[1 + \frac{h}{2np}(2n-1)\right]}$$

$$= \frac{\left[1 + \frac{h}{2np}(2n+1)\right]}{\left[1 + \frac{h}{2np}(2n-1)\right]}$$

$$= \frac{\left[1 + \frac{h}{p}\left(1 + \frac{1}{2n}\right)\right]}{\left[1 + \frac{h}{p}\left(1 - \frac{1}{2n}\right)\right]}$$

$$= \left[1 + \frac{h}{p}\left(1 + \frac{1}{2n}\right)\right]\left[1 + \frac{h}{p}\left(1 - \frac{1}{2n}\right)\right]^{-1}$$

$$= \left[1 + \frac{h}{p}\left(1 + \frac{1}{2n}\right)\right]\left[1 - \frac{h}{p}\left(1 - \frac{1}{2n}\right)\right]$$

$$=\left[1-\frac{h}{p}\left(1-\frac{1}{2n}\right)+\frac{h}{p}\left(1+\frac{1}{2n}\right)\right]$$

$$=\left[1-\frac{h}{p}+\frac{h}{2np}+\frac{h}{p}+\frac{h}{2np}\right]$$

$$=\left[1+\frac{h}{2np}\right]$$

$$R.H.S=\left[\frac{p+q}{2q}\right]^{\frac{1}{x}}$$

$$=\left[\frac{p+p-h}{2(p-h)}\right]^{\frac{1}{x}}$$

$$=\left[\frac{2p-2h}{2p-2h}\right]^{\frac{1}{x}}$$

$$=\left[\frac{2p\left(1-\frac{h}{2p}\right)}{2p\left(1-\frac{h}{2p}\right)}\right]^{\frac{1}{x}}$$

$$=\left(1-\frac{h}{2p}\right)^{\frac{1}{x}}\left(1-\frac{h}{p}\right)^{\frac{1}{x}}$$

$$=\left(1+\frac{h}{2np}\right)\left(1-\frac{h}{2np}\right)$$

$$=1-\frac{h}{np}+\frac{h}{2np}$$

$$=1+\frac{h}{2np}$$

$$=L.H.S$$

Hence proved

$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q}\gg\left[\frac{p+q}{2q}\right]^{\frac{1}{x}}$$

8. Show that $\left[\frac{n}{2(n+N)}\right]^{\frac{1}{2}} \approx \frac{8n}{9n-N} - \frac{n+N}{4n}$ where n and N are nearly equal.

Solution:

Let N=n+h, where 'h' is so small that its square and higher powers can be neglected.

Therefore by putting N=n+h

Equation becomes:

$$L.H.S = \left[\frac{n}{2(n+N)}\right]^{\frac{1}{2}} = \left[\frac{n}{2(n+n+h)}\right]^{\frac{1}{2}}$$

$$= \left[\frac{n}{2.2n\left(1+\frac{h}{2n}\right)}\right]^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[1+\frac{h}{2n}\right]^{\frac{-1}{2}}$$

$$= \frac{1}{2}\left[1+\left(\frac{-1}{2}\right)\left(\frac{h}{2n}\right)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\frac{-1}{2}\right)}{2!}\left(\frac{-1}{2}\right)^{2}+\ldots\ldots\right]$$

Neglected square term.

$$= \frac{1}{2}\left[1-\left(\frac{h}{4n}\right)\right]$$

$$= \frac{1}{2}-\frac{h}{8n}$$

By putting N=n+h in equation

R.H.S

$$\frac{8n}{9n-N}-\frac{n+N}{4n}=\frac{8n}{9n-(n+h)}-\frac{n+(n+h)}{4n}$$

$$=\frac{8n}{9n-n-h}-\frac{n+n+h}{4n}$$
$$=\frac{8n}{8n\left(1-\frac{h}{8n}\right)}-\left(\frac{2n}{4n}+\frac{h}{4n}\right)$$
$$=\left(1+\frac{h}{8n}\right)^{-1}-\left(\frac{1}{2}+\frac{h}{4n}\right)$$
$$=\left(1+\frac{h}{8n}\right)-\left(\frac{1}{2}+\frac{h}{4n}\right)$$
$$=1+\frac{h}{8n}-\frac{1}{2}-\frac{h}{4n}$$
$$=\frac{1}{2}-\frac{h}{8n}$$
$$=L.H.S$$

Hence Proved

$$\left[\frac{n}{2(n+N)}\right]^{\frac{1}{2}}\approx\frac{8n}{9n-N}-\frac{n+N}{4n}$$

9. Identify the following series as binomial expansion and find the sum in each case.

i. $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2.4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)+\ldots..$

ii. $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{214}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{381}\left(\frac{1}{4}\right)^3+\ldots..$

iii. $1-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+\ldots..$

iv. $1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+\ldots..$

i. $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2.4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)+.....$

Solution:

Let $(1+x)^n$= $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2.4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)+.....$

Or

$$1+nx+\frac{n(n-1)}{2!}x^2+....=1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!4}\left(\frac{1}{4}\right)^2+...$$

Comparing we get

$$nx=-\frac{1}{2}\left(\frac{1}{4}\right)$$

$$\frac{n(n-1)}{2!}x^2=\frac{1.3}{2!4}\left(\frac{1}{4}\right)^2$$

Mulltiplying both sides by '21'

$$\Rightarrow \quad n(n-1)x^2=\frac{1.3.2!}{2!4}\left(\frac{1}{4}\right)^2$$

$$n(n-1)x^2=\frac{3}{4\times16}$$

$$nx=-\frac{1}{8}$$

squaring

$$n^2x^2=\frac{1}{6^4}$$

Dividing them

$$\frac{n(n-1)}{n}=3$$

$$n-3n=1$$

$$-2n=1$$

$$n=\frac{-1}{2}$$

$$x=-\frac{1}{8}\times\frac{1}{n}$$

$$x=-\frac{1}{8}\times\frac{1}{\frac{-1}{2}}=\frac{1}{4}$$

Therefore

$$x = \frac{1}{4} \, and \; n = \frac{-1}{2}$$

$$(1+x)^n = \left(1+\frac{1}{4}\right)^{\frac{-1}{2}}$$

$$= \left(\frac{4+1}{4}\right)^{\frac{-1}{2}}$$

$$= \left(\frac{5}{4}\right)^{\frac{-1}{2}}$$

$$= \left(\frac{5}{4}\right)^{\frac{-1}{2}} = \sqrt{\frac{4}{5}}$$

Hence;

$$1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1.3}{2.4}\left(\frac{1}{4}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right) + ... = \sqrt{\frac{4}{5}}$$

ii. $1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1.3}{214}\left(\frac{1}{4}\right)^2 - \frac{1.3.5}{381}\left(\frac{1}{4}\right)^3 + ...$

Solution:

Let $(1+x)^n = 1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1.3}{214}\left(\frac{1}{4}\right)^2 - \frac{1.3.5}{381}\left(\frac{1}{4}\right)^3 + ...$

Or

$$1 + nx + \frac{n(n-1)}{2!}x^2 + ... = 1 - \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1.3}{2.4}\left(\frac{1}{2}\right)^2 - ...$$

By comparing, we get

$$nx = -\frac{1}{2}\left(\frac{1}{2}\right) \quad ....(1)$$

and $\frac{n(n-1)}{2!}X^2 = \frac{1.3}{2.4}\left(\frac{1}{2}\right)^2$

Multiplying both sides by '2!'

$\Rightarrow \quad n(n-1)x^2 = \frac{3}{16} \quad ...(2)$

Squaring equation (i), we get

$$n^2x^2 = \frac{1}{16}\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Dividing (2) by (3), we get

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{16}\times\frac{16}{1}$$

$$\frac{n-1}{n} = 3$$

$\Rightarrow$ n-1=3n

$\Rightarrow$ 3n-n=-1

$\Rightarrow$ 2n=-1

$\Rightarrow$ $n=-\frac{1}{2}$

Put $n=-\frac{1}{2}$ **in equation (1)**

$$\left(-\frac{1}{2}\right)x = -\frac{1}{2}\left(\frac{1}{2}\right)$$

Multiplying both sides by '-2'

$\Rightarrow$ $x=\frac{1}{2}$

Putting values of $x = \frac{1}{2}$ *and* $n=-\frac{1}{2}$ *in* $(1+x)^n$

i.e.

$$(1+x)^n = \left(1+\frac{1}{2}\right)^{\frac{-1}{2}} = \left(\frac{2+1}{2}\right)^{\frac{-1}{2}}$$

$$= \left(\frac{3}{2}\right)^{\frac{-1}{2}} = \left(\frac{2}{3}\right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$$

Hence ;

$$1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{214}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{381}\left(\frac{1}{4}\right)^3+\ldots..\infty = \sqrt{\frac{2}{3}}$$

iii. $1-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+\ldots..$

Solution:

Let $(1+x)^n = 1-\frac{3}{4}+\frac{3.8}{4.8}-\frac{1.3.7}{4.8.12}+\ldots..$

Or

$$1+nx+\frac{n(n-1)}{2!}x^2+...=1+\frac{3}{4}+\frac{3.5}{4.8}+...$$

By comparing, we get

$nx=\frac{3}{4}$ ....(i)

and $\frac{n(n-1)}{2!}x^2=\frac{3.5}{4.8}$

Multiplying both sides by '2!'

==> $n(n-1)x^2=\frac{15}{16}$............................(ii)

Squaring equation (i) we get

$n^2x^2=\frac{9}{16}$.......................(iii)

Dividing (ii) by (iii) , we get

$$\frac{n(n-1)x^2}{n^2x^2}=\frac{15}{16}\times\frac{16}{9}$$

$$\frac{n-1}{n}=\frac{5}{3}$$

$$\Rightarrow 3(n-1)=5n$$

$$\Rightarrow 5n-3n=-3$$

$$\Rightarrow 2n=-3$$

$$\Rightarrow n=-\frac{3}{2}$$

Put $n=-\frac{3}{2}$ **in equation(i)**

$$\left(-\frac{3}{2}\right)x=\frac{3}{4}$$

Multiplying both sides by $-\frac{3}{2}$ **we get**

$$x=\frac{2}{3}\left(\frac{3}{4}\right)=-\frac{1}{2}$$

Putting values of $x=-\frac{1}{2}$ *and* $n=-\frac{3}{2}$ **in (1+x)ⁿ**

i.e. $(1+x)^n=\left(1-\frac{1}{2}\right)^{\frac{-3}{2}}$

$$=\left(\frac{2-1}{2}\right)^{\frac{-3}{2}}=\left(\frac{1}{2}\right)^{\frac{-3}{2}}$$

$$=(2)^{\frac{3}{2}}=\left(2^3\right)^{\frac{1}{2}}=2\sqrt{2}$$

Hence;

$$1-\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+....\infty=2\sqrt{2}$$

iv. $1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+.....$

Solution:

Let $(1+x)^n=1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+.....\infty$

Or

$$1+nx+\frac{n(n-1)}{2!}x^2+....=1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-......\infty$$

By comparing, we get

$nx=-\frac{1}{2}.\frac{1}{3}$ ......(i)

and

$$\frac{n(n-1)}{2!}x^2=\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2$$

Multiplying both sides by 2!

$n(n-1)x^2=\frac{1}{4}$ ......(ii)

Squaring equation (1), we get

$n^2x^2=\frac{1}{36}$ ....(iii)

Dividing (ii) by (iii),we get

$$\frac{n(n-1)x^2}{n^2x^2}=\frac{1}{12}\times\frac{36}{1}$$

$$\frac{n-1}{n}=3$$

$$n-1=3n$$

$$3n - n = -1$$

$$n = -\frac{1}{2}$$

Put $n = -\frac{1}{2}$ in eq. (i), we get

$$\left(-\frac{1}{2}\right)x = -\frac{1}{6}$$

Multiplying both sides by '-2'we get

$$x = (-2)\left(\frac{-1}{6}\right) = \frac{1}{3}$$

Putting values of x $= \frac{1}{3}$ AND n=$-\frac{1}{2} in (1+x)^n$

$$(1+x)^n = \left(1+\frac{1}{3}\right)^{\frac{-1}{2}} = \left(\frac{3+1}{3}\right)^{\frac{-1}{2}}$$

$$= \left(\frac{4}{3}\right)^{\frac{-1}{2}} = \left(\frac{3}{4}\right)^{\frac{1}{2}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Hence;

$$1 - \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}\left(\frac{1}{3}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3 + ... \infty = \frac{\sqrt{3}}{2}$$

10. Use binomial theorem to show that

$$1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + ... = \sqrt{2}$$

Solution:

Let the given series be identical with the expansion of (1+x)n for $|x| < 1$ when n is not a positive integer. We know that;

$$(1+x)n = 1 = nx = \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

Comparing the series use get

$$nx = \frac{1}{4} or\ x = \frac{1}{4n} \quad \text{........ (i)}$$

And $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{4.8}$ .... (ii)

By subtracting $x = \frac{1}{4n}$ in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2 = \frac{1.3}{4.8}$$

Or $\frac{n(n-1)}{2}\left(\frac{1}{16n^2}\right) = \frac{3}{32} \Rightarrow n-1 = 3n$

$$\Rightarrow n = -\frac{1}{2}$$

By putting $n = -\frac{1}{2}$ in (i)

$$x = \frac{1}{4 \times \left[-\frac{1}{2}\right]} = -\frac{1}{2}$$

Now

$$(1+x)^n = \left[1+\left(-\frac{1}{2}\right)\right]^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{-\frac{1}{2}} = (2)^{\frac{1}{2}}$$

Hence Proved

$$1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + ... = \sqrt{2}$$

11. If $y = \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + ...$, prove that $y^2+2y-2=0$

Solution:

$$y = \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + ...(A)$$

By identify the series

$$1 + \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + ...$$

With

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

We have

$$nx = \frac{1}{3} \text{ or } x = \frac{1}{3n} \quad ......(i)$$

and $$\frac{n(n-1)}{2!}\left(\frac{1}{3n}\right)^2 = \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 ....(ii)$$

Subtracting $x = \frac{1}{3n}$ in eq (ii) gives

$$\frac{n(n-1)}{2!}\left(\frac{1}{3n}\right)^2 = \frac{1.3}{2!}\left(\frac{1}{3}\right)^2$$

or

$$\frac{n(n-1)}{2} \times \frac{1}{9n^2} = \frac{3}{2} \times \frac{1}{9} \Rightarrow \frac{n-1}{n} = 3$$

$$n - 1 = 3n$$
$$n - 3n = 1$$
$$-2n = 1$$
$$n = -\frac{1}{2}$$

By putting $n = -\frac{1}{2}$ in eq. (i) we get

$$X = \frac{1}{3\left(-\frac{1}{2}\right)} = -\frac{2}{3}$$

Now

$$(1+n)^n = \left[1+\left(-\frac{2}{3}\right)\right]^{-\frac{1}{2}} + \left(\frac{1}{3}\right)^{-\frac{1}{2}}$$

$$= (3)^{-\frac{1}{2}} = \sqrt{3}$$

Therefore eq. (A) becomes

$$(1+y) = 1+\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+....\sqrt{3}$$

$$1+y=\sqrt{3}$$

Squaring both the sides of eq. (ii)

We get $(1+y)2=\left(\sqrt{3}\right)^2$

Or

$$1+2y+y^2=3 \Rightarrow y^2+2y-2=0$$

Hence Proved.

12. **If** $2y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+.....$ **proved that $4y^2+4y-1=0$**

Solution:

$$1+2y=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+.....(A)$$

Let

$$=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+.....$$

It will be identical with the expansion.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Where $|x| < 1$ and n is not a positive integer.

By comparing this with the right hand side of eq. (A)

We get

$$nx = \frac{1}{2^2} \text{ or } x = \frac{1}{4n} \quad ....(1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!}.\frac{1}{2^4} \quad ....(2)$$

Subtracting $x = \frac{1}{4n}$ in eq.(2) we get

$$\frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2 = \frac{1.3}{2!}.\frac{1}{2^4}$$

$$\frac{n(n-1)}{2}.\frac{1}{16n^2} = \frac{3}{2}.\frac{1}{16}$$

$$\frac{n-1}{n} = 3$$

$$n-1 = 3n$$

$$-1 = 3n - n$$

$$2n = -1$$

$$n = \frac{-1}{2}$$

Putting $n = \frac{-1}{2}$ in eq. (1) we get

$$x = \frac{1}{4\left(-\frac{1}{2}\right)} = -\frac{1}{2}$$

Now

$$(1+x)^n = \left[1+\left(-\frac{1}{2}\right)\right]^{-\frac{1}{2}} = (2)^{\frac{1}{2}} = \sqrt{2}$$

Therefore

$$=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+.....=\sqrt{2}$$

And eq. (A) becomes

$$1+2y=\sqrt{2} \quad .....(B)$$

Squaring both the sides of eq. (B) we get.

$$(1+2y)^2=(\sqrt{2})^2$$

Hence proved.

$4y^2+4y-1=2 \rightarrow 4y^2+4y-1=0$

13. If $y=\frac{2}{5}+\frac{1.3}{2!}.\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+......$ **prove that $y^2+2y-4=0$**

Solution:

$$1+y=1+\frac{2}{5}+\frac{1.3}{2!}.\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+......(A)$$

Identifying the series on the right-hand side of eq. (A) with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+.....\infty$$

Where $|x|<1$ n is not a positive integer, we get

$nx = \frac{2}{5}$ *or* $x = \frac{2}{5n}$ ....(1)

and

$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!}\left(\frac{2}{5}\right)^2$ ....(2)

By putting $x = \frac{2}{5n}$ in eq. (2) we get

$\frac{n(n-1)}{2!}\left(\frac{2}{5}\right)^2 = \frac{1.3}{2!}\left(\frac{2}{5}\right)^2$

or

$\frac{n(n-1)}{2}.\frac{4}{25n^2} = \frac{3}{2}.\frac{4}{25}$

$\Rightarrow n-1 = 3n \Rightarrow n = -\frac{1}{2}$

Putting $n = -\frac{1}{2}$ in eq. (1), we get

$x = \frac{2}{5\left(-\frac{1}{2}\right)} = -\frac{4}{5}$

Now

$(1+x)^n = \left[1+\left(-\frac{4}{5}\right)\right]^{-\frac{1}{2}} = \left(\frac{1}{5}\right)^{-\frac{1}{2}} = 5^{\frac{1}{2}} = \sqrt{5}$

therefore

$1+\frac{2}{5}+\frac{1.3}{2!}.\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+..... = \sqrt{5}$

And eq. (A) changes as

$1+y = \sqrt{5}$ ......(B)

By squaring both sides of eq. (B) we get

$$(1+y)^2 = \left(\sqrt{5}\right)^2$$

Or

$$1+2y+y^2 = 5 \Rightarrow y^2+2y-4=0$$

Hence proved.