

Symmetric functions of the roots of a quadratic equation:

Define symmetric functions of the roots of a quadratic equation:

Definition:

Symmetric functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged.

For example, if

$$\begin{aligned} f(\alpha, \beta) &= \alpha^2 + \beta^2, \text{ then} \\ f(\beta, \alpha) &= \beta^2 + \alpha^2 = \alpha^2 + \beta^2 && (\because \beta^2 + \alpha^2 = \alpha^2 + \beta^2) \\ &= f(\alpha, \beta) \end{aligned}$$

Exercise 2.4

1. If α, β are the roots of the equation $x^2+px+q=0$, then evaluate

(i) $\alpha^2 + \beta^2$

Solution:

$$\alpha^2 + \beta^2$$

$$x^2 + px + q = 0$$

Here $a=1, b=p, c=q$

Let α, β be the roots of the given equation

$$\text{Then } \alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

$$= -\frac{p}{1} \quad \alpha\beta = \frac{q}{1}$$

$$\begin{aligned}
 \text{Now } \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\
 &= (-p)^2 - 2(q) \\
 &= p^2 - 2q
 \end{aligned}$$

$$(ii) \alpha^3\beta + \alpha\beta^3$$

Solution:

$$\alpha^3\beta + \alpha\beta^3$$

$$x^2 + px + q = 0$$

Here $a=1$, $b=p$, $c=q$

Let α , β be the roots of the given equation

$$\begin{aligned}
 \text{Then } \alpha + \beta &= -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \\
 &= -\frac{p}{1} \quad \alpha\beta = \frac{q}{1} \\
 &= -p \quad \quad \quad = q
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \alpha^3\beta + \alpha\beta^3 &= \alpha\beta(\alpha^2 + \beta^2) \\
 &= \alpha\beta[(\alpha + \beta)^2 - 2\alpha\beta] \\
 &= q[(-p)^2 - 2q] \\
 &= q(p^2 - 2q)
 \end{aligned}$$

$$(iii) \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

Solution:

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$x^2 + px + q = 0$$

Here $a=1$, $b=p$, $c=q$

Let α, β be the roots of the given equation

$$\begin{aligned} \text{Then } \alpha + \beta &= -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \\ &= -\frac{p}{1} \quad \alpha\beta = \frac{q}{1} \\ &= -p \quad = q \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{\alpha}{\beta} + \frac{\beta}{\alpha} &= \frac{\alpha^2 + \beta^2}{\alpha\beta} \\ &= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} \\ &= \frac{(-p)^2 - 2q}{q} = \frac{1}{q}(p^2 - 2q) \end{aligned}$$

2. If α, β are the roots of the equation $4x^2 - 5x + 6 = 0$, then find the values of

$$(i) \frac{1}{\alpha} + \frac{1}{\beta}$$

Solution:

$$\frac{1}{\alpha} + \frac{1}{\beta}$$

$$4x^2 - 5x + 6 = 0$$

$$\text{Here } a=4, b=-5, c=6$$

Let α, β be the roots of the given equation

$$\begin{aligned} \text{Then } \alpha + \beta &= -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \\ &= -\frac{(-5)}{4} \quad \alpha\beta = \frac{6}{4} \\ &= \frac{5}{4} \quad = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\alpha + \beta}{\alpha\beta} \\ &= \frac{\frac{5}{4}}{\frac{3}{2}} = \frac{5}{4} \times \frac{2}{3} = \frac{5}{6} \end{aligned}$$

(ii) $\alpha^2 \beta^2$

Solution:

$$\alpha^2 \beta^2$$

$$4x^2 - 5x + 6 = 0$$

$$\text{Here } a=4, b=-5, c=6$$

Let α, β be the roots of the given equation

$$\begin{aligned} \text{Then } \alpha + \beta &= -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \\ &= -\frac{(-5)}{4} \quad \alpha\beta = \frac{6}{4} \\ &= \frac{5}{4} \quad \quad \quad = \frac{3}{2} \end{aligned}$$

$$\text{Now } \alpha^2 \beta^2 = (\alpha\beta)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

(iii) $\frac{1}{\alpha^2 \beta} + \frac{1}{\alpha \beta^2}$

Solution:

$$4x^2 - 5x + 6 = 0$$

$$\text{Here } a=4, b=-5, c=6$$

Let α, β be the roots of the given equation

$$\begin{aligned} \text{Then } \alpha + \beta &= -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \\ &= -\frac{(-5)}{4} \quad \alpha\beta = \frac{6}{4} \\ &= \frac{5}{4} \quad \quad \quad = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{1}{\alpha^2 \beta} + \frac{1}{\alpha \beta^2} &= \frac{\alpha + \beta}{\alpha^2 \beta^2} = \frac{\alpha + \beta}{(\alpha\beta)^2} \\ &= \frac{\frac{5}{4}}{\left(\frac{3}{2}\right)^2} = \frac{\frac{5}{4}}{\frac{9}{4}} = \frac{5}{4} \times \frac{4}{9} = \frac{5}{9} \end{aligned}$$

$$(iv) \quad \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$$

Solution:

$$4x^2 - 5x + 6 = 0$$

$$\text{Here } a=4, b=-5, c=6$$

Let α, β be the roots of the given equation

$$\begin{aligned} \text{Then } \alpha + \beta &= -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \\ &= -\frac{(-5)}{4} \quad \alpha\beta = \frac{6}{4} \\ &= \frac{5}{4} \quad \quad \quad = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} &= \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta} \\ &= \frac{\left(\frac{5}{4}\right)^3 - 3\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)}{\frac{3}{2}} = \frac{\frac{125}{64} - \frac{45}{8}}{\frac{3}{2}} \\ &= \frac{125 - 360}{64} \times \frac{2}{3} = -\frac{235}{64} \times \frac{2}{3} \\ &= -\frac{235}{96} \end{aligned}$$

3. If α, β are the roots of the equation $lx^2+mx+n=0$ ($l \neq 0$), then find the values of:

$$(i) \quad \alpha^3\beta^2 + \alpha^2\beta^3$$

Solution:

$$lx^2 + mx + n = 0$$

$$\text{Here } a=l, b=m, c=n$$

$$\begin{aligned} \text{Then } \alpha + \beta &= -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \\ &= -\frac{m}{l} \quad \alpha\beta = \frac{n}{l} \end{aligned}$$

$$\begin{aligned} \text{Now } \alpha^3\beta^2 + \alpha^2\beta^3 &= \alpha^2\beta^2(\alpha + \beta) = (\alpha\beta)^2(\alpha + \beta) \\ &= \left(\frac{n}{l}\right)^2 \left(-\frac{m}{l}\right) = -\frac{mn^2}{l^3} \end{aligned}$$

$$\text{(ii) } \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

Solution:

$$lx^2 + mx + n = 0$$

Here $a=l$, $b=m$, $c=n$

Let α, β be the roots of the given equation

$$\begin{aligned} \text{Then } \alpha + \beta &= -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \\ &= -\frac{m}{l} \quad \alpha\beta = \frac{n}{l} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{1}{\alpha^2} + \frac{1}{\beta^2} &= \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} \\ &= \frac{\left(-\frac{m}{l}\right)^2 - 2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2} = \frac{\frac{m^2}{l^2} - \frac{2n}{l}}{\frac{n^2}{l^2}} \\ &= \frac{m^2 - 2nl}{l^2} \times \frac{l^2}{n^2} = \frac{m^2 - 2nl}{n^2} \\ &= \frac{1}{n^2} [m^2 - 2nl] \end{aligned}$$

