Chapter 4

Partial Fractions

In this unit, students will learn how to:

- Define proper, improper and rational fraction.
- Resolve an algebraic fraction into partial fractions when its denominator consists of
  - Non-repeated linear factors,
  - Repeated linear factors,
  - Non-repeated quadratic factors,
  - Repeated quadratic factors.

Fraction:

The quotient of two numbers or algebraic expressions is called a fraction. The quotient is indicated by a bar (—). We write, the dividend on top of the bar and the divisor below the bar. For example, $\frac{x^2+2}{x-2}$ is a fraction with $x - 2 \neq 0$. If $x - 2 = 0$, then the fraction is not defined because $x - 2 = 0 \Rightarrow x = 2$ which make the denominator of the fraction zero.

Rational Fraction:

An expression of the form $\frac{N(x)}{D(x)}$, where N (x) and D (x) are polynomials in x with real coefficients, is called a rational fraction. The polynomial $D(x) \neq 0$ in the expression. For example, $\frac{x^2+3}{(x+1)^2(x+2)}$ *and* $\frac{2x}{(x-1)(x+2)}$ are rational fractions.

Proper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ with D (x) ≠ 0 is called a proper fraction if degree of the polynomial N (x) in the numerator is less than the degree of the polynomial D (x) in the denominator. For example, $\frac{2}{x+1}, \frac{2x-3}{x^2+4}$ and $\frac{2x}{(x-1)(x+2)}$ are proper fractions.

Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ with D (x) ≠ 0 is called an improper fraction if degree of the polynomial N(x) is greater or equal to the degree of the polynomial D (x).

e.g., $\frac{5x}{x+2}, \frac{3x^2+2}{x^2+7x+12}, \frac{6x^4}{x^3+1}$ are improper fractions.

Every improper fraction can be reduced by division to the sum of a polynomial and a proper fraction. This means that if degree of the numerator is greater or equal to the degree of the denominator, then we can divide N (x) by D (x) obtaining a quotient polynomial Q (x) and a remainder polynomial R(x), whose degree is less than the degree of D (x).

Thus $\frac{N(x)}{D(x)}$= Q (x)+ $\frac{R(x)}{D(x)}$ with D (x) ≠ 0. Where Q (x) is quotient polynomial and $\frac{R(x)}{D(x)}$ is a proper fraction. For example, $\frac{x^2+1}{x+1}$ is an improper fraction.

$\frac{x^2+1}{x+1} = (x-1) + \frac{2}{x+1}$, i.e., an improper fraction, $\frac{x^2+1}{x+1}$ has been resolved to a quotient polynomial Q (x) = x - 1 and a proper fraction $\frac{2}{x+1}$.

Resolution of Fraction into Partial Fractions:

Consider $\frac{1}{x-1}, \frac{-2}{x+1}, \frac{4}{x}$ a set of three fractions each of which is prefixed by a positive or negative sign. It is easy to find a single fraction, which is equal to the sum of these fractions.

Thus $\frac{1}{x-1} - \frac{2}{x+1} + \frac{4}{x} = \frac{x(x+1) - 2x(x-1) + 4(x-1)(x+1)}{x(x-1)(x+1)}$

$$= \frac{x^2 + x - 2x^2 + 2x + 4x^2 - 4}{x(x-1)(x+1)}$$

$$= \frac{3x^2 + 3x - 4}{x(x-1)(x+1)}$$

The single fraction $\frac{3x^2 + 3x - 4}{x(x-1)(x+1)}$ is the simplified form of the given fractions and is known as resultant fraction. The given fractions $\frac{1}{x-1}, \frac{-2}{x+1}$ and $\frac{4}{x}$ are called components or partial fraction.

Every proper fraction $\frac{N(x)}{D(x)}$ with $D(x) \neq 0$ can be resolved into an algebraic sum of partial fractions as follows:

Resolution of an algebraic fraction into partial fractions, when D(x) consists of non-repeated linear factors:

Rule I:

If linear factor (ax + b) occurs as a factor of D(x), then there is a partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be found.

In $\frac{N(x)}{D(x)}$, the polynomial D(x) may be written as,

$D(x) = (a_1x + b_1)(a_2x+b_2).......(a_nx + b_n)$ with all factors distinct.

We have, $\frac{N(x)}{D(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3} + ... + \frac{A_n}{a_nx+b_n}$,

where $A_1, A_2, \ldots A_n$ are constants to-be determined.

Note:

General method applicable to resolve all rational fractions of the form $\frac{N(x)}{D(x)}$ is as follows:

(i) The numerator N(x) must be of lower degree than the denominator D(x).

(ii) If degree of N(x) is greater than the degree of D(x), then division is used and the remainder fraction R(x) can be broken into partial fractions.

(iii) Make substitution of constants accordingly

(iv) Multiply both the sides by L.C.M.

(v) Arrange the terms on both sides in descending order.

(vi) Equate the coefficients of like powers of x on both sides, we get as many as equations as there are constants in assumption.

(vii) Solving these equations, we can find the values of constants.