Find 3rd, 4th, mean and continued proportion:

We are already familiar with proportions that if quantities a, b, c and d are in proportion, then a : b :: c : d

i.e., product of extremes = product of means

Third Proportional

If three quantities a, b and c are related as a : b :: b : c, then c is called the third proportion.

Fourth Proportional

If four quantities a, b, c and d are related as a: b :: c : d

Then d is called the fourth proportional.

Mean Proportional

If three quantities a, b and c are related as a : b :: b : c, then b is called the mean proportional.

Continued Proportion

If three quantities a, b and c are related as

a : b :: b : c

where a is first, b is the mean and c is the third proportional, then a, b and c are in continued proportion.

Exercise 3.3

(i) 6, 12

Solution:

Let C be the third proportional, then

6:12 :: 12:C

∴ Product of extremes = Product of means

6C = 12 x 12

6C = 144

$C = \frac{144}{6}$

C = 24

(ii) $a^3, 3a^2$

Solution:

Let C be the third proportional, then

$$a^3 : 3a^2 :: 3a^2 : C$$

∴ Product of extremes = Product of means

$$(C)(a^3) = (3a^2)(3a^2)$$

$$C = \frac{(3a^2)(3a^2)}{a^3}$$

$$C = \frac{9a^4}{a^3}$$

$$C = 9a$$

(iii) $a^2 - b^2, a - b$

Solution:

Let C be the third proportional, then

$\therefore$ Product of extremes = Product of means

$$\left(a^2-b^2\right)(C)=(a-b)(a-b)$$

$$C=\frac{(a-b)(a-b)}{\left(a^2-b^2\right)}$$

$$C=\frac{(a-b)(a-b)}{(a-b)(a+b)}$$

$$C=\frac{(a-b)}{(a+b)}$$

(iv) $(x-y)^2, x^3-y^3$

Solution:

Let C be the third proportional, then

$$(x-y)^2 : x^3-y^3 :: x^3-y^3 : C$$

$\therefore$ Product of extremes = Product of means

$$(C)(x-y)^2=\left(x^3-y^3\right)\left(x^3-y^3\right)$$

$$C=\frac{(x-y)\left(x^2+xy+y^2\right)(x-y)\left(x^2+xy+y^2\right)}{(x-y)^2}$$

$$C=\frac{(x-y)^2\left(x^2+xy+y^2\right)^2}{(x-y)^2}$$

$$C=\left(x^2+xy+y^2\right)^2$$

(v) $(x+y)^2, x^2-xy-2y^2$

Solution:

Let C be the third proportional, then

$$(x+y)^2 : x^2-xy-2y^2 :: x^2-xy-2y^2 : C$$

$$C=\frac{\left(x^2-xy-2y^2\right)^2}{(x+y)^2}$$

$$C=\frac{\left(x^2-2xy+xy-2y^2\right)^2}{(x+y)^2}$$

$$C=\frac{\left[x(x-2y)+y(x-2y)\right]^2}{(x+y)^2}$$

$$C=\frac{\left[(x-2y)(x+y)\right]^2}{(x+y)^2}$$

$$C=\frac{(x-2y)^2(x+y)^2}{(x+y)^2}$$

$$C=(x-2y)^2$$

(vi) $\dfrac{p^2-q^2}{p^3+q^3}, \dfrac{p-q}{p^2-pq+q^2}$

Solution:

Let C be the third proportional, then

$$\frac{p^2-q^2}{p^3+q^3}:\frac{p-q}{p^2-pq+q^2}::\frac{p-q}{p^2-pq+q^2}:C$$

$\therefore$ Product of extremes = Product of means

$$(C)\left(\frac{p^2-q^2}{p^3+q^3}\right)=\left(\frac{p-q}{p^2-pq+q^2}\right)\left(\frac{p-q}{p^2-pq+q^2}\right)$$

$$C=\frac{(p-q)^2}{\left(p^2-pq+q^2\right)^2}\times\frac{p^3+q^3}{p^2-q^2}$$

$$C=\frac{(p-q)^2}{\left(p^2-pq+q^2\right)^2}\times\frac{(p+q)\left(p^2-pq+q^2\right)}{(p+q)(p-q)}$$

$$C=\frac{p-q}{p^2-pq+q^2}$$

2. Find a fourth proportional to

(i) 5, 8, 15

Solution:

Let x be the fourth proportional, then

$$5 : 8 :: 15 : x$$

$\therefore$ Product of extremes = Product of means

$$(5)(x) = (8)(15)$$

$$x = \frac{(8)(15)}{(5)}$$

$$x = 8 \times 3 = 24$$

(ii) $4x^4, 2x^3, 18x^5$

Solution:

Let C be the fourth proportional, then

$$4x^4 : 2x^3 :: 18x^5 : C$$

$\therefore$ Product of extremes = Product of means

$$(4x^4)(C) = (2x^3)(18x^5)$$

$$C = \frac{36x^8}{4x^4}$$

$$C = 9x^4$$

(iii) $15a^5b^6, 10a^2b^5, 21a^3b^3$

Solution:

Let x be the fourth proportional, then

$$15a^5b^6 : 10a^2b^5 :: 21a^3b^3 : x$$

$\therefore$ Product of extremes = Product of means

$$x(15a^5b^6) = (10a^2b^5)(21a^3b^3)$$

$$x = 2 \times 7b^2$$
$$x = 14b^2$$

(iv) $x^2 - 11x + 24, (x-3), 5x^4 - 40x^3$

Solution:

Let x be the fourth proportional, then

$$x^2 - 11x + 24 : (x-3) :: 5x^4 - 40x^3 : C$$

$\therefore$ Product of extremes = Product of means

$$C(x^2 - 11x + 24) = (x-3)(5x^4 - 40x^3)$$
$$C = \frac{(x-3)[5x^3(x-8)]}{x^2 - 11x + 24}$$
$$C = \frac{5x^3(x-3)(x-8)}{x^2 - 8x - 3x + 24}$$
$$C = \frac{5x^3(x-3)(x-8)}{x(x-8) - 3(x-8)}$$
$$C = \frac{5x^3(x-3)(x-8)}{(x-3)(x-8)}$$
$$C = 5x^3$$

(v) $p^3 + q^3, p^2 - q^2, p^2 - pq + q^2$

Solution:

Let C be the fourth proportional, then

$$p^3 + q^3 : p^2 - q^2 :: p^2 - pq + q^2 : C$$

$\therefore$ Product of extremes = Product of means

$$C(p^3 + q^3) = (p^2 - q^2)(p^2 - pq + q^2)$$
$$C = \frac{(p-q)(p+q)(p^2 - pq + q^2)}{(p^3 + q^3)}$$

$$C = \frac{(p-q)(p^3+q^3)}{(p^3+q^3)}$$

$$C = p - q$$

(vi) $(p^2-q^2)(p^2+pq+q^2), p^3+q^3, p^3-q^3$

Solution:

Let x be the fourth proportional, then

$$(p^2-q^2)(p^2+pq+q^2) : p^3+q^3 :: p^3-q^3 : x$$

$\therefore$ Product of extremes = Product of means

$$x(p^2-q^2)(p^2+pq+q^2) = (p^3+q^3)(p^3-q^3)$$

$$x = \frac{(p^3+q^3)(p^3-q^3)}{(p^2-q^2)(p^2+pq+q^2)}$$

$$x = \frac{(p^3+q^3)(p^3-q^3)}{(p+q)(p-q)(p^2+pq+q^2)}$$

$$C = \frac{(p+q)(p^2-pq+q^2)(p^3-q^3)}{(p+q)(p^3-q^3)}$$

$$C = p^2 - pq + q^2$$

3. Find a mean proportional between

(i) 20,45

Solution:

Let m be the mean proportional, then

$$20 : m :: m : 45$$

$\therefore$ Product of means = Product of extremes

$$m \times m = 20 \times 45$$

$$m^2 = 900$$

(ii) $20x^3y^5, 5x^7y$

Solution:

Let m be the mean proportional, then

$$20x^3y^5 : m :: m : 5x^7y$$

$\therefore$ Product of means = Product of extremes

$$m \times m = 20x^3y^5 \times 5x^7y$$
$$m^2 = 100x^{10}y^6$$
$$m = \pm\sqrt{100x^{10}y^6}$$
$$m = \pm\left(100x^{10}y^6\right)^{\frac{1}{2}}$$
$$m = \pm\left(10^2\right)^{\frac{1}{2}}\left(x^{10}\right)^{\frac{1}{2}}\left(y^6\right)^{\frac{1}{2}}$$
$$m = \pm 10x^5y^3$$

(iii) $15p^4qr^3, 135q^5r^7$

Solution:

Let m be the mean proportional, then

$$15p^4qr^3 : m :: m : 135q^5r^7$$

$\therefore$ Product of means = Product of extremes

$$m \times m = 15p^4qr^3 \times 135q^5r^7$$
$$m \times m = 15 \times 135p^4q^6r^{10}$$
$$m^2 = 2025p^4q^6r^{10}$$
$$m = \pm\sqrt{2025p^4q^6r^{10}}$$
$$m = \pm\left(2025p^4q^6r^{10}\right)^{\frac{1}{2}}$$
$$m = \pm\left(45^2\right)^{\frac{1}{2}}\left(p^4\right)^{\frac{1}{2}}\left(q^6\right)^{\frac{1}{2}}\left(r^{10}\right)^{\frac{1}{2}}$$
$$m = \pm 45p^2q^3r^5$$

Solution:

Let m be the mean proportional, then

$$x^2 - y^2 : m :: m : \frac{x-y}{x+y}$$

$\therefore$ Product of means = Product of extremes

$$m \times m = x^2 - y^2 \times \frac{x-y}{x+y}$$

$$m^2 = (x-y)(x+y) \times \frac{x-y}{x+y}$$

$$m^2 = (x-y)(x-y)$$

$$m^2 = (x-y)^2$$

$$m = \pm\sqrt{(x-y)^2}$$

$$m = \pm(x-y)$$

4. Find the values of the letter involved in the following continued proportions.

(i) 5, p, 45

Solution:

Since 5, p and 45 are in continued proportions.

$$5 : p :: p : 45$$

$\therefore$ Product of means = Product of extremes

$$p \times p = 5 \times 45$$

$$p^2 = 225$$

$$p = \pm\sqrt{225}$$

$$p = \pm 15$$

(ii) 8, x, 18

Solution:

$\therefore$ Product of means = Product of extremes

$$x \times x = 8 \times 18$$
$$x^2 = 144$$
$$x = \pm\sqrt{144}$$
$$x = \pm 12$$

(iii) 12, 3p-6, 27

Solution:

Since $12, 3p-6$ and 27 are in continued proportions.

$$12 : 3p-6 :: 3p-6 : 27$$

$\therefore$ Product of means = Product of extremes

$$(3p-6)(3p-6) = 12 \times 27$$
$$(3p-6)^2 = 324$$
$$\sqrt{(3p-6)^2} = \pm\sqrt{324}$$
$$3p-6 = \pm 18$$
$$3p-6 = -18 \quad or \quad 3p-6 = 18$$
$$3p = 6-18 \quad or \quad 3p = 18+6$$
$$3p = -12 \quad or \quad 3p = 24$$
$$p = -4 \quad or \quad p = 8$$

(iv) 7, m-3, 28

Solution:

Since $7, \mathrm{m}-3$ and 28 are in continued proportions.

$$7 : \mathrm{m}-3 :: \mathrm{m}-3 : 28$$

$\therefore$ Product of means = Product of extremes

$$(\mathrm{m}-3)(\mathrm{m}-3) = 7 \times 28$$
$$(\mathrm{m}-3)^2 = 196$$
$$\sqrt{(\mathrm{m}-3)^2} = \pm\sqrt{196}$$
$$\mathrm{m}-3 = \pm 14$$

$$m = 3 - 14 \quad or \quad m = 3 + 14$$
$$m = -11 \quad or \quad m = 17$$