Cube roots of unity and their properties

The cube roots of unity:

Let a number x be the cube root of unity,

i.e. $x=(1)^{\frac{1}{3}}$

Solution:

or $x^3=1$

$\Rightarrow$ $x^3-1=0$

$(x)^3-(1)^3=0$

$(x-1)(x^2+x+1)=0$ $\left[\text{using } a^3-b^3=(a-b)(a^2+ab+b^2)\right]$

Either $x-1=0$ *or* $x^2+x+1=0$

$\Rightarrow$ $x=1$ *or* $x=\dfrac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}$

$$x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm i\sqrt{3}}{2}$$

Three cube roots of unity are

$1, \dfrac{-1+i\sqrt{3}}{2}$ *and* $\dfrac{-1-i\sqrt{3}}{2}$, where $i=\sqrt{-1}$

Recognize complex cube roots of unity as ω and ω^2 :

The two complex cube roots of unity are $\dfrac{-1+\sqrt{-3}}{2}$ and $\dfrac{-1-\sqrt{-3}}{2}$.

If we name anyone of these as w (pronounced as omega), then the other is ω^2

Properties of cube roots of unity:

(a) Prove that each of the complex cube roots of unity is the square of the other.

Proof:

The complex cube roots of unity are $\dfrac{-1+\sqrt{-3}}{2}$ and $\dfrac{-1-\sqrt{-3}}{2}$.

We prove that

Solution :

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{-1-\sqrt{-3}}{2}, \quad \text{and} \quad \left(\frac{-1-\sqrt{-3}}{2}\right)^2=\frac{-1+\sqrt{-3}}{2}$$

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{1+(-3)-2\sqrt{-3}}{4} \qquad \left(\frac{-1-\sqrt{-3}}{2}\right)^2=\frac{-1+(-3)+2\sqrt{-3}}{4}$$

$$=\frac{-2-2\sqrt{-3}}{4} \qquad =\frac{-2+2\sqrt{-3}}{4}$$

$$=\frac{2\left(-1-\sqrt{-3}\right)}{4} \qquad =\frac{2\left(-1+\sqrt{-3}\right)}{4}$$

$$=\frac{-1-\sqrt{-3}}{2} \qquad =\frac{-1+\sqrt{-3}}{2}$$

Thus, each of the complex cube root of unity is the square of the other, that is,

$if\ \omega=\frac{-1+\sqrt{-3}}{2}, then\ \omega^2=\frac{-1-\sqrt{-3}}{2}$ and if $\omega=\frac{-1-\sqrt{-3}}{2}$ *then*

$\omega^2=\frac{-1+\sqrt{-3}}{2}$

(b) Prove that the product of three cube roots of unity is one.

Proof:

Solution :

Three cube roots of unity are

$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$

The product of cube roots of unity $=(1)\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)$

$$=\frac{(-1)^2-\left(\sqrt{-3}\right)^2}{4}=\frac{-1-(-3)}{4}=\frac{1+3}{4}=\frac{4}{4}=1$$

i.e., $(1)(\omega)(\omega^2)=1\ or\ \omega^3=1$

(c) Prove that each complex cube root of unity is reciprocal of the other.

Proof:

We know that $\omega^3 = 1 \Rightarrow \omega.\omega^2 = 1, so$

$$\omega = \frac{1}{\omega^2} \quad or \quad \omega^2 = \frac{1}{\omega}$$

Thus, each complex cube root of unity is reciprocal of the other.

(b) Prove that the sum of all the cube roots of unity is zero.

i.e., $1+\omega+\omega^2 = 0$

Proof:

The cube roots of unity are

$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$

$if\ \omega = \frac{-1+\sqrt{-3}}{2}, then\ \omega^2 = \frac{-1-\sqrt{-3}}{2}$

The sum o all the roots $= 1+\omega+\omega^2$

$$= 1 + \frac{-1+\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2}$$

$$= \frac{2-1+\sqrt{-3}-1-\sqrt{-3}}{2} = \frac{0}{2} = 0$$

Thus $1+\omega+\omega^2 = 0$

We can easily deduce the following results, that is

$(i)\quad 1+\omega^2 = -\omega \qquad (ii)\quad 1+\omega = -\omega^2 \qquad (iii)\quad \omega+\omega^2 = -1$

Exercise 2.2

1. Find the cube roots of -1,8, -27,64.

(i) The three cube roots of -1

Solution:

Let $x^3 = -1$

$(x)^3 + (1)^3 = 0$

$(x+1)(x^2 - x + 1) = 0$

Either $x + 1 = 0$ *or* $x^2 - x + 1 = 0$

$x = -1$ *Here* $a = 1, b = -1, c = 1$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{1 + \sqrt{-3}}{2} \quad or \quad x = \frac{1 - \sqrt{-3}}{2}$$

$$= -\left(\frac{-1 - \sqrt{-3}}{2}\right) \qquad = -\left(\frac{-1 + \sqrt{-3}}{2}\right)$$

$$= -\omega^2 \qquad = -\omega$$

Three cube roots of -1 are -1, $-\omega, -\omega^2$

(ii) The three cube roots of 8

Solution:

Let $x^3 = 8$

$x^3 - 8 = 0$

$(x)^3 - (2)^3 = 0$

$(x-2)(x^2 + 2x + 4) = 0$

$Either\ \ x-2=0 \qquad or \qquad x^2+2x+4=0$

$x=2 \qquad Here\ a=1, b=2, c=4$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(4)}}{2(1)}$$

$$x=\frac{-2\pm\sqrt{4-16}}{2}$$

$$x=\frac{-2\pm\sqrt{-12}}{2}$$

$$x=\frac{-2\pm2\sqrt{-3}}{2}$$

$$x=\frac{2\left(-1\pm i\sqrt{3}\right)}{2} \because i=\sqrt{-1}$$

$$x=2\left(\frac{-1+i\sqrt{3}}{2}\right) \ or \ x=2\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$=2\omega \qquad =2\omega^2$$

Three cube roots of 8 are $2, 2\omega, 2\omega^2$

(iii) The three cube roots of -27

Solution:

Let $x^3=-27$

$x^3+27=0$

$(x)^3+(3)^3=0$

$(x+3)(x^2-3x+9)=0$

$Either\ \ x+3=0 \qquad or \qquad x^2-3x+9=0$

$x = -3$ $\qquad$ $Here\ a = 1, b = -3, c = 9$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{3 \pm \sqrt{-27}}{2}$$

$$x = \frac{3 \pm 3\sqrt{-3}}{2}$$

$$x = \frac{3\left(1 \pm i\sqrt{3}\right)}{2} \because i = \sqrt{-1}$$

$$x = 3\left(\frac{1 + i\sqrt{3}}{2}\right) \ or \ x = 3\left(\frac{1 - i\sqrt{3}}{2}\right)$$

$$x = -3\left(\frac{-1 - i\sqrt{3}}{2}\right) \ or \ x = -3\left(\frac{-1 + i\sqrt{3}}{2}\right)$$

$$= -3\omega^2 \qquad\qquad = -3\omega$$

Three cube roots of -27 are $-3, -3\omega, -3\omega^2$

(iv) The three cube roots of 64

Solution:

Let $x^3 = 64$

$x^3 - 64 = 0$

$(x)^3 - (4)^3 = 0$

$(x - 4)\left(x^2 + 4x + 16\right) = 0$

$$\text{Either } x-4=0 \qquad or \qquad x^2+4x+16=0$$
$$x=4 \qquad\qquad Here\ a=1, b=4, c=16$$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-(4)\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$$
$$x=\frac{-4\pm\sqrt{16-64}}{2}$$
$$x=\frac{-4\pm\sqrt{-48}}{2}$$
$$x=\frac{-4\pm 4\sqrt{-3}}{2}$$
$$x=\frac{4\left(-1\pm i\sqrt{3}\right)}{2} \because i=\sqrt{-1}$$
$$x=4\left(\frac{-1+i\sqrt{3}}{2}\right) \quad or \quad x=4\left(\frac{-1-i\sqrt{3}}{2}\right)$$
$$=4\omega \qquad\qquad =4\omega^2$$

Three cube roots of 64 are $4, 4\omega, 4\omega^2$

2. Evaluate

(i) $\left(1-\omega-\omega^2\right)^7$

Solution:

$$\left(1-\omega-\omega^2\right)^7=\left[1-\left(\omega+\omega^2\right)\right]^7$$
$$=\left[1-(-1)\right]^7 \qquad \because \omega+\omega^2=-1$$
$$=(1+1)^7$$
$$=2^7=128$$

(ii) $\left(1-3\omega-3\omega^2\right)^5$

Solution:

$$\left(1-3\omega-3\omega^2\right)^5=\left[1-3\left(\omega+\omega^2\right)\right]^5$$
$$=\left[1-3(-1)\right]^5 \because \omega+\omega^2=-1$$
$$=(1+3)^5$$
$$=4^5=1024$$

(iii) $\left(9+4\omega+4\omega^2\right)^3$

Solution:

$$\left(9+4\omega+4\omega^2\right)^3=\left[9+4\left(\omega+\omega^2\right)\right]^3$$
$$=\left[9+4(-1)\right]^3 \because \omega+\omega^2=-1$$
$$=(9-4)^3$$
$$=5^3=125$$

(iv) $\left(2+2\omega-2\omega^2\right)\left(3-3\omega+3\omega^2\right)$

Solution:

$$\left(2+2\omega-2\omega^2\right)\left(3-3\omega+3\omega^2\right)$$
$$=\left[2(1+\omega)-2\omega^2\right]\left[3\left(1+\omega^2\right)-3\omega\right]$$
$$=\left[2\left(-\omega^2\right)-2\omega^2\right]\left[3(-\omega)-3\omega\right] \qquad \because 1+\omega=-\omega^2$$
$$=\left(-2\omega^2-2\omega^2\right)(-3\omega-3\omega)$$

$= \left(-4\omega^2\right)\left(-6\omega\right)$

$= \left(-4\right)\left(-6\right)\left(\omega^2.\omega\right)$

$= 24\omega^3$

$= 24\left(1\right) \qquad \because \omega^3 = 1$

$= 24$

(v) $\left(-1+\sqrt{-3}\right)^6 + \left(-1-\sqrt{-3}\right)^6$

Solution:

$\left(-1+\sqrt{-3}\right)^6 + \left(-1-\sqrt{-3}\right)^6$

$= \left(2\omega^6\right) + \left(2\omega^2\right)^6 \qquad \because \omega = \dfrac{-1+\sqrt{-3}}{2} \;\; and \;\; \omega^2 = \dfrac{-1-1\sqrt{-3}}{2}$

$= 2^6\left(\omega^6\right) + 2^6\left(\omega^{12}\right) \qquad 2\omega = -1+\sqrt{-3} \;\; and \;\; 2\omega^2 = -1-1\sqrt{-3}$

$= 2^6\left[\left(\omega^3\right)^2\right] + 2^6\left[\left(\omega^3\right)^4\right]$

$= 2^6\left[\left(1\right)^2\right] + 2^6\left[\left(1\right)^4\right] \qquad \because \omega^3 = 1$

$= 2^6\left[1+1\right]$

$= 2^6.2 = 2^{6+1} = 2^7$

$= 128$

(vi) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9 + \left(\dfrac{-1-\sqrt{-3}}{2}\right)^9$

Solution:

$\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9 + \left(\dfrac{-1-\sqrt{-3}}{2}\right)^9$

$= \omega^9 + \left(\omega^2\right)^9 \qquad \because \omega = \frac{-1+\sqrt{-3}}{2} \text{ and } \omega^2 = \frac{-1-1\sqrt{-3}}{2}$

$= \omega^9 + \omega^{18}$

$= \left(\omega^3\right)^3 + \left(\omega^3\right)^6$

$= (1)^3 + (1)^6 \qquad \because \omega^3 = 1$

$= 1 + 1 = 2$

(vii) $\omega^{37} + \omega^{38} - 5$

Solution:

$\omega^{37} + \omega^{38} - 5$

$= \omega^{37} + \omega^{38} - 5$

$= \omega^{36}.\omega + \omega^{36}.\omega^2 - 5$

$= \left(\omega^3\right)^{12}.\omega + \left(\omega^3\right)^{12}.\omega^2 - 5$

$= (1)^{12}.\omega + (1)^{12}.\omega^2 - 5 \qquad \because \omega^3 = 1$

$= \omega + \omega^2 - 5$

$= -1 - 5 \qquad \because \omega + \omega^2 = -1$

$= -6$

(viii) $\omega^{-13} + \omega^{-17}$

Solution:

$\omega^{-13} + \omega^{-17}$

$= \omega^{-13} + \omega^{-17}$

$= \omega^{-12-1} + \omega^{-15-2}$

$= \omega^{-12}.\omega^{-1} + \omega^{-15}.\omega^{-2}$

$= \left(\omega^3\right)^{-4}.\omega^{-1} + \left(\omega^3\right)^{-5}.\omega^{-2}$

$= (1)^{-4} . \omega^{-1} + (1)^{-5} . \omega^{-2} \qquad \because \omega^3 = 1$

$= \omega^{-1} + \omega^{-2}$

$= \frac{1}{\omega} + \frac{1}{\omega^2}$

$= \frac{\omega^2 + \omega}{\omega^3} \qquad \because \omega + \omega^2 = -1 \text{ and } \omega^3 = 1$

$= \frac{-1}{1}$

$= -1$

3. Prove that $x^3 + y^3 = (x+y)(x+\omega y)(x+\omega^2 y)$.

Solution:

$x^3 + y^3 = (x+y)(x+\omega y)(x+\omega^2 y)$

$$
\begin{aligned}
R.H.S &= (x+y)(x+\omega y)(x+\omega^2 y) \\
&= (x+y)\left[x(x+\omega^2 y)+\omega y(\omega+\omega^2 y)\right] \\
&= (x+y)\left[x^2+\omega^2 xy+\omega xy+\omega^3 y^2\right] \\
&= (x+y)\left[x^2+(\omega^2+\omega)xy+(1)y^2\right] \quad \because \omega^3 = 1 \\
&= (x+y)\left[x^2+(-1)xy+y^2\right] \qquad \because \omega^2+\omega = -1 \\
&= (x+y)(x^2-xy+y^2) \\
&= x^3+y^3 \\
&= L.H.S
\end{aligned}
$$

Hence Proved.

4. Prove that $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$.

Solution:

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$$

$$\begin{aligned}
R.H.S &= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)\\
&= (x+y+z)\left[x(x+\omega^2 y+\omega z)+\omega y(x+\omega^2 y+\omega z)+\omega^2 z(x+\omega^2 y+\omega z)\right]\\
&= (x+y+z)\left[x^2+\omega^2 xy+\omega xz+\omega xy+\omega^3 y^2+\omega^2 yz+\omega^2 xz+\omega^4 yz+\omega^3 z^2\right]\\
&= (x+y+z)\left[x^2+\omega^2 xy+\omega xz+\omega xy+\omega^3 y^2+\omega^2 yz+\omega^2 xz+\omega^3.\omega yz+\omega^3 z^2\right]\\
&= (x+y+z)\left[x^2+\omega^2 xy+\omega xy+\omega^2 yz+(1)\omega yz+\omega^2 xz+\omega xz+(1)y^2+(1)z^2\right] \qquad \because \omega^3=1\\
&= (x+y+z)\left[x^2+(\omega^2+\omega)xy+(\omega^2+\omega)yz+(\omega^2+\omega)xz+y^2+z^2\right]\\
&= (x+y+z)\left[x^2+(-1)xy+(-1)yz+(-1)xz+y^2+z^2\right] \qquad \because \omega^2+\omega=-1
\end{aligned}$$

$$\begin{aligned}
&= (x+y+z)\left[x^2+y^2+z^2-xy-yz-zx\right]\\
&= x^3+y^3+z^3-3xz\\
&= L.H.S
\end{aligned}$$

Hence Proved.

5. Prove that $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8).....2\text{n factors}=1$.

Solution:

$$\begin{aligned}
L.H.S &= (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8).....2\text{n factors}\\
&= (1+\omega)(1+\omega^2)(1+\omega^3.\omega)(1+\omega^2.\omega^6).....2\text{n factors}\\
&= (1+\omega)(1+\omega^2)(1+\omega^3.\omega)\left(1+\omega^2.(\omega^3)^2\right).....2\text{n factors}\\
&= (1+\omega)(1+\omega^2)(1+(1)\omega)\left(1+\omega^2(1)^2\right).....2\text{n factors} \because \omega^3=1\\
&= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2).....2\text{n factors}\\
&= (-\omega^2)(1+\omega^2)(-\omega^2)(1+\omega^2).....2\text{n factors} \qquad \because 1+\omega=-\omega^2\\
&= \left[(-\omega^2)(-\omega)\right]\left[(-\omega^2)(-\omega)\right].....\text{n factors} \qquad \because 1+\omega^2=-\omega
\end{aligned}$$

$= [\omega^3][\omega^3]$.....n factors

$= (1)(1)$.....n factors $\quad \because \omega^3 = 1$

$= (1)^n$

$= 1$

$= R.H.S$

Hence Proved.