Resolution of fraction when D (x) consists of non-repeated irreducible quadratic factors.

Rule III:

If a quadratic factor ($ax^2 + bx + c$) with $a \neq 0$ occur once as a factor of D(x), the partial fraction is of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be found.

Exercise 4.3

Resolve into partial fractions.

(1) $\frac{3x-11}{(x+3)(x^2+1)}$

Solution:

$$\frac{3x-11}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$3x-11 = A(x^2+1) + (Bx+C)(x+3)$ ....(1)

$3x-11 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$

$3x-11 = Ax^2 + Bx^2 + 3Bx + Cx + A + 3C$ ....(2)

To find A, we put x + 3 = 0 $\Rightarrow x = -3$ in eq.(1), we get

$3(-3) - 11 = A((-3)^2 + 1) + (B(-3) + C)(-3+3)$

$-9 - 11 = A(9+1) + (-3B + C)(0)$

$-20 = 10A$

$or\ 10A = -20$

Dividing both sides by '10', we get
$\Rightarrow A = -2$

To find B and C, equating coefficient of x^2 and constant on both sides of eq.(2), we get

$$A + B = 0$$
$$-2 + B = 0$$
$$\Rightarrow B = 2$$

And $A + 3C = -11$

$$-2 + 3C = -11$$
$$3C = -11 + 2$$
$$3C = -9$$

Dividing both sides by '3', we get

$$C = -3$$

Thus required partial fractions are $\frac{-2}{x+3} + \frac{2x+(-3)}{x^2+1}$

Hence, $\frac{3x-11}{(x+3)(x^2+1)} = \frac{-2}{x+3} + \frac{2x-3}{x^2+1}$

$(2)\ \frac{3x+7}{(x^2+1)(x+3)}$

Solution:

$$\frac{3x+7}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3}$$

Multiplying both sides by $(x^2+1)(x+3)$, we get

$$3x+7=(Ax+B)(x+3)+C(x^2+1) \quad \text{.......}(1)$$
$$3x+7=Ax^2+3Ax+Bx+3B+Cx^2+C$$
$$3x+7=Ax^2+Cx^2+3Ax+Bx+3B+C \quad \text{........}(2)$$

To find C, we put x + 3 = 0 $\Rightarrow x=-3$ in eq.(1), we get

$$3(-3)+7=(A(-3)+B)(-3+3)+C((-3)^2+1)$$
$$-9+7=(-3A+B)(0)+C(9+1)$$
$$-2=10C$$
$$or\ 10C=-2$$

Dividing both sides by '10', we get

$$\Rightarrow C=-\frac{1}{5}$$

To find A and B, equating coefficient of x^2 and constant on both sides of (2), we get

$$A+C=0$$
$$A+\left(-\frac{1}{5}\right)=0$$
$$\Rightarrow A=\frac{1}{5}$$

And $3B+C=7$

$$3B+\left(-\frac{1}{5}\right)=7$$
$$3B=7+\frac{1}{5}$$
$$3B=\frac{36}{5}$$
$$B=\frac{36}{5}\times\frac{1}{3}$$
$$B=\frac{12}{5}$$

Thus required partial fractions are $\dfrac{\frac{1}{5}x+\frac{12}{5}}{x^2+1}+\dfrac{\frac{-1}{5}}{x+3}$

Hence, $\frac{3x+7}{(x^2+1)(x+3)} = \frac{x+12}{5(x^2+1)} - \frac{1}{5(x+3)}$

(3) $\frac{1}{(x+1)(x^2+1)}$

Solution:

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+1)(x^2+1)$, we get

$$1 = A(x^2+1) + (Bx+C)(x+1) \qquad ....(1)$$

$$1 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$1 = Ax^2 + Bx^2 + Bx + Cx + A + C \qquad ....(2)$$

To find A, we put x + 1 = 0 $\Rightarrow x = -1$ in eq.(1), we get

$$1 = A((-1)^2+1) + (B(-1)+C)(-1+1)$$

$$1 = A(1+1) + (-B+C)(0)$$

$$1 = A(2)$$

$$or\ 2A = 1$$

Dividing both sides by '2', we get

$$\Rightarrow A = \frac{1}{2}$$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$$A + B = 0$$

$$\frac{1}{2} + B = 0 \qquad \because A = \frac{1}{2}$$

$$\Rightarrow B = -\frac{1}{2}$$

And $A+C=1$

$$\frac{1}{2}+C=1 \qquad \because A=\frac{1}{2}$$

$$C=1-\frac{1}{2}$$

$$C=\frac{1}{2}$$

Thus required partial fractions are $\dfrac{\frac{1}{2}}{x+1}+\dfrac{\frac{-1}{2}x+\frac{1}{2}}{x^2+1}$

Hence, $\dfrac{1}{(x+1)(x^2+1)}=\dfrac{1}{2(x+1)}-\dfrac{x-1}{2(x^2+1)}$

(4) $\dfrac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$9x-7=A(x^2+1)+(Bx+C)(x+3)$ ....(1)

$9x-7=Ax^2+A+Bx^2+3Bx+Cx+3C$

$9x-7=Ax^2+Bx^2+3Bx+Cx+A+3C$ ....(2)

To find A, we put x + 3 = 0 $\Rightarrow x=-3$ in eq.(1), we get

$9(-3)-7=A((-3)^2+1)+(B(-3)+C)(-3+3)$

$-27-7=A(9+1)+(-3B+C)(0)$

$-34=10A$

$or\ 10A=-34$

Dividing both sides by '10', we get

$\Rightarrow A = \frac{-34}{10} = -\frac{17}{5}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$$A + B = 0$$

$$-\frac{17}{5} + B = 0 \quad \because A = -\frac{17}{5}$$

$$\Rightarrow B = \frac{17}{5}$$

And $A + 3C = -7$

$$-\frac{17}{5} + 3C = -7 \qquad \because A = -\frac{17}{5}$$

$$3C = -7 + \frac{17}{5}$$

$$3C = -\frac{18}{5}$$

$$C = -\frac{18}{5} \times \frac{1}{3}$$

$$C = -\frac{6}{5}$$

Thus required partial fractions are $\frac{\frac{-17}{5}}{x+1} + \frac{\frac{17}{5}x - \frac{6}{5}}{x^2+1}$

Hence, $\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17}{5(x+1)} + \frac{17x-6}{5(x^2+1)}$

(5) $\frac{3x+7}{(x+3)(x^2+4)}$

Solution:

$$Let \quad \frac{3x+7}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$$

Multiplying both sides by $(x+3)(x^2+4)$, we get

$3x+7 = A(x^2+4) + (Bx+C)(x+3)$ ....(1)

$3x+7 = Ax^2 + 4A + Bx^2 + 3Bx + Cx + 3C$

$3x+7 = Ax^2 + Bx^2 + 3Bx + Cx + 4A + 3C$ ....(2)

To find A, we put x + 3 = 0 $\Rightarrow x = -3$ in eq.(1), we get

$3(-3)+7 = A((-3)^2+4) + (B(-3)+C)(-3+3)$

$-9+7 = A(9+4) + (-3B+C)(0)$

$-2 = 13A$

$or\ 13A = -2$

Dividing both sides by '13', we get

$\Rightarrow A = -\frac{2}{13}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A + B = 0$

$-\frac{2}{13} + B = 0 \quad \because A = -\frac{2}{13}$

$\Rightarrow B = \frac{2}{13}$

And $4A + 3C = 7$

$4\left(-\frac{2}{13}\right) + 3C = 7 \quad \because A = -\frac{2}{13}$

$-\frac{8}{13} + 3C = 7$

$3C = 7 + \frac{8}{13}$

$3C = \frac{99}{13}$

$C = \frac{99}{13} \times \frac{1}{3}$

$C = \frac{33}{13}$

Thus required partial fractions are $\frac{\frac{-2}{13}}{x+3} + \frac{\frac{2}{13}x + \frac{33}{13}}{x^2+4}$

Hence, $\frac{3x+7}{(x+3)(x^2+4)} = \frac{-2}{13(x+3)} + \frac{2x+33}{13(x^2+4)}$

(6) $\frac{x^2}{(x+2)(x^2+4)}$

Solution:

Let $\frac{x^2}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

Multiplying both sides by $(x+2)(x^2+4)$, we get

$x^2 = A(x^2+4) + (Bx+C)(x+2)$ ....(1)

$x^2 = Ax^2 + 4A + Bx^2 + 2Bx + Cx + 2C$

$x^2 = Ax^2 + Bx^2 + 2Bx + Cx + 4A + 2C$ ....(2)

To find A, we put x + 2 = 0 $\Rightarrow x = -2$ in eq.(1), we get

$(-2)^2 = A((-2)^2 + 4) + (B(-2) + C)(-2+2)$

$4 = A(4+4) + (-2B+C)(0)$

$4 = 8A$

$or\ 8A = 4$

Dividing both sides by '8', we get

$\Rightarrow A = \frac{4}{8} = \frac{1}{2}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$$A+B=1$$

$$\frac{1}{2}+B=1 \qquad \because A=\frac{1}{2}$$

$$B=1-\frac{1}{2}$$

$$\Rightarrow B=\frac{1}{2}$$

And $\quad 4A+2C=0$

$$4\left(\frac{1}{2}\right)+2C=0 \qquad \because A=\frac{1}{2}$$

$$2+2C=0$$

$$2C=-2$$

$\Rightarrow \quad C=-1$

Thus required partial fractions are $\dfrac{\frac{1}{2}}{x+2}+\dfrac{\frac{1}{2}x-1}{x^2+4}$

Hence, $\dfrac{x^2}{(x+2)(x^2+4)}=\dfrac{1}{2(x+2)}+\dfrac{x-2}{2(x^2+4)}$

(7) $\dfrac{1}{x^3+1}$

Solution:

$$\frac{1}{x^3+1}=\frac{1}{(x)^3+(1)^3}=\frac{1}{(x+1)(x^2-x+1)}$$

Let $\quad \dfrac{1}{(x+1)(x^2-x+1)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2-x+1}$

Multiplying both sides by $(x+1)(x^2-x+1)$, we get

$$1=A(x^2-x+1)+(Bx+C)(x+1) \qquad ....(1)$$

$1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$

$1 = Ax^2 + Bx^2 - Ax + Bx + Cx + A + C$ ....(2)

To find A, we put $x + 1 = 0 \Rightarrow x = -1$ in eq.(1), we get

$1 = A\left((-1)^2 - (-1) + 1\right) + \left(B(-1) + C\right)(-1+1)$

$1 = A(1+1+1) + (-B+C)(0)$

$1 = A(3) + (-B+C)(0)$

$or\ 3A = 1$

Dividing both sides by '3', we get

$\Rightarrow A = \frac{1}{3}$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$A + B = 0$

$\frac{1}{3} + B = 0 \qquad \because A = \frac{1}{3}$

$B = -\frac{1}{3}$

And $A + C = 1$

$\frac{1}{3} + C = 1 \qquad \because A = \frac{1}{2}$

$C = 1 - \frac{1}{3}$

$C = \frac{2}{3}$

Thus required partial fractions are $\frac{\frac{1}{3}}{x+1} + \frac{\frac{-1}{3}x + \frac{2}{3}}{x^2 - x + 1}$

Hence, $\frac{1}{x^3+1} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$

$$(8)\ \frac{x^2+1}{x^3+1}$$

Solution:

$$\frac{x^2+1}{x^3+1}=\frac{x^2+1}{(x)^3+(1)^3}=\frac{x^2+1}{(x+1)(x^2-x+1)}$$

$$Let\qquad \frac{x^2+1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

Multiplying both sides by $(x+1)(x^2-x+1)$, we get

$$x^2+1=A(x^2-x+1)+(Bx+C)(x+1) \qquad ....(1)$$

$$x^2+1=Ax^2-Ax+A+Bx^2+Bx+Cx+C$$

$$x^2+1=Ax^2+Bx^2-Ax+Bx+Cx+A+C \qquad ....(2)$$

To find A, we put x + 1 = 0 $\Rightarrow x=-1$ in eq.(1), we get

$$(-1)^2+1=A((-1)^2-(-1)+1)+(B(-1)+C)(-1+1)$$

$$1+1=A(1+1+1)+(-B+C)(0)$$

$$2=A(3)+(-B+C)(0)$$

$$2=A(3)$$

$$or\ 3A=2$$

$$\Rightarrow A=\frac{2}{3}$$

To find B and C, equating coefficient of x^2 and constant on both sides of (2), we get

$$A+B=1$$

$$\frac{2}{3}+B=1 \qquad \because A=\frac{2}{3}$$

$$B=1-\frac{2}{3}$$

$$B=\frac{1}{3}$$

And $\quad A+C=1$

$$\frac{2}{3}+C=1 \qquad \because A=\frac{2}{3}$$

$$C=1-\frac{2}{3}$$

$$C=\frac{1}{3}$$

Thus required partial fractions are $\dfrac{\frac{2}{3}}{x+1}+\dfrac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}$

Hence, $\dfrac{x^2+1}{x^3+1}=\dfrac{2}{3(x+1)}+\dfrac{x+1}{3(x^2-x+1)}$