Roots and co-efficient of a quadratic equation:

We know that $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$ are roots of the equation $ax^2+bx+c=0$ where a, b are coefficients of x^2 and x respectively. While c is the constant term.

Relation between roots and co-efficient of a quadratic equation:

If $\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

then we can find the sum and the product of the roots as follows.

Sum of the roots = $\alpha+\beta$

$$=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}$$

Product of the roots = $\alpha\beta$

$$=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}=\frac{b^2-\left(b^2-4ac\right)}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$$

If we denote the sum of roots and product of roots by S and P respectively,then

$$S=-\frac{b}{a}=-\frac{\text{Co-efficient of x}}{\text{Co-efficient of x}^2} \quad and \quad P=\frac{c}{a}=\frac{\text{Constant term}}{\text{Co-efficient of x}^2}$$

Exercise 2.3

1. Without solving, find the sum and the product of the following quadratic equations.

(i) $x^2 - 5x + 3 = 0$

Solution:

$x^2 - 5x + 3 = 0$

Here a=1, b=-5, c=3

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha + \beta = -\frac{b}{a} = -\frac{(-5)}{1} = 5$

And product of roots = $\alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$

(ii) $3x^2 + 7x - 11 = 0$

Solution:

$3x^2 + 7x - 11 = 0$

Here a=3, b=7, c=-11

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha + \beta = -\frac{b}{a} = -\frac{7}{3}$

And product of roots = $\alpha\beta = \frac{c}{a} = -\frac{11}{3}$

(iii) $px^2 - qx + r = 0$

Solution:

$px^2 - qx + r = 0$

Here a=p, b=-q, c=r

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{(-q)}{p} = \dfrac{q}{p}$

And product of roots = $\alpha\beta = \dfrac{c}{a} = \dfrac{r}{p}$

(iv) $(a+b)x^2 - ax + b = 0$

Solution:

$(a+b)x^2 - ax + b = 0$

Here a=$a+b$, b=-a, c=b

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{(-a)}{a+b} = \dfrac{a}{a+b}$

And product of roots = $\alpha\beta = \dfrac{c}{a} = \dfrac{b}{a+b}$

(v) $(l+\mathrm{m})x^2 + (m+n)x + n - l = 0$

Solution:

$(l+\text{m})x^2+(m+n)x+n-l=0$

Here a=$l+\text{m}$, b=$m+n$, c=$n-l$

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{m+n}{l+\text{m}}$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{n-l}{l+\text{m}}$

(vi) $7x^2-5mx+9n=0$

Solution:

$7x^2-5mx+9n=0$

Here a=7, b=$-5m$, c=9n

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-5m)}{7}=\frac{5m}{7}$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{9n}{7}$

2. Find the value of k, if

(i) Sum of the roots of the equation $2kx^2$-3x+4k=0 is twice the product of the roots.

Solution:

$2kx^2-3x+4k=0$

Here a=2k, b=-3, c=4k

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-3)}{2k}=\frac{3}{2k}$

And product of roots $= \alpha\beta = \frac{c}{a} = \frac{4k}{2k} = 2$

As sum of the roots is twice the product of the roots, so

$\alpha + \beta = 2\alpha\beta$

$\frac{3}{2k} = 2(2)$

$\frac{3}{2k} = 4$

$or\ k = \frac{3}{8}$

(ii) Sum of the roots of the equation $x^2 + (3k-7)x + 5k = 0$ is $\frac{3}{2}$ times the product of the roots.

Solution:

$x^2 + (3k-7)x + 5k = 0$

Here a=1, b=$3k-7$, c=5k

Let α and β be the roots of the given equation

Then Sum of roots $= \alpha + \beta = -\frac{b}{a} = -\frac{(3k-7)}{1} = -3k + 7$

And product of roots $= \alpha\beta = \frac{c}{a} = \frac{5k}{1} = 5k$

As sum of the roots is $\frac{3}{2}$ of the product of the roots, so

$\alpha + \beta = \frac{3}{2}\alpha\beta$

$-3k + 7 = \frac{3}{2}(5k)$

$-3k + 7 = \frac{15k}{2}$

$-6k + 14 = 15k$

$15k + 6k = 14$

$21k = 14$

$k = \frac{14}{21}$

$k = \frac{2}{3}$

3. Find k, if

(i) Sum of the squares of the roots of the equation $4kx^2 + 3kx - 8 = 0$ is 2.

Solution:

$4kx^2 + 3kx - 8 = 0$

Here a=4k, b=$3k$, c=-8

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha + \beta = -\frac{b}{a} = -\frac{3k}{4k} = -\frac{3}{4}$

And product of roots = $\alpha\beta = \frac{c}{a} = \frac{-8}{4k}$

As sum of the square of roots is twice the product of the roots, so

$\alpha^2 + \beta^2 = 2$

$(\alpha + \beta)^2 - 2\alpha\beta = 2 \qquad \because (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$\left(-\frac{3}{4}\right)^2 - 2\left(\frac{-8}{4k}\right) = 2 \qquad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\frac{9}{16} + \frac{16}{4k} = 2$

$\frac{16}{4k} = 2 - \frac{9}{16} \quad \rightarrow \quad \frac{16}{4k} = \frac{32-9}{16}$

$\frac{16}{4k} = \frac{23}{16} \quad \rightarrow \quad 23 \times 4k = 16 \times 16$

$k = \frac{16 \times 16}{23 \times 4} \quad \rightarrow \quad k = \frac{64}{23}$

(ii) Sum of the squares of the roots of the equation $x^2-2kx+(2k+1)=0$ is 6.

Solution:

$x^2-2kx+(2k+1)=0$

Here a=1, b=-2k, c=2k+1

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-2k)}{1}=2k$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{2k+1}{1}=2k+1$

As sum of the square of roots is 6 to the product of the roots, so

$\alpha^2+\beta^2=6$

$(\alpha+\beta)^2-2\alpha\beta=6 \qquad \because (\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$

$(2k)^2-2(2k+1)=6 \qquad \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$4k^2-4k-2=6$

$4k^2-4k-2-6=0$

$4k^2-4k-8=0$

$4(k^2-k-2)=0$

$\Rightarrow k^2-k-2=0$

$k^2-2k+k-2=0$

$k(k-2)+1(k-2)=0$

$(k-1)(k-2)=0$

Either $k+1=0$ *or* $k-2=0$

$k=-1 \qquad k=2$

4. Find p, if

(i) The roots of the equation $x^2-x+p^2=0$ differ by unity.

Solution:

$x^2 - x + p^2 = 0$

Here a=1, b=-1, c=p^2

Let α and α-1 be the roots of the given equation

Then $\alpha+\alpha\text{-}1 = -\dfrac{b}{a}$ *and* $\alpha(\alpha\text{-}1) = \dfrac{c}{a}$

$2\alpha - 1 = -\dfrac{(-1)}{1}$ $\quad$ $\alpha^2\text{-}1 = \dfrac{p^2}{1}$

$2\alpha - 1 = 1$ $\quad$ $\alpha^2\text{-}1 = p$

$2\alpha = 1+1$ $\quad$ *put* $\alpha = 1$ in above eq.,we get

$2\alpha = 2$ $\quad$ $(1)^2 - 1 = p$

$\alpha = 1$ $\quad$ $1 - 1 = p$

$p = 0$

(ii) The roots of the equation x^2 +3x+p-2=0 differ by 2.

Solution:

$x^2 + 3x + p - 2 = 0$

Here a=1, b=3, c=$p - 2$

Let α and $\alpha - 2$ be the roots of the given equation

Then $\alpha+\alpha - 2 = -\dfrac{b}{a}$ *and* $\alpha(\alpha - 2) = \dfrac{c}{a}$

$2\alpha - 2 = -\dfrac{3}{1}$ $\quad$ $\alpha^2 - 2\alpha = \dfrac{p-2}{1}$

$2\alpha - 2 = -3$ $\quad$ $\alpha^2 - 2\alpha = p - 2$

$2\alpha = -3 + 2$ $\quad$ *put* $\alpha = -\dfrac{1}{2}$ in above eq.,we get

$\alpha = -\dfrac{1}{2}$ $\quad$ $\left(-\dfrac{1}{2}\right)^2 - 2\left(-\dfrac{1}{2}\right) = p - 2$

$$\frac{1}{4}+1=p-2$$
$$\frac{1}{4}=p-2-1$$
$$\frac{1}{4}=p-3$$
$$4p-12=1$$
$$4p=1+12$$
$$4p=13$$
$$p=\frac{13}{4}$$

5. Find m, if

(i) The roots of the equation $x^2-7x+3m-5=0$ satisfy the relation $3\alpha+2\beta=4$.

Solution:

$x^2-7x+3m-5=0$

Here a=1, b= −7, c=$3m-5$

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{(-7)}{1}=7$

And product of roots = $\alpha\beta=\frac{c}{a}=\frac{3m-5}{1}=3m-5$

Now $\alpha+\beta=7$

$\beta=7-\alpha$ ....(i) and $\alpha\beta=3m-5$ ....(ii)

Since $3\alpha+2\beta=4$ ....(iii)

Put β in eq(iii), we have

$3\alpha+2(7-\alpha)=4$

$3\alpha+14-2\alpha=4$

$3\alpha-2\alpha+14=4$

$\alpha=4-14$

$\alpha=-10$

Put $\alpha = -10$ in eq.(i), we get

$$\beta = 7 + 10$$
$$\beta = 17$$

Put $\alpha = -10, \beta = 17$ in eq.(ii), we get

$$(-10)(17) = 3m - 5$$
$$5 - 170 = 3m$$

or $3m = -165$

$$m = -55$$

(ii) The roots of the equation $x^2 + 7x + 3m - 5 = 0$ satisfy the relation $3\alpha - 2\beta = 4$.

Solution:

$x^2 + 7x + 3m - 5 = 0$

Here a=1, b=7, c=$3m - 5$

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{(7)}{1} = -\dfrac{7}{1} = -7$

And product of roots = $\alpha\beta = \dfrac{c}{a} = \dfrac{3m-5}{1} = 3m - 5$

Now $\alpha + \beta = -7$

$\beta = -7 - \alpha$ ....(i) *and* $\alpha\beta = 3m - 5$ ....(ii)

Since $3\alpha - 2\beta = 4$ ....(iii)

Put β in eq(iii), we have

$$3\alpha - 2(-7 - \alpha) = 4$$
$$3\alpha + 14 + 2\alpha = 4$$
$$3\alpha + 2\alpha = 4 - 14$$
$$5\alpha = -10$$
$$\alpha = -2$$

Put $\alpha = -2$ in eq.(i), we get

$$\beta = -7-(-2)$$
$$\beta = -7+2$$
$$\beta = -5$$

Put $\alpha = -2$ *and* $\beta = -5$ in eq.(iii), we get

$$(-2)(-5) = 3m-5$$
$$10 = 3m-5$$
or $$3m = 10+5$$
$$3m = 15$$
$$m = 5$$

(iii) The roots of the equation $3x^2$ -2x+7m+2=0 satisfy the relation $7\alpha - 3\beta = 18$.

Solution:

$3x^2 - 2x + 7m + 2 = 0$

Here a=3, b=-2, c=$7m+2$

Let α and β be the roots of the given equation

Then Sum of roots = $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$$\alpha + \beta = -\frac{(-2)}{3} \qquad \alpha\beta = \frac{7m+2}{3} \quad ....(ii)$$

$$\alpha + \beta = \frac{2}{3}$$

$$\beta = \frac{2}{3} - \alpha \qquad ....(i)$$

Since $7\alpha - 3\beta = 18$ $........(iii)$

Put $\beta = \frac{2}{3} - \alpha$ in eq(iii), we have

$$7\alpha - 3\left(\frac{2}{3} - \alpha\right) = 18$$
$$7\alpha - 2 + 3\alpha = 18$$
$$7\alpha + 3\alpha = 18 + 2$$

$$10\alpha = 20$$
$$\alpha = 2$$

Put $\alpha = 2$ in eq.(i), we get

$$\beta = \frac{2}{3} - 2$$
$$\beta = \frac{2-6}{3}$$
$$\beta = -\frac{4}{3}$$

Put $\alpha = 2$ *and* $\beta = -\frac{4}{3}$ in eq.(ii), we get

$$(2)\left(-\frac{4}{3}\right) = \frac{7m+2}{3}$$
$$-\frac{8}{3} = \frac{7m+2}{3}$$
$$-\frac{8}{3} \times 3 = 7m+2$$
$$-8 = 7m+2$$
$$7m = -8-2$$

or $7m = -10$

$$m = -\frac{10}{7}$$

6. Find m, if sum and product of the roots of the following equations is equal to a given number λ.

(i) $(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Solution:

$(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Here a=2m+3, b=7m – 5, c=$3m-10$

Let α and β be the roots of the given equation

Then $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$$\alpha+\beta=-\frac{(7m-5)}{2m+3} \qquad \alpha\beta=\frac{3m-10}{2m+3} \quad ....(ii)$$

As $\quad \alpha+\beta=\alpha\beta=\lambda$

So $\quad \lambda=-\frac{(7m-5)}{2m+3} \quad ....(i)$ *and* $\lambda=\frac{3m-10}{2m+3} \quad ....(ii)$

Comparing eq.(i) and eq (ii), we get

$$-\frac{(7m-5)}{2m+3}=\frac{3m-10}{2m+3}$$
$$-(7m-5)=\frac{3m-10}{2m+3}\times(2m+3)$$
$$-7m+5=3m-10$$
$$3m+7m=5+10$$
$$10m=15$$
$$m=\frac{15}{10}$$
$$m=\frac{3}{2}$$

(ii) $4x^2-(3+5m)x-(9m-17)=0$

Solution:

$4x^2-(3+5m)x-(9m-17)=0$

Here a=4, b= $-(3+5m)$, c= $-(9m-17)$

Let $\quad \alpha$ and β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$$\alpha+\beta=-\frac{[-(3+5m)]}{4} \qquad \alpha\beta=\frac{-(9m-17)}{4} \quad ....(ii)$$
$$\alpha+\beta=\frac{3+5m}{4}$$

As $\quad \alpha+\beta=\alpha\beta=\lambda$

So $\quad \lambda=\frac{3+5m}{4} \quad ....(i)$ *and* $\lambda=\frac{-(9m-17)}{4} \quad ....(ii)$

Comparing eq.(i) and eq(ii), we get

$$\frac{3+5m}{4} = \frac{-(9m-17)}{4}$$

$$4(3+5m) = -4(9m-17)$$

$$3+5m = -(9m-17)$$

$$3+5m = -9m+17$$

$$5m+9m = 17-3$$

$$14m = 14$$

$$\Rightarrow \quad m = 1$$