Resolution of a fraction when D (x) has repeated irreducible quadratic factors.

Rule IV:

If a quadratic factor ($ax^2 + bx + c$) with $a \neq 0$, occurs twice in the denominator, the corresponding partial fractions are

$$\frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2}$$

The constants A, B, C and D are found in the usual way.

Exercise 4.4

Resolve into partial fraction.

(1) $\dfrac{x^3}{(x^2+4)^2}$

Solution:

$$Let \qquad \frac{x^3}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

Multiplying both sides by $(x^2+4)^2$, we get

$$x^3 = (Ax+B)(x^2+4) + Cx + D \qquad ....(1)$$

$$x^3 = Ax^3 + 4Ax + Bx^2 + 4B + Cx + D \qquad ....(2)$$

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(2)

We get

Coefficient of x^3 : $A = 1$

Coefficient of x^2 : $B = 0$

Coefficient of x : $4A + C = 0$ ....(3)

Constant: $4B + D = 0$ ....(4)

Put $A = 1$ in eq.(3), we get

$4(1) + C = 0$

$4 + C = 0$

$C = -4$

Put $B = 0$ in eq.(4), we get

$4(0) + D = 0$

$D = 0$

Thus required partial fractions are $\frac{(1)x+0}{x^2+4} + \frac{(-4)x+0}{(x^2+4)^2}$

Hence, $\frac{x^3}{(x^2+4)^2} = \frac{x}{x^2+4} - \frac{4x}{(x^2+4)^2}$

(2) $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$

Solution:

Let $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

Multiplying both sides by $(x+1)(x^2+1)^2$, we get

$x^4 + 3x^2 + x + 1 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1)$ ....(1)

$$x^4+3x^2+x+1=A\left(x^4+2x^2+1\right)+\left(Bx+C\right)\left(x^3+x^2+x+1\right)+Dx^2+Dx+Ex+E$$

$$x^4+3x^2+x+1=Ax^4+2Ax^2+A+Bx^4+Bx^3+Bx^2+Bx+Cx^3+Cx^2$$
$$+Cx+C+Dx^2+Dx+Ex+E$$

$$x^4+3x^2+x+1=Ax^4+Bx^4+Bx^3+Cx^3+2Ax^2+Bx^2+Cx^2$$
$$+Dx^2+Bx+Cx+Dx+Ex+A+C+E \quad ....(2)$$

To find A, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$(-1)^4+3(-1)^2+(-1)+1=A\left((-1)^2+1\right)^2$

$1+3-1+1=A(1+1)^2$

$4=A(2)^2$

$4=4A$

$4A=4$

$\Rightarrow \ A=1$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4 :	$A+B=1$	.......(3)
Coefficient of x^3 :	$B+C=0$	.......(4)
Coefficient of x^2 :	$2A+B+C+D=3$	........(5)
Coefficient of x :	$B+C+D+E=1$	........(6)

Put $A=1$ in eq.(3), we get

$1+B=1$

$B=1-1$

$B=0$

Put $B=0$ in eq.(4), we get

$0+C=0$

$C=0$

Put A=1, B = 0, C=0 in eq.(5), we get

$2(1)+0+0+D=3$

$2+D=3$

$D = 3 - 2$

$D = 1$

Put $B = 0, C = 0, D = 1$ in eq.(6), we get

$0 + 0 + 1 + E = 1$

$1 + E = 1$

$E = 1 - 1$

$E = 0$

Thus required partial fractions are $\frac{1}{x+1} + \frac{(0)x+(0)}{x^2+1} + \frac{(1)x+(0)}{(x^2+1)^2}$

Hence, $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{1}{x+1} + \frac{x}{(x^2+1)^2}$

$(3)\ \frac{x^2}{(x+1)(x^2+1)^2}$

Solution:

Let $\frac{x^2}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

Multiplying both sides by $(x+1)(x^2+1)^2$, we get

$x^2 = A(x^2+1)^2 + (Bx+C)(x^2+1)(x+1) + (Dx+E)(x+1)$ ....(1)

$x^2 = A(x^4+2x^2+1) + (Bx+C)(x^3+x^2+x+1)$
$+Dx^2 + Dx + Ex + E$

$x^2 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^3 + Bx^2 + Bx + Cx^3$
$+Cx^2 + Cx + C + Dx^2 + Dx + Ex + E$

$x^2 = Ax^4 + Bx^4 + Bx^3 + Cx^3 + 2Ax^2 + Bx^2 + Cx^2 + Dx^2$
$+Bx + Cx + Dx + Ex + A + C + E$ ....(2)

To find A, we put x+1 = 0 $\Rightarrow x = -1$ in eq.(1), we get

$(-1)^2 = A\left((-1)^2 + 1\right)^2$

$1 = A(1+1)^2$

$1 = A(2)^2$

$1 = 4A$

$or\ 4A = 1$

$\Rightarrow\ A = \frac{1}{4}$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4 : $A + B = 0$ ....(3)

Coefficient of x^3 : $B + C = 0$ ....(4)

Coefficient of x^2 : $2A + B + C + D = 1$ ....(5)

Coefficient of x : $B + C + D + E = 0$ ....(6)

Put $A = \frac{1}{4}$ in eq.(4), we get

$\frac{1}{4} + B = 0$

$B = -\frac{1}{4}$

Put B $= -\frac{1}{4}$ in eq.(4), we get

$-\frac{1}{4} + C = 0$

$C = \frac{1}{4}$

Put A $= \frac{1}{4}$, B $= -\frac{1}{4}$, C $= \frac{1}{4}$ in eq.(5), we get

$2\left(\frac{1}{4}\right) + \left(-\frac{1}{4}\right) + \left(\frac{1}{4}\right) + D = 1$

$\frac{2}{4}-\frac{1}{4}+\frac{1}{4}+D=1$

$\frac{1}{2}+D=1$

$D=1-\frac{1}{2}$

$D=\frac{1}{2}$

Put $B=-\frac{1}{4}, C=\frac{1}{4}, D=\frac{1}{2}$ in eq.(6), we get

$\left(-\frac{1}{4}\right)+\left(\frac{1}{4}\right)+\left(\frac{1}{2}\right)+E=0$

$-\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+E=0$

$E=-\frac{1}{2}$

Thus required partial fractions are $\frac{\frac{1}{4}}{x+1}+\frac{\left(\frac{-1}{4}\right)x+\left(\frac{1}{4}\right)}{x^2+1}+\frac{\left(\frac{1}{2}\right)x+\left(\frac{-1}{2}\right)}{\left(x^2+1\right)^2}$

Hence, $\frac{x^2}{(x+1)\left(x^2+1\right)^2}=\frac{1}{4(x+1)}-\frac{x-1}{4\left(x^2+1\right)}+\frac{x-1}{2\left(x^2+1\right)^2}$

(4) $\frac{x^2}{(x-1)\left(x^2+1\right)^2}$

Solution:

Let $\frac{x^2}{(x-1)\left(x^2+1\right)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$

Multiplying both sides by $(x-1)(x^2+1)^2$, we get

$$x^2 = A(x^2+1)^2 + (Bx+C)(x^2+1)(x-1) + (Dx+E)(x-1) \text{ ....(1)}$$

$$x^2 = A(x^4+2x^2+1) + (Bx+C)(x^3-x^2+x-1)$$
$$+Dx^2 - Dx + Ex - E$$

$$x^2 = Ax^4 + 2Ax^2 + A + Bx^4 - Bx^3 + Bx^2 - Bx + Cx^3$$
$$-Cx^2 + Cx - C + Dx^2 - Dx + Ex - E$$

$$x^2 = Ax^4 + Bx^4 - Bx^3 + Cx^3 + 2Ax^2 + Bx^2 - Cx^2 + Dx^2$$
$$-Bx + Cx - Dx + Ex + A - C - E \qquad \text{.......(2)}$$

To find A, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$$(1)^2 = A((1)^2+1)^2$$
$$1 = A(1+1)^2$$
$$1 = A(2)^2$$
$$1 = 4A$$
$$or\ 4A = 1$$
$$\Rightarrow A = \frac{1}{4}$$

To find B,C,D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq.(2)

We get

Coefficient of x^4 : $A+B=0$ ....(3)

Coefficient of x^3 : $-B+C=0$ ....(4)

Coefficient of x^2 : $2A+B-C+D=1$ ....(5)

Coefficient of x : $-B+C-D+E=0$ ....(6)

Put $A=\frac{1}{4}$ in eq.(4), we get

$$\frac{1}{4}+B=0$$
$$B=-\frac{1}{4}$$

Put $B = -\frac{1}{4}$ in eq.(4), we get

$$-\left(-\frac{1}{4}\right) + C = 0$$

$$\frac{1}{4} + C = 0$$

$$C = -\frac{1}{4}$$

Put $A = \frac{1}{4}, B = -\frac{1}{4}, C = -\frac{1}{4}$ in eq.(5), we get

$$2\left(\frac{1}{4}\right) + \left(-\frac{1}{4}\right) - \left(-\frac{1}{4}\right) + D = 1$$

$$\frac{2}{4} - \frac{1}{4} + \frac{1}{4} + D = 1$$

$$\frac{1}{2} + D = 1$$

$$D = 1 - \frac{1}{2}$$

$$D = \frac{1}{2}$$

Put $B = -\frac{1}{4}, C = -\frac{1}{4}, D = \frac{1}{2}$ in eq.(6), we get

$$-\left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right) - \frac{1}{2} + E = 0$$

$$\frac{1}{4} - \frac{1}{4} - \frac{1}{2} + E = 0$$

$$E = \frac{1}{2}$$

Thus required partial fractions are $\frac{\frac{1}{4}}{x-1} + \frac{\left(\frac{-1}{4}\right)x + \left(\frac{-1}{4}\right)}{x^2+1} + \frac{\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)}{\left(x^2+1\right)^2}$

Hence, $\frac{x^2}{(x-1)\left(x^2+1\right)^2} = \frac{1}{4(x-1)} - \frac{x+1}{4\left(x^2+1\right)} + \frac{x+1}{2\left(x^2+1\right)^2}$

$$(5)\ \frac{x^4}{\left(x^2+2\right)^2}$$

Solution:

$$\frac{x^4}{\left(x^2+2\right)^2}=\frac{x^4}{x^4+4x^2+4}$$

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & 1 \\ x^4+4x^2+4 & x^4 \\ & \underline{\pm x^4 \pm 4x^2 \pm 4} \\ & -4x^2-4 \\ & -\left(4x^2+4\right) \end{array}$$

$$\frac{x^4}{\left(x^2+2\right)^2}=1-\frac{4x^2+4}{\left(x^2+2\right)^2}$$

$$Let \quad \frac{4x^2+4}{\left(x^2+2\right)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}$$

Multiplying both sides by $\left(x^2+2\right)^2$, we get

$4x^2+4=(Ax+B)\left(x^2+2\right)+(Cx+D)$

$4x^2+4=Ax^3+2Ax+Bx^2+2B+Cx+D$

$4x^2+4=Ax^3+Bx^2+2Ax+Cx+2B+D \quad ....(1)$

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(1), We get

Coefficient of x^3 : $A=0$

Coefficient of x^2 : $B=4$

Coefficient of x : $2A+C=0 \quad ....(2)$

Constant: $2B+D=4 \quad ....(3)$

Put $A=0$ in eq.(2), we get

$2(0)+C=0$

$C = 0$

Put B = 4 in eq.(3), we get

$2(4) + D = 4$

$8 + D = 4$

$D = 4 - 8$

$D = -4$

Thus required partial fractions are $\frac{(0)x + (4)}{x^2 + 2} + \frac{(0)x + (-4)}{(x^2 + 2)^2}$

$$\text{Hence, } \frac{x^4}{(x^2 + 2)^2} = 1 - \left[\frac{4}{x^2 + 2} - \frac{4}{(x^2 + 2)^2}\right]$$
$$= 1 - \frac{4}{x^2 + 2} + \frac{4}{(x^2 + 2)^2}$$

(6) $\frac{x^5}{(x^2 + 1)^2}$

Solution:

$$\frac{x^5}{\left(x^2+1\right)^2}=\frac{x^5}{x^4+2x^2+1}$$

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & x \\ x^4+2x^2+1 & x^5 \\ & \underline{\pm x^5 \pm 2x^3 \pm x} \\ & -2x^3-x \\ & -\left(2x^3+x\right) \end{array}$$

$$\frac{x^5}{\left(x^2+1\right)^2}=x-\frac{2x^3+x}{\left(x^2+1\right)^2}$$

Let $\quad \dfrac{2x^3+x}{\left(x^2+1\right)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{\left(x^2+1\right)^2}$

Multiplying both sides by $\left(x^2+1\right)^2$, we get

$2x^3+x=\left(Ax+B\right)\left(x^2+1\right)+\left(Cx+D\right)$

$2x^3+x=Ax^3+Ax+Bx^2+B+Cx+D$

$2x^3+x=Ax^3+Bx^2+Ax+Cx+B+D \qquad ....(1)$

To find A,B,C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq.(1). We get

Coefficient of x^3 : $\quad A=2$

Coefficient of x^2 : $\quad B=0$

Coefficient of x : $\quad A+C=1 \qquad ....(2)$

Constant: $\quad B+D=0 \qquad ....(3)$

Put $A=2$ in eq.(2), we get

$A+C=1$

$2+C=1$

$C=1-2$

$C=-1$

Put B $=0$ in eq.(3), we get

$0+D=0$

$D=0$

Thus required partial fractions are $\dfrac{2x+0}{x^2+1}+\dfrac{(-1)x+0}{\left(x^2+1\right)^2}$

Hence, $\dfrac{x^5}{\left(x^2+1\right)^2}=x-\left[\dfrac{2x}{x^2+1}+\dfrac{-x}{\left(x^2+1\right)^2}\right]$