Resolution of a fraction when D(x) consists of repeated linear factors:

Rule II:

If a linear factor (ax + b) occurs n times as a factor of D(x), then there are n partial fractions of the form.

$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+......+\frac{A_n}{(ax+b)^n}$ where A_1 , A_2.......A_n are constants and $n \geq 2$ is a positive integer.

$$\frac{N(x)}{D(x)}=\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+......+\frac{A_n}{(ax+b)^n}$$

Exercise 4.2

Resolve into partial fraction.

(1) $\frac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution:

Let $\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2$ ....(1)

$x^2-3x+1=A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$

$x^2-3x+1=Ax^2-3Ax+2A+Bx-2B+Cx^2-2Cx+C$

$x^2-3x+1=Ax^2+Cx^2-3Ax+Bx-2Cx+2A-2B+C$ ....(2)

To find C, we put $x-2=0 \Rightarrow x=2$ in eq.(1), we get

$$(2)^2-3(2)+1=A(2-1)(2-2)+B(2-2)+C(2-1)^2$$
$$4-6+1=A(1)(0)+B(0)+C(1)^2$$
$$5-6=A(0)+B(0)+C$$
$$-1=C$$
$$or\ C=-1$$

To find B, we put $(x-1)^2=0 \Rightarrow x-1=0 \Rightarrow x=1$ in eq.(1), we get

$$(1)^2-3(1)+1=A(1-1)(1-2)+B(1-2)+C(1-1)^2$$
$$1-3+1=A(0)(-1)+B(-1)+C(0)$$
$$2-3=A(0)+B(-1)+C(0)$$

$$-1=-B$$
$$\Rightarrow B=1$$

To find A, equating coefficient of x^2 on both sides of (2), we get

$$A+C=1$$
$$A+(-1)=1$$
$$A=1+1$$
$$A=2$$

Thus required partial fractions are $\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{-1}{x-2}$

Hence, $\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{2}{x-1}+\frac{1}{(x-1)^2}-\frac{1}{x-2}$

(2) $\frac{x^2+7x+11}{(x+2)^2(x+3)}$

Solution:

$$Let \quad \frac{x^2+7x+11}{(x+2)^2(x+3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+3}$$

Multiplying both sides by $(x+2)^2(x+3)$, we get

$x^2+7x+11 = A(x+2)(x+3) + B(x+3) + C(x+2)^2$ ....(1)

$x^2+7x+11 = A(x^2+5x+6) + B(x+3) + C(x^2+4x+4)$

$x^2+7x+11 = Ax^2+5Ax+6A+Bx+3B+Cx^2+4Cx+4C$

$x^2+7x+11 = Ax^2+Cx^2+5Ax+Bx+4Cx+6A+3B+4C$ ....(2)

To find C, we put $x+3=0 \Rightarrow x=-3$ in eq.(1), we get

$(-3)^2+7(-3)+11 = A(-3+2)(-3+3) + B(-3+3) + C(-3+2)^2$

$9-21+11 = A(-1)(0) + B(0) + C(-1)^2$

$20-21 = A(0) + B(0) + C(1)$

$-1 = C$

$or\ C = -1$

To find B, we put $(x+2)^2 = 0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$(-2)^2+7(-2)+11 = A(-2+2)(-2+3) + B(-2+3) + C(-2+2)^2$

$4-14+11 = A(0)(1) + B(1) + C(0)^2$

$15-14 = A(0) + B(1) + C(0)$

$1 = B$

$\Rightarrow B = 1$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A + C = 1$

$A + (-1) = 1$

$A = 1+1$

$A = 2$

Thus required partial fractions are $\frac{2}{x+2} + \frac{1}{(x+2)^2} + \frac{-1}{x+3}$

Hence, $\frac{x^2+7x+11}{(x+2)^2(x+3)} = \frac{2}{x+2} + \frac{1}{(x+2)^2} - \frac{1}{x+3}$

$$(3)\ \frac{9}{(x-1)(x+2)^2}$$

Solution:

$$Let \quad \frac{9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Multiplying both sides by $(x-1)(x+2)^2$, we get

$$9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \quad ....(1)$$

$$9 = A(x^2+4x+4) + B(x^2+x-2) + C(x-1)$$

$$9 = Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C$$

$$9 = Ax^2 + Bx^2 + 4Ax + Bx + Cx + 4A - 2B - C \quad ....(2)$$

To find A, we put $\mathrm{x} - 1 = 0 \Rightarrow x = 1$ in eq.(1), we get

$$9 = A(1+2)^2 + B(1-1)(1+2) + C(1-1)$$

$$9 = A(3)^2 + B(0)(3) + C(0)$$

$$9 = A(9) + B(0) + C(0)$$

$$9 = 9A$$

$$or\ 9A = 9$$

Dividinge the both sides by '9', we get

A = 1

To find C, we put $(x+2)^2=0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$$9=A(-2+2)^2+B(-2-1)(-2+2)+C(-2-1)$$
$$9=A(0)^2+B(-3)(0)+C(-3)$$
$$9=A(0)+B(0)+C(-3)$$
$$9=-3C$$
$$or\ -3C=9$$

Dividing both sides by '-3', we get

$C=-3$

To find B, equating coefficient of x^2 on both sides of (2), we get

$$A+B=0$$
$$1+B=0$$
$$B=-1$$

Thus required partial fractions are $\frac{1}{x-1}+\frac{-1}{x+2}+\frac{-3}{(x+2)^2}$

Hence, $\frac{9}{(x-1)(x+2)^2}=\frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+2)^2}$

(4) $\frac{x^4+1}{x^2(x-1)}$

Solution:

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & x+1 \\ x^3-x^2 & x^4+1 \\ & \underline{\pm x^4 \mp x^3} \\ & x^3+1 \\ & \underline{\pm x^3 \mp x^2} \\ & x^2+1 \end{array}$$

$$\frac{x^4+1}{x^2(x-1)} = x+1+\frac{x^2+1}{x^2(x-1)}$$

$$Let \quad \frac{x^2+1}{x^2(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$$

Multiplying both sides by $x^2(x-1)$, we get

$x^2+1 = A(x)(x+1)+B(x-1)+Cx^2$ ....(1)

$x^2+1 = Ax^2-Ax+Bx-B+Cx^2$

$x^2+1 = Ax^2+Cx^2-Ax+Bx-B$ ....(2)

To find C, we put $x-1=0 \Rightarrow x=1$ in eq.(1), we get

$(1)^2+1 = A(1)(1-1)+B(1-1)+C(1)^2$

$1+1 = A(1)(0)+B(0)+C(1)$

$2 = A(0)+B(0)+C(1)$

$2 = C$

$or\ C = 2$

To find B, we put $x^2=0 \Rightarrow x=0$ in eq.(1), we get

$(0)^2+1 = A(0)(0-1)+B(0-1)+C(0)^2$

$1 = A(0)(-1)+B(-1)+C(0)$

$1 = -B$

$or\ B = -1$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A+C=1$

$A+2=1$

$A=1-2$

$A=-1$

Thus required partial fractions are $\frac{-1}{x}+\frac{-1}{x^2}+\frac{2}{x-1}$

Hence, $\frac{x^4+1}{x^2(x-1)} = x+1-\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x-1}$

(5) $\frac{7x+4}{(3x+2)(x+1)^2}$

Solution:

$$Let \qquad \frac{7x+4}{(3x+2)(x+1)^2} = \frac{A}{(3x+2)}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

Multiplying both sides by $(3x+2)(x+1)^2$, we get

$$7x+4 = A(x+1)^2 + B(3x+2)(x+1) + C(3x+2) \quad ....(1)$$

$$7x+4 = A(x^2+2x+1) + B(3x^2+5x+2) + C(3x+2)$$

$$7x+4 = Ax^2+2Ax+A+3Bx^2+5Bx+2B+3Cx+2C$$

$$7x+4 = Ax^2+3Bx^2+2Ax+5Bx+3Cx+A+2B+2C \quad ....(2)\backslash$$

To find A, we put 3x + 2 = 0 $\Rightarrow 3x=-2 \Rightarrow x=\frac{-2}{3}$ in eq.(1), we get

$$7\left(\frac{-2}{3}\right)+4 = A\left(\frac{-2}{3}+1\right)^2 + B\left(3\left(\frac{-2}{3}\right)+2\right)\left(\frac{-2}{3}+1\right)+C\left(3\left(\frac{-2}{3}\right)+2\right)$$

$$-\frac{14}{3}+4 = A\left(\frac{1}{3}\right)^2 + B(-2+2)\left(\frac{1}{3}\right)+C(-2+2)$$

$$-\frac{2}{3} = A\left(\frac{1}{9}\right)+B(0)+C(0)$$

$$-\frac{2}{3} = \frac{1}{9}A$$

$$or\ \frac{1}{9}A = -\frac{2}{3}$$

$$A = -\frac{2}{3}\times\frac{9}{1}$$

$$A = -6$$

To find C, we put $(x+1)^2=0 \Rightarrow x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$7(-1)+4=A(-1+1)^2+B(3(-1)+2)(-1+1)+C(3(-1)+2)$
$-7+4=A(0)^2+B(-3+2)(0)+C(-3+2)$
$-3=A(0)+B(0)+C(-1)$
$-3=-C$
$or\ -C=-3$
$\Rightarrow C=3$

To find B, equating coefficient of x^2 on both sides of (2), we get
$A+3B=0$
$-6+3B=0$
$3B=6$
$\Rightarrow B=2$

Thus required partial fractions are $\frac{-6}{3x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$

Hence, $\frac{7x+4}{(3x+2)(x+1)^2}=\frac{-6}{3x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$

(6) $\frac{1}{(x-1)^2(x+1)}$

Solution:

Let $\frac{1}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$

Multiplying both sides by $(x-1)^2(x+1)$, we get
$1=A(x-1)(x+1)+B(x+1)+C(x-1)^2$ ....(1)
$1=A(x^2-1)+B(x+1)+C(x^2-2x+1)$
$1=Ax^2-A+Bx+B+Cx^2-2Cx+C$
$1=Ax^2+Cx^2+Bx-2Cx-A+B+C$ ....(2)

To find C, we put $x+1=0 \Rightarrow x=-1$ in eq.(1), we get

$$1=A(-1-1)(-1+1)+B(-1+1)+C(-1-1)^2$$
$$1=A(-2)(0)+B(0)+C(-2)^2$$
$$1=A(0)+B(0)+C(4)$$
$$1=4C$$
$$or\ 4C=1$$
$$\Rightarrow C=\frac{1}{4}$$

To find B, we put $(x-1)^2=0 \Rightarrow x-1=0 \Rightarrow x=1$ in eq.(1), we get

$$1=A(1-1)(1+1)+B(1+1)+C(1-1)^2$$
$$1=A(0)(2)+B(2)+C(0)^2$$
$$1=A(0)+B(2)+C(0)$$
$$1=2B$$
$$or\ 2B=1$$
$$\Rightarrow B=\frac{1}{2}$$

To find A, equating coefficient of x^2 on both sides of (2), we get

$$A+C=0$$
$$A+\frac{1}{4}=0$$
$$\Rightarrow A=-\frac{1}{4}$$

Thus required partial fractions are $\frac{-1/4}{x-1}+\frac{1/2}{(x-1)^2}+\frac{1/4}{x+1}$

Hence, $\frac{1}{(x-1)^2(x+1)}=\frac{-1}{4(x-1)}+\frac{1}{2(x-1)^2}+\frac{1}{4(x+1)}$

(7) $\frac{3x^2+15x+16}{(x+2)^2}$

Solution:

$$=\frac{3x^2+15x+16}{x^2+4x+4}$$

This is an improper fraction so by long division, we have

$$\begin{array}{r|l} & 3 \\ x^2+4x+4 & 3x^2+15x+16 \\ & \pm 3x^2 \pm 12x \pm 12 \\ \hline & 3x+4 \end{array}$$

$$\frac{3x^2+15x+16}{x^2+4x+4}=3+\frac{3x+4}{(x+2)^2}$$

$$Let \quad \frac{3x+4}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$$

Multiplying both sides by $(x+2)^2$, we get

$$3x+4=A(x+2)+B \qquad ....(1)$$

$$3x+4=Ax+2A+B \qquad ....(2)$$

To find B, we put $(x+2)^2=0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq.(1), we get

$$3(-2)+4=A(-2+2)+B$$

$$-6+4=A(0)+B$$

$$-2=B$$

$$or \ B=-2$$

To find A, equating coefficient of x on both sides of (2), we get

$$A=3$$

Thus required partial fractions are $\frac{3}{x+2}+\frac{-2}{(x+2)^2}$

Hence, $\frac{3x^2+15x+16}{(x+2)^2}=3+\frac{3}{x+2}-\frac{2}{(x+2)^2}$

(8) $\dfrac{1}{(x^2-1)(x+1)}$

Solution:

$$=\frac{1}{(x-1)(x+1)(x+1)}$$
$$=\frac{1}{(x-1)(x+1)^2}$$

$$Let\quad \frac{1}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

Multiplying both sides by $(x-1)(x+1)^2$, we get

$$1=A(x+1)^2+B(x-1)(x+1)+C(x-1)\qquad ....(1)$$
$$1=A(x^2+2x+1)+B(x^2-1)+C(x-1)$$
$$1=Ax^2+2Ax+A+Bx^2-B+Cx-C$$
$$1=Ax^2+Bx^2+2Ax+Cx+A-B-C\qquad ....(2)$$

To find A, we put $x-1=0\Rightarrow x=1$ in eq.(1), we get

$$1=A(1+1)^2+B(1-1)(1+1)+C(1-1)$$
$$1=A(2)^2+B(0)(2)+C(0)$$
$$1=A(4)+B(0)+C(0)$$
$$1=4A$$
$$or\ 4A=1$$
$$\Rightarrow A=\frac{1}{4}$$

To find C, we put $(x+1)^2=0\Rightarrow x+1=0\Rightarrow x=-1$ in eq.(1), we get

$$1=A(-1+1)^2+B(-1-1)(-1+1)+C(-1-1)$$
$$1=A(0)^2+B(-2)(0)+C(-2)$$
$$1=A(0)+B(0)+C(-2)$$

$$1 = -2C$$
$$or\ -2C = 1$$
$$\Rightarrow C = -\frac{1}{2}$$

To find B, equating coefficient of x^2 on both sides of (2), we get

$$A + B = 0$$
$$\frac{1}{4} + B = 0$$
$$\Rightarrow B = -\frac{1}{4}$$

Thus required partial fractions are $\frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{(x+1)^2}$

Hence, $\frac{1}{(x^2-1)(x+1)} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}$